Two-Dimensional Burst Identification Codes and Their Use in Burst Correction

KHALED A. S. ABDEL-GHAFFAR, ROBERT J. MCELIECE, FELLOW, IEEE, AND HENK C. A. VAN TILBORG, SENIOR MEMBER, IEEE

***Abstract*—A new class of codes, called burst identification codes, is defined and studied. These codes can be used to determine the patterns of burst errors. Two-dimensional burst correcting codes can be easily constructed from burst identification codes. The resulting class of codes is simple to implement and has lower redundancy than other comparable codes.**

I. Introduction and Summary

In most memory devices, information is stored in two dimensions. In such cases, errors usually take the form of two-dimensional bursts. For example, VLSI RAM chips are sensitive to alpha particles and other radiation effects which can cause two-dimensional bursts of errors. Two-dimensional burst correcting codes can be used to combat such errors. Elspas [1] has shown that the product of two cyclic codes is a burst correcting code. Later, Imai [2], [3] has constructed a class of two-dimensional burst correcting codes based on generalizing one-dimensional Fire codes [4]. Interleaving techniques were also used in [2] to construct two-dimensional burst correcting codes.

We construct a new class of two-dimensional burst correcting codes. The main idea behind this class is what we call "burst identification codes." Two-dimensional codes, as well as burst identification codes, are defined in Section II. A detailed study of one-dimensional and two-dimensional burst identification codes is given in Sections III, IV, and V. In Section VI two-dimensional burst correcting codes are constructed using burst identification codes. To compare our two-dimensional codes with other codes mentioned in the literature, a criterion, which we call the "excess redundancy," is used to estimate the redundancy in each class of codes. The codes constructed in Section VI are generally better, in terms of excess redundancy, than the other classes of codes. Although the codes constructed in Section VI are noncyclic, the complexity of their encoders and decoders, which are described in Section VII, is comparable to the complexity of the encoders and decoders of cyclic codes.

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K. A. S. Abdel-Ghaffar and R. J. McEliece are with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125.

H. C. A. van Tilborg is with the Department of Mathematics and Computing Science, Eindhoven University of Technology, 5600 MB, Eindhoven, The Netherlands.

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II. Preliminaries and Definitions

An $n_1 \times n_2$ array, where n_1 and n_2 are positive integers, is an array of n_1 rows and n_2 columns. A *binary two-dimensional code* of area $n_1 \times n_2$ is a set of $n_1 \times n_2$ binary arrays, whose elements are called *codewords*. A *binary linear two-dimensional code* $\mathscr{C}$ of area $n_1 \times n_2$ is a subspace of the $n_1 n_2$-dimensional space of $n_1 \times n_2$ arrays over $\boldsymbol{F}_2$. Let k be the dimension of $\mathscr{C}$, and $[g_{i,j}^{(1)}],\cdots,[g_{i,j}^{(k)}]$, where $0 \le i < n_1$, $0 \le j < n_2$, be a basis for $\mathscr{C}$. The $n_1 \times n_2$ matrix $G=[\boldsymbol{g}_{i,j}]$, where $\boldsymbol{g}_{i,j}=(g_{i,j}^{(1)},\cdots,g_{i,j}^{(k)})$ is called a *generator matrix* of $\mathscr{C}$. The *dual code* of $\mathscr{C}$, denoted by $\mathscr{C}^{\perp}$, is the null space of $\mathscr{C}$. If the $n_1 \times n_2$ matrix $H=[\boldsymbol{h}_{i,j}]$ is a generator for $\mathscr{C}^{\perp}$, then H is called a *parity check matrix* of the code $\mathscr{C}$. The elements of H are elements in the r-dimensional vector space over $\boldsymbol{F}_2$, where $r=n_1n_2-k$ is called the *redundancy* of the code. The *syndrome* of a binary array $[a_{i,j}]$ of area $n_1 \times n_2$, with respect to the parity check matrix H of $\mathscr{C}$, is defined as $\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}a_{i,j}\boldsymbol{h}_{i,j}$. Thus a binary array of area $n_1 \times n_2$ is a codeword in $\mathscr{C}$ if and only if its syndrome is zero with respect to any given parity check matrix.

The map $[a_{i,j}] \mapsto a(x,y)=\sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1}a_{i,j}x^iy^j$ defines an isomorphism between the n_1n_2-dimensional vector space of $n_1 \times n_2$ arrays over $\boldsymbol{F}_2$ and the vector space of bivariate polynomials $\{p(x,y) \in \boldsymbol{F}_2[x,y]\colon \deg_x p(x,y) < n_1, \deg_y p(x,y) < n_2\}$. We will frequently identify each array with its image under this isomorphism.

A binary two-dimensional linear code $\mathscr{C}$ is said to be *cyclic* if $xc(x,y)$ and $yc(x,y)$, both $\bmod(x^{n_1}+1, y^{n_2}+1)$, are in $\mathscr{C}$ for each $c(x,y) \in \mathscr{C}$. Thus a cyclic code of area $n_1 \times n_2$ is an ideal in the residue class ring $\boldsymbol{F}_2[x,y]/(x^{n_1}+1, y^{n_2}+1)$.

The pairs of positive integers will be partially ordered by saying that (b_1,b_2) is *less* than (n_1,n_2) if $b_1 \le n_1$, $b_2 \le n_2$, and $b_1b_2 \ne n_1n_2$. We denote this by $(b_1,b_2) \prec (n_1,n_2)$. A $b_1 \times b_2$-*burst*, $(b_1,b_2) \preccurlyeq (n_1,n_2)$, is a nonzero $n_1 \times n_2$ binary array whose nonzero components are confined to a rectangle of area $b_1 \times b_2$. Let $\{(i,j)\colon u_1 \le i < u_1+b_1', u_2 \le j < u_2+b_2'\}$, where $0 \le u_1 \le n_1-b_1'$, $0 \le u_2 \le n_2-b_2'$, be the smallest rectangle containing the nonzero components of the $b_1 \times b_2$-burst $B=[a_{i,j}], 0 \le i < n_1$, $0 \le j < n_2$. Then

B is said to have area $b_1' \times b_2'$. For $b_1' \le b_1'' \le b_1$ and $b_2' \le b_2'' \le b_2$, we say that B has the *pattern* $[a_{i,j}]$, $u_1 \le i < u_1 + b_1''$, $u_2 \le j < u_2 + b_2''$, starting at *position* (u_1, u_2). In the following, it is more convenient to speak about "the" pattern of B by considering $[a_{i,j}]$, $u_1 \le i < u_1 + b_1''$, $u_2 \le j < u_2 + b_2''$, to represent the same pattern for all $b_1' \le b_1'' \le b_1$ and $b_2' \le b_2'' \le b_2$. By this convention, it is to be noted that the pattern and the starting position of any burst are unique. Thus $[a_{i,j}]$ is a $b_1 \times b_2$-burst if and only if $a(x, y) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} a_{i,j}x^i y^j = x^{u_1}y^{u_2}b(x, y)$, for some $0 \le u_1 \le n_1 - b_1$, $0 \le u_2 \le n_2 - b_2$, and $b(x, y) \in \mathscr{B}_{b_1 b_2}$, where

$$\mathscr{B}_{b_1 b_2} = \{ p(x, y) \in \boldsymbol{F}_2[x, y] : \deg_x p(x, y) < b_1, \deg_y p(x, y) < b_2, p(x,0) \ne 0, p(0, y) \ne 0 \}.$$

In such case the burst pattern is given by the polynomial $b(x, y)$ and its starting position is (u_1, u_2). The array $[a_{i,j}]$, $0 \le i < n_1$, $0 \le j < n_2$, is called a $b_1 \times b_2$-*cyclic burst* if $a(x, y) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} a_{i,j}x^i y^j \equiv x^{u_1}y^{u_2}b(x, y)$ (mod $x^{n_1}+1, y^{n_2}+1$), for some $0 \le u_1 < n_1$, $0 \le u_2 < n_2$, and $b(x, y) \in \mathscr{B}_{b_1 b_2}$. Thus a $b_1 \times b_2$-burst is a $b_1 \times b_2$-cyclic burst, but the converse does not always hold. The starting position (u_1, u_2) of the cyclic burst and its pattern, which is given by $b(x, y)$, are not necessarily unique. This will be considered in the following lemma.

Lemma 1: A necessary and sufficient condition for all $b_1 \times b_2$-cyclic bursts $[a_{i,j}]$, $0 \le i < n_1$, $0 \le j < n_2$, to have unique patterns and starting positions is $n_1 \ge 2b_1 - 1$ and $n_2 \ge 2b_2 - 1$.

Proof: If $n_1 < 2b_1 - 1$, then the burst $[a_{i,j}]$, defined by $a_{i,j} = 1$ if and only if $(i, j) \in \{(0,0), (b_1 - 1, 0)\}$ has starting positions $(0,0)$ and $(b_1 - 1, 0)$ since

$$1 + x^{b_1 - 1} \equiv x^{b_1 - 1}(1 + x^{n_1 - b_1 + 1}) \pmod{x^{n_1}+1, y^{n_2}+1}$$

and $\deg_x (1 + x^{n_1 - b_1 + 1}) = n_1 - b_1 + 1 < b_1$. Thus $n_1 \ge 2b_1 - 1$ is a necessary condition. By similarity, $n_2 \ge 2b_2 - 1$ is also necessary. Conversely, suppose $n_1 \ge 2b_1 - 1$, $n_2 \ge 2b_2 - 1$, and

$$x^{u_1}y^{u_2}b'(x, y) \equiv x^{v_1}y^{v_2}b''(x, y) \pmod{x^{n_1}+1, y^{n_2}+1}$$

where $0 \le u_1, v_1 < n_1$, $0 \le u_2, v_2 < n_2$, $b'(x, y), b''(x, y) \in \mathscr{B}_{b_1 b_2}$. Multiplying both sides by $x^{n_1 - u_1}y^{n_2 - u_2}$, we obtain

$$b'(x, y) + x^{v_1 - u_1}y^{v_2 - u_2}b''(x, y) \equiv 0 \pmod{x^{n_1}+1, y^{n_2}+1}.$$

From $b''(0, y) \ne 0$, it follows that $0 \le v_1 - u_1 < b_1$. From $b'(0, y) \ne 0$, it follows that $v_1 - u_1 \le 0$ or $v_1 - u_1 > n_1 - b_1$. Since $n_1 \ge 2b_1 - 1$, we have $u_1 = v_1$. Similarly, $u_2 = v_2$ and hence $b'(x, y) = b''(x, y)$. □

The following lemma gives the number of distinct $b_1 \times b_2$-burst patterns, which we denote by $N(b_1, b_2)$.

