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# Two-component Fermi systems:

## II. Superfluid coupled cluster theory

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In this second paper of a series the coupled cluster method (CCM) or exp($S$) formalism is applied to two-component Fermi superfluids using a Bardeen-Cooper-Schrieffer (BCS) ground state as a zeroth-order approximation. We concentrate on developing the formalism necessary for carrying out eventual numerical calculations on realistic superconducting systems. We do this by generalising the one-component formalism in an appropriate manner and by using the results in the first paper of this series, where we studied two-component Fermi fluids. We stress the previous successes of the CCM, both from the point of view of analytic and numerical results, and we further indicate its potential for studying superconductivity. We restrict ourselves here to a so-called ring plus single particle energy (RING+SPE) approximation for general potentials and show how it can be formulated as a set of four coupled, bilinear integral equations for the cluster-integrated amplitudes. These latter amplitudes are themselves derived from the four-point functions of the system which provide a measure of the two-particle/two-hole component in the true ground-state wavefunction with respect to the BCS model state. We indicate how to obtain possible analytic solutions.

## 1. Introduction

Theories dealing with two-component systems, i.e. those possessing two different types of particles, are of great theoretical interest as generalisations of one-component systems. Furthermore they are of considerable practical importance because such physical systems as real metals and electron-hole plasmas in photo-excited semiconductors, and such phenomena as superconductivity, can all be described most generally within a two-component picture.

Among many-body theories available for treating strongly correlated systems, the coupled cluster method (CCM) or exp($S$) formalism stands out as a particularly powerful and accurate tool [1] with numerous extremely successful earlier applications to one-component systems. Building on these demonstrably solid foundations, our aim here is to extend the CCM to two-component Fermi systems by developing the formalism for two-component fluid and superfluid coupled cluster theory.

The Fermi fluid coupled cluster theory of two-component systems has been extensively developed in a previous paper [2], henceforth referred to as I. In that paper, a comprehensive review of the one-component CCM is given, including details of its applications to closed-shell atomic nuclei [3–5], the electron gas [6–8], and examples from quantum chemistry [9–13]. The various CCM formalisms in use, namely the so-called ground state and excited state formalisms and the extended exp($S$) formalism of Arponen [14], are extensively discussed with special emphasis placed on the choice of starting wavefunctions (model states).

By their nature, the CCM equations must be truncated to be of use, and four such schemes which have

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been developed for this purpose are outlined in I: the SUB$n$ truncation [6, 8, 15, 16]; the $\chi_n$ or Bochum truncation used for systems interacting via hard-core potentials, such as closed-shell atomic nuclei [3–5]; the Offermann, Kümmel and Ey (OKE) approximation used in open-shell atomic nuclei [17–20]; and the super-SUB$n$ truncation scheme developed to tackle liquid $^4$He [21]. Special emphasis has been made on the choice of decomposition of the equations into a natural hierarchy in each case.

Having given a review of the method and the methodology, the extension of the CCM to two-component normal Fermi fluids is described in I, leaving two-component Fermi superfluids to be dealt with in the present paper, henceforth referred to as II. To treat Fermi fluids, the formalism associated with the choice of the so-called fluid state and excited state as a model state is developed, aiming to describe systems such as real metals and electron-hole plasmas in photo-excited semiconductors. Attention is focussed in the first place on the high-density limit: the random-phase approximation (RPA) in the ground state and its excited state analogue. In the former case the equations are particularly analysed with respect to their dependence on the dimensionless parameter formed by the ratio of the masses of the two species, and exact analytic solutions are obtained. For coulombic potentials ($V_{11} V_{22} = V_{12}^2$) it is shown that the exact analytic solution is unique and an expression for the correlation energy is obtained. For the more general case of non-coulombic potentials ($V_{11} V_{22} \neq V_{12}^2$) we simply indicate how to obtain a possible analytic solution. The excited state RPA-like treatment is also discussed.

As stated in I there is every reason to expect that these analytic results will provide the first step towards a full two-component calculation as accurate as that in the one-component case [5, 6, 8], where results for the correlation energy accurate to better than 1% over the entire metallic density range have been obtained for the electron gas.

It was also explained in I how in the fluid excited states formalism of Emrich [15, 16] one could obtain so-called "de-excited states", i.e. states with lower energy than the so-called "ground state". We described how this could arise when our particular choice of model state (namely, the filled Fermi sea $|\Phi_F\rangle$) was not necessarily the best choice for approximating the true ground state. The intimate connection of this phenomenon in the CCM with phase transitions in the corresponding physical system has been stressed elsewhere [23]. Intuitively, one may feel that the closer is the model state to the true ground state and, correspondingly therefore, the "smaller" is the correlation operator $S$, the better that choice of model state is likely to be at a given level of approximation [22]. Alternatively, we may intuit that if the model state shares the same underlying symmetries as the exact ground state, the better it is likely to be [23].

If the model state is not an eigenfunction of some operator $\Theta$ which commutes with the Hamiltonian, then it must be a linear superposition of such eigenfunctions. Often such a general wavefunction has components mimicking higher order correlations. A famous example arises in the theory of pair effects where one uses a wavefunction for which particle number is not conserved [24]. It can be shown that if we wish to mimic higher order correlations we are rather naturally led to consider breaking the particle number conservation [22]. The new model state $|\Phi_{\mathrm{BCS}}\rangle$ is built from the usual number-conserving filled Fermi sea by means of a Bogoliubov transformation [25] which explicitly introduces the violation of particle number conservation and corresponding new annihilation operators for which $|\Phi_{\mathrm{BCS}}\rangle$ is the vacuum. It can further be shown that $|\Phi_{\mathrm{BCS}}\rangle$ is the most general determinantal state not orthogonal to $|\Phi_F\rangle$ [22].

It is well known that although the screened coulomb potential of a degenerate (high density) electron gas is repulsive [26], there is also a virtual electron-electron interaction arising in real metals from the exchange of phonons associated with the crystal lattice. Furthermore, this interaction is attractive for electrons near the Fermi surface, thereby providing a physical basis for the attractive interparticle potential in Cooper's model [27] of superconductivity. This model can be regarded as a special case of the physical situation where we have two different types of particles and the repulsive potential between like particles is much weaker than the attractive potential between the unlike particles so that an effective attractive interaction results between the weakly interacting particles. To have BCS superconducting solutions requires just such an effective attractive interparticle potential. One way an attractive potential of this kind can arise is by the introduction of a second particle into the system.

We thus proceed to develop the CCM formalism for treating two-component Fermi superfluids by choosing $|\Phi_{\mathrm{BCS}}\rangle$ as our model state. Our aim is basically to provide a more complete description of superconductivity than is afforded by the standard one-component approach where one assumes the interparticle potential to be attractive from the outset [24, 25, 28]. In the ambit of superconductivity and the CCM, we are interested both in comparing the different model states formulated, and in investigating which approximations can yield superconducting solutions within a given formalism. Specific to the study of superconductivity, it is of considerable practical interest to inquire what materials can be superconductors and if so what is the size of their associated energy gap. The recent discoveries of superconductors at higher temperatures than previously found and the consequent search for materials which are superconducting at even higher temperatures makes this inquiry extremely important. It is our belief that the CCM formalism, following excellent numerical results in the one-component case for a $|\Phi_F\rangle$ model state [7], and a $|\Phi_{\mathrm{BCS}}\rangle$ model state [8], plus the availability of analytic results in the two-component case for Fermi fluids [2], provides a potentially formidable tool for studying superconductivity and dealing with strong phonon interactions. This has become particularly clear following recent work by Emrich [29], Emrich and Zabolitzky [8, 30, 31], and in particular the very recent suggestion by Emrich et al. [32, 33] of a specific experiment, based on their CCM work, designed to settle the question of whether BaLaCuO and YBaCuO are or are not phonon-induced superconductors.

In this particular paper we attempt to lay the basis for future work which will look at superconductivity in inhomogeneous systems. Consequently, this paper is the analogue of the first paper by Bishop and Lührmann [6] which looked at a particular approximation (the random phase approximation (RPA)) to the full two-body one-component CCM equation and established a basis for studying realistic systems [7] and extending the formalism [2, 8, 14, 21].

In Sect. 2 of this paper we derive the equations appropriate for treating two-component Fermi superfluids and provide a brief discussion of the resulting gap equations. In Sect. 3 we apply a so-called ring plus single-particle energy (RING+SPE) treatment in a SUB2 approximation to coulombic potentials ($V_{11}V_{22}=V_{12}^2$). This has been shown, *a posteriori*, to be an accurate approximation (viz., within 5% over the metallic density range) to the extremely accurate theoretical (SUB4) correlation energies for one-component systems [8], in the case of bare coulombic potentials. We thereby obtain a set of coupled nonlinear integral equations with generalised energy denominators, and furthermore we show how these may be simultaneously exactly linearised and decoupled. By replacing the general (self-consistent) energy denominators with their bare kinetic energy components, the RING+SPE equations for pure Coulombic potentials are shown to reduce to the ground state RPA equations [2], a result which may be expected from the particle-hole symmetry of the RPA equations. In Sect. 4 the same RING+SPE treatment is applied to non-coulombic potentials ($V_{11}V_{22}\neq V_{12}^2$). Section 5 concerns conclusions and future work.

## 2. Superconducting solutions in two-component homogeneous Fermi systems

Consider a fermion system with two species of particles, "1" and "2", with masses $m_1$ and $m_2$ respectively, and with Hamiltonian:

$$H=\sum_{i=1,2}\left(\sum_{k=1}^{N_i}T_i(x_k)+\tfrac{1}{2}\sum_{l=1}^{N_i}\sum_{k<l}V_{ii}(x_k,x_l)\right)+\sum_{k=1}^{N_1}\sum_{l=1}^{N_2}V_{12}(x_k,x_l) \tag{2.1}$$

where the index "$i$" refers to a particle of species "$i$", $N_i$ to the number of particles of species "$i$" with $N_1+N_2=N$, and $T$ and $V$ are the kinetic and potential terms respectively.

