Two-Dimensional Cellular Automata

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A largely phenomenological study of two-dimensional cellular automata is reported. Qualitative classes of behavior similar to those in one-dimensional cellular automata are found. Growth from simple seeds in two-dimensional cellular automata can produce patterns with complicated boundaries, characterized by a variety of growth dimensions. Evolution from disordered states can give domains with boundaries that execute effectively continuous motions. Some global properties of cellular automata can be described by entropies and Lyapunov exponents. Others are undecidable.

1. INTRODUCTION

Cellular automata are mathematical models for systems in which many simple components act together to produce complicated patterns of behavior. One-dimensional cellular automata have now been investigated in several ways (Ref. 1 and references therein). This paper presents an exploratory study of two-dimensional cellular automata.[2] The extension to two dimensions is significant for comparisons with many experimental results on pattern formation in physical systems. Immediate applications include dendritic crystal growth,[6] reaction-diffusion systems, and turbulent flow patterns. (The Navier–Stokes equations for fluid flow appear to admit turbulent solutions only in two or more dimensions.)

A cellular automaton consists of a regular lattice of sites. Each site takes on k possible values, and is updated in discrete time steps according

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[2] Some aspects of two-dimensional cellular automata were discussed in Refs. 2 and 3, and mentioned in Ref. 4. Additive two-dimensional cellular automata were considered in Ref. 5.

to a rule ϕ that depends on the value of sites in some neighborhood around it. The value a_i of a site at position i in a one-dimensional cellular automata with a rule that depends only on nearest neighbors thus evolves according to

$$a_i^{(t+1)} = \phi[a_{i-1}^{(t)}, a_i^{(t)}, a_{i+1}^{(t)}] \tag{1.1}$$

There are several possible lattices and neighborhood structures for two-dimensional cellular automata. This paper considers primarily square lattices, with the two neighborhood structures illustrated in Fig. 1. A five-neighbor square cellular automaton then evolves in analogy with Eq. (1.1) according to

$$a_{i,j}^{(t+1)} = \phi[a_{i,j}^{(t)}, a_{i,j+1}^{(t)}, a_{i+1,j}^{(t)}, a_{i,j-1}^{(t)}, a_{i-1,j}^{(t)}] \tag{1.2}$$

Here we often consider the special class of totalistic rules, in which the value of a site depends only on the sum of the values in the neighborhood:

$$a_{i,j}^{(t+1)} = f[a_{i,j}^{(t)} + a_{i,j+1}^{(t)} + a_{i+1,j}^{(t)} + a_{i,j-1}^{(t)} + a_{i-1,j}^{(t)}] \tag{1.3}$$

These rules are conveniently specified by a code[7]

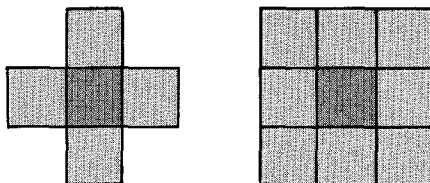$$C = \sum_n f(n)k^n \tag{1.4}$$

Fig. 1. Neighborhood structures considered for two-dimensional cellular automata. In the cellular automaton evolution, the value of the center cell is updated according to a rule that depends on the values of the shaded cells. Cellular automata with neighborhood (a) are termed "five-neighbor square;" those with neighborhood (b) are termed "nine-neighbor square." (These neighborhoods are sometimes referred to as the von Neumann and Moore neighborhoods, respectively.) Totalistic cellular automaton rules take the value of the center site to depend only on the sum of the values of the sites in the neighborhood. With outer totalistic rules, sites are updated according to their previous values, and the sum of the values of the other sites in the neighborhood. Triangular and hexagonal lattices are also possible, but are not used in the examples given here. Notice that five-neighbor square, triangular, and hexagonal cellular automaton rules may all be considered as special cases of general nine-neighbor square rules.

Table 1. Numbers of Possible Rules of Various Kinds for Cellular Automata with Two States Per Site, and Neighborhoods of the Form Shown in Fig. 1 [a]

Rule type	5-neighbor square	9-neighbor square	Hexagonal
General	$2^{32} \simeq 4 \times 10^9$	$2^{512} \simeq 10^{154}$	$2^{128} \simeq 3 \times 10^{38}$
Rotationally symmetric	$2^{12} = 4096$	$2^{140} \simeq 10^{42}$	$2^{64} \simeq 2 \times 10^{19}$
Reflection symmetric	$2^{24} \simeq 2 \times 10^7$	$2^{288} \simeq 5 \times 10^{86}$	$2^{80} \simeq 10^{24}$,
			$2^{74} \simeq 2 \times 10^{22}$
Completely symmetric	$2^{12} = 4096$	$2^{102} \simeq 5 \times 10^{30}$	$2^{28} \simeq 3 \times 10^8$
Outer totalistic	$2^{10} = 1024$	$2^{18} \simeq 3 \times 10^5$	$2^{14} = 16384$
Totalistic	$2^5 = 32$	$2^9 = 512$	$2^7 = 128$

[a] The two entries for reflectional symmetries of the hexagonal lattice refer to reflections across a cell and across a boundary, respectively. The number of quiescent rules (defined to leave the null configuration invariant) is always half the total number of rules of a given kind.

We also consider outer totalistic rules, in which the value of a site depends separately on the sum of the values of sites in a neighborhood, and on the value of the site itself:

$$a_{i,j}^{(t+1)} = \tilde{f}(a_{i,j}^{(t)}, a_{i,j+1}^{(t)} + a_{i+1,j}^{(t)} + a_{i,j-1}^{(t)} + a_{i-1,j}^{(t)}) \qquad (1.5)$$

Such rules are specified by a code

$$\tilde{C} = \sum_n \tilde{f}[a, n] k^{kn+a} \qquad (1.6)$$

This paper considers two-dimensional cellular automata with values 0 or 1 at each site, corresponding to $k = 2$. Table I gives the number of possible rules of various kinds for such cellular automata. A notorious example of an outer totalistic nine-neighbor square cellular automaton is the "Game of Life",[8] with a rule specified by code $\tilde{C} = 224$.

Despite the simplicity of their construction, cellular automata are found to be capable of very complicated behavior. Direct mathematical analysis is in general of little utility in elucidating their properties. One must at first resort to empirical means. This paper gives a phenomenological study of typical two-dimensional cellular automata. Its approach is largely experimental in character: cellular automaton rules are selected and their evolution from various initial states is traced by direct simulation.[3] The emphasis is on generic properties. Typical initial states are

[3] Several computer systems were used. The first was the spcial-purpose pipelined TTL machine built by the M.I.T. Information Mechanics group.[9] This machine updates all sites on a 256×256 square cellular automaton lattice 60 times per second. It is controlled by a

chosen. Except for some restricted kinds of rules, Table I shows that the number of possible cellular automaton rules is far too great for each to be investigated explicitly. For the most part one must resort to random sampling, with the expectation that the rules so selected are typical. The phenomena identified by this experimental approach may then be investigated in detail using analytical approximations, and by conventional mathematical means. Generic properties are significant because they are independent of precise details of cellular automaton construction, and may be expected to be universal to a wide class of systems, including those that occur in nature.

Empirical studies strongly suggest that the qualitative properties of one-dimensional cellular automata are largely independent of such features of their construction as the number of possible values for each site, and the size of the neighborhood. Four qualitative classes of behavior have been identified in one-dimensional cellular automata.[7] Starting from typical initial configurations, class-1 cellular automata evolve to homogeneous final states. Class-2 cellular automata yield separated periodic structures. Class-3 cellular automata exhibit chaotic behavior, and yield aperiodic patterns. Small changes in initial states usually lead to linearly increasing regions of change. Class-4 cellular automata exhibit complicated localized and propagating structures. Cellular automata may be considered as information-processing systems, their evolution performing some computation on the sequence of site values given as the initial state. It is conjectured that class-4 cellular automata are generically capable of universal computation, so that they can implement arbitrary information-processing procedures.

Dynamical systems theory methods may be used to investigate the global properties of cellular automata. One considers the set of configurations generated after some time from any possible initial configuration. Most cellular automaton mappings are irreversible (and not surjective), so that the set of configurations generated contracts with time. Class-1 cellular automata evolve from almost all initial states to a unique final state, analogous to a fixed point. Class-2 cellular automata evolve to collections of periodic structures, analogous to limit cycles. The contraction

of the set of configurations generated by a cellular automaton is reflected in a decrease in its entropy or dimension. Starting from all possible initial configurations (corresponding to a set defined to have dimension one), class-3 cellular automata yield sets of configurations with smaller, but positive, dimensions. These sets are directly analogous to the chaotic (or "strange") attractors found in some continuous dynamical systems (e.g., Ref. 10).