Lemma 2: The number of distinct $b_1 \times b_2$-burst patterns $N(b_1, b_2)$ is given by

$$N(b_1, b_2) = 2^{b_1 b_2 - 1} + (2^{b_1 - 1} - 1)(2^{b_2 - 1} - 1) \times 2^{(b_1 - 1)(b_2 - 1)}.$$

Hence

$$2^{b_1 b_2 - 1} \le N(b_1, b_2) < 2^{b_1 b_2}$$

and the equality holds if and only if b_1 or b_2 is 1.

Proof: From the definitions, it follows that $N(b_1, b_2)$ is the total number of binary $b_1 \times b_2$-arrays with the property that their first row and column are nonzero. The first and second terms give the number of arrays satisfying this property with "1" and "0" at position (0,0), respectively. □

In this paper, we will consider binary linear codes only. If $n_1 = 1$, we say that the code is a *one-dimensional code* of length n_2. In such case, the first dimension will be suppressed. Hence, from Lemma 2, it follows that the number of distinct b-burst patterns is given by $N(b) = 2^{b-1}$. According to our notation, the parity check matrix of a one-dimensional code of length n and redundancy r is $1 \times n$ matrix $[\boldsymbol{h}_0, \cdots, \boldsymbol{h}_{n-1}]$, where $\boldsymbol{h}_0, \cdots, \boldsymbol{h}_{n-1} \in \boldsymbol{F}_2^r$.

A two-dimensional linear code $\mathscr{C}$ is said to be a $b_1 \times b_2$-*burst identification code* if no codeword is a $b_1 \times b_2$-burst, or a sum of two $b_1 \times b_2$-bursts of different patterns. Equivalently, the code $\mathscr{C}$ is a $b_1 \times b_2$-burst identification code if and only if the syndromes of the $b_1 \times b_2$-bursts with respect to any given parity check matrix of $\mathscr{C}$ are nonzero and distinct for distinct burst patterns.

If a $b_1 \times b_2$-burst identification code is used over a channel that may add to any transmitted codeword a $b_1 \times b_2$-burst, then the receiver can determine the burst pattern added by the channel. It is important to note that the receiver may not be able uniquely to determine the burst position. Hence the transmitted codeword may not be uniquely determined. Thus a $b_1 \times b_2$-burst correcting code is a $b_1 \times b_2$-burst identification code, but the converse does not always hold. In other words, a $b_1 \times b_2$-burst identification code may contain a codeword which is the sum of two $b_1 \times b_2$-bursts of the same pattern.

We define $r_{n_1 \times n_2}(b_1, b_2)$ to be the minimum redundancy required to construct a $b_1 \times b_2$-burst identification code of area $n_1 \times n_2$. Clearly, $r_{n_1 \times n_2}(b_1, b_2)$ is a non-decreasing function in n_1 and n_2. In Section IV, we will derive an upper bound on $r_{n_1 \times n_2}(b_1, b_2)$ which is independent of n_1 and n_2. Thus $r_{n_1 \times n_2}(b_1, b_2)$ is constant for n_1 and n_2 sufficiently large. This constant will be denoted by $r(b_1, b_2)$. In the next section, we start studying burst identification codes by considering the one-dimensional case.

III. One-Dimensional Burst Identification Codes

A one-dimensional code is a b-burst identification code if and only if the syndromes of the b-bursts are nonzero and distinct for distinct burst patterns. Since the number of different burst patterns is $N(b) = 2^{b-1}$, it follows that the minimum redundancy required to construct a b-burst identification code of arbitrarily large length is bounded by $r(b) \ge \lceil \log_2(1 + 2^{b-1}) \rceil$, which implies $r(b) \ge b$. It is obvious that $r(1) = 1$, which is achieved by a code whose parity check matrix is $H=[1,1,\cdots,1]$. The bound $r(b)\geq b$ follows also from the following lemma which is an immediate consequence of the definition.

Lemma 3: Let H be a parity check matrix of a b-burst identification code. Then, every b consecutive elements of H are linearly independent.

The next lemma shows that a stronger bound holds for $b\geq 3$.

Lemma 4: Let H be a parity check matrix of a b-burst identification code of length $n\geq 2b-2$. Then every $2b-2$ consecutive elements of H are linearly independent.

Proof: If $b\leq 2$, the result follows from Lemma 3. So, we assume in the following that $b\geq 3$. Let $H=[\boldsymbol{h}_0,\boldsymbol{h}_1,\cdots,\boldsymbol{h}_{n-1}]$ and suppose that $\sum_{i=0}^{u-1}a_i\boldsymbol{h}_{l+i}=\boldsymbol{0}$, where $a_i\in\boldsymbol{F}_2$, $a_0=a_{u-1}=1, 1\leq u\leq 2b-2$, and $0\leq l\leq n-u$. Lemma 3 implies that $u\geq b+1$. The burst pattern $(1, a_1+1, a_b, a_{b+1},\cdots,a_{u-1})$ starting at position $l+b-2$ has syndrome $\boldsymbol{h}_{l+b-2}+(a_1+1)\boldsymbol{h}_{l+b-1}+\sum_{i=b}^{u-1}a_i\boldsymbol{h}_{l+i}=\sum_{i=0}^{b-3}a_i\boldsymbol{h}_{l+i}+(a_{b-2}+1)\boldsymbol{h}_{l+b-2}+(a_1+a_{b-1}+1)\boldsymbol{h}_{l+b-1}$. However, the burst pattern $(1, a_1,\cdots,a_{b-3}, a_{b-2}+1, a_1+a_{b-1}+1)$ starting at position l has the same syndrome, but obviously a different pattern (compare the second coordinate in the burst patterns). This contradicts the assumption that the code is a b-burst identification code. □

It follows from this lemma that $r(b)\geq 2b-2$. The following theorem gives a construction of b-burst identification codes, for $b\geq 2$, of arbitrarily large lengths and with redundancy $2b-2$. First, we define $\boldsymbol{e}_i=(e_{i,0},e_{i,1},\cdots,e_{i,2b-3})$, $0\leq i<2b-2$, by $e_{i,i}=1$, and $e_{i,j}=0$ for $i\neq j$. So $\boldsymbol{e}_i$ is the ith canonical vector of length $2b-2$.

Theorem 5: Let $b\geq 2$ and $\boldsymbol{e}_i$ be the ith canonical vector of length $2b-2$. Then the code $\mathscr{C}$ of length n whose parity check matrix is given by $H=[\boldsymbol{h}_0,\boldsymbol{h}_1,\cdots,\boldsymbol{h}_{n-1}]$, where $\boldsymbol{h}_i=\boldsymbol{e}_{i \bmod (2b-2)}$ is a b-burst identification code with redundancy $2b-2$.

Proof: Consider a burst of length $b'\leq b$. Let $(a_0=1, a_1,\cdots,a_{b'-1}=1)$ be its pattern and l its position, where $0\leq l\leq n-b'$. The syndrome of this burst is

$$s=\sum_{i=0}^{b'-1}a_i\boldsymbol{h}_{l+i}=\sum_{i=0}^{b'-1}a_i\boldsymbol{e}_{l+i \bmod (2b-2)}.$$

The vectors $\boldsymbol{e}_{l+i \bmod (2b-2)}$ for $0\leq i<b'$ are distinct since $b'\leq b\leq 2b-2$. Hence the weight of the vector $\boldsymbol{s}$, i.e., the number of its nonzero components, is equal to the weight of the burst pattern. This ends the proof for $b=2$ since the burst patterns are either (1) or (1,1). Now, let $b\geq 3$. Then $\boldsymbol{s}=(s_0,s_1,\cdots,s_{2b-3})$ is a cyclic shift of the $(2b-2)$-tuple $(a_0,a_2,\cdots,a_{b'-1},0,\cdots,0)$. Hence if $\boldsymbol{s}$ has a unique cyclic string of consecutive zeros of length $\geq b-2$, then the burst pattern $(a_0,a_1,\cdots,a_{b'-1})$ can be uniquely deduced from $\boldsymbol{s}$. If this is not the case, then $\boldsymbol{s}$ has two cyclic strings of $b-2$ zeros each, which occurs if and only if $b'=b$ and the burst pattern is $(a_0=1,0,0,\cdots,0,a_{b-1}=1)$. Also in this case the burst pattern is uniquely determined from the syndrome. □

Combining Lemma 4 and Theorem 5, along with the fact that $r(1)=1$, Theorem 6 follows.

Theorem 6: $r(1)=1$, and $r(b)=2b-2$ for $b\geq 2$.

IV. Two-Dimensional Burst Identification Codes

A two-dimensional code is a $b_1\times b_2$-burst identification code if and only if the syndromes of the $b_1\times b_2$-bursts are nonzero and distinct for distinct burst patterns. Since the number of different patterns is $N(b_1,b_2)$, as given in Lemma 2, it follows that the minimum redundancy required to construct a $b_1\times b_2$-burst identification code of arbitrarily large area is bounded by $r(b_1,b_2)\geq\lceil\log_2(1+N(b_1,b_2))\rceil$, which implies $r(b_1,b_2)\geq b_1b_2$. In this section, we will prove that $2b_1b_2-2\leq r(b_1,b_2)\leq 2b_1b_2$. The following two lemmas are the two-dimensional versions of Lemmas 3 and 4.

Lemma 7: Let $H=[\boldsymbol{h}_{i,j}]$, $0\leq i<n_1, 0\leq j<n_2$, be a parity check matrix of a $b_1\times b_2$-burst identification code. Then, for every pair of integers (u_1,u_2), such that $0\leq u_1\leq n_1-b_1, 0\leq u_2\leq n_2-b_2$, the vectors $\boldsymbol{h}_{i,j}$, for $u_1\leq i<u_1+b_1, u_2\leq j<u_2+b_2$ are linearly independent.

The proof of the previous lemma follows immediately from the definition.

Lemma 8: Let $H=[\boldsymbol{h}_{i,j}]$, $0\leq i<n_1$, $0\leq j<n_2$, be a parity check matrix of a $b_1\times b_2$-burst identification code. Let $0\leq u_1,v_1\leq n_1-b_1, 0\leq u_2,v_2\leq n_2-b_2, I_{u_1,u_2}=\{(i,j): u_1\leq i<u_1+b_1, u_2\leq j<u_2+b_2\}$, and define I_{v_1,v_2} similarly. If $|I_{u_1,u_2}\cap I_{v_1,v_2}|\geq 2$, then the vectors $h_{i,j}$, $(i,j)\in I_{u_1,u_2}\cup I_{v_1,v_2}$ are linearly independent.

Proof: Without loss of generality, assume that $u_1\leq v_1$ and $u_2\leq v_2$. Let $J=I_{u_1,u_2}\cap I_{v_1,v_2}=\{(i,j): v_1\leq i<u_1+b_1, v_2\leq j<u_2+b_2\}$, and suppose $|J|\geq 2$. We may assume that $u_1+b_1-v_1\geq 2$, otherwise interchange i and j (see Fig. 1). Now, suppose that $\sum_{(i,j)\in I_{u_1,u_2}\cup I_{v_1,v_2}}a_{i,j}\boldsymbol{h}_{i,j}=\boldsymbol{0}$, $a_{i,j}\in\boldsymbol{F}_2$, and not all are zero.

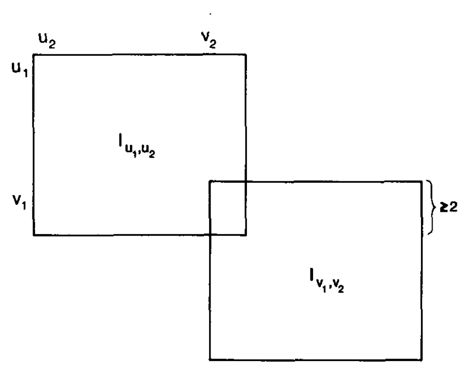

Fig. 1. I_{u_1,u_2} and I_{v_1,v_2}.

Let x be an indeterminate. Define the array $[c_{i,j}]$, $0\leq i<b_1$, $0\leq j<b_2$, by $c_{0,0}=1$, $c_{1,0}=x$, and $c_{i,j}=a_{v_1+i,v_2+j}$ otherwise. The array $[c_{i,j}]$ defines the pattern of a $b_1\times b_2$-burst starting at position (v_1,v_2). Let B_1 be this

burst. The syndrome of B_1 is

$$\sum_{i=0}^{b_1-1}\sum_{j=0}^{b_2-1} c_{i,j}\boldsymbol{h}_{v_1+i,v_2+j}$$

$$= h_{v_1,v_2} + x\boldsymbol{h}_{v_1+1,v_2}$$

$$+ \sum_{(i,j)\in I_{v_1,v_2}-\{(v_1,v_2),(v_1+1,v_2)\}} a_{i,j}\boldsymbol{h}_{i,j}$$

$$= \left(1+a_{v_1,v_2}\right)\boldsymbol{h}_{v_1,v_2} + \left(x+a_{v_1+1,v_2}\right)\boldsymbol{h}_{v_1+1,v_2}$$

$$+ \sum_{(i,j)\in I_{v_1,v_2}} a_{i,j}\boldsymbol{h}_{i,j}$$

$$= \sum_{(i,j)\in I_{u_1,u_2}-J} a_{i,j}\boldsymbol{h}_{i,j} + \left(1+a_{v_1,v_2}\right)\boldsymbol{h}_{v_1,v_2}$$

$$+ \left(x+a_{v_1+1,v_2}\right)\boldsymbol{h}_{v_1+1,v_2}.$$

Define the array $[d_{i,j}]$, $0 \le i < b_1, 0 \le j < b_2$, as

$$d_{i,j} = \begin{cases} a_{u_1+i,u_2+j}, & \text{if } (u_1+i, u_2+j) \in I_{u_1,u_2} - J; \\ 1 + a_{v_1,v_2}, & \text{if } (u_1+i, u_2+j) = (v_1, v_2); \\ x + a_{v_1+1,v_2}, & \text{if } (u_1+i, u_2+j) = (v_1+1, v_2); \\ 0, & \text{otherwise.} \end{cases}$$

Let B_2 be the burst whose pattern is defined by the array $[d_{i,j}]$, $0 \le i < b_1$, $0 \le j < b_2$, and whose starting position is (u_1', u_2'), where $u_1' = u_1 + t_1$, $u_2' = u_2 + t_2$, and t_1, t_2 are the maximum values for which the rectangle $\{(i,j)\colon t_1 \le i < b_1, t_2 \le j < b_2\}$ contains all the nonzero components of the array $[d_{i,j}]$, $0 \le i < b_1$, $0 \le j < b_2$. From Lemma 7, it follows that $(u_1', u_2') \in I_{u_1,u_2} - J$. The syndrome of B_2, which is $\sum_{i=0}^{b_1-1}\sum_{j=0}^{b_2-1} d_{i,j}\boldsymbol{h}_{u_1+i,u_2+j}$, is the same as the syndrome of B_1. Hence B_1 and B_2 should have the same pattern. This implies that $d_{u_1'+i,u_2'+j} = c_{i,j}$ for $0 \le i < b_1 - u_1'$, $0 \le j < b_2 - u_2'$ (see Fig. 2). In particular, it implies that $d_{u_1'+1,u_2'} = x$, independent of our choice of x! However, $(u_1', u_2') \in I_{u_1,u_2} - J$ implies $(u_1'+1, u_2') \ne (v_1+1, v_2)$. Since d_{v_1+1,v_2} is the only element in the array $[d_{i,j}]$, $0 \le i < b_1$, $0 \le j < b_2$, that depends on x, it follows that $d_{u_1'+1,u_2'}$ does not depend on x. By choosing $x = 1 + d_{u_1'+1,u_2'}$, we get a contradiction. □

It follows from the previous lemma that $r(b_1, b_2) \ge 2b_1b_2 - 2$.

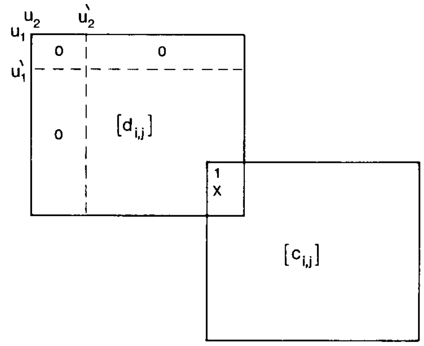

Fig. 2. Arrays $[c_{i,j}]$ and $[d_{i,j}]$.

Before presenting a construction of a $b_1 \times b_2$-burst identification code with a redundancy $2b_1b_2$, we have to say something about the notation that we shall use. Let $\mathcal{Q}$ be the set $\{0', 1', \cdots, (b_1b_2-1)'\} \cup \{0'', 1'', \cdots, (b_1b_2-1)''\}$. A vector $\boldsymbol{h} \in \boldsymbol{F}_2^{2b_1b_2}$ can be represented as $\boldsymbol{h} = (h_{0'}, h_{1'}, \cdots, h_{(b_1b_2-1)'}, h_{0''}, h_{1''}, \cdots, h_{(b_1b_2-1)''})$. We associate with every vector $\boldsymbol{h} \in \boldsymbol{F}_2^{2b_1b_2}$ its characteristic set $\mathcal{H} = \{q \in \mathcal{Q}\colon h_q = 1\}$. In particular, the parity check matrix $H = [\boldsymbol{h}_{i,j}]$ can be represented by the sets $\mathcal{H}_{i,j}$ instead of the vectors $\boldsymbol{h}_{i,j}$.

In the following construction of burst identification codes, we define a matrix $\hat{H} = [\hat{\boldsymbol{h}}_{i,j}]$, $0 \le i < 2b_1$, $0 \le j < 2b_2$. The parity check matrix of a code of area $n_1 \times n_2$, denoted by $H = [\boldsymbol{h}_{i,j}]$, $0 \le i < n_1$, $0 \le j < n_2$, is then defined periodically by $\boldsymbol{h}_{i,j} = \hat{\boldsymbol{h}}_{i \bmod 2b_1, j \bmod 2b_2}$. The matrix $\hat{H}$ is called the *building block* of the code.

Theorem 9: Let the elements of the $2b_1 \times 2b_2$ building block $\hat{\mathcal{H}}$ be defined as

$$\hat{\mathcal{H}}_{i,j} = \{(ib_2+j)'\}$$

$$\hat{\mathcal{H}}_{i,j+b_2} = \{(ib_2+j)', (ib_2+j)''\}$$

$$\hat{\mathcal{H}}_{i+b_1,j} = \{(ib_2+j)', (((i+1) \bmod b_1)b_2+j)''\}$$

$$\hat{\mathcal{H}}_{i+b_1,j+b_2} = \{(ib_2+j)''\}$$

where $0 \le i < b_1$, $0 \le j < b_2$. Then, $\hat{\mathcal{H}}$ is the building block of a $b_1 \times b_2$-burst identification code of redundancy $2b_1b_2$.

Before giving the proof, the following is an example of the construction of the building block for $b_1 = b_2 = 3$.

Example: With $b_1 = b_2 = 3$, the building block $\hat{\mathcal{H}}$ is

$$\begin{bmatrix} \{0'\} & \{1'\} & \{2'\} & \{0',0''\} & \{1',1''\} & \{2',2''\} \\ \{3'\} & \{4'\} & \{5'\} & \{3',3''\} & \{4',4''\} & \{5',5''\} \\ \{6'\} & \{7'\} & \{8'\} & \{6',6''\} & \{7',7''\} & \{8',8''\} \\ \{0',3''\} & \{1',4''\} & \{2',5''\} & \{0''\} & \{1''\} & \{2''\} \\ \{3',6''\} & \{4',7''\} & \{5',8''\} & \{3''\} & \{4''\} & \{5''\} \\ \{6',0''\} & \{7',1''\} & \{8',2''\} & \{6''\} & \{7''\} & \{8''\} \end{bmatrix}. \qquad (1)$$

Proof: It is clear from the construction that the redundancy is $2b_1b_2$. Let B be a $b_1 \times b_2$-burst whose pattern is $[a_{i,j}]$, $0 \le i < b_1, 0 \le j < b_2$, starting at position (u_1, u_2). Its syndrome is given by

$$s = \sum_{i=0}^{b_1-1} \sum_{j=0}^{b_2-1} a_{i,j} \boldsymbol{h}_{u_1+i, u_2+j}.$$

Let $J = \{(u_1+i, u_2+j): a_{i,j} = 1\}$ be the set of positions of the nonzero elements of B. The projection of the burst B on the building block is a cyclic burst $\hat{B}$, where the positions of its nonzero components are given by the set $\hat{J} = \{i \bmod 2b_1, j \bmod 2b_2: (i,j) \in J\}$. Since the area of building block is $2b_1 \times 2b_2$, it follows from Lemma 1 that the cyclic burst $\hat{B}$ has unique burst pattern and starting position. Hence, it suffices to show that the burst pattern of $\hat{B}$ can be uniquely determined from the syndrome s. In fact, we will show that from s we can even uniquely determine $\hat{J}$, except for few cases in which $\hat{J}$ is determined up to a shift of b_1 and b_2, in the vertical and horizontal directions, respectively.

From the definition of $\hat{\mathscr{H}}$, it follows that $\hat{\mathscr{H}}_{i,j}$ and $\hat{\mathscr{H}}_{i',j'}$ are disjoint if $(i,j) \ne (i',j')$, $|i-i'| < b_1$, and $|j-j'| < b_2$. So, if $\mathscr{S}$ denotes the characteristic set of the syndrome s, then from the construction of the parity check matrix of the code we have

$$\mathscr{S} = \biguplus_{(i,j) \in \hat{J}} \hat{\mathscr{H}}_{i,j} \tag{2}$$

where $\uplus$ denotes the union of disjoint sets. It also follows from the definition of $\hat{\mathscr{H}}$ that if $\hat{\mathscr{H}}_{i,j}$ contains l' or l'', then $l \equiv j \pmod{b_2}$. Let $\hat{J}_l = \{(i,j) \in \hat{J}: j \equiv l \pmod{b_2}\}$, $\mathscr{S}_l' = \{s' \in \mathscr{S}: s \equiv l \pmod{b_2}\}$, $\mathscr{S}_l'' = \{s'' \in \mathscr{S}: s \equiv l \pmod{b_2}\}$, and $\mathscr{S}_l = \mathscr{S}_l' \uplus \mathscr{S}_l''$, where $0 \le l < b_2$. Thus $\hat{J}_l$ is the restriction of $\hat{J}$ to columns l and $l + b_2$. Hence, $\hat{J} = \bigcup_{0 \le l < b_2} \hat{J}_l$ and $\mathscr{S} = \biguplus_{0 \le l < b_2} \mathscr{S}_l$, where $\mathscr{S}_l = \biguplus_{(i,j) \in \hat{J}_l} \hat{\mathscr{H}}_{i,j}$ for $0 \le l < b_2$, which follows from (2). It is sufficient to show that from each individual $\mathscr{S}_l$, where $0 \le l < b_2$, we can determine uniquely the burst pattern defined on $\hat{J}_l$, which is a $b_1 \times 1$-burst. This is demonstrated only for $l = 0$ since the other values of l can be treated similarly.

Notice from the construction of $\hat{\mathscr{H}}$ that the number of elements contained in $\hat{\mathscr{H}}_{i,0}$ from the set $\{0', 1', \cdots, (b_1b_2-1)\}$ is at least equal to the number of elements contained in $\hat{\mathscr{H}}_{i,0}$ from the set $\{0'', 1'', \cdots, (b_1b_2-1)''\}$. Of course, the same holds for disjoint unions of $\hat{\mathscr{H}}_{i,0}$. The converse holds for $\hat{\mathscr{H}}_{i,b_2}$. Equality occurs only in $\hat{\mathscr{H}}_{i,0}$ for $b_1 \le i < 2b_1$, and in $\hat{\mathscr{H}}_{i,b_2}$ for $0 \le i < b_1$.

Hence we have the following set of rules for identifying the $b_1 \times 1$-burst pattern defined by $\hat{J}_0$ from $\mathscr{S}_0$. For other values of l, these rules are also applicable after obvious modifications.

Rule 1, If $|\mathscr{S}_0'| > |\mathscr{S}_0''|$ *or* $|\mathscr{S}_0'| = |\mathscr{S}_0''|$ *and* $\mathscr{S}_0'' \ne \{(ib_2)'': (ib_2)' \in \mathscr{S}_0'\}$*:* In this case, the elements of $\hat{J}_0$ are of the form $(i, 0)$. In fact, $\hat{J}_0 = \{(i,0): (ib_2)' \in \mathscr{S}_0', (((i+1) \bmod b_1)b_2)'' \notin \mathscr{S}_0''\} \cup \{(i+b_1, 0): (ib_2)' \in \mathscr{S}_0', (((i+1) \bmod b_1)b_2)'' \in \mathscr{S}_0''\}$.

Rule 2, If $|\mathscr{S}_0'| < |\mathscr{S}_0''|$*, or* $|\mathscr{S}_0'| = |\mathscr{S}_0''|$ *and* $\mathscr{S}_0'' \ne \{(((i+1) \bmod b_1)b_2)'': (ib_2)' \in \mathscr{S}_0'\}$*:* In this case, the elements of $\hat{J}_0$ are of the form (i, b_2). In fact, $\hat{J}_0 = \{(i, b_2): (ib_2)' \in \mathscr{S}_0', (ib_2)'' \in \mathscr{S}_0''\} \cup \{(i+b_1, b_2): (ib_2)' \notin \mathscr{S}_0', (ib_2)'' \in \mathscr{S}_0''\}$.

Rule 3, If $\mathscr{S}_0'' = \{(ib_2)'': (ib_2)' \in \mathscr{S}_0'\} = \{(((i+1) \bmod b_1)b_2)'': (ib_2)' \in \mathscr{S}_0'\}$*:* In this case, either $\mathscr{S}_0' = \mathscr{S}_0'' = \mathscr{S}_0 = \phi$, the null set, which implies $\hat{J}_0 = \phi$, or $\mathscr{S}_0' = \{(ib_2)': 0 \le i < b_1\}$ and $\mathscr{S}_0'' = \{(ib_2)'': 0 \le i < b_1\}$. The latter case implies that $\hat{J}_0 = \{(i,0): b_1 \le i < 2b_1\}$ or $\hat{J}_0 = \{(i, b_2): 0 \le i < b_1\}$. However, these two possibilities for $\hat{J}_0$ give the same pattern.

By applying this algorithm to $\mathscr{S}_l$ for $l = 1, 2, \cdots, b_2 - 1$, this ambiguity will be resolved unless when rule 3 is applicable for all values of l. In the latter situation, two possibilities for $\hat{J}$ can be deduced from the syndrome $\mathscr{S}$. However, both give the same burst pattern. □

Example (continued): Let $\mathscr{S} = \{2', 5', 6', 7', 8', 4'', 5'', 6'', 7'', 8''\}$ be a syndrome with respect to the building block of the 3×3-burst identification code given in (1). Then $\mathscr{S}_0' = \{6'\}, \mathscr{S}_0'' = \{6''\}$. Rule 2 applies, and we find $\hat{J}_0 = \{(2,3)\}$. Similarly, $\mathscr{S}_1' = \{7'\}, \mathscr{S}_1'' = \{4'', 7''\}$, and rule 2 yields $\hat{J}_1 = \{(2,4),(4,4)\}$. Finally, $\mathscr{S}_2' = \{2', 5', 8'\}, \mathscr{S}_2'' = \{5'', 8''\}$, and rule 1 yields $\hat{J}_2 = \{(2,2),(3,2),(4,2)\}$. Thus the cyclic burst $\hat{B}$ deduced from $\hat{J}_0, \hat{J}_1, \hat{J}_2$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and its pattern is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Example (continued): Let $\mathscr{S} = \{0', 1', 3', 6', 0'', 3'', 4'', 6'', 8''\}$ be a syndrome with respect to the building block of the 3×3-burst identification code given in (1). Then $\mathscr{S}_0' = \{0', 3', 6'\}$, $\mathscr{S}_0'' = \{0'', 3'', 6''\}$, and rule 3 yields $\hat{J}_0 = \{(3,0),(4,0),(5,0)\}$ or $\{(0,3),(1,3),(2,3)\}$. We also have $\mathscr{S}_1' = \{1'\}, \mathscr{S}_1'' = \{4''\}$, and rule 1 yields $\hat{J}_1 = \{(3,1)\}$. Since the burst is assumed to be confined to a rectangle of area 3×3, then $\hat{J}_0 = \{(3,0),(4,0),(5,0)\}$. Finally, $\mathscr{S}_2' = \phi, \mathscr{S}_2'' = \{8''\}$, and rule 2 yields $\hat{J}_2 = \{(5,5)\}$. So, the cyclic burst $\hat{B}$ deduced from $\hat{J}_0, \hat{J}_1, \hat{J}_2$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and its pattern is given by

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The next theorem, which is the most important result of this section, follows from Lemma 8 and Theorem 9.

Theorem 10: Let $r(b_1, b_2)$ be the minimum redundancy required to construct a $b_1 \times b_2$-burst identification code of arbitrarily large area. Then

$$2b_1b_2 - 2 \le r(b_1, b_2) \le 2b_1b_2.$$

V. Some Specific Burst Identification Codes

In this section we will consider $b_1 \times b_2$-burst identification codes for some specific values of b_1 and b_2. We note that if H is a parity check matrix of a $b_1 \times b_2$-burst identification code, then the transpose of H is a parity check matrix of a $b_2 \times b_1$-burst identification code, and so $r(b_1, b_2) = r(b_2, b_1)$.

A. $1 \times b$-Burst Identification Codes

Obviously, $r(1,1) = 1$ which is achieved by a code whose parity check matrix is composed of ones. So, in the following we take $b > 1$. The next theorem gives an explicit construction for $1 \times b$-burst identification code with $2b-2$ redundant bits.

Theorem 11: Let $b > 1$ and let $\boldsymbol{e}_i$ be the ith canonical vector of length $2b-2$. Then, the code of area $n_1 \times n_2$ whose parity check matrix is given by $H = [\boldsymbol{h}_{i,j}]$, $0 \le i < n_1, 0 \le j < n_2$, where $\boldsymbol{h}_{i,j} = \boldsymbol{e}_{j \bmod (2b-2)}$, is a $1 \times b$-burst identification code of redundancy $2b-2$.

Proof: The patterns of the $1 \times b$-bursts are the same as those of the one-dimensional b-bursts. In Theorem 5, a construction is given of a one-dimensional b-burst identification code. Hence the code defined in Theorem 11, which is simply the code defined in Theorem 5 repeated n_1 times, is a $1 \times b$-burst identification code. The redundancy is obviously $2b-2$. □

Combining this result with Theorem 10, we obtain the following theorem.

Theorem 12: $r(1,1) = 1$, and $r(1,b) = r(b,1) = 2b-2$ for $b > 1$.

B. 2×2-Burst Identification Codes

From Theorem 10, we know that $6 \le r(2,2) \le 8$. Here, we will prove that $r(2,2) = 7$. First we will show that $r(2,2) > 6$. Suppose that H is a 3×4 submatrix of a parity check matrix of a 2×2-burst identification code with redundancy 6. By studying the structure of H, we will establish a contradiction. By Lemma 8, we may assume, without loss of generality, that H has the form

$$\begin{bmatrix} \boldsymbol{p} & \boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{u} \\ \boldsymbol{q} & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} \\ \boldsymbol{t} & \boldsymbol{e}_4 & \boldsymbol{e}_5 & \boldsymbol{w} \end{bmatrix}$$

where $\boldsymbol{e}_i$ is the ith canonical vector of length 6, and $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{t}$, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors of length 6. We shall write $\boldsymbol{p} = (p_0, p_1, \cdots, p_5)$, and the same notation holds for the other vectors.

Lemma 13: $q_1 = q_5 = v_0 = v_4 = 1$.

Proof: Suppose that $q_1 = 0$. By Lemma 8, applied to $\{(i,j)\colon i = 1,2,\ j = 0,1,2\}$, it follows that $q_0 = 1$. The burst

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1+q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has syndrome $(0,0,1,q_3,q_4,q_5)$, as does the burst

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & q_3 & 0 \\ 0 & q_4 & q_5 & 0 \end{bmatrix},$$

while obviously these two bursts have different patterns. This contradiction proves that $q_1 = 1$. For reasons of symmetry, $q_5 = v_0 = v_4 = 1$. □

Lemma 14: $\boldsymbol{p} = \boldsymbol{e}_5$, $\boldsymbol{t} = \boldsymbol{e}_1$, $\boldsymbol{u} = \boldsymbol{e}_4$, and $\boldsymbol{w} = \boldsymbol{e}_0$.

Proof: Let x be an indeterminate. Since $q_1 = 1$ by Lemma 13, it follows that the burst B_1, given by

$$\begin{bmatrix} 1 & p_0 + p_1q_0 & 0 & 0 \\ p_1 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

has syndrome $(0,0,p_2 + p_1q_2 + x, p_3 + p_1q_3, p_4 + p_1q_4, p_5 + p_1q_5)$, as does the burst B_2 given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_2 + p_1q_2 + x & p_3 + p_1q_3 & 0 \\ 0 & p_4 + p_1q_4 & p_5 + p_1q_5 & 0 \end{bmatrix}.$$

Hence these two bursts should have the same pattern. By taking $x = 1$, it follows that $p_2 + p_1q_2 = 0$, $p_0 + p_1q_0 = p_3 + p_1q_3$, $p_1 = p_4 + p_1q_4$, and $p_5 + p_1q_5 = 1$. Hence

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x & p_0 + p_1q_0 & 0 \\ 0 & p_1 & 1 & 0 \end{bmatrix}.$$

If we now take $x = 0$, and compare B_1 with B_2, it follows that $p_1 = p_0 = 0$. Substituting this in the previous equations, we get $p_2 = p_3 = p_4 = 0$ and $p_5 = 1$. Hence $\boldsymbol{p} = \boldsymbol{e}_5$. By symmetry, we get $\boldsymbol{t} = \boldsymbol{e}_1$, $\boldsymbol{u} = \boldsymbol{e}_4$, and $\boldsymbol{w} = \boldsymbol{e}_0$. □

By Lemma 14, H has the form

$$\begin{bmatrix} \boldsymbol{e}_5 & \boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_4 \\ \boldsymbol{q} & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{v} \\ \boldsymbol{e}_1 & \boldsymbol{e}_4 & \boldsymbol{e}_5 & \boldsymbol{e}_0 \end{bmatrix}.$$

Furthermore, by Lemma 13 we have $q_1 = q_5 = 1$ and $v_0 = v_4 = 1$. The proof of the next lemma contradicts the assumption $r(2,2) = 6$.

Lemma 15: $r(2,2) > 6$.

Proof: By Lemma 8, applied to $\{(i,j)\colon i = 0,1,\ j = 0,1,2\}$, $\{(i,j)\colon i = 1,2,\ j = 0,1,2\}$, and $\{(i,j)\colon i = 0,1,2,\ j = 0,1\}$, we get $q_4 = q_0 = q_3 = 1$. Hence $\boldsymbol{q} = (1,1,q_2,1,1,1)$. By symmetry, $\boldsymbol{v} = (1,1,1,v_3,1,1)$. However, the burst

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1+q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has syndrome $(0,1,1,1,1,1)$, as does the burst

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1+v_3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

while, obviously, these bursts have different patterns. This contradiction proves the lemma. □

The following theorem gives a 2×2-burst identification code with redundancy 7. As in Theorem 9, the parity check matrix $H=[\boldsymbol{h}_{i,j}]$, $0\le i<n_1$, $0\le j<n_2$, is defined periodically by the 4×4 building block $\hat{H}=[\hat{\boldsymbol{h}}_{i,j}]$, $0\le i<4$, $0\le j<4$, as $\boldsymbol{h}_{i,j}=\hat{\boldsymbol{h}}_{i \bmod 4, j \bmod 4}$, where $\hat{\boldsymbol{h}}_{i,j}\in \boldsymbol{F}_2^7$.

Theorem 16: Let

$$\hat{H}=\begin{bmatrix} \boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \boldsymbol{e}_4 & \boldsymbol{e}_5 & \boldsymbol{e}_6 & \boldsymbol{1} \\ \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_0 & \boldsymbol{e}_1 \\ \boldsymbol{e}_6 & \boldsymbol{1} & \boldsymbol{e}_4 & \boldsymbol{e}_5 \end{bmatrix}$$

where $\boldsymbol{e}_i$ is the ith canonical vector of length 7, and $\boldsymbol{1}=(1,1,1,1,1,1,1)$. Then $\hat{H}$ is the building block of a 2×2-burst identification code of redundancy 7.

Proof: It is clear from the construction that the redundancy is 7. Since the building block has size 4×4, it suffices, as we have demonstrated in the proof of Theorem 9 by using Lemma 1, to show that the burst pattern of any 2×2-cyclic burst $\hat{B}$ on the building block can be uniquely determined from its syndrome.

Let $\hat{B}$ be a 2×2-cyclic burst, and let $\boldsymbol{s}$ be its syndrome, whose weight is denoted by $w(\boldsymbol{s})$. Let $\hat{J}$ be the set of positions of the nonzero components of $\hat{B}$. From the construction of $\hat{H}$, it follows that each vector $\boldsymbol{u}\in \{\boldsymbol{e}_0,\boldsymbol{e}_1,\cdots,\boldsymbol{e}_6,\boldsymbol{1}\}$ occurs twice in $\hat{H}$, namely at positions (i,j) and $(i+2 \bmod 4, j+2 \bmod 4)$, for some $0\le i,j<4$. Since $\hat{B}$ is assumed to be of area 2×2, or less, it follows that no vector $\boldsymbol{u}$ can contribute twice to $\boldsymbol{s}$. The weight $w(\boldsymbol{s})=4$ if and only if the pattern of $\hat{B}$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. For all other burst patterns, $w(\boldsymbol{s})>4$ if and only if $\hat{J}$ contains $(1,3)$ or $(3,1)$. So we may replace $\boldsymbol{1}$ by $\boldsymbol{e}_7$ in the burst identification algorithm, where we now view the $\boldsymbol{e}_i$ as the canonical vectors of length 8. The weight of the burst is now equal to $w(\boldsymbol{s})$. Let $\boldsymbol{u}$ be one of the vectors that contributed to $\boldsymbol{s}$. The nonzero components of the burst $\hat{B}$ are contained in a 3×3 subarray, which corresponds to the 3×3 submatrix

$$\hat{H}(\boldsymbol{u})=\begin{bmatrix} \boldsymbol{u}_0 & \boldsymbol{u}_1 & \boldsymbol{u}_2 \\ \boldsymbol{u}_3 & \boldsymbol{u} & \boldsymbol{u}_4 \\ \boldsymbol{u}_2 & \boldsymbol{u}_5 & \boldsymbol{u}_0 \end{bmatrix}$$

where the vectors $\boldsymbol{u}_i$, $0\le i\le 5$, and $\boldsymbol{u}$ are all different and from the set $\{\boldsymbol{e}_0,\boldsymbol{e}_1,\cdots,\boldsymbol{e}_7\}$. Note that this 3×3 submatrix is the same for the two positions of $\boldsymbol{u}$ in $\hat{H}$. The burst pattern of $\hat{B}$ can now be easily determined from $\boldsymbol{s}$ and $\hat{H}(\boldsymbol{u})$. □

Combining Lemma 15 and Theorem 16, we arrive at the following theorem.

Theorem 17: $r(2,2)=7$.

C. 3×2-Burst Identification Codes

The following theorem is stated here without its lengthy proof which can be found in [5].

Theorem 18: $11\le r(3,2)\le 12$.

VI. Burst Correcting Codes

A two-dimensional linear code $\mathscr{C}$ is said to be a $b_1\times b_2$-*burst correcting code* if no codeword, except the all-zero codeword, is a $b_1\times b_2$-burst or a sum of two $b_1\times b_2$-bursts. Equivalently, the code $\mathscr{C}$ is a $b_1\times b_2$-burst correcting code if and only if the syndromes of the $b_1\times b_2$-bursts with respect to any given parity check matrix of $\mathscr{C}$ are nonzero and distinct.

We are mainly interested in $b_1\times b_2$-burst correcting codes whose areas are much larger than $b_1\times b_2$. In this section, we will construct a class of burst correcting codes using burst identification codes. To compare this class of codes with other classes mentioned in the literature, the following measure is used to estimate the redundancy required in each class. Consider an infinite class $\mathscr{S}$ of $b_1\times b_2$-burst correcting codes, and suppose that for every positive integer n, the subset $\mathscr{S}(n)$ of codes in $\mathscr{S}$ of area $n_1\times n_2$ for some $(n_1,n_2)\succ(n,n)$ is nonempty. For each $\mathscr{C}\in\mathscr{S}$, let $n_{1\mathscr{C}}\times n_{2\mathscr{C}}$ and $r_{\mathscr{C}}$ denote the area and redundancy of $\mathscr{C}$, respectively. Then, we define the *excess redundancy* of the class $\mathscr{S}$ as

$$\bar{r}_{\mathscr{S}}(b_1,b_2)=\lim_{n\to\infty}\ \inf_{\mathscr{C}\in\mathscr{S}(n)}\left(r_{\mathscr{C}}-\log_2(n_{1\mathscr{C}}n_{2\mathscr{C}})\right),$$

if such limit exists. The concept of excess redundancy is a modified version of an earlier measure of efficiency of one-dimensional burst correcting codes developed by Fire [4]. Note that $\bar{r}_{\mathscr{S}}(b_1,b_2)$ exists if and only if $\inf_{\mathscr{C}\in\mathscr{S}(n)}(r_{\mathscr{C}}-\log_2(n_{1\mathscr{C}}n_{2\mathscr{C}}))$ is bounded as a function of n since it is a nondecreasing function. If this function is unbounded, we take $\bar{r}_{\mathscr{S}}(b_1,b_2)=\infty$. The definition of excess redundancy may need some clarification. A $b_1\times b_2$-burst correcting code of area $n_1\times n_2$ and redundancy r must have distinct syndromes for all distinct 1×1-bursts. Since there are n_1n_2 such bursts, it follows that r should be at least $\log_2(n_1n_2)$. This explains the term "excess" used to describe $\bar{r}_{\mathscr{S}}(b_1,b_2)$. It follows from the definition of excess redundancy that if $\bar{r}_{\mathscr{S}}(b_1,b_2)$ is finite, then for every $\epsilon>0$ and every positive integer n, there exists a $b_1\times b_2$-burst correcting code in $\mathscr{S}$ of area $n_1\times n_2$, for some $(n_1,n_2)\succ(n,n)$, whose redundancy is less than $\bar{r}_{\mathscr{S}}(b_1,b_2)+\log_2(n_1n_2)+\epsilon$.

For a code of area $n_1 \times n_2$, it can be easily shown that the total number of $b_1 \times b_2$-bursts is

$$(n_1 - b_1)(n_2 - b_2)N(b_1, b_2) + (n_1 - b_1)\sum_{b_2'=1}^{b_2} N(b_1, b_2') + (n_2 - b_2)\sum_{b_1'=1}^{b_1} N(b_1', b_2) + \sum_{b_1'=1}^{b_1}\sum_{b_2'=1}^{b_2} N(b_1', b_2'),$$

where $N(b_1, b_2)$ is the number of distinct patterns of $b_1 \times b_2$-bursts. Hence if the code is a $b_1 \times b_2$-burst correcting code of redundancy r, then $2^r - 1$ should be at least equal to this number. This implies the following result.

Theorem 19: Let $\bar{r}(b_1, b_2)$ denote the excess redundancy of the class of all $b_1 \times b_2$-burst correcting codes. Then,

$$\bar{r}(b_1, b_2) \geq \log_2 N(b_1, b_2).$$

Our main aim is to develop two-dimensional burst correcting codes whose excess redundancy is small. Before doing that, it may be illuminating to consider one-dimensional codes whose theory is better understood.

Hamming codes are 1-burst correcting codes whose excess redundancy is 0. For 2-burst correcting codes, Abramson codes [6] have excess redundancy 1. The excess redundancies of these two classes of codes satisfy the minimum bound of the one-dimensional version of Theorem 19 with equality. For b-burst correcting codes, with $b \geq 3$, the best known class until recently in terms of excess redundancy was Fire codes [4]. The excess redundancy of this class is $(2b-1) - \log_2(2b-1)$. However, it has been shown recently [7] that for every positive integer $b \geq 3$, a class of cyclic b-burst correcting codes exists whose excess redundancy is $b-1$. This class satisfies the lower bound of the one-dimensional version of Theorem 19 with equality.

We return to two-dimensional burst correcting codes. The first class of such codes ever reported in the literature is due to Elspas [1]. The codes in this class are products of cyclic codes. The excess redundancy of these codes is infinite for all values of b_1 and b_2. However, note that these codes have other error correcting capabilities, in addition to correcting two-dimensional bursts. Moreover, the excess redundancy is useful in measuring the efficiency of a $b_1 \times b_2$-burst correcting code only if its area is much larger than $b_1 \times b_2$. For example, codes obtained by interleaving have infinite excess redundancy. However, some of these codes are asymptotically optimum as b_1 and b_2 tend to infinity [2].

The $\gamma\beta$-codes developed by Nomura *et al.* [8], are cyclic 1×1-burst correcting codes whose excess redundancy is 0, which meets the lower bound of Theorem 19 with equality. The class of two-dimensional Fire codes [2], [3] has excess redundancy $(2b_1 - 1)(2b_2 - 1) - \log_2(2b_1 - 1)(2b_2 - 1)$. Apart from the codes developed in this paper, this excess redundancy is the best known value in case $(b_1, b_2) \succ (1,1)$.

Let $\mathscr{C}_I$ be a $b_1 \times b_2$-burst identification code of area $n_1 \times n_2$. Then $\mathscr{C}_I$ has no nonzero codeword which is a $b_1 \times b_2$-burst or a sum of two $b_1 \times b_2$-bursts of different patterns. Suppose that $\mathscr{C}_L$ is a code of area $n_1 \times n_2$ that has no nonzero codeword which is a $b_1 \times b_2$-burst or a sum of two $b_1 \times b_2$-bursts of the same pattern. Then the subspace $\mathscr{C} = \mathscr{C}_I \cap \mathscr{C}_L$ is a code of area $n_1 \times n_2$ that has no nonzero codeword which is a $b_1 \times b_2$-burst or a sum of two $b_1 \times b_2$-bursts. In other words, $\mathscr{C}$ is a $b_1 \times b_2$-burst correcting code. The code $\mathscr{C}_L$ is called a *$b_1 \times b_2$-burst locating code* since it can determine the location of any single burst if its pattern is known. The redundancy of $\mathscr{C}$ is at most equal to the sum of the redundancies of $\mathscr{C}_I$ and $\mathscr{C}_L$. A code whose redundancy is the sum of the redundancies of $\mathscr{C}_I$ and $\mathscr{C}_L$ and which is a subspace of $\mathscr{C}_I \cap \mathscr{C}_L$ will be called a burst identification and locating (BIL) code.

Starting with the one-dimensional burst identification codes of Theorem 5, it is possible to construct a family of one-dimensional BIL codes whose excess redundancy is $r(b)$ as given in Theorem 6. The required burst locating codes are simply Hamming codes. One-dimensional BIL codes have a simple structure although they are not cyclic. However, their excess redundancy, which is $2b-2$ for $b \geq 2$, is slightly larger than the excess redundancy of the class of cyclic Fire codes which is $(2b-1) - \log_2(2b-1)$.

Obviously, the redundancy r of a $b_1 \times b_2$-burst locating code of area $n_1 \times n_2$ must satisfy $2^r - 1 \geq n_1 n_2$. In the following theorem, a technique which is due to Nomura *et al.* [8] is given to construct burst locating codes which satisfy this bound with equality.

Theorem 20: Let m_1 and m_2 be positive integers. Let α be a primitive element in $\boldsymbol{F}_{2^{m_1 m_2}}$. Let n_1 and n_2 be positive integers such that the following conditions are satisfied:

1) $n_1 n_2 = 2^{m_1 m_2} - 1$.
2) m_1 is the multiplicative order of 2 modulo n_1.
3) $\mathrm{GCD}(n_1, n_2) = 1$.

Let $\gamma = \alpha^{n_2}$ and $\beta = \alpha^{n_1}$. Then,

i) The orders of γ and β are n_1 and n_2, respectively.
ii) The minimal polynomial of γ over $\boldsymbol{F}_2$ is of degree m_1, and the minimal polynomial of β over $\boldsymbol{F}_{2^{m_1}}$ is of degree m_2.
iii) The elements $\gamma^i\beta^j$, for $0 \leq i < m_1$, $0 \leq j < m_2$, are linearly independent over $\boldsymbol{F}_2$.
iv) $\gamma^i\beta^j = 1$ if and only if $n_1 | i$ and $n_2 | j$.
v) The matrix $[\gamma^i\beta^j]$, $0 \leq i < n_1$, $0 \leq j < n_2$, is a parity check matrix of a cyclic $m_1 \times m_2$-burst locating code of area $n_1 \times n_2$ and redundancy $m_1 m_2$.

Proof: Part i) immediately follows from condition 1). From condition 2) it follows that the minimal polynomial of γ over $\boldsymbol{F}_2$ is of degree m_1. The degree of the minimal polynomial of β over $\boldsymbol{F}_{2^{m_1}}$ is the least positive integer d such that $2^{m_1 d} \equiv 1 \pmod{n_2}$. Conditions 1), 2), and 3) imply that for such d, we have $2^{m_1 d} \equiv 1 \pmod{2^{m_1 m_2} - 1}$ and hence $d = m_2$. This proves ii).

Now, suppose that

$$\sum_{j=0}^{m_2-1}\sum_{i=0}^{m_1-1} a_{i,j}\gamma^i\beta^j = 0$$

where $a_{i,j} \in F_2$. The fact that the minimal polynomial of β over $F_{2^{m_1}}$ is of degree m_2 implies $\sum_{i=0}^{m_1-1} a_{i,j}\gamma^i = 0$ for all $0 \le j < m_2$, which implies $a_{i,j} = 0$ for all $0 \le i < m_1$ and $0 \le j < m_2$ as the minimal polynomial of γ over F_2 is of degree m_1. This proves iii).

To prove iv), note that $n_1|i$ and $n_2|j$ implies $\gamma^i\beta^j = 1$ from i). On the other hand, if $\gamma^i\beta^j = 1$, then $\gamma^{in_2} = \beta^{-jn_2} = 1$, which gives $n_1|in_2$. This implies $n_1|i$ by (3). Hence $\beta^j = 1$, which gives $n_2|j$.

Next, we prove v). Let $[c_{i,j}]$, $0 \le i < n_1$, $0 \le j < n_2$, be an array over F_2. The syndrome of this array is given by

$$s = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} c_{i,j}\gamma^i\beta^j.$$

Thus the array $[c_{i,j}]$ is a codeword if and only if $c(\gamma, \beta) = 0$, where $c(x, y) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} c_{i,j}x^iy^j$. Hence the code is an ideal in $F_2[x, y]/(x^{n_1}+1, y^{n_2}+1)$, and thus is cyclic. Hence to show that it is an $m_1 \times m_2$-burst locating code, it suffices to prove that if

$$c(x, y) \equiv b(x, y) + x^{u_1}y^{u_2}b(x, y) \pmod{x^{n_1}+1, y^{n_2}+1}$$

is a codeword, where $b(x, y) \in \mathscr{B}_{m_1m_2}$, then $n_1|u_1$ and $n_2|u_2$. Suppose $c(x, y)$ is a codeword. Then $b(\gamma, \beta) + \gamma^{u_1}\beta^{u_2}b(\gamma, \beta) = 0$. However, from part iii), it follows that $b(\gamma, \beta) \ne 0$, which implies $\gamma^{u_1}\beta^{u_2} = 1$. Part iv) gives $n_1|u_1$ and $n_2|u_2$.

Thus the code is indeed a cyclic $m_1 \times m_2$-burst locating code of area $n_1 \times n_2$ and redundancy $\le m_1m_2$. However from iii), it follows that the redundancy is exactly m_1m_2. □

A code whose construction is as given in Theorem 20 will be called a *γβ-code* with parameters (m_1, m_2). The γβ-codes presented here are the duals of the codes studied in [8], which are called γβ-array codes.

In the following, we will show that for all positive integers b_1 and b_2, there exists an infinite number of $b_1 \times b_2$-burst locating codes within the class of γβ-codes. The basic argument in the proof is due to Gordon [9].

Theorem 21: For every pair of positive integers (m_1, m_2) such that $m_1 \ne 6$ and $m_1 \ge m_2$, there exists a γβ-code with parameters (m_1, m_2).

Proof: Let $m_1 \ne 6$ and $m_1 \ge m_2$. From a result in [10], it follows that a prime p exists such that m_1 is the multiplicative order of 2 modulo p. This implies $m_1 \le p-1$ by Fermat's theorem. Let $p^a \| 2^{m_1}-1$ for some positive integer a, i.e., $p^a | 2^{m_1}-1$ but $p^{a+1} \nmid 2^{m_1}-1$. Assume that $p | (2^{m_1m_2}-1)/(2^{m_1}-1)$. Then

$$\underbrace{2^{(m_2-1)m_1} + 2^{(m_2-2)m_1} + \cdots + 2^{m_1} + 1}_{m_2 \text{ terms}} \equiv 0 \pmod{p}.$$

Since $2^{m_1} \equiv 1 \pmod{p}$, it follows that $p|m_2$. Hence $m_2 \ge p \ge m_1 + 1$, which contradicts $m_1 \ge m_2$. Thus $p \nmid (2^{m_1m_2}-1)/(2^{m_1}-1)$, which implies $p \nmid (2^{m_1m_2}-1)/p^a$. Let $n_1 = p^a$ and $n_2 = (2^{m_1m_2}-1)/p^a$. Then, conditions 1)–3) of Theorem 20 are satisfied, and hence there is a γβ-code with parameters (m_1, m_2). □

Corollary 22: If b_1, b_2, and n are positive integers, then there exists a $b_1 \times b_2$-burst locating code which is a γβ-code of area greater than $n \times n$ with parameters (m_1, m_2) for all sufficiently large m_1 and m_2.

Proof: If $(b_1, b_2) \preccurlyeq (m_1, m_2)$, then an $m_1 \times m_2$-burst locating code is a $b_1 \times b_2$-burst locating code. The corollary now follows from Theorem 21 and conditions 1) and 2) of Theorem 20. □

Some practical applications may require the areas of the burst locating codes to be squares or close to squares. In the construction given in Theorem 20, it follows that $n_1 \le 2^{m_1} - 1$ and

$$n_2 = \frac{2^{m_1m_2}-1}{n_1} \ge \frac{2^{m_1m_2}-1}{2^{m_1}-1} \ge 2^{m_1(m_2-1)}.$$

Thus, if n_1 and n_2 are required to be large and close in value, then m_2 is restricted to be less or equal to 2. However, this may restrict the γβ-code to be a $b_1 \times b_2$-burst locating code with $b_2 = 1$ or 2 only.

In the following, we construct $b_1 \times b_2$-burst locating codes of square areas for all positive integers b_1 and b_2.

Theorem 23: Let b_1, b_2, t_1, t_2, and m be positive integers, $m > 1$. Let α and β be primitive elements in F_{2^m}, not necessarily distinct. Suppose that the following conditions hold.

1) The elements α^{it_1+j}, $0 \le i < b_1$, $0 \le j < b_2$, are linearly independent over F_2.
2) The elements β^{i+jt_2}, $0 \le i < b_1$, $0 \le j < b_2$, are linearly independent over F_2.
3) $\mathrm{GCD}(t_1t_2 - 1, 2^m - 1) = 1$.

Then the code with parity check matrix $[\boldsymbol{h}_{i,j}]$, $0 \le i, j < 2^m - 1$, given by $\boldsymbol{h}_{i,j} = (\alpha^{it_1+j}, \beta^{i+jt_2})$, is a cyclic $b_1 \times b_2$-burst locating code of area $2^m - 1 \times 2^m - 1$, and redundancy $2m$.

Proof: Let $n = 2^m - 1$. An array $[c_{i,j}]$, $0 \le i, j < n$, over F_2 is a codeword if and only if its syndrome is zero, i.e., if and only if

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} c_{i,j}\alpha^{it_1+j} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} c_{i,j}\beta^{i+jt_2} = 0.$$

Let $c(x, y) = \sum_{i=0}^{n_1-1}\sum_{j=0}^{n_2-1} c_{i,j}x^iy^j$, then it follows that $c(x, y)$ is a codeword if and only if $c(\alpha^{t_1}, \alpha) = c(\beta, \beta^{t_2}) = 0$. Thus the code is an ideal in $F_2[x, y]/(x^n+1, y^n+1)$, and hence is cyclic. To prove that the code is a $b_1 \times b_2$-burst locating code, it suffices to show that if

$$c(x, y) \equiv b(x, y) + x^{u_1}y^{u_2}b(x, y) \pmod{x^n+1, y^n+1}$$

is a codeword, where $b(x, y) \in \mathscr{B}_{b_1b_2}$, then $u_1 \equiv u_2 \equiv 0 \pmod{n}$. Suppose $c(x, y)$ is a codeword, then

$$b(\alpha^{t_1}, \alpha) + \alpha^{u_1t_1+u_2}b(\alpha^{t_1}, \alpha) = b(\beta, \beta^{t_2}) + \beta^{u_1+u_2t_2}b(\beta, \beta^{t_2}) = 0.$$

From conditions 1) and 2), it follows that $b(\alpha^{t_1}, \alpha)$ and $b(\beta, \beta^{t_2})$ are nonzero, which implies

$$u_1t_1 + u_2 \equiv u_1 + u_2t_2 \equiv 0 \pmod{n}.$$

From condition 3) it follows that $u_1 \equiv u_2 \equiv 0 \pmod{n}$. Thus the code is a $b_1 \times b_2$-burst locating code of area $2^m - 1 \times 2^m - 1$, and redundancy $2m$ at most. Since the redundancy of a burst locating code of area $n \times n$ should be at least $\log_2(n^2+1)$, it follows that the redundancy is exactly $2m$. □

A code whose construction is as given in Theorem 23 will be called an *$\alpha\beta$-code*. These codes have the largest possible areas among all burst locating codes of square areas.

Conditions 1) and 2) of Theorem 23 may be tedious to check. In the following, we will give a systematic technique to satisfy these conditions if m is large with respect to b_1 and b_2.

Lemma 24: Let b_1, b_2, t_1, t_2, and m be positive integers such that

i) $t_1 \geq b_2$ and $m \geq (b_1 - 1)t_1 + b_2$.
ii) $t_2 \geq b_1$ and $m \geq (b_2 - 1)t_2 + b_1$.

Then, the following holds for any primitive elements α and β in $\boldsymbol{F}_{2^m}$:

1) The elements α^{it_1+j}, $0 \leq i < b_1$, $0 \leq j < b_2$, are linearly independent over $\boldsymbol{F}_2$.
2) The elements β^{i+jt_2}, $0 \leq i < b_1$, $0 \leq j < b_2$, are linearly independent over $\boldsymbol{F}_2$.

Proof: It suffices, by symmetry, to prove that condition i) implies 1). From i), it follows that the numbers $it_1 + j$, where $0 \leq i < b_1$, $0 \leq j < b_2$, are distinct and lie between 0 and $m-1$. Since the minimal polynomial of α has degree m, then condition 1) holds. □

The following is an immediate corollary of the previous lemma.

Corollary 25: If b_1 and b_2 are positive integers, then a $b_1 \times b_2$-burst locating code exists which is an $\alpha\beta$-code of area $2^m - 1 \times 2^m - 1$, for all sufficiently large m.

In the beginning of this section, we have shown a technique to construct BIL codes which are burst correcting codes using burst identification codes and burst locating codes. Using the class of $\gamma\beta$-codes or the class of $\alpha\beta$-codes, it can be shown easily that the excess redundancy of the class of BIL codes is given by

$$\bar{r}_{\mathrm{BIL}}(b_1, b_2) = r(b_1, b_2),$$

which is the minimum redundancy required to construct a $b_1 \times b_2$-burst identification code of arbitrarily large area as defined in Section II. From Theorem 10, it follows that

$$2b_1b_2 - 2 \leq \bar{r}_{\mathrm{BIL}}(b_1, b_2) \leq 2b_1b_2.$$

From Section V, we also have $\bar{r}_{\mathrm{BIL}}(1, b) = 2b - 2$ for $b > 1$ and $\bar{r}_{\mathrm{BIL}}(2,2) = 7$. It follows that the excess redundancy of the class of BIL codes is less than the excess redundancy of the class of two-dimensional Fire codes for all values of b_1 and b_2 except if b_1 or b_2 is 1, or $b_1 = b_2 = 2$, and possibly $(b_1, b_2) = (2,3)$ or $(3,2)$.

The excess redundancy $\bar{r}_{\mathrm{BIL}}(b_1, b_2)$ is about twice the lower bound of Theorem 19 which is $\log_2 N(b_1, b_2)$. It is shown in [5] that a class of cyclic burst correcting codes exists whose excess redundancy is $\lceil \log_2 N(b_1, b_2) \rceil$. The proof of the existence of this class of codes is a two-dimensional generalization of the proof of the existence of the one-dimensional optimum cyclic burst correcting codes as given in [7]. This proof depends on some results from algebraic geometry, namely, Weil's estimates for character sums. However, these codes are hard to find and may not exist unless the areas are very large.

VII. Encoding and Decoding BIL Codes

A BIL code, as explained in the previous section, is constructed from a burst identification code and a burst locating code. Now we will assume that the burst locating codes are cyclic. As explained in Section VI, the classes of $\gamma\beta$-codes and $\alpha\beta$-codes are cyclic burst locating codes which can be used to construct BIL codes whose excess redundancy is minimum. We do not assume that the BIL codes considered in this section are necessarily cyclic.

The techniques presented here are generalizations of encoding and decoding techniques of one-dimensional cyclic burst correcting codes. Both one-dimensional encoding and decoding techniques were generalized by Imai in [3], [11] to two-dimensional cyclic codes. We will briefly describe these techniques and modify them to be suitable for BIL codes.

Let $H = [\boldsymbol{h}_{i,j}]$, $0 \leq i < n_1$, $0 \leq j < n_2$, be the parity check matrix of a $b_1 \times b_2$-burst correcting code of redundancy r. Let $\Omega = \{(i,j)\colon 0 \leq i < n_1, 0 \leq j < n_2\}$. From the definition of $b_1 \times b_2$-burst correcting codes, it follows that no codeword is a $b_1 \times b_2$-burst. Hence a set of parity check positions Π of cardinality r exists such that $\{(i,j)\colon 0 \leq i < b_1, 0 \leq j < b_2\} \subseteq \Pi$.

A. Encoding and Decoding Cyclic Burst Correcting Codes

Shift registers are commonly used to encode and decode one-dimensional cyclic codes [12]. A two-dimensional shift register is described by Imai [11] in which $|\Pi|$ storage devices are arranged in the form of the parity check positions given by Π. The connections of the register are determined according to the parity check matrix of the cyclic code. We represent the contents of the shift register by the polynomial $\sigma(x,y) = \sum_{(i,j) \in \Pi} \sigma_{i,j} x^i y^j$, where $\sigma_{i,j} \in \boldsymbol{F}_2$ is the content of the storage device at position (i,j). The register can be shifted in the x and y directions.

First, we will describe the encoding of two-dimensional cyclic codes. Let $m_{i,j}$, where $(i,j) \in \Omega - \Pi$, be the information bits. Let $m(x,y) = \sum_{(i,j) \in \Omega} m_{i,j} x^i y^j$, where $m_{i,j} = 0$ for $(i,j) \in \Pi$. The coefficients of $m(x,y)$ are fed into the register. After a number of shifts in the x and y directions, the contents of the register, represented by $\sigma(x,y)$, give the parity check bits of the codeword $c(x,y)$ corresponding to the information polynomial $m(x,y)$, i.e., $c(x,y) = m(x,y) + \sigma(x,y)$.

Now, we describe the decoding process. Consider a cyclic $b_1 \times b_2$-burst correcting code. Let $v(x,y) = c(x,y) + e(x,y)$ be the received word, where $e(x,y)$ is a $b_1 \times b_2$-

burst of error added to the transmitted codeword $c(x, y)$. Then $e(x, y) = x^{u_1}y^{u_2}b(x, y)$ for some $(u_1, u_2) \in \Omega$ and $b(x, y) \in \mathscr{B}_{b_1, b_2}$.

Imai [11] has given a two-dimensional version of the error trapping algorithm which is well-known for decoding one-dimensional cyclic burst correcting codes [12]. The received word $v(x, y)$ is fed into the two-dimensional shift register. The register is then shifted in the x and y directions without an input until its contents, represented by $\sigma(x, y)$, display a pattern of a $b_1 \times b_2$-burst, i.e., $\sigma(x, y) \in \mathscr{B}_{b_1, b_2}$. In this case, $b(x, y) = \sigma(x, y)$ and the burst position (u_1, u_2) is determined by the number of shifts in the x and y directions. Hence $e(x, y) = x^{u_1}y^{u_2}b(x, y)$ is determined.

B. Encoding and Decoding BIL Codes

Consider a $b_1 \times b_2$-burst correcting code which is a BIL code. Its parity check matrix can be written as $(\boldsymbol{h}'_{i,j}, \boldsymbol{h}''_{i,j})$, $0 \le i < n_1$, $0 \le j < n_2$, where $[\boldsymbol{h}'_{i,j}]$ is the parity check matrix of a cyclic burst locating code, and $[\boldsymbol{h}''_{i,j}]$ is the parity check matrix of a burst identification code.

Let Π be parity check positions for the BIL code such that $(i, j) \in \Pi$ for $0 \le i < b_1$, $0 \le j < b_2$. The information bits are represented by the polynomial $m(x, y) = \sum_{(i,j) \in \Omega} m_{i,j}x^iy^j$, where $m_{i,j} = 0$ for $(i, j) \in \Pi$. Let $m(x, y)$ be encoded for the cyclic burst locating code using the two-dimensional shift register described before. Let $c'(x, y)$ be the codeword corresponding to $m(x, y)$ in the burst locating code. Naturally, $c'(x, y)$ may not be a codeword in the burst identification code. Let s_B denote its syndrome with respect to the burst identification code. We will obtain a codeword $c''(x, y) = \sum_{(i,j) \in \Pi} c''_{i,j}x^iy^j$ in the cyclic code whose syndrome with respect to the burst identification code is s_B. The polynomial $c''(x, y)$ is determined by

$$\sum_{(i,j) \in \Pi} c''_{i,j}\boldsymbol{h}'_{i,j} = 0$$

$$\sum_{(i,j) \in \Pi} c''_{i,j}\boldsymbol{h}''_{i,j} = s_B.$$

From the definition of Π, it follows that the elements $(\boldsymbol{h}''_{i,j}, \boldsymbol{h}''_{i,j})$, where $(i, j) \in \Pi$, are independent. Hence these equations can be solved to obtain $c''(x, y)$. Now, $c(x, y) = c'(x, y) + c''(x, y)$ is the codeword in the burst correcting code corresponding to the information bits represented by the polynomial $m(x, y)$.

Now, we describe a technique to decode BIL codes. We use the burst identification code to determine the burst pattern $b(x, y)$. Then we use the two-dimensional shift register applied to the cyclic burst locating code until the contents of the register display the burst pattern, i.e., $\sigma(x, y) = b(x, y)$. The burst position (u_1, u_2) is determined by the number of shifts in the x and y directions.

Note that in many cases encoding and decoding of BIL codes are not computationally difficult. The reason is that the burst identification codes as given in Section V and VI are periodic with periods in order of $b_1 \times b_2$.

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