The Fermi sea model state in the normal CCM is given by:

$$|\Phi_F\rangle=\prod_{\mathbf{k}_1}^{k_{F_1}}\prod_{\mathbf{k}_2}^{k_{F_2}}c^\dagger_{\mathbf{k}_1 s_1}c^\dagger_{\mathbf{k}_2 s_2}|0\rangle;\qquad s_i=\uparrow,\downarrow \tag{2.2}$$

where $c^\dagger_{\mathbf{k}_i s_i}$ are fermion creation operators for the particle of species "$i$" with momentum $\mathbf{k}_i$ and spin projection $s_i$, and $|0\rangle$ is the bare vacuum; i.e. $c_{\mathbf{k}s}|0\rangle=0$ for all $\mathbf{k}$. Equation (2.2) implies:

$$c^\dagger_{|\mathbf{k}_i|\leq k_{F_i}s_i}|\Phi_F\rangle=c_{|\mathbf{k}_i|>k_{F_i}s_i}|\Phi_F\rangle=0\qquad i=1,2. \tag{2.3}$$

We now explicitly break the conservation of particle number to choose a new model state. This we do by defining a Bogoliubov transformation [25]:

$$|\Phi_{\mathrm{BCS}}\rangle=\prod_{\mathbf{k}_1}^{k_{F_1}}\prod_{\mathbf{k}_2}^{k_{F_2}}\alpha_{\mathbf{k}_1}\beta_{-\mathbf{k}_1}\alpha_{\mathbf{k}_2}\beta_{-\mathbf{k}_2}|0\rangle \tag{2.4}$$

with

$$\begin{aligned}\alpha_{\mathbf{k}_i}&=u_{k_i}c_{\mathbf{k}_i\uparrow}-v_{k_i}c^\dagger_{-\mathbf{k}_i\downarrow}\\ \beta_{-\mathbf{k}_i}&=u_{k_i}c_{-\mathbf{k}_i\downarrow}+v_{k_i}c^\dagger_{\mathbf{k}_i\uparrow},\end{aligned} \tag{2.5}$$

$$\alpha_{\mathbf{k}_i}|\Phi_{\mathrm{BCS}}\rangle=\beta_{-\mathbf{k}_i}|\Phi_{\mathrm{BCS}}\rangle=0. \tag{2.6}$$

The requirement that the Bogoliubov transformation (2.5) be canonical implies $u_{k_i}^2+v_{k_i}^2=1$ [26]. The exact ground state of the fully interacting system is expressed by means of the $\exp(S)$ ansatz:

$$|\Psi\rangle=\exp(S)|\Phi_{\mathrm{BCS}}\rangle \tag{2.7a}$$

$$=\exp\left(\sum_{n=2}^{\infty}S_n\right)|\Phi_{\mathrm{BCS}}\rangle, \tag{2.7b}$$

$$S_n = \sum_{n_1=0}^{n} [(n_1)!(n_2)!]^{-2}\, \delta_{n_1+n_2,n} \cdot \sum_{\substack{\mathbf{k}^1 \ldots \mathbf{k}^n \\ \mathbf{l}^1 \ldots \mathbf{l}^n}} \langle \mathbf{k}^1 \ldots \mathbf{k}^n |\, S_n \,| \mathbf{l}^1 \ldots \mathbf{l}^n \rangle_A \cdot \beta^{\dagger}_{-\mathbf{k}^1} \ldots \beta^{\dagger}_{-\mathbf{k}^n}\, \alpha^{\dagger}_{\mathbf{l}^n} \ldots \alpha^{\dagger}_{\mathbf{l}^1} \tag{2.7c}$$

where we have excluded the $\langle\, | S_1 |\, \rangle$ term for spatially homogeneous systems, as conservation of momentum requires that it be zero. The suffix $A$ on the ket state in Eq. (2.7) indicates a complete antisymmetrization with respect to both sets of arguments, referring to each species of particle, exactly as in I. In Eq. (2.7) we have three different types of matrix elements $\langle\, | S_2 |\, \rangle$: namely for species "1"−"1", "2"−"2", and "1"−"2". In Eq. (2.7) the symbols $\mathbf{k}^1, \mathbf{k}^2, \ldots, \mathbf{k}^n$ refer to bra states, and the symbols $\mathbf{l}^1, \mathbf{l}^2, \ldots, \mathbf{l}^n$ refer to ket states, with $n = n_1 + n_2$.

The Schrödinger equation for the ground state $H |\Psi_g\rangle = E_g |\Psi_g\rangle$ may then be decomposed into a set of coupled equations. The ground state energy and the correlation energy are obtained in the standard CCM manner [8, 34]. At the SUB2 level of approximation largely considered in the remainder of this paper, all correlation operators $S_n$ with $n > 2$ are set to zero.

We now introduce the operator of the thermodynamic potential of the two-component fermion system [26] to enable us to treat assemblies with an indefinite number of particles:

$$\begin{aligned} K = &\sum_{i=1,2} (H_{ii} - \mu_i N_i) + H_{12} \\ = &\sum_{i=1,2} \Bigg\{ \sum_{\mathbf{k}_i s_i} c^{\dagger}_{\mathbf{k}_i s_i} c_{\mathbf{k}_i s_i} \left[ \frac{\hbar^2 k_i^2}{2 m_i} - \mu_i \right] \\ &+ \tfrac{1}{2} \sum_{\substack{\mathbf{k}_i \mathbf{k}'_i \mathbf{q} \\ s_i s'_i}} \langle \mathbf{k}_i + \mathbf{q}, s_i; \mathbf{k}'_i - \mathbf{q}, s'_i |\, V_{ii} \,| \mathbf{k}_i, s_i; \mathbf{k}'_i, s'_i \rangle \\ &\cdot c^{\dagger}_{\mathbf{k}_i+\mathbf{q}, s_i} c^{\dagger}_{\mathbf{k}'_i-\mathbf{q}, s'_i} c_{\mathbf{k}'_i s'_i} c_{\mathbf{k}_i s_i} \Bigg\} \\ &+ \sum_{\substack{\mathbf{k}_1 \mathbf{k}_2 \mathbf{q} \\ s_1 s_2}} \langle \mathbf{k}_1 + \mathbf{q}, s_1; \mathbf{k}_2 - \mathbf{q}, s_2 |\, V_{12} \,| \mathbf{k}_1, s_1; \mathbf{k}_2, s_2 \rangle \\ &\cdot c^{\dagger}_{\mathbf{k}_1+\mathbf{q}, s_1} c^{\dagger}_{\mathbf{k}_2-\mathbf{q}, s_2} c_{\mathbf{k}_2 s_2} c_{\mathbf{k}_1 s_1} \end{aligned} \tag{2.8}$$

where we assume overall momentum conservation, and where $\mu_i$ is the chemical potential for particle species $i$.

The thermodynamic potential can be put into normal-ordered form with respect to the transformed creation and destruction operators, and hence rearranged as a sum of zeroth-order and one-body terms, plus the normal ordering of the two-body potential terms [26]. The one-body terms, i.e. those terms bilinear in the new operators $\alpha$, $\alpha^{\dagger}$, $\beta$, $\beta^{\dagger}$, are then extracted from the thermodynamic potential after the canonical Bogoliubov transformation (2.5) and a normal ordering of the new operators is made [26, 34].

The resulting one-body terms are, for particles of type "1":

$$\begin{aligned} H_1^{(1)} &= \sum_{\mathbf{k}_1} (\alpha^{\dagger}_{\mathbf{k}_1} \alpha_{\mathbf{k}_1} + \beta^{\dagger}_{-\mathbf{k}_1} \beta_{-\mathbf{k}_1}) \cdot [a_{k_1} (\xi^{V_{11}}_{k_1} + \xi^{V_{12}}_{k_1}) + b_{k_1} \Delta^{V_{11}}_{k_1}] \\ H_2^{(1)} &= \sum_{\mathbf{k}_1} (\alpha^{\dagger}_{\mathbf{k}_1} \beta^{\dagger}_{-\mathbf{k}_1} + \beta_{-\mathbf{k}_1} \alpha_{\mathbf{k}_1}) \cdot [b_{k_1} (\xi^{V_{11}}_{k_1} + \xi^{V_{12}}_{k_1}) - a_{k_1} \Delta^{V_{11}}_{k_1}] \end{aligned} \tag{2.9}$$

and for particles of type "2":

$$\begin{aligned} H_1^{(2)} &= \sum_{\mathbf{k}_2} (\alpha^{\dagger}_{\mathbf{k}_2} \alpha_{\mathbf{k}_2} + \beta^{\dagger}_{-\mathbf{k}_2} \beta_{-\mathbf{k}_2}) \cdot [a_{k_2} (\xi^{V_{22}}_{k_2} + \xi^{V_{21}}_{k_2}) + b_{k_2} \Delta^{V_{22}}_{k_2}] \\ H_2^{(2)} &= \sum_{\mathbf{k}_2} (\alpha^{\dagger}_{\mathbf{k}_2} \beta^{\dagger}_{-\mathbf{k}_2} + \beta_{-\mathbf{k}_2} \alpha_{\mathbf{k}_2}) \cdot [b_{k_2} (\xi^{V_{22}}_{k_2} + \xi^{V_{21}}_{k_2}) - a_{k_2} \Delta^{V_{22}}_{k_2}] \end{aligned} \tag{2.10}$$

where $H_1$ refers to the number conserving terms and $H_2$ to the number non-conserving terms.

In Eqs. (2.9–10) we have introduced the notation $(i, j = 1, 2;\ i \neq j)$:

$$\begin{aligned} &a_{k_i} \equiv u^2_{k_i} - v^2_{k_i}; \qquad b_{k_i} \equiv 2 u_{k_i} v_{k_i}; \qquad T_{k_i} \equiv \hbar^2 k_i^2 / 2 m_i \\ &\xi^{V_{ii}}_{k_i} \equiv T_{k_i} - \mu_i + \sum_{\mathbf{k}'_i} \langle \mathbf{k}_i \mathbf{k}'_i |\, \bar{V}_{ii} \,| \mathbf{k}_i \mathbf{k}'_i \rangle\, v^2_{k'_i} \\ &\xi^{V_{ij}}_{k_i} \equiv \sum_{\mathbf{k}_j} \langle \mathbf{k}_i \mathbf{k}_j |\, \bar{V}_{ij} \,| \mathbf{k}_i \mathbf{k}_j \rangle\, v^2_{k_j}; \\ &\Delta^{V_{ii}}_{k_i} \equiv -\sum_{\mathbf{k}'_i} \langle \mathbf{k}_i - \mathbf{k}'_i |\, V_{ii} \,| \mathbf{k}_i - \mathbf{k}'_i \rangle\, \tfrac{1}{2} b_{k'_i} \\ &\langle \mathbf{k}_i \mathbf{k}'_i |\, \bar{V}_{ii} \,| \mathbf{k}_i \mathbf{k}'_i \rangle \\ &\equiv \langle \mathbf{k}_i \mathbf{k}'_i |\, V_{ii} \,| \mathbf{k}_i \mathbf{k}'_i \rangle + \langle \mathbf{k}_i - \mathbf{k}'_i |\, V_{ii} \,| \mathbf{k}_i - \mathbf{k}'_i \rangle \\ &\quad - \langle \mathbf{k}_i \mathbf{k}'_i |\, V_{ii} \,| \mathbf{k}'_i \mathbf{k}_i \rangle \\ &\langle \mathbf{k}_i \mathbf{k}_j |\, \bar{V}_{ij} \,| \mathbf{k}_i \mathbf{k}_j \rangle \\ &\equiv \langle \mathbf{k}_i \mathbf{k}_j |\, V_{ij} \,| \mathbf{k}_i \mathbf{k}_j \rangle + \langle \mathbf{k}_i - \mathbf{k}_j |\, V_{ij} \,| \mathbf{k}_i - \mathbf{k}_j \rangle. \end{aligned} \tag{2.11}$$

Following Fetter and Walecka [26] we require that the number non-conserving terms vanish. This requires:

$$b_{k_i} [\xi^{V_{ii}}_{k_i} + \xi^{V_{ij}}_{k_i}] = a_{k_i} \Delta^{V_{ii}}_{k_i} \qquad (i, j = 1, 2;\ i \neq j). \tag{2.12}$$

The resulting gap equations always have zero gap solutions and may or may not have non-zero gap solutions. If they do not have non-zero gap solutions the system at this level of starting approximation becomes a two-component Fermi fluid and is described by the formalism of I. We now assume that non-zero gap solutions to Eq. (2.10) exist for the purposes of the remainder of this paper.

From Eqs. (2.9–12) we see that the term $V_{12}$ ($\mathbf{q}=0$) plays an important role. This is the only contribution from $V_{12}$ to the one-body terms. If $V_{12}(0)=0$ we see that the equations for the one-body terms (2.9–10) reduce to two one-component equations, one for each type of particle. For repulsive potentials $V_{ii}$ ($i=1, 2$) we see that, provided we require the energy be bounded from below [26], the only possible solutions are $\Delta_{k_i}^{V_{ii}}=0$ ($i=1, 2$), i.e. we have no superconducting model solutions. Clearly, a system with standard bare coulombic potentials cannot support such superconducting solutions.

For potentials $V_{12}(0)\neq 0$ we can have the possibility of a non-zero model gap function, and hence also superconducting model solutions if:

$$b_{k_i}=-\Delta_{k_i}^{V_{ii}}/E_{k_i}^{V_{ii}}; \qquad a_{k_i}=-[\xi_{k_i}^{V_{ii}}+\xi_{k_i}^{V_{ij}}]/E_{k_i}^{V_{ii}}$$

$$E_{k_i}^{V_{ii}}\equiv([\Delta_{k_i}^{V_{ii}}]^2+[\xi_{k_i}^{V_{ii}}+\xi_{k_i}^{V_{ij}}]^2)^{1/2} \qquad i, j=1, 2;\ i\neq j. \tag{2.13}$$

This zeroth-order result suggests that the system has become unstable and a change of phase from fluid to superfluid is taking place.

Physically, we can interpret this instability as being due to the potential between unlike particles being much stronger than either $V_{11}$ or $V_{22}$, with the result that particles of type "1" or "2" experience an effective attractive interaction. For this to be the case we require, from (2.11) and (2.13) that $(\xi_{k_i}^{V_{ii}}+\xi_{k_i}^{V_{ij}})<0$.

The above physical situation is associated with $V_{12}(\mathbf{q}=0)\neq 0$, i.e. the long range limit of the interspecies interparticle potential. This indicates that a necessary condition for particles of species "$i$" to experience an effective attractive interaction via type "$j$" ($j\neq i$) particles is that each type "$i$" particle sees all type "$j$" particles.

Equation (2.12) suggests that there might be two-component systems which support two gaps, one for each type of particle. It is possible that this may be of relevance towards explaining high-$T_c$ superconductors. However, it is clear that further work needs to be done to include the effect of correlations before firm conclusions can be drawn on this matter. One of the purposes of the remainder of this paper is to begin this process.

The fluid ground state analysis performed in Paper I [2] was only valid for potentials without a $\mathbf{q}=0$ component. Clearly, the presence of the $\mathbf{q}=0$ component is intimately connected with the presence of superconducting solutions and we proceed to use this to analyse the case $\mathbf{q}=0$ for the fluid ground state. This we do by taking $\lim_{q\to 0}\langle\, | S_2 |\, \rangle$ in Eq. (3.18) of Paper I [2]. By requiring the system to have normal solutions (gap $\Delta=0$) we see we must have $V_{11}(0)=V_{22}(0)=V_{12}(0)=0$ [34] i.e. the fluid ground state cannot yield superconducting solutions in the random phase approximation (RPA). There remains the possibility of these solutions when one goes beyond RPA; however, if they exist they remain to be found. The BCS CCM formalism, superior to the fluid CCM formalism from the point of view of numerical accuracy [8], is, not surprisingly, also superior from the point of view of obtaining superconducting solutions.

We now proceed in the remainder of this work to begin to employ the CCM to pass beyond the BCS starting approximation, and to include the full effects of the remaining correlations in the system. This process is begun in the next section by an approximate treatment, which has already been shown to be extremely accurate in the very demainding application to such a strongly-correlated system as the electron gas in the metallic density regime.

## 3. RING + SPE treatment of two-body correlations in SUB2 approximation for coulombic potentials

In this section we investigate the case where the potentials in question are Coulomb-like viz. $V_{11}V_{22}=V_{12}^2$. (N.B. Bare coulombic potentials are a special case of this, so the analysis applies to potentials other than bare coulombic). We first write down the full two-component two-body equation.

The exact two-component CCM equations of interest can readily be obtained from their one-component analogues by introducing the appropriate Hamiltonian terms corresponding to a second particle, viz. $T_2$, $V_{22}$ and $V_{12}$, and modifying the equations in an appropriate manner [2]. The full two-component two-body equation for an overall neutral system of spin-1/2 particles consisting of two species $i=1, 2$ with masses $m_i$ and charges $e_i$ is given by [2, 8, 34]:

$$T_{\mathrm{RING}}^S+T_{\mathrm{RINGEX}}^S+T_{\mathrm{SPE}}^S+T_{\mathrm{PH}}^S+T_{\mathrm{SLAD}}^S$$
$$+T_{\mathrm{CS3}}^S+T_{\mathrm{CS4}}^S+T_{\mathrm{EEX}}^S+T_{\mathrm{QLAD}}^S=0$$

$$T_{\mathrm{RING}}^Z+T_{\mathrm{SPE}}^Z+T_{\mathrm{PH}}^Z+T_{\mathrm{SLAD}}^Z$$
$$+T_{\mathrm{CS3}}^Z+T_{\mathrm{CS4}}^Z+T_{\mathrm{EEX}}^Z+T_{\mathrm{QLAD}}^Z=0 \tag{3.1}$$

where the superscripts $X$, $Y$, $Z$ refer to the three four-point functions $X_2$, $Y_2$, $Z_2$ which provide a measure of the two-particle/two-hole component of the true ground state wavefunction, and are defined to be:

$$X_{2;\mathbf{k}_1'\mathbf{l}_1'}^{\mathbf{k}_1\mathbf{l}_1} \equiv \langle \mathbf{k}_1, s_1; \mathbf{l}_1, s_1' | S_2 | \mathbf{k}_1', s_1; \mathbf{l}_1', s_1' \rangle_A$$

$$Y_{2;\mathbf{k}_2'\mathbf{l}_2'}^{\mathbf{k}_2\mathbf{l}_2} \equiv \langle \mathbf{k}_2, s_2; \mathbf{l}_2, s_2' | S_2 | \mathbf{k}_2', s_2; \mathbf{l}_2', s_2' \rangle_A$$

$$Z_{2;\mathbf{k}_1'\mathbf{l}_2'}^{\mathbf{k}_1\mathbf{l}_2} \equiv \langle \mathbf{k}_1, s_1; \mathbf{l}_2, s_2 | S_2 | \mathbf{k}_1', s_1; \mathbf{l}_2', s_2 \rangle. \tag{3.2}$$

$X_2$ and $Y_2$ thus refer to the exact two-particle/two-hole amplitudes for particles of type "1" alone and type "2" alone respectively; while $Z_2$ is the comparable mixed function for two (linked) particle-hole excitations, with one for each of types "1" and "2" together. These four-point functions have been defined in Eq. (3.13) of Bishop and Lahoz [2].

The subscripts in the terms in Eq. (3.1) indicate that the terms have been grouped in a manner analogous to that in the one-component case. In the description that follows the equation numbers refer to Emrich and Zabolitzky [8]. $T_{\mathrm{RING}}^S$ is the ring diagrams term (cf. Eq. (25)); $T_{\mathrm{RINGEX}}^S$ is the exchange term corresponding to the ring term (cf. Eq. (26)); $T_{\mathrm{SPE}}^S$ is the self-consistent single particle energy term (cf. Eq. (27)); $T_{\mathrm{PH}}^S$ is the particle-hole term (cf. Eq. (29)); $T_{\mathrm{SLAD}}^S$ is the spin-ladder term, which includes a term corresponding to the scattering of spin-parallel electrons and a term corresponding to the scattering of spin-antiparallel electrons (cf. Eq. (30)); $T_{\mathrm{CS3}}^S$ is the coupling term involving the six-point function $S_3$ which provides a measure of the three-particle/three-hole component of the true ground state wave function (cf. Eq. (31)); $T_{\mathrm{CS4}}^S$ is the coupling term involving the eight-point function $S_4$ (cf. Eq. (32)); $T_{\mathrm{EEX}}^S$ is the extra exchange term corresponding to the ring term (cf. Eq. (33)); $T_{\mathrm{QLAD}}^S$ is the quadratic ladder term (cf. Eq. (34)). The various $T_i^Z$ terms are precisely the mixed "1" – "2" analogues of the above single-species terms, except that $T_{\mathrm{RINGEX}}^Z$ now does not appear, since such a term arises only from the exchange of like particles.

The ring plus single particle energy (RING + SPE) approximation to Eq. (3.1) is:

$$T_{\mathrm{RING}}^S + T_{\mathrm{SPE}}^S = 0; \qquad S = \{X, Y, Z\} \tag{3.3}$$

where, for a non-diffuse Fermi surface, the RING terms correspond to the random phase approximation (RPA) terms in I, and the SPE (single particle energy) terms correspond to the kinetic energy, complete hole potential and complete particle potential terms as given in I and by [34].

The particles $i = 1, 2$ of our system interact via the following pair-wise spin-independent local potentials: $V_{ii}$ between species "$i$" – "$i$" and $V_{12}$ between species "1" – "2". These potentials are written in momentum space as follows:

$$\langle \mathbf{k}_i, s_i; \mathbf{l}_i, s_i' | V_{ii} | \mathbf{k}_i', s_i''; \mathbf{l}_i', s_i''' \rangle$$
$$= V_{ii}(\mathbf{k}_i - \mathbf{k}_i')\, \delta_{\mathbf{k}_i + \mathbf{l}_i, \mathbf{k}_i' + \mathbf{l}_i'}\, \delta_{s_i s_i''}\, \delta_{s_i' s_i'''},$$

$$\langle \mathbf{k}_1, s_1; \mathbf{l}_2, s_2 | V_{12} | \mathbf{k}_1', s_1'; \mathbf{l}_2', s_2' \rangle$$
$$= V_{12}(\mathbf{k}_1 - \mathbf{k}_1')\, \delta_{\mathbf{k}_1 + \mathbf{l}_2, \mathbf{k}_1' + \mathbf{l}_2'}\, \delta_{s_1 s_1'}\, \delta_{s_2 s_2'}.$$

The explicit form of Eq. (3.3) for $S = X$ in the SUB2 approximation, i.e. where we put $S_n = 0$ for $n > 2$ and, where we assume momentum conservation, is:

$$\begin{aligned}
&T_{\mathrm{RING}}^X(\mathbf{k}_1, \mathbf{k}_1', \mathbf{k}_1'', \mathbf{k}_1''') \\
&= V_{11}(q)\Big[b_{k_1 k_1'}\, b_{k_1'' k_1'''} + b_{k_1 k_1'} \sum_{\mathbf{l}_1 \mathbf{l}_1'} b_{l_1 l_1'}\, X_{2;\mathbf{l}_1'\mathbf{k}_1'''}^{\mathbf{l}_1\mathbf{k}_1''} \\
&\quad + b_{k_1'' k_1'''} \sum_{\mathbf{l}_1 \mathbf{l}_1'} b_{l_1 l_1'}\, X_{2;\mathbf{k}_1'\mathbf{l}_1'}^{\mathbf{k}_1\mathbf{l}_1} \\
&\quad + \sum_{\mathbf{l}_1 \mathbf{l}_1'} \sum_{\mathbf{l}_1'' \mathbf{l}_1'''} b_{l_1 l_1'}\, b_{l_1'' l_1'''}\, X_{2;\mathbf{k}_1'\mathbf{l}_1'}^{\mathbf{k}_1\mathbf{l}_1}\, X_{2;\mathbf{l}_1'''\mathbf{k}_1'''}^{\mathbf{l}_1''\mathbf{k}_1''}\Big] \\
&\quad + V_{12}(q)\Big[b_{k_1 k_1'} \sum_{\mathbf{l}_2 \mathbf{l}_2'} b_{l_2 l_2'}\, Z_{2;\mathbf{l}_2'\mathbf{k}_1'''}^{\mathbf{l}_2\mathbf{k}_1''} \\
&\quad + b_{k_1'' k_1'''} \sum_{\mathbf{l}_2 \mathbf{l}_2'} b_{l_2 l_2'}\, Z_{2;\mathbf{k}_1'\mathbf{l}_2'}^{\mathbf{k}_1\mathbf{l}_2}\Big] \\
&\quad + \sum_{\mathbf{l}_1 \mathbf{l}_1'} \sum_{\mathbf{l}_2 \mathbf{l}_2'} b_{l_1 l_1'}\, b_{l_2 l_2'} \\
&\quad \cdot \Big[X_{2;\mathbf{k}_1'\mathbf{l}_1'}^{\mathbf{k}_1\mathbf{l}_1}\, Z_{2;\mathbf{l}_2'\mathbf{k}_1'''}^{\mathbf{l}_2\mathbf{k}_1''} + Z_{2;\mathbf{k}_1'\mathbf{l}_2'}^{\mathbf{k}_1\mathbf{l}_2}\, X_{2;\mathbf{l}_1'\mathbf{k}_1'''}^{\mathbf{l}_1\mathbf{k}_1''}\Big] \\
&\quad + V_{22}(q) \sum_{\mathbf{l}_2 \mathbf{l}_2'} \sum_{\mathbf{l}_2'' \mathbf{l}_2'''} b_{l_2 l_2'}\, b_{l_2'' l_2'''}\, Z_{2;\mathbf{k}_1'\mathbf{l}_2'}^{\mathbf{k}_1\mathbf{l}_2}\, Z_{2;\mathbf{l}_2'''\mathbf{k}_1'''}^{\mathbf{l}_2''\mathbf{k}_1''};
\end{aligned}$$

$$q \equiv |\mathbf{k}_i - \mathbf{k}_i'|; \qquad i = 1, 2, \tag{3.4}$$

$$\begin{aligned}
T_{\mathrm{SPE}}^X(\mathbf{k}_1, \mathbf{k}_1', \mathbf{k}_1'', \mathbf{k}_1''') = X_{2;\mathbf{k}_1'\mathbf{k}_1'''}^{\mathbf{k}_1\mathbf{k}_1''}\big[&E_{\mathrm{SUB2}}(k_1) + E_{\mathrm{SUB2}}(k_1') \\
&+ E_{\mathrm{SUB2}}(k_1'') + E_{\mathrm{SUB2}}(k_1''')\big],
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
E_{\mathrm{SUB2}}(k_1) = a_{k_1}[\varepsilon_a^{\mathrm{HF}}(k_1) - \mu_1] &+ b_{k_1}\, \varepsilon_b^{\mathrm{HF}}(k_1) \\
&+ \sum_{\mathbf{l}_1 \mathbf{l}_1' \mathbf{l}_1''} b_{k_1 l_1''}\, b_{l_1 l_1'}\, V_{11}(|\mathbf{k}_1 - \mathbf{l}_1''|)\, X_{2;\mathbf{l}_1''\mathbf{l}_1'}^{\mathbf{k}_1\mathbf{l}_1} \\
&+ \sum_{\mathbf{l}_2 \mathbf{l}_2' \mathbf{l}_1''} b_{k_1 l_1''}\, b_{l_2 l_2'}\, V_{12}(|\mathbf{k}_1 - \mathbf{l}_1''|)\, Z_{2;\mathbf{l}_1''\mathbf{l}_2'}^{\mathbf{k}_1\mathbf{l}_2}
\end{aligned}$$

$$\varepsilon_a^{\mathrm{HF}}(k_1) = T_{k_1} - \sum_{\mathbf{l}_1} V_{11}(|\mathbf{k}_1 - \mathbf{l}_1|)\, v_{l_1}^2$$

$$\varepsilon_b^{\mathrm{HF}}(k_1) = -\sum_{\mathbf{l}_1} V_{11}(|\mathbf{k}_1 - \mathbf{l}_1|)\, \tfrac{1}{2} b_{l_1}$$

$$T_{k_1} = \hbar^2 k_1^2 / 2 m_1 \tag{3.6}$$

where $b_{kk'} = u_k v_{k'} + u_{k'} v_k$, and where the quantities $u_k$, $v_k$ have been introduced in Eqs. (2.5–6), and $a_k$ and $b_k$ in Eq. (2.11). The equations for $S = Y$ can be derived from those for $S=X$ by changing particles of type "1" to type "2" and vice versa, and modifying the potentials and four-point functions accordingly. The equations for $S=Z$ are similarly obtained. The full equations can be found in [34]. As in the one-component case [8], we can associate diagrams with Eq. (3.1) [34] and write down the correlation energy for general potentials:

$$\begin{aligned} E_c = E_g - E_{\mathrm{HF}} &= \langle \Phi_{\mathrm{BCS}} | \left[ V_{11} + V_{22} + V_{12} \right] S_2 | \Phi_{\mathrm{BCS}} \rangle \\ &= \sum_{i=1,2} \Big[ \sum_{\mathbf{k}_i \mathbf{k}_i'} \sum_{\mathbf{l}_i \mathbf{l}_i'} V_{ii}(|\mathbf{k}_i - \mathbf{k}_i'|)\, S^{\mathbf{k}_i \mathbf{l}_i}_{2;\mathbf{k}_i' \mathbf{l}_i'} \Big] \\ &\quad + \sum_{\mathbf{k}_1 \mathbf{k}_1'} \sum_{\mathbf{l}_2 \mathbf{l}_2'} V_{12}(|\mathbf{k}_1 - \mathbf{k}_1'|)\, Z^{\mathbf{k}_1 \mathbf{l}_2}_{2;\mathbf{k}_1' \mathbf{l}_2'} \end{aligned} \tag{3.7}$$

where $E_{\mathrm{HF}}$ is the Hartree-Fock energy (cf. Fetter and Walecka [26]).

To solve Eq. (3.3) we introduce the two-component screening functions $X^b$, $Y^b$, $Z^b$, $\hat{Z}^b$ defined by the equations:

$$\sum_{\mathbf{l}_1 \mathbf{l}_1'} b_{l_1 l_1'}\, X^{\mathbf{k}_1 \mathbf{l}_1}_{2;\mathbf{k}_1' \mathbf{l}_1'} = b_{k_1 k_1'} \left[ -1 + q\, X^b(\mathbf{k}_1, \mathbf{k}_1') \right], \tag{3.8}$$

$$\sum_{\mathbf{l}_2 \mathbf{l}_2'} b_{l_2 l_2'}\, Y^{\mathbf{k}_2 \mathbf{l}_2}_{2;\mathbf{k}_2' \mathbf{l}_2'} = b_{k_2 k_2'} \left[ -1 + q\, Y^b(\mathbf{k}_2, \mathbf{k}_2') \right], \tag{3.9}$$

$$\sum_{\mathbf{l}_2 \mathbf{l}_2'} b_{l_2 l_2'}\, Z^{\mathbf{k}_1 \mathbf{l}_2}_{2;\mathbf{k}_1' \mathbf{l}_2'} = b_{k_1 k_1'}\, q\, Z^b(\mathbf{k}_1, \mathbf{k}_1'), \tag{3.10}$$

$$\sum_{\mathbf{l}_1 \mathbf{l}_1'} b_{l_1 l_1'}\, Z^{\mathbf{l}_1 \mathbf{k}_2}_{2;\mathbf{l}_1' \mathbf{k}_2'} = b_{k_2 k_2'}\, q\, \hat{Z}^b(\mathbf{k}_2, \mathbf{k}_2'). \tag{3.11}$$

We now concentrate on Coulomb-like potentials, i.e. $V_{11} V_{22} = V_{12}^2$. Substituting Eqs. (3.8–11) in Eq. (3.3) [34] and converting the resulting equations to integral equations by changing the sums to integrals using the standard relation $\sum_{\mathbf{k}_i} \to \Omega/(2\pi)^3 \int \mathrm{d}\mathbf{k}_i$, yields the coupled, non-linear integral equations:

$$\begin{aligned} &q\, G_1^b(\mathbf{k}_1, \mathbf{k}_1') \\ &= V_{11}(q) - \frac{q\, G_1^b(\mathbf{k}_1, \mathbf{k}_1')}{2\pi \tilde{\beta}_1} \left\{ \int \mathrm{d}\mathbf{l}_1' \frac{b^2_{l_1 l_1'}\, q\, G_1^b(\mathbf{l}_1, \mathbf{l}_1')}{\bar{E}(\mathbf{l}_1'; \mathbf{q}) + \bar{E}(\mathbf{k}_1'; \mathbf{q})} \right. \\ &\quad \left. + \int \mathrm{d}\mathbf{l}_2' \frac{b^2_{l_2 l_2'}\, q\, H_1^b(\mathbf{l}_2, \mathbf{l}_2')}{\bar{E}(\mathbf{l}_2'; \mathbf{q}) + \bar{E}(\mathbf{k}_1'; \mathbf{q})} \right\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &q\, H_1^b(\mathbf{k}_2, \mathbf{k}_2') \\ &= V_{22}(q) - \frac{q\, H_1^b(\mathbf{k}_2, \mathbf{k}_2')}{2\pi \tilde{\beta}_1} \left\{ \int \mathrm{d}\mathbf{l}_2' \frac{b^2_{l_2 l_2'}\, q\, H_1^b(\mathbf{l}_2, \mathbf{l}_2')}{\bar{E}(\mathbf{l}_2'; \mathbf{q}) + \bar{E}(\mathbf{k}_2'; \mathbf{q})} \right. \\ &\quad \left. + \int \mathrm{d}\mathbf{l}_1' \frac{b^2_{l_1 l_1'}\, q\, G_1^b(\mathbf{l}_1, \mathbf{l}_1')}{\bar{E}(\mathbf{l}_1'; \mathbf{q}) + \bar{E}(\mathbf{k}_2'; \mathbf{q})} \right\}. \end{aligned} \tag{3.13}$$

In Eqs. (3.12–13) we have introduced the three-point functions $G_1^b$, $G_2^b$, $H_1^b$ and $H_2^b$ defined by:

$$\begin{aligned} &\begin{bmatrix} q\, G_1^b(\mathbf{k}_1, \mathbf{k}_1') & q\, G_2^b(\mathbf{k}_1, \mathbf{k}_1') \\ q\, H_2^b(\mathbf{k}_2, \mathbf{k}_2') & q\, H_1^b(\mathbf{k}_2, \mathbf{k}_2') \end{bmatrix} \\ &\equiv \begin{bmatrix} q\, X^b(\mathbf{k}_1, \mathbf{k}_1') & q\, Z^b(\mathbf{k}_1, \mathbf{k}_1') \\ q\, \hat{Z}^b(\mathbf{k}_2, \mathbf{k}_2') & q\, Y^b(\mathbf{k}_2, \mathbf{k}_2') \end{bmatrix} \begin{bmatrix} V_{11}(q) & V_{12}(q) \\ V_{12}(q) & V_{22}(q) \end{bmatrix}. \end{aligned}$$

We have also used the relation satisfied by coulombic potentials: $V_{11} V_{22} = V_{12}^2$ to simplify the Eqs. (3.12–13). In (3.12–13) we have $\mathbf{k}_i = \mathbf{k}_i' - \mathbf{q}$, $\mathbf{l}_i = \mathbf{l}_i' + \mathbf{q}$ $(i=1, 2)$ and have rendered the momenta dimensionless by scaling them with $k_{F_1}$. In this manner we define the new variables:

$$\bar{E}(\mathbf{k}_i'; \mathbf{q}) \equiv E_{\mathrm{SUB}\,2}(k_{F_1} k_i) + E_{\mathrm{SUB}\,2}(k_{F_1} k_i'); \qquad i=1, 2 \tag{3.14}$$

and the dimensionless factor:

$$\tilde{\beta}_1 \equiv 4\pi^2/\Omega\, k_{F_1}^3 .$$

Equations (3.12–13) may now be linearised and decoupled in the standard manner [2, 34]. This is accomplished [34] by introducing functions $(I_0^b)^{-1}$, $(J_0^b)^{-1}$ which satisfy linear integral equations analogous to Eqs. (4.9–10) in Bishop and Lahoz [2], and dielectric type functions $K^b$, $L^b$ which are analogous to those defined by Eq. (4.5) in Bishop and Lahoz [2], and which played an important role in the derivation of analytic solutions in the random phase approximation (RPA) for coulombic potentials.

Explicitly, to accomplish this exact linearisation we introduce the dimensionless parameter $M \equiv m_2/m_1$, and the dimensionless variables $E_i$, $E_i'$ $(i=1, 2)$ given by

$$E_i \equiv \frac{2 m_i}{\hbar^2 k_{F_1}^2} \bar{E}(\mathbf{k}_i'; \mathbf{q}); \qquad E_i' \equiv \frac{2 m_i}{\hbar^2 k_{F_1}^2} \bar{E}(\mathbf{l}_i'; \mathbf{q}). \tag{3.15}$$

For illustrative purposes we restrict ourselves to standard bare coulombic potentials:

$$V_{ii}(q) = 4\pi e_1^2/\Omega q^2; \qquad V_{12}(q) = 4\pi e_1 e_2/\Omega q^2; \quad q \neq 0$$

$$V_{11}(0) = V_{22}(0) = V_{12}(0) = 0$$

and assume that the ratio of the charges $e_2/e_1 = -1$, and that we have equal numbers of the two types of particles so that the respective Fermi momenta are equal, i.e. $k_{F_1} = k_{F_2} = k_F$.

We first multiply Eq. (3.12) by the factor $b^2_{k_1 k_1'}/[E_1+E_1''']$, where:

$$E_1''' \equiv \frac{2m_i}{\hbar^2 k_F^2}\,\bar{E}(\mathbf{k}_1''';\mathbf{q})$$

$$=\frac{2m_1}{\hbar^2 k_F^2}\,[E_{\mathrm{SUB}\,2}(k_F\,k_1''')+E_{\mathrm{SUB}\,2}(k_{F_1}\,k_1'')];$$

$$\mathbf{k}_1''=\mathbf{k}_1'''-\mathbf{q}$$

and then integrate with respect to $\mathbf{k}_1'$. We similarly multiply Eq. (3.13) by the factor $b^2_{k_2 k_2'}/[E_2+M E_1''']$ and then integrate with respect to $\mathbf{k}_2'$. We then employ a generalisation of the trick used in the one-component case [6] and I. The equations resulting from this rearrangement are combined in a suitable manner (cf. [34]; Sect. 4 in I) and simplified using the definition of the dielectric type functions:

$$K^b(\mathbf{k}_1,\mathbf{k}_1')$$

$$\equiv K^b(\mathbf{k}_1,\mathbf{k}_1';E_1)=1+\frac{1}{2\pi\tilde{\beta}}\int \mathrm{d}\,\mathbf{k}_1'''\,b^2_{k_1''k_1'''}$$

$$\cdot\left(\frac{1}{E_1'''+E_1}+\frac{1}{E_1'''-E_1}\right)+\frac{M}{2\pi\tilde{\beta}}\int \mathrm{d}\,\mathbf{k}_2'''\,b^2_{k_2''k_2'''}$$

$$\cdot\left(\frac{1}{E_2'''+ME_1}+\frac{1}{E_2'''-ME_1}\right) \tag{3.16}$$

$$L^b(\mathbf{k}_2,\mathbf{k}_2')\equiv K^b(\mathbf{k}_1,\mathbf{k}_1';E_2/M) \tag{3.17}$$

to yield a linear, coupled integral equation involving $q\,G_1^b(\mathbf{k}_1,\mathbf{k}_1')\,K^b(\mathbf{k}_1,\mathbf{k}_1')$, which is the BCS analogue of Eq. (4.7) in I. Similarly, multiplying Eq. (3.12) by the factor $b^2_{k_1k_1'}/[E_1+E_2''']$ and integrating with respect to $\mathbf{k}_1'$ and multiplying Eq. (3.13) by the factor $b^2_{k_2k_2'}/[E_2+E_2''']$ and integrating with respect to $\mathbf{k}_2'$ yields a linear, coupled integral equation involving $q\,H_1^b(\mathbf{k}_2,\mathbf{k}_2')\,L^b(\mathbf{k}_2,\mathbf{k}_2')$ which is the BCS analogue of Eq. (4.8) in I (cf. [34]).

To decouple the above equations a similar trick to that used in I (cf. Sects. 4, 5) and for linearising Eqs. (3.12–13) is employed. To achieve this we introduce linear integral equations satisfied by functions $(I_0^b)^{-1}$, $(J_0^b)^{-1}$ (cf. Eqs. (4.9–10) in I):

$$I_0^b(\mathbf{k}_1,\mathbf{k}_1')=1+\frac{I_0^b(\mathbf{k}_1,\mathbf{k}_1')}{2\pi\tilde{\beta}}$$

$$\cdot\int \mathrm{d}\,\mathbf{k}_1'''\,\frac{1}{I_0^b(\mathbf{k}_1'',\mathbf{k}_1''')\,K^b(\mathbf{k}_1'',\mathbf{k}_1''')(E_1'''-E_1)}$$

$$J_0^b(\mathbf{k}_2,\mathbf{k}_2')=1+\frac{J_0^b(\mathbf{k}_2,\mathbf{k}_2')}{2\pi\tilde{\beta}}$$

$$\cdot\int \mathrm{d}\,\mathbf{k}_2'''\,\frac{M}{J_0^b(\mathbf{k}_2'',\mathbf{k}_2''')\,L^b(\mathbf{k}_2'',\mathbf{k}_2''')(E_2'''-E_2)}$$

and multiply the linear integral equation involving $q\,G_1^b(\mathbf{k}_1,\mathbf{k}_1')\,K^b(\mathbf{k}_1,\mathbf{k}_1')$ by the factor

$b^2_{k_1k_1'}/[K^b(\mathbf{k}_1,\mathbf{k}_1')\,I_0^b(\mathbf{k}_1,\mathbf{k}_1')(E_1-E_2'''/M)]$

and integrate with respect to $\mathbf{k}_1'$, and the linear integral equation involving $q\,H_1^b(\mathbf{k}_2,\mathbf{k}_2')\,L^b(\mathbf{k}_2,\mathbf{k}_2')$ by the factor $b^2_{k_2k_2'}/[L^b(\mathbf{k}_2,\mathbf{k}_2')\,J_0^b(\mathbf{k}_2,\mathbf{k}_2')(E_2-ME_1''')]$ and integrate with respect to $\mathbf{k}_2'$. The first integration yields after using the integral equation satisfied by $(J_0^b)^{-1}$:

$$q\,G^b(\mathbf{k}_1,\mathbf{k}_1')\,K^b(\mathbf{k}_1,\mathbf{k}_1')$$

$$=1+\frac{1}{2\pi\tilde{\beta}}\int \mathrm{d}\,\mathbf{k}_1'''\,\frac{b^2_{k_1''k_1'''}\,q\,G^b(\mathbf{k}_1'',\mathbf{k}_1''')}{E_1'''-E_1}$$

$$\mathbf{k}_1''=\mathbf{k}_1'''-\mathbf{q};\quad \mathbf{k}_1=\mathbf{k}_1'-\mathbf{q};\quad b_{k_1''k_1'''}\neq 0. \tag{3.18}$$

Similarly, the second integration yields:

$$q\,H^b(\mathbf{k}_2,\mathbf{k}_2')\,L^b(\mathbf{k}_2,\mathbf{k}_2')$$

$$=1+\frac{M}{2\pi\tilde{\beta}}\int \mathrm{d}\,\mathbf{k}_2'''\,\frac{b^2_{k_2''k_2'''}\,q\,H^b(\mathbf{k}_2'',\mathbf{k}_2''')}{E_2'''-E_2}$$

$$\mathbf{k}_2''=\mathbf{k}_2'''-\mathbf{q};\quad \mathbf{k}_2=\mathbf{k}_2'-\mathbf{q};\quad b_{k_2''k_2'''}\neq 0. \tag{3.19}$$

In Eqs. (3.16–19) we have introduced the factor:

$$\tilde{\beta}\equiv 2\pi^2\hbar^2/[m_1\,\Omega\,k_F\,V_{11}(q\,k_F)]$$

and in Eqs. (3.18–19) the functions:

$$G^b(\mathbf{k}_1,\mathbf{k}_1')\equiv\frac{G_1^b(\mathbf{k}_1,\mathbf{k}_1')}{V_{11}(q\,k_F)}\left[1-\frac{M}{2\pi\tilde{\beta}}\int \mathrm{d}\,\mathbf{k}_2'''\,b^2_{k_2''k_2'''}\middle/\right.$$

$$\left.[J_0^b(\mathbf{k}_2'',\mathbf{k}_2''')\,L^b(\mathbf{k}_2'',\mathbf{k}_2''')(E_2'''-ME_1)]\right], \tag{3.20}$$

$$H^b(\mathbf{k}_2,\mathbf{k}_2')\equiv\frac{H_1^b(\mathbf{k}_2,\mathbf{k}_2')}{V_{22}(q\,k_F)}\left[1-\frac{1}{2\pi\tilde{\beta}}\int \mathrm{d}\,\mathbf{k}_1'''\,b^2_{k_1''k_1'''}\middle/\right.$$

$$\left.[I_0^b(\mathbf{k}_1'',\mathbf{k}_1''')\,K^b(\mathbf{k}_1'',\mathbf{k}_1''')(E_1'''-E_2/M)]\right]. \tag{3.21}$$

Clearly, Eqs. (3.20–21) enable us to simplify considerably the linear, uncoupled integral equations resulting from the above analysis.

An analysis similar to the one above also applies to more general potentials satisfying $V_{11}V_{22}=V_{12}^2$. The resulting integral equations have the same structure as (3.18–19).

It is easy to show [34] that for each of the functions $f\equiv\{G^b, H^b, b, K^b, L^b, (I_0^b)^{-1}, (J_0^b)^{-1}\}$ we have the symmetry property:

$$f(\mathbf{k},\mathbf{k}')=f(\mathbf{k}',\mathbf{k});\quad f_{kk'}=f_{k'k}. \tag{3.22}$$

To make further progress we introduce the more symmetric momentum variables $\boldsymbol{\kappa}_i$, $\boldsymbol{\kappa}'_i$ by the relations: $\mathbf{k}'_i = \boldsymbol{\kappa}_i + \mathbf{q}/2$, $\mathbf{l}'_i = \boldsymbol{\kappa}'_i - \mathbf{q}/2$, with $\mathbf{k}_i = \boldsymbol{\kappa}_i - \mathbf{q}/2$, $\mathbf{l}_i = \boldsymbol{\kappa}'_i + \mathbf{q}/2$ $(i=1, 2)$. Use of the symmetry property (3.22) allows us to interpret a general function $f(\mathbf{k}, \mathbf{k}')$, $f_{kk'}$ as:

$$f(\mathbf{k}, \mathbf{k}') \equiv f(\boldsymbol{\kappa} + \tfrac{1}{2}\mathbf{q}, \boldsymbol{\kappa} - \tfrac{1}{2}\mathbf{q}); \qquad f_{kk'} \equiv f_{|\boldsymbol{\kappa}+\frac{1}{2}\mathbf{q}|\,|\boldsymbol{\kappa}-\frac{1}{2}\mathbf{q}|}\,.$$

It is clear from Eqs. (3.18–19) and their counterparts for $K^b$, $L^b$ that the functions satisfying them must depend only on a scalar variable. The most convenient choice of scalar is the energy $E$. Thus $q\,G^b$ has a solution: $q\,G^b(\mathbf{k}_1, \mathbf{k}'_1) = q\,G(E_1)$ where we continue to suppress any further explicit dependence on the variable $q$ also. Similarly for the other functions of interest. We now introduce the transformation (cf. Eq. (4.15) in Bishop and Lahoz [2]):

$$\int_{\Gamma_i} \mathrm{d}\boldsymbol{\kappa}_i\, f(E_i) = 2\pi q \int_{L_E} \mathrm{d}E_i\, T_0(E_i)\, f(E_i) \tag{3.23}$$

where $f(E_i)$ is an arbitrary function, $T_0(E_i)$ and $L_E$ are to be determined and:

$$\boldsymbol{\kappa}_i \in \Gamma_i \Rightarrow \begin{cases} |\boldsymbol{\kappa}_i - \frac{1}{2}\mathbf{q}| < 1 \\ |\boldsymbol{\kappa}_i + \frac{1}{2}\mathbf{q}| > 1. \end{cases} \tag{3.24}$$

We now forego further analytic treatment of the full RING+SPE equations for standard bare coulombic potentials in this section and concentrate on the particular case where the energy denominators are replaced by their bare kinetic energy components. The motivation for this is to check that the BCS-based formalism does simplify in the correct manner.

We now substitute Eq. (3.23) in (3.18). If we replace the energy denominators by their bare kinetic energy components and use the result that for bare coulombic potentials, where the $\mathbf{q}=0$ contribution from the potentials is zero we have no superconducting solutions (cf. Sect. 2), we obtain

$$E_{\mathrm{SUB\,2}}(k_i) = |T_{k_i} - \mu_i|,$$

$$\begin{aligned} b_{k_i k'_i} = {} & \theta(|\boldsymbol{\kappa}_i + \tfrac{1}{2}\mathbf{q}| - 1)\,\theta(1 - |\boldsymbol{\kappa}_i - \tfrac{1}{2}\mathbf{q}|) \\ & + \theta(|\boldsymbol{\kappa}_i - \tfrac{1}{2}\mathbf{q}| - 1)\,\theta(1 - |\boldsymbol{\kappa}_i + \tfrac{1}{2}\mathbf{q}|) \end{aligned}$$

where $\theta(x)$ is the usual Heaviside unit step function. We now split the integral into an integration over $\Gamma_i$ (cf. Eq. (3.24)) and $\hat{\Gamma}_i$ where

$$\boldsymbol{\kappa}_i \in \hat{\Gamma}_i \Rightarrow \begin{cases} |\boldsymbol{\kappa}_i + \frac{1}{2}\mathbf{q}| < 1 \\ |\boldsymbol{\kappa}_i - \frac{1}{2}\mathbf{q}| > 1. \end{cases} \tag{3.25}$$

Using the result $\boldsymbol{\kappa}_i \in \hat{\Gamma}_i \Leftrightarrow -\boldsymbol{\kappa}_i \in \Gamma_i$, and that for bare kinetic energy $E(\kappa_i) = E(-\kappa_i)$, we can rewrite (3.18) in terms of an integral over $L_E$. We now assume the energy $E_i$ has the form $E_i = E_i(x_i)$ where $x_i = \boldsymbol{\kappa}_i \cdot \hat{q}$, again suppressing dependence on the variable $q$ which enters only parametrically. By changing the integration variable in (3.23) to $x_i \equiv \boldsymbol{\kappa}_i \cdot \hat{q}$, and using the result (cf. Eq. (4.41) in I)

$$\int_{\Gamma_i} \mathrm{d}\boldsymbol{\kappa}_i = 2\tilde{\beta} q \int_{\tilde{L}_i} \mathrm{d}x_i\, M^{1-i}\, \mathrm{Im}\{K^b_+(M^{1-i} x_i)\}; \qquad i = 1, 2$$

where we have taken account of the overlap between the integration regimes for the different particles, one obtains the relation [34]:

$$q\,G^b(x_1)\,K^b(x_1) = 1 + \frac{1}{\tilde{\beta}} \int_{L_E} \mathrm{d}x'_1\, \frac{b^2(x'_1)\,q\,G^b(x'_1)}{x'_1 - x_1} \tag{3.26}$$

where we have used the above energy expression and the property (3.22).

Comparison with Eq. (4.17) of I shows that (3.26) is simply the random phase approximation (RPA) equation for the three-point function $G$ [2, 34] with $b^2(x_1) = [\tilde{N}_1(x_1) + M\tilde{N}_2(Mx_1)]$, $L_E = \tilde{L}_1$ and where $\tilde{N}_i$, $\tilde{L}_i$ $(i=1, 2)$ have been defined in (4.15) of I. The same result is obtained for the three-point function $H$. This result, expected on the physical grounds that the RPA possesses particle-hole symmetry, is also true in the one-component case [34] and shows that the BCS-based formalism simplifies in the correct manner.

## 4. RING+SPE treatment of two-body correlations in SUB2 approximation for non-coulombic potentials

We now consider the case where the potentials are non-coulombic, i.e. $V_{11}V_{22} \neq V_{12}^2$. As in I, the equations appropriate to this case are more complicated than the coulombic ones, although the techniques for solving them are conceptually the same. In this case the equations for the quantities $T_{\mathrm{RING}}$ and $T_{\mathrm{SPE}}$ are as the ones written out in Sect. 3, viz. Eqs. (3.2–6) except for the following two differences:

*(i)* In general $V_{11}(0)$, $V_{22}(0)$, $V_{12}(0) \neq 0$. However, we allow situations where we make one of the $V_{ii}$ $(i=1, 2)$ or $V_{12}$ as small as we like. The limit $V_{ii} \to 0$ $(i=1, 2)$ with $|V_{12}| \gg V_{ii}$ $(i=1, 2)$ can be seen to allow the possibility of superconducting solutions (cf. Sect. 2), whereas the limit $V_{12} \to 0$, under the conditions described in Sect. 2, does not give superconducting solutions.

*(ii)* In order to emphasise the possibility of nonzero model gap solutions Eq. (3.6) is replaced by:

$$\begin{aligned} E_{\mathrm{SUB\,2}}(k_i) = {} & a_{k_i}(\xi^{V_{ii}}_{k_i} + \xi^{V_{ij}}_{k_i}) + b_{k_i}\,\Delta^{V_{ii}}_{k_i} \\ & + \sum_{\mathbf{l}''_i} b_{k_i l''_i} \{\sum_{\mathbf{l}_i \mathbf{l}'_i} b_{l_i l'_i} \langle \mathbf{k}_i\, \mathbf{l}_i |\, V_{ii}\, | \mathbf{l}''_i\, \mathbf{l}'_i \rangle\, S^{\mathbf{k}_i \mathbf{l}_i}_{2;\mathbf{l}''_i \mathbf{l}'_i} \end{aligned}$$

$$+\sum_{\mathbf{l}_j\mathbf{l}_j'} b_{l_j l_j'} \langle \mathbf{k}_i\, \mathbf{l}_j |\, V_{ij} \,| \mathbf{l}_i''\, \mathbf{l}_j' \rangle\, Z^{\mathbf{k}_i \mathbf{l}_j}_{2;\mathbf{l}_i''\mathbf{l}_j'}\};$$

$$i \neq j;\ i,j=1,2 \tag{4.1}$$

where we have used Eq. (2.11). (Strictly speaking we can have non-zero model gap solutions when $V_{12}(0) \neq 0$, see Sect. 2).

We now substitute the two-component screening Eqs. (3.8–11) in the suitably modified RING+SPE equations. To simplify the resulting equations we use the three-point functions $G_i^b$, $H_i^b$ ($i=1,2$) introduced in Sect. 3 and take into account the fact that the potentials we are considering now in general contain a non-zero $\mathbf{q}=0$ component, by introducing the new three-point functions $G_i$, $H_i$ ($i=1,2$), defined in the same manner as $G_i^b$, $H_i^b$ with:

$$F=\{X, Y, Z, \hat{Z}\}; \qquad F_q \equiv q\, F^b$$

such that $\lim\limits_{q\to 0} q\, F^b = F_0$.

We thus obtain the coupled, non-linear equations:

$$X_q(\mathbf{k}_1, \mathbf{k}_1') = 1 - X_q(\mathbf{k}_1, \mathbf{k}_1') \sum_{\mathbf{l}_1\mathbf{l}_1'} \frac{b^2_{l_1 l_1'}\, G_1(\mathbf{l}_1, \mathbf{l}_1')}{\bar{E}(\mathbf{l}_1'; \mathbf{q}) + \bar{E}(\mathbf{k}_1'; \mathbf{q})}$$
$$- Z_q(\mathbf{k}_1, \mathbf{k}_1') \sum_{\mathbf{l}_1\mathbf{l}_1'} \frac{b^2_{l_1 l_1'}\, G_2(\mathbf{l}_1, \mathbf{l}_1')}{\bar{E}(\mathbf{l}_1'; \mathbf{q}) + \bar{E}(\mathbf{k}_1'; \mathbf{q})}, \tag{4.2}$$

$$Z_q(\mathbf{k}_1, \mathbf{k}_1') = - X_q(\mathbf{k}_1, \mathbf{k}_1') \sum_{\mathbf{l}_2\mathbf{l}_2'} \frac{b^2_{l_2 l_2'}\, H_2(\mathbf{l}_2, \mathbf{l}_2')}{\bar{E}(\mathbf{l}_2'; \mathbf{q}) + \bar{E}(\mathbf{k}_1'; \mathbf{q})}$$
$$- Z_q(\mathbf{k}_1, \mathbf{k}_1') \sum_{\mathbf{l}_2\mathbf{l}_2'} \frac{b^2_{l_2 l_2'}\, H_1(\mathbf{l}_2, \mathbf{l}_2')}{\bar{E}(\mathbf{l}_2'; \mathbf{q}) + \bar{E}(\mathbf{k}_1'; \mathbf{q})}, \tag{4.3}$$

where the energy $\bar{E}$ has been defined in Eq. (3.14).

The equations for the remaining four-point functions viz. $Y$, $\hat{Z}$ are easily obtained from those for $X$, $Z$ by suitable interchange of the indices for particles of type "1" and "2" and by modifying the functions appropriately.

Equations (4.2)–(4.3) and their $Y$, $\hat{Z}$ counterparts are the basic RING+SPE equations for non-coulombic potentials. In a manner analogous to that employed in Paper I (cf. Sects. 4, 5) we convert these equations to integral equations. These are:

$$X_q(\mathbf{k}_1, \mathbf{k}_1') = 1 - \frac{X_q(\mathbf{k}_1, \mathbf{k}_1')}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_1''' \frac{b^2_{k_1'' k_1'''}\, G_1(\mathbf{k}_1'', \mathbf{k}_1''')}{E_1''' + E_1}$$
$$- \frac{Z_q(\mathbf{k}_1, \mathbf{k}_1')}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_1''' \frac{b^2_{k_1'' k_1'''}\, G_2(\mathbf{k}_1'', \mathbf{k}_1''')}{E_1''' + E_1}, \tag{4.4}$$

$$Z_q(\mathbf{k}_1, \mathbf{k}_1') = - \frac{X_q(\mathbf{k}_1, \mathbf{k}_1')}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_2''' \frac{b^2_{k_2'' k_2'''}\, H_2(\mathbf{k}_2'', \mathbf{k}_2''')}{E_2''' + M E_1}$$
$$- \frac{Z_q(\mathbf{k}_1, \mathbf{k}_1')}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_2''' \frac{b^2_{k_2'' k_2'''}\, H_1(\mathbf{k}_2'', \mathbf{k}_2''')}{E_2''' + M E_1}, \tag{4.5}$$

and similarly for $Y_q$ and $\hat{Z}_q$. These equations have the same form as Eqs. (3.26–29) in I. Consequently, the same analysis used in I to linearise and decouple these equations is applicable to Eqs. (4.4–5). This yields the uncoupled linear integral equations:

$$G_1(E_1)\, F(E_1) = 1 + \frac{1}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_1''' \frac{b^2_{k_1'' k_1'''}\, G_1(E_1''')}{E_1''' - E_1}$$
$$+ \frac{J(E_1)}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_1''' \frac{b^2_{k_1'' k_1'''}\, G_1(E_1''')\, \hat{V}(E_1''')}{E_1''' - E_1}, \tag{4.6}$$

$$H_2(E_2)\, \hat{F}(E_2) = -2\pi\tilde{\beta}\, I(E_2) + \frac{1}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_2'''$$
$$\frac{b^2_{k_2'' k_2'''}\, M H_2(E_2''')}{E_2''' - E_2}$$
$$+ \frac{I(E_2)}{2\pi\tilde{\beta}} \int \mathrm{d}\,\mathbf{k}_2''' \frac{b^2_{k_2'' k_2'''}\, M H_2(E_2''')\, \hat{Y}(E_2''')}{E_2''' - E_2} \tag{4.7}$$

where we have defined:

$$G_1(E_1) \equiv G_1(\mathbf{k}_1, \mathbf{k}_1'); \qquad H_2(E_2) \equiv H_2(\mathbf{k}_2, \mathbf{k}_2')$$

$$\underset{\sim}{W}(E_1) \equiv \begin{bmatrix} W_0(E_1) & W_1(E_1) \\ W_2(ME_1) & W_3(ME_1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2\pi\tilde{\beta}}$$
$$\cdot \left[\begin{array}{c:c} \int \mathrm{d}\,\mathbf{k}_1'''\, b^2_{k_1'' k_1'''} \left( \frac{1}{E_1''' + E_1} + \frac{1}{E_1''' - E_1} \right) & 0 \\ \hdashline 0 & \int \mathrm{d}\,\mathbf{k}_2'''\, M\, b^2_{k_2'' k_2'''} \left( \frac{1}{E_2''' + ME_1} + \frac{1}{E_2''' - ME_1} \right) \end{array}\right]$$
$$\cdot \begin{bmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{bmatrix}, \tag{4.8}$$

$$W(E_1) \equiv \det[\underset{\sim}{W}(E_1)]/T(E_1)$$
$$T(E_1) \equiv V_{11}\, W_3(ME_1) - V_{12}\, W_2(ME_1)$$
$$\hat{W}(E_2) \equiv \det[\underset{\sim}{W}(E_2/M)]/U(E_2)$$
$$U(E_2) \equiv V_{22}\, W_0(E_2/M) - V_{12}\, W_1(E_2/M)$$
$$F(E_1) \equiv \frac{\det[\underset{\sim}{W}(E_1)]}{[T(E_1) + (V_{12}/2\pi\tilde{\beta})\, \hat{V}(E_1)\, J_0(ME_1)]}$$
$$J(E_1) \equiv -\frac{V_{12}}{2\pi\tilde{\beta}}\, \frac{J_0(ME_1)\, F(E_1)}{\det[\underset{\sim}{W}(E_1)]}$$
$$\hat{V}(E_1) \equiv \int \mathrm{d}\,\mathbf{k}_2'''\, b^2_{k_2'' k_2'''}$$
$$\cdot \frac{M V_{12}}{(E_2''' - ME_1)\, \hat{W}(E_2''')\, J_0(E_2''')\, U(E_2''')}$$
$$\hat{F}(E_2) \equiv \frac{\det[\underset{\sim}{W}(E_2/M)]}{[U(E_2) + (V_{12}/2\pi\tilde{\beta})\, \hat{Y}(E_2)\, I_0(E_2/M)]}$$
$$I(E_2) \equiv -\frac{V_{12}}{2\pi\tilde{\beta}}\, \frac{I_0(E_2/M)\, \hat{F}(E_2)}{\det[\underset{\sim}{W}(E_2/M)]}$$
$$\hat{Y}(E_2) \equiv \int \mathrm{d}\,\mathbf{k}_1'''\, b^2_{k_1'' k_1'''}$$
$$\cdot \frac{V_{12}}{(E_1''' - E_2/M)\, W(E_1''')\, I_0(E_1''')\, T(E_1''')}. \tag{4.9}$$

In the above equations we have introduced the dimensionless variable $E_i$ (cf. Eq. (3.15)) and the factor $\underset{\sim}{\beta}$ (cf. Sect. 3).

In particular, it can be seen that $F^q$ plays the role of a dielectric-like function, and that functions such as $J^q$ and $\hat{V}^q$ depend on $V_{12}$. This result suggests ways in which Eqs. (4.4–5) could be simplified for ($i=1$ and/or 2), (e.g., if $|V_{12}/V_{ii}| \ll 1$, or $|V_{12}/V_{ii}| \gg 1$), and thus enable one to obtain solutions for $G_1$ and the other three-point functions which can be used to check numerical calculations. The equations for the remaining three-point functions are obtained in a similar manner. As in Sect. 3, one can introduce the more symmetric momentum variables $\boldsymbol{\kappa}_i$, $\boldsymbol{\kappa}'_i$ ($i=1, 2$).

Similarly (cf. Eq. (5.9) in I), we can obtain the following relation:

$$\det[\underset{\sim}{W}(E_1)] = \frac{(\det \underset{\sim}{V})^2}{\det[\underset{\sim}{H}(E_1)]\,\det[\underset{\sim}{H}(-E_1)]} \tag{4.10}$$

$$\underset{\sim}{H}(E_1) \equiv \begin{bmatrix} G_1(E_1) & G_2(E_1) \\ H_2(ME_1) & H_1(ME_1) \end{bmatrix}; \qquad \underset{\sim}{V} \equiv \begin{bmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{bmatrix}.$$

By considering the equation for the energy (4.1) we see that the superconducting case has one more degree of freedom than the normal case, namely the gap function. A consequence of this is that the RING + SPE four-point functions $\langle \,|\, S_2 \,|\, \rangle$ may have associated with them a multiplicity of non-zero gap functions. This can be seen by realising that the single particle energy $E_{\mathrm{SUB}\,2}(k_i)$ (cf. Eq. (3.6)) contains a term associated with the gap function viz. $b_{k_i}\,\varepsilon_b^{\mathrm{HF}}(k_i)$ plus three other terms. Different values of the non-zero gap function may then be associated with the same energy $E_{\mathrm{SUB}\,2}(k_i)$ provided that the sum of these four terms remains constant, leading to the above-mentioned multiplicity. In contrast, the normal case would only allow the gap $\Delta_{k_i}^{V_{ii}} = 0$ and no multiplicity. The structure of the full two-body Eq. (3.1) where the energy $E_{\mathrm{SUB}\,2}(k_i)$ only appears in the $T_{\mathrm{SPE}}$ terms, suggests this result is true for the full two-body equation as well.

A possible method of solving the resulting decoupled linear equations analytically is that employed in I, where the theory of Muskhelishvili-Omnès integral equations [35, 36] is used. The value of any analytic solutions obtained will be in providing checks for subsequent numerical calculations. They will thus serve as a starting point for these same numerical calculations, as in the case of the fluid one-component CCM [6, 7].

As described in Sect. 1, the RING + SPE approximation has been shown *a posteriori* to be a very accurate approximation over the metallic density range for one-component systems in the case of bare coulombic potentials [8]. It is reasonable to expect that the two-component analogue of the RING + SPE will also prove an accurate approximation for determining the two-component correlation energies and investigating two-component superconducting solutions, as well as providing a stepping stone towards realistic calculations (see Sect. 1).

## 5. Conclusions and further work

This paper concerns the further development of the two-component coupled cluster method (CCM) formalism based on a bare noninteracting model state of BCS type as a zeroth approximation. We aim to provide the foundation for studying Fermi superfluids and in particular the phenomenon of superconductivity in a quite general way. It is our belief that the enormous success enjoyed by the CCM from the point of view both of analytic results (in the one- and two-component cases) and numerical results (in the one-component case) makes it a potentially very useful tool for undertaking this task.

The new model state is chosen from the filled Fermi sea by means of a Bogoliubov transformation which explicitly breaks conservation of particle number. The usual exp($S$) ansatz is then made, to correlate fully (in principle exactly, when no truncations are made) this model state into the exact system ground state. The resulting two-component equations and expressions for the ground state and correlation energy are seen to be generalisations of their one-component analogues.

To investigate the possibility of superconducting solutions we work with the thermodynamic potential and extract the one-body terms therefrom. We then require that the number non-conserving components of these terms vanish and obtain the two-component gap equations in the standard BCS prescription. These gap equations are analysed under certain conditions and the role of $V_{12}(0)$, associated with the long range limit $q \to 0$, where all particles of type "$i$" can see all type "$j$" particles and thus may experience an effective attractive interaction via the particles of type "$j$" ($j \neq i$), is discussed.

The above investigation leads us to conclude that the ground state random phase approximation (RPA) cannot yield superconducting solutions, and that the BCS formalism is superior to the fluid CCM formalism both from the point of view of superconductivity as well as providing better numerical accuracy.

The analysis of the gap equations arising from the two-component formalism allows the possibility of two-component systems having two gaps associated with them, one for each type of particle. This circumstance may be of relevance towards explaining high-$T_c$ superconducting materials, although, further numerical work is needed before firm conclusions can be reached.

A test on the formalism is performed by analysing the RING + SPE equations for bare coulombic potentials and showing that these equations reduce to the fluid ground state RPA equations when the energy denominators are replaced by their bare kinetic energy content. This result is expected due to the particle-hole symmetry of the RPA. The RING + SPE equations for general potentials are also derived and simplified by the introduction of the so-called screening equations, analogous to the fluid ground state three-point functions whose solution has been obtained in the RPA.

By comparison of the gap equation for the above simple case and the screening equations derived, we discuss normal and superconducting solutions in terms of an extra degree of freedom, viz. the gap function. This leads us to venture the possibility of a multiplicity of gap functions for a given set of physical quantities in the superconducting phase, with only the trivial solution for the gap function, $\Delta = 0$, in the normal phase.

Future work ought to concern the development of numerical schemes for solving the general RING + SPE and gap equations and for obtaining quantitative results analogous to those available in the one-component case. Eventually, a treatment of inhomogeneous systems employing suitable generalisations of the formalism developed in this paper will be undertaken.

An ultimate aim of developing this CCM theory of superconductivity is to answer the question of whether a particular substance is superconducting and if so what is its associated energy gap. It is hoped that the above numerical work will go some way towards answering this question. The CCM theory of superconductivity developed by Emrich et al. [8, 29, 30] and in particular their recent suggestion of an experiment (Emrich et al. [32, 33]), based on their CCM work, and designed to inquire whether BaLaCuO and YBaCuO are phonon-induced superconductors, has already clearly shown the practical potential of the CCM for studying superconductivity and, in particular, these recently discovered high-$T_c$ superconductors. In this area, the relevance of a second gap in the two-component superfluid system towards explaining high temperature superconductors and the possibility of multiple gap solutions, both of great theoretical and practical interest, could also be more thoroughly investigated by, for example, extending the present formalism to finite temperatures, and inhomogeneous systems.

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