Entropy or dimension gives only a coarse characterization of sets of cellular automaton configurations. Formal language theory (e.g. Ref. 11) provides a more complete and detailed characterization.[12] Configurations may be considered as words in a formal language; sets of configurations are specified by the grammatical rules of the language. The set of configurations generated after any finite number of time steps in the evolution of a one-dimensional cellular automaton can be shown to form a regular language: the possible configurations thus correspond to possible paths through a finite graph. For most class-3 and -4 cellular automata, the complexity of this graph grows rapidly with time, so that the limit set is presumably not a regular language (cf. Ref. 13).

This paper reports evidence that certain global properties of two-dimensional cellular automata are very similar to those of one-dimensional cellular automata. Many of the local phenomena found in two-dimensional cellular automata also have analogs in one dimension. However, there are a variety of phenomena that depend on the geometry of the two-dimensional lattice. Many of these phenomena involve complicated boundaries and interfaces, which have no direct analog in one dimension.

Section 2 discusses the evolution of two-dimensional cellular automata from simple "seeds," consisting of a few nonzero initial sites. Just as in one dimension, some cellular automata give regular and self-similar patterns; others yield complicated and apparently random patterns. A new feature in two dimensions is the generation of patterns with dendritic boundaries, much as observed in many natural systems. Most two-dimensional patterns generated by cellular automaton growth have a polytopic boundary that reflects the structure of the neighborhood in the cellular automaton rule (cf. Ref. 14). Some rules, however, yield slowly growing patterns that tend to a circular shape independent of the underlying cellular automaton lattice.

Section 3 considers evolution from typical disordered initial states. Some cellular automata evolve to stationary structures analogous to crystalline forms. The boundaries between domains of different phases may behave as if they carry a surface tension: positive surface tensions lead to large smooth-walled domains; negative surface tensions give rise to labyrinthine structures with highly convoluted walls. Other cellular automata yield chaotic, class-3, behavior. Small changes in their initial con-
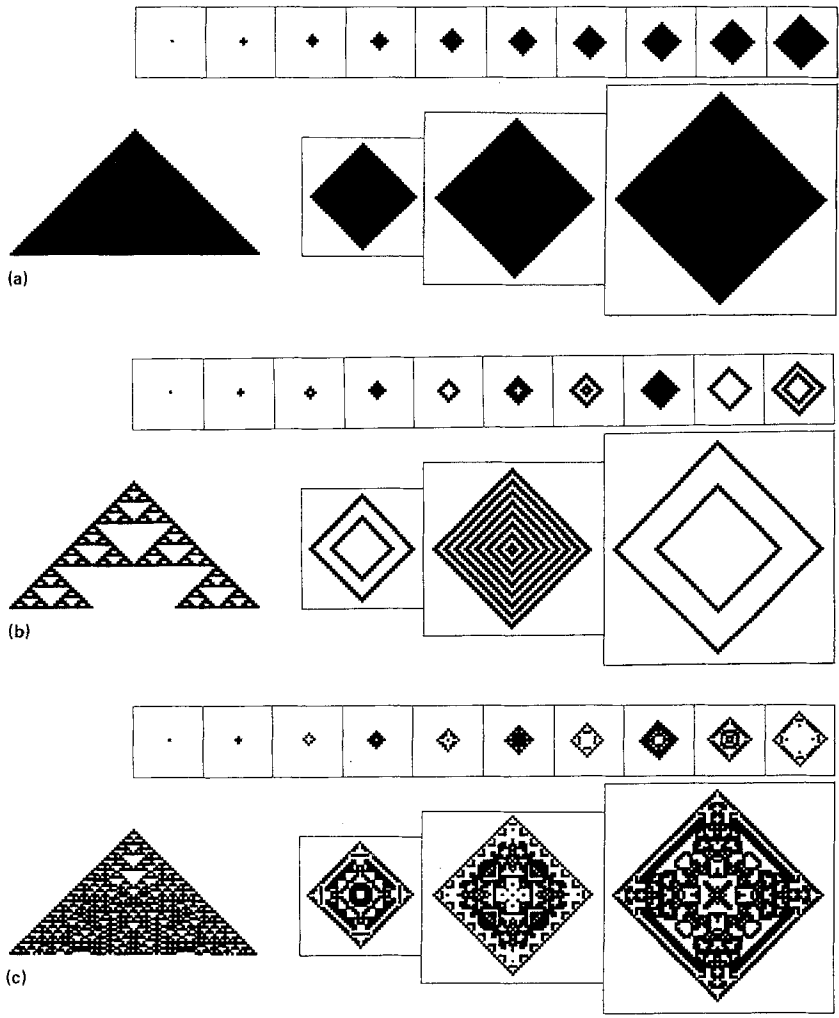
Fig. 2. Examples of classes of patterns generated by evolution of two-dimensional cellular automata from a single-site seed. Each part corresponds to a different cellular automaton rule. All the rules shown are both rotation and reflection symmetric. For each rule, a sequence of frames shows the two-dimensional configurations generated by the cellular automaton evolution after the indicated number of time steps. Black squares represent sites with value 1; white squares sites with value 0. On the left is a space-time section showing the time evolution of the center horizontal line of sites in the two-dimensional lattice. Successive lines correspond to successive time steps. The cellular automaton rules shown are five-neighbor square outer totalistic, with codes (a) 1022, (b) 510, (c) 374, (d) 614 (sum modulo 2 rule), (e) 174, (f) 494.
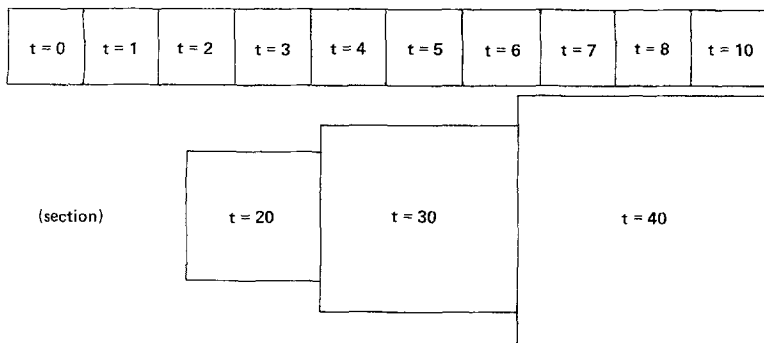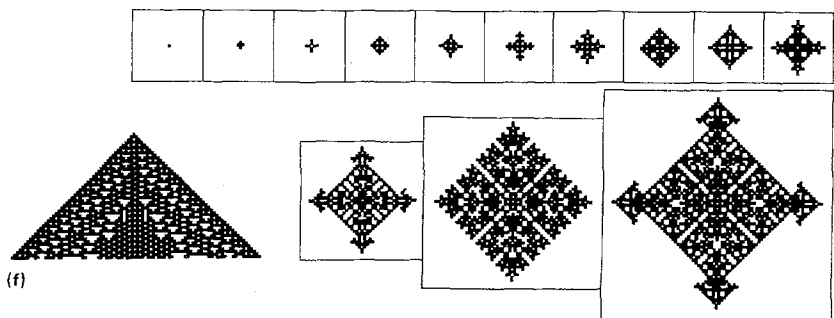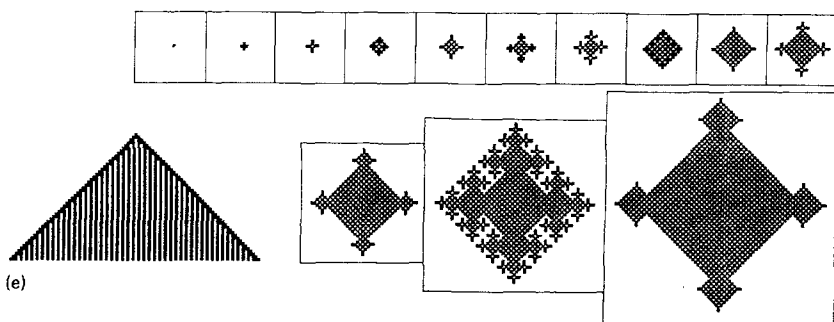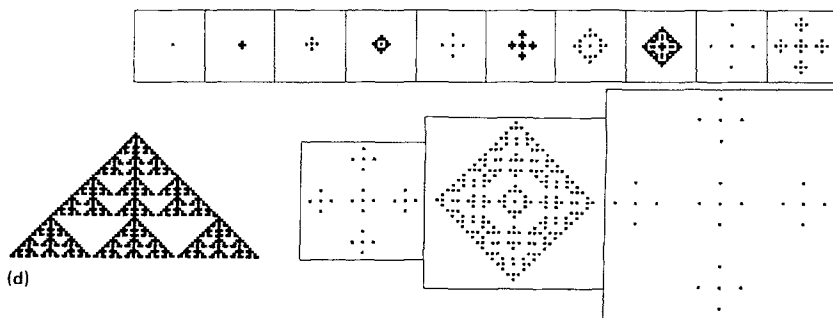
(d)

(e)

(f)

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7	t = 8	t = 10

(section)

t = 20

t = 30

t = 40

Fig. 2 (*continued*)

figurations lead to linearly increasing regions of change, usually circular or at least rounded.

Section 4 discusses some quantitative characterizations of the global properties of two-dimensional cellular automata. Many definitions are carried through directly from one dimension, but some results are rather different. In particular, the sets of configurations that can be generated after a finite number of time steps of cellular automaton evolution are no longer described by regular languages, and may in fact be nonrecursive. As a consequence, several global properties that are decidable for one-dimensional cellular automata become undecidable in two dimensions (cf. Ref. 15).

2. EVOLUTION FROM SIMPLE SEEDS

This section discusses patterns formed by the evolution of cellular automata from simple seeds. The seeds consist of single nonzero sites, or small regions containing a few nonzero sites, in a background of zero sites. The growth of cellular automata from such initial conditions should provide models for a variety of physical and other phenomena. One example is crystal growth.[6] The cellular automaton lattice corresponds to the crystal lattice, with nonzero sites representing the presence of atoms or regions of the crystal. Different cellular automaton rules are found to yield both faceted (regular) and dendritic (snowflake-like) crystal structures. In other systems the seed may correspond to a small initial disturbance, which grows with time to produce a complicated structure. Such a phenomenon presumably occurs when fluid turbulence develops downstream from an obstruction or orifice.[4]

Figure 2 shows some typical examples of patterns generated by the evolution of two-dimensional cellular automata from initial states containing a single nonzero site. In each case, the sequence of two-dimensional patterns formed is shown as a succession of "frames." A space-time "section" is also shown, giving the evolution of the center horizontal line in the two-dimensional lattice with time. Fig. 3 shows a view of the complete three-dimensional structures generated. Fig. 4 gives some examples of space-time sections generated by typical one-dimensional cellular automata.

With some cellular automaton rules, simple seeds always die out, leaving the null configuration, in which all sites have value zero. With other rules, all or part of the initial seed may remain invariant with time, yielding a fixed pattern, independent of time. With many cellular automaton rules, however, a growing pattern is produced.

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[4] A cellular automaton approximation to the Euler equations is given in Ref. 16.
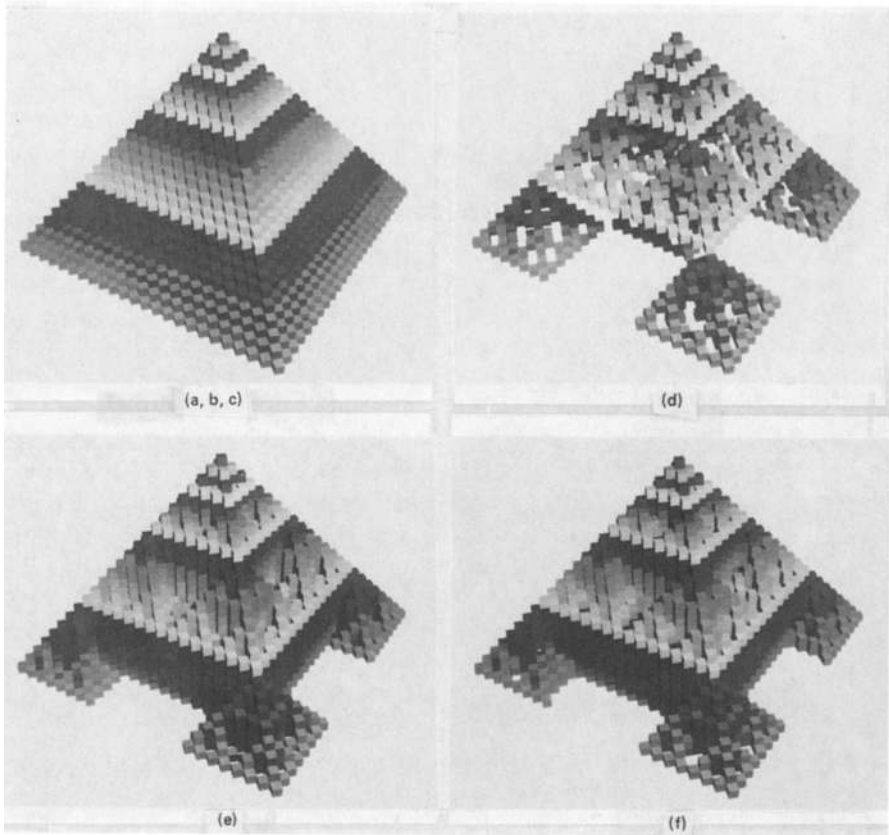
Fig. 3. View of three-dimensional structures formed from the configurations generated in the first 24 time steps of the evolution of the two-dimensional cellular automata shown in Fig. 2. Rules (a), (b), and (c) all give rise to configurations with regular, faceted, boundaries. Rules (d), (e), and (f) yield dendritic patterns. In this and other three-dimensional views, the shading ranges periodically from light to dark when the number of time steps increases by a factor of two. The three-dimensional graphics here and in Figs. 10 and 14 is courtesy of M. Prueitt at Los Alamos National Laboratory.

Rule (a) in Figs. 2 and 3 is an example of the simple case in which the growing pattern is uniform. At each time step, a regular pattern with a fixed density of nonzero sites is produced. The boundary of the pattern consists of flat (linear) "facets," and traces out a pyramid in space-time, whose edges lie along the directions of maximal growth. Sections through this pyramid are analogous to the space-time pattern generated by the one-dimensional cellular automaton of Fig. 4(a).

Cellular automaton rule (b) in Figs. 2 and 3 yields a pattern whose boundary again has a simple faceted form, but whose interior is not
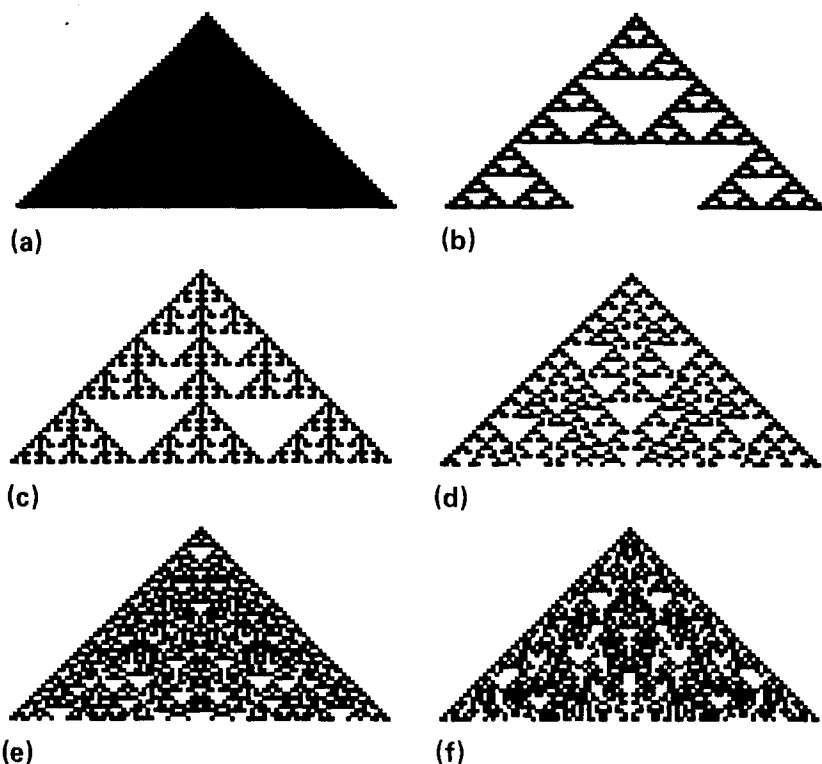
Fig. 4. Examples of classes of patterns generated by evolution of one-dimensional cellular automata from a single-site seed. Successive time steps are shown on successive lines. Nonzero sites are shown black. The cellular automaton rules shown are totalistic nearest-neighbor $(r = 1)$, with k possible values at each site: (a) $k = 2$, code 14, (b) $k = 2$, code 6, (c) $k = 2$, code 10, (d) $k = 3$, code 21, (e) $k = 3$, code 102, (f) $k = 3$, code 138. Irregular patterns are also generated by some $k = 2$, $r = 2$ rules (such as that with totalistic code 10), and by asymmetric $k = 2$, $r = 1$ rules (such as that with rule number 30).

uniform. Space-time sections through the pattern exhibit an asymptotically self-similar or fractal form: pieces of the pattern, when magnified, are indistinguishable from the whole. Figure 4(b) shows a one-dimensional cellular automaton that yields sections of the same form. The density of nonzero sites in these sections tends asymptotically to zero. The pattern of nonzero sites in the sections may be characterized by a Hausdorff or fractal dimension that is found by a simple geometrical construction to have value $\log_2 3 \simeq 1.59$.

Self-similar patterns are generated in cellular automata that are invariant under scale or blocking transformations.[17,18] Particular blocks of sites in a cellular automaton often evolve according to a fixed effective

cellular automaton rule. The overall behavior of the cellular automaton is then left invariant by a replacement of each block with a single site and of the original cellular automaton rule by the effective rule. In some cases, the effective rule may be identical to the original rule. Then the patterns generated must be invariant under the blocking transformation, and are therefore self-similar. (All the rules so far found to have this property are additive.) In many cases, the effective rule obtained after several blocking transformations with particular blocks may be invariant under further blocking transformations. Then if the initial state contains only the appropriate blocks, the patterns generated must be self-similar, at least on sufficiently large length scales.

Cellular automaton (c) gives patterns that are not homogeneous, but appear to have a fixed nonzero asymptotic density. The patterns have a complex, and in some respects random, appearance. It is remarkable that simple rules, even starting from the simple initial conditions shown, can generate patterns of such complexity. It seems likely that the iteration of the cellular automaton rule is essentially the simplest procedure by which these patterns may be specified. The cellular automaton rule is thus "computationally irreducible" (cf. Ref. 19).

Cellular automata (a), (b), and (c) in Figs 2 and 3 all yield patterns whose boundaries have a simple faceted form. Cellular automata (d), (e), and (f) give instead patterns with corrugated, dendritic, boundaries. Such complicated boundaries can have no analog in one-dimensional cellular automata: they are a first example of a qualitative phenomenon in cellular automata that requires two or more dimensions.

Cellular automaton (d) follows the simple additive rule that takes the value of each site to be the sum modulo two of the previous values of all sites in its five-site neighborhood. The space-time pattern generated by this rule has a fractal form. The fractal dimension of this pattern, and its analogs on d-dimensional lattices, is given by[4]: $\log_2\{d[(1 + 4/d)^{1/2} + 1]\}$, or approximately 2.45 for $d = 2$. The average density of nonzero sites in the pattern tends to zero with time.

Rules (e) and (f) give patterns with nonzero asymptotic densities. The boundaries of the patterns obtained at most time steps are corrugated, and have fractal forms analogous to Koch curves. The patterns grow by producing "branches" along the four lattice directions. Each of these branches then in turn produces side branches, which themselves produce side branches, and so on. This recursive process yields a highly corrugated boundary. However, as the process continues, the side branches grow into each other, forming an essentially solid region. In fact, after each 2^j time steps the boundary takes on an essentially regular form. It is only between such times that a dendritic boundary is present.

Cellular automaton (e) is an example of a "solidification" rule,[6] in which any site, once it attains value one, never reverts to value zero. Such rules are of significance in studies of processes such as crystal growth. Notice that although the interior of the pattern takes on a fixed form with time, the possibility of a simple one-dimensional cellular automaton model for the boundary alone is precluded by nonlocal effects associated with interactions between different side branches.

The boundaries of the patterns generated by cellular automata (a), (b), and (c) expand with time, but maintain the same faceted form. So after a rescaling in linear dimensions by a factor of t, the boundaries take on a fixed form: the pattern obtained is a fixed point of the product of the cellular automaton mapping and the rescaling transformation (cf. Refs. 20 and 21). The boundaries of Figs. 2(d, e, f) and 3(d, e, f) continually change with time; a fixed limiting form after rescaling can be obtained only by considering a particular sequence of time steps, such as those of the form 2^j. The result depends critically on the sequence considered: some sequences yield dendritic limiting forms, while other yield faceted forms. The complete space-time patterns illustrated in Figs 3(d, e, f) again approach a fixed limiting form after rescaling only when particular sequences of times are considered. It appears, however, that the forms obtained with different sequences have the same overall properties: they are asymptotically self-similar and have definite fractal dimensions.

The limiting structure of patterns generated by the growth of cellular automata from simple seeds can be characterized by various "growth dimensions." Two general types may be defined. The first, denoted generically D, depend on the overall space-time pattern. The second, denoted $\bar{D}$, depend only on the boundary of the pattern. The boundary may be defined as the set of sites that can be reached by some path on the lattice that begins at infinity and does not cross any nonzero sites. The boundary can thus be found by a simple recursive procedure (cf. Ref. 22). For rules that depend on more than nearest-neighboring sites, paths that pass within the range of the rule of any nonzero site are also excluded, and so no paths can enter any "pores" in the surface of the pattern.

Growth dimensions in general describe the logarithmic asymptotic scaling of the total sizes of patterns with their linear dimensions. For example, the spatial growth dimension D_x is defined in terms of the total number of sites n (interior and boundary) contained in patterns generated by a cellular automaton as a function of time t by the limit of $\log n/\log t$ as $t \to \infty$. Figure 5 shows the behavior of $\log n$ as a function of $\log t$ for the cellular automata of Figs 2 and 3. For those with faceted boundaries, $D_x = \log n/\log t = 2$ for all sufficiently large t: the total size of the patterns scales as the square of the parameter t that determines their linear dimensions.

When the boundaries can be dendritic, however, $\log n$ varies irregularly with $\log t$. In case (d), for example, $\log n$ depends on the number of non-zero digits in the binary decomposition of the integer t (cf. Ref. 4): $\log n/\log t$ is thus maximal when $t = 2^j - 1$, and is minimal when $t = 2^j$. One may define upper and lower spatial growth dimensions D_x^+ and D_x^- in terms of the upper and lower limits (lim *sup* and lim *inf*) of $\log n/\log t$ as $t \to \infty$. For case (d), $D_x^+ = 2$, while $D_x^- = 0$. For cases (e) and (f), $\log n/\log t$ oscillates with time, achieving its maximal value at $t = 2^j - 1$, and its minimal value at or near $t = 3/2 \times 2^j$. However, in these cases numerical results suggest that the upper and lower growth dimensions are in fact equal, and in both cases have a values $\simeq 2$.

An alternative definition of the spatial growth dimension includes only nonzero sites in computing the total sizes of patterns generated by cellular automaton evolution. With this definition, the spatial growth dimension has no definite limit even for cellular automata such as that of case (b) which give patterns with faceted boundaries.

The spatial growth dimensions $\bar{D}_x$ for the boundaries of patterns generated by cellular automata are obtained from the limits of $\log \bar{n}/\log t$ at large t, where $\bar{n}$ gives the number of sites in the boundary at time t (cf. Ref. 23). Figure 5 shows the behavior of $\log \bar{n}$ with $\log t$ for the cellular automata of Figs. 2 and 3. For the faceted boundary cases (a), (b), and (c), $\bar{D}_x = 1$. In cases (d), (e), and (f), where dendritic boundaries occur, $\log \bar{n}$ varies irregularly with $\log t$. $\log \bar{n}/\log t$ is minimal when $t = 2^j$ and the boundary is faceted, and is maximal when the boundary is maximally dendritic, typically at $t = 2^j - 1$. No unique limit for $\bar{D}_x$ exists. In case (d), $\bar{D}_x^+ = 1.62 \pm 0.02$, while $\bar{D}_x^- = 0$. In case (e), $\bar{D}_x^+ = 1.65 \pm 0.02$ and $\bar{D}_x^- = 1$, while in case (f), $\bar{D}_x^+ = 1.53 \pm 0.02$ and $\bar{D}_x^- = 1$.

The limiting forms obtained after rescaling for the spatial patterns generated by the dendritic cellular automata (d), (e), and (f) depend on the sequences of time steps used in the limiting procedure, so that there are no unique values for their spatial growth dimensions. On the other hand, the overall forms of the complete space-time patterns generated by these cellular automata do have definite limits, so that the growth dimensions that characterize them have definite values. The total growth dimensions D and $\bar{D}$[5] may be defined as $\lim_{T \to \infty} \log N/\log T$ and $\lim_{T \to \infty} \log \bar{N}/\log T$, where N is the total number of sites contained in the space-time pattern generