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# TWO CONSTRUCTIONS OF COMPATIBLE RELATIONS

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Besides congruences, other significant relations on algebras which have many applications are the so called compatible tolerances. This is the reason for investigating tolerances on algebras (see [4], [5]) and for finding conditions under which these tolerances are not congruences (see [6], [7]). The aim of this paper is to give a method of construction of compatible tolerances which are not congruences from compatible relations (especially from congruences) and show that on varieties these constructions characterize the existence of tolerances. In the second part of the paper we introduce a method of construction of compatible relations from orderings.

## 1. INDUCED TOLERANCES

Let $\mathfrak{A} = (A, F)$ be an algebra. A binary relation $R$ on $A$ is said to be *compatible* (*with* $\mathfrak{A}$) if for each $n$-ary $f \in F$ and arbitrary $a_i, b_i \in A$ $(i = 1, \ldots, n)$ the following implication is valid:

$$\langle a_i, b_i \rangle \in R \quad \text{for} \quad i = 1, \ldots, n \Rightarrow \langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R \,.$$

A binary relation $T$ on a set $M$ is called *tolerance* (or *tolerance relation*) provided it is reflexive and symmetric. Hence every equivalence is a tolerance and every congruence is a compatible tolerance. Denote by $\Delta$ the identical relation on $A$, i.e. $\langle a, b \rangle \in \Delta$ if and only if $a = b$. Let $\mathfrak{A} = (A, F)$ be an algebra. Following [1], denote by $\mathscr{C}(\mathfrak{A})$ the lattice of all congruences on $\mathfrak{A}$ and by $\vee$ the lattice join in $\mathscr{C}(\mathfrak{A})$. Further, if $\emptyset \neq R \subseteq A \times A$, then

$$\Theta(R) = \bigcap\{\Theta \in \mathscr{C}(\mathfrak{A});\ R \subseteq \Theta\}\,, \quad \text{i.e. } \Theta(R) \text{ is the } \textit{congruence generated by } R\,.$$

**Theorem 1.** *Let $\mathfrak{A} = (A, F)$ be an algebra and $R$, $S$ reflexive compatible relations on $\mathfrak{A}$. Put $T = R \,.\, S \cap S \,.\, R$. Then*

(a) *$T$ is a compatible tolerance on $\mathfrak{A}$;*

(b) *if $\Theta(R) . \Theta(S) \neq \Theta(S) . \Theta(R)$, then $T$ is not a congruence.*

Proof. The symmetry of $T$ is evident. We prove the compatibility of $T$. Let $\langle a_i, b_i \rangle \in T$ for $i = 1, \ldots, n$ and let $f \in F$ be an $n$-ary operation. Then there exist $c_i, d_i \in A$ with $\langle a_i, c_i \rangle \in R$, $\langle c_i, b_i \rangle \in S$, $\langle a_i, d_i \rangle \in S$, $\langle d_i, b_i \rangle \in S$. Put $a = f(a_1, \ldots, a_n)$, $b = f(b_1, \ldots, b_n)$, $c = f(c_1, \ldots, c_n)$, $d = f(d_1, \ldots, d_n)$. As $R, S$ are compatible, the above implies

$$\langle a, c \rangle \in R\,, \quad \langle c, b \rangle \in S \quad \text{and} \quad \langle a, d \rangle \in S\,, \quad \langle d, b \rangle \in R\,,$$

thus $\langle a, b \rangle \in R . S \cap S . R = T$. The compatibility is proved. Clearly, the reflexivity of $R, S$ implies the reflexivity of $T$, thus $T$ is a compatible tolerance.

Further, suppose that $\Theta(R) . \Theta(S) \neq \Theta(S) . \Theta(R)$ and $T$ is a congruence on $\mathfrak{A}$. The reflexivity of $R, S$ implies $R, S \subseteq T$. As $T \in \mathscr{C}(\mathfrak{A})$, also $\Theta(R) \subseteq T$, $\Theta(S) \subseteq T$. Hence

$$T = R . S \cap S . R \subseteq \Theta(R) . \Theta(S) \cap \Theta(S) . \Theta(R) \subseteq \Theta(R) \vee \Theta(S) \subseteq T\,,$$

$$\text{i.e.} \quad \Theta(R) . \Theta(S) \cap \Theta(S) . \Theta(R) = \Theta(R) \vee \Theta(S)$$

and by the formula for $\vee$ in $\mathscr{C}(\mathfrak{A})$ we obtain

$$\Theta(R) \vee \Theta(S) = \Theta(R) . \Theta(S) = \Theta(S) . \Theta(R)$$

contrary to the assumptions. The proof is complete.

**Remark.** The statement (b) of Theorem 1 cannot be converted, i.e. the permutability of $\Theta(R)$, $\Theta(S)$ does not imply that $T$ is a congruence. This is shown by the following example:

**Example 1.** Let $\mathfrak{G} = (\{a, b, c\}; \{.\})$ be a groupoid with the following Cayley's table:

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $a$ |
| $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $a$ |

and $R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle, \langle c, c \rangle\}$, $S = \{\langle a, a \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle c, c \rangle\}$. Clearly, $R, S$ are reflexive and compatible with $\mathfrak{G}$. Let $T = R . S \cap S . R$. Since $R . S = S . R = R \cup S$, we find that $T = R \cup S$ is not a congruence, because it is not symmetric. Nonetheless, $\Theta(R)$, $\Theta(S)$ are permutable, because they are equal to the Cartesian square of $\{a, b, c\}$.

Let us call $T$ the *induced (by $R$, $S$) relation*, provided $R$, $S$ are compatible relations on $\mathfrak{A}$ and $T = R \,.\, S \cap S \,.\, R$. For the case of congruences, the converse assertion to (b) of Theorem 1 is true:

**Corollary.** *Let $R$, $S$ be congruences on $\mathfrak{A}$. The compatible tolerance $T$ induced by $R$, $S$ is a congruence on $\mathfrak{A}$ if and only if $R$, $S$ are permutable.*

Proof. If $R$, $S$ are permutable, then $T = R \,.\, S$ is a congruence. If $R$, $S$ are not permutable, then by Theorem 1, $T$ is not a congruence.

We say that an *algebra $\mathfrak{A}$ admits (induced) tolerances*, if there exists an (induced by relations different from $\Delta$, $T$) compatible tolerance $T$ on $\mathfrak{A}$ which is not a congruence. We say that a *variety of algebras $\mathscr{V}$ admits (induced) tolerances*, if there exists $\mathfrak{A} \in \mathscr{V}$ which admits (induced) tolerances.

**Theorem 2.** *A variety $\mathscr{V}$ of algebras admits tolerances if and only if it admits induced tolerances.*

Proof. If $\mathscr{V}$ admits tolerances, then, by Werner's Theorem from [3], there exists $\mathfrak{A} \in \mathscr{V}$ and congruences $\Theta$, $\Phi \in \mathscr{C}(\mathfrak{A})$ which are not permutable. By Corollary, the tolerance $T$ induced by $\Theta$, $\Phi$ is not a congruence. Hence $\mathscr{V}$ admits induced tolerances. The converse assertion is evident.

**Remark.** If $\mathscr{V}$ is a variety, then, by Werner's Theorem, $\mathscr{V}$ admits tolerances if and only if $\mathscr{V}$ has no permutable congruences. By Corollary, if an algebra $\mathfrak{A}$ has no permutable congruences, it admits tolerances. However, the converse statement for algebras is not true in the general case, as is illustrated by the following example:

**Example 2.** Let $\mathfrak{A} = (\{a, b, c\}; \{\cdot\})$ be a commutative groupoid with Cayley's table

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ |
| $b$ | $b$ | $a$ | $b$ |
| $c$ | $b$ | $b$ | $a$ |

Then the relation $R = \{\langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle\} \cup \Delta$ is a compatible tolerance which is not a congruence. However, $\mathfrak{A}$ has only three congruences, namely $\Delta$, the Cartesian square of $\{a, b, c\}$ and $C$ given by the partition $\{a, b\}$, $\{c\}$, hence $\mathfrak{A}$ has permutable congruences.

## 2. CONTRACTIONS

H. L. SKALA introduced the concept of contraction for pseudo-ordered sets and proved that every pseudo-ordered set is isomorphic to a contraction of a partially ordered set (see [2]). This concept can be generalized also to arbitrary relations and modified also for compatible relations.

If $R$ is a binary relation on a set $M$, then a *global relation* $R(M)$ is a relation on $2^M$ defined by:

$$\langle X, Y\rangle \in R(M) \quad \text{iff there exist} \quad a \in X\,, \quad b \in Y \quad \text{with} \quad \langle a, b\rangle \in R\,.$$

Let $\mathfrak{A} = (A, F)$ be a partial algebra. Denote by $\mathfrak{A}(G)$ the so called *global partial algebra*, i.e. $\mathfrak{A}(G) = (2^A, F)$, where

$$f(X_1, \ldots, X_n) = \emptyset \quad \text{whenever} \quad X_i = \emptyset \quad \text{for some} \quad i \in \{1, \ldots, n\} \quad \text{and}$$

$$f(X_1, \ldots, X_n) = \{f(a_1, \ldots, a_n);\ a_i \in X_i \text{ for } i = 1, \ldots, n\}$$

in the opposite case.

Hence, if $\mathfrak{A} = (A, F)$ is an algebra and $\mathfrak{A}(G)$ its global partial algebra, then clearly $\mathfrak{B} = (2^A - \{\emptyset\}, F)$ is a subalgebra (everywhere defined) of $\mathfrak{A}(G)$.

For a relation $R$ on $A$ and $B \subseteq A$ the restriction of $R$ on $B$, i.e. $R \cap (B \times B)$, will be denoted by the same symbol $R$.

Let $\mathfrak{A} = (A, F)$, $\mathfrak{B} = (B, G)$ be algebras and $R$ (or $S$) a compatible relation on $\mathfrak{A}$ (or $\mathfrak{B}$, respectively). We call $\langle \mathfrak{A}, R\rangle$ *isomorphic with* $\langle \mathfrak{B}, S\rangle$ provided $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic algebras and $\langle A, R\rangle$ and $\langle B, S\rangle$ are isomorphic relational systems. For a partial algebra $\mathfrak{A} = (A, F)$, $R$ is compatible on $\mathfrak{A}$ whenever $\langle a_i, b_i\rangle \in R$ implies

$$\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in R \quad \text{provided} \quad f(a_1, \ldots, a_n),\ f(b_1, \ldots, b_n)$$

exist.

**Definition.** Let $R$ be a compatible relation on an algebra $\mathfrak{A}$, $\mathfrak{B} = (B, F)$ a partial algebra of the same type as $\mathfrak{A}$ and $P$ a compatible relation on $\mathfrak{B}$. We call $\langle \mathfrak{A}, R\rangle$ a *contraction of* $\langle \mathfrak{B}, P\rangle$ provided $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}(G)$ with mutually disjoint subsets as elements and $R$ is the restriction of $P(B)$.

A partial algebra $\mathfrak{B} = (B, F)$ is *partially ordered* if there exists a partial ordering $P$ on $B$ and $P$ is compatible with $\mathfrak{B}$.

**Theorem 3.** *Every algebra $\mathfrak{A}$ with a reflexive compatible relation $R$ is isomorphic with a contraction of a partially ordered partial algebra.*

Proof. Let $\mathfrak{A} = (A, F)$ be an algebra and $R$ a reflexive compatible relation on $\mathfrak{A}$. Put $C = A \times \{0, 1\}$.

1°. Suppose $f \in F$ is $n$-ary and $(a_1, i_1), \ldots, (a_n, i_n) \in C$. Define $f((a_1, i_1), \ldots \ldots, (a_n, i_n)) = (f(a_1, \ldots, a_n), i_1)$ provided $i_1 = \ldots = i_n$ only. Evidently, $\mathfrak{C} = (C, F)$ is a partial algebra of the same type as $\mathfrak{A}$.

2°. Introduce $P$ on $C$ by the rule:

$\langle (a, i), (b, j) \rangle \in P$ if and only if $(a, i) = (b, j)$ or $\langle a, b \rangle \in R$ and $i < j$. Clearly $P$ is reflexive. If $\langle (a, i), (b, j) \rangle \in P$ and $\langle (b, j), (a, i) \rangle \in P$, then we have only one possbility $(a, i) = (b, j)$, hence $P$ is antisymmetric. The transitivity of $P$ can be proved analogously, thus $P$ is a partial ordering on $C$.

3°. Let $\langle (a_r, i_r), (b_r, j_r) \rangle \in P$ for $r = 1, \ldots, n$ and $f \in F$ be $n$-ary. If $f((a_1, i_1), \ldots \ldots, (a_n, i_n))$, $f((b_1, j_1), \ldots, (b_n, j_n))$ are defined, then $i_1 = \ldots = i_n$, $j_1 = \ldots = j_n$ and $f((a_1, i_1), \ldots, (a_n, i_n)) = (f(a_1, \ldots, a_n), i_1)$, $f((b_1, j_1), \ldots, (b_n, j_n)) = (f(b_1, \ldots \ldots, b_n), j_1)$. If $i_1 < j_1$, then $i_r = 0$ and $j_r = 1$ for every $r$, hence $\langle a_r, b_r \rangle \in R$ and the compatibility of $R$ implies $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R$, thus $\langle (f(a_1, \ldots, a_n), i_1), (f(b_1, \ldots, b_n), j_1) \rangle \in P$. If $i_1 = j_1$, then $i_r = j_r$ for every $r$, hence $(a_r, i_r) = (b_r, j_r)$. From the reflexivity of $P$ we have also

$$(f(a_1, \ldots, a_n), i_1) = (f(b_1, \ldots, b_n), j_1), \quad \text{i.e. also}$$
$$\langle (f(a_1, \ldots, a_n), i_1), (f(b_1, \ldots, b_n), j_1) \rangle \in P.$$

We have no other possibility, thus the compatibility of $P$ on $\mathfrak{C}$ is proved.

4°. Let $\mathfrak{C}(G)$ or $P(C)$ be a global partial algebra or relation, respectively, and $B = \{\{(a, 0), (a, 1)\};\ a \in A\}$. Then $B \subseteq 2^C$ and $B$ contains only mutually disjoint subsets. Denote $a^* = \{(a, 0), (a, 1)\}$. If $a_1^*, \ldots, a_n^* \in B$ and $f \in F$ and $f(a_1, \ldots, a_n) = a$, then clearly $f(a_1^*, \ldots, a_n^*)$ exists and $f(a_1^*, \ldots, a_n^*) = a^*$, thus $\mathfrak{B} = (B, F)$ is a subalgebra (everywhere defined) of $\mathfrak{C}(G)$ and $\varphi : a \to a^*$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

5°. If $\langle a, b \rangle \in R$, then $\langle (a, 0), (b, 1) \rangle \in P$ and $\langle a^*, b^* \rangle \in P(C)$. Conversely, if $\langle a^*, b^* \rangle \in P(C)$, then $\langle (a, i), (b, j) \rangle \in P$ for some $(a, i) \in a^*$, $(b, j) \in b^*$. Thus either $(a, i) = (b, j)$ or $\langle a, b \rangle \in R$ and $i = 0$, $j = 1$. In both cases we have $\langle a, b \rangle \in R$. Therefore also the relational systems $\langle A, R \rangle$, $\langle B, P(C) \rangle$ are isomorphic which completes the proof.

A relation $P$ on a set $M$ is called a *strict ordering* provided $P$ is acyclic and transitive.

**Theorem 4.** *Every algebra $\mathfrak{A}$ with a compatible relation $R$ is isomorphic with a contraction of a strictly ordered partial algebra.*

The proof is analogous to that of Theorem 3, only the relation $P$ on $C$ is defined by the rule:

$$\langle (a, i), (b, j) \rangle \in P \quad \text{if and only if} \quad \langle a, b \rangle \in R \quad \text{and} \quad i < j.$$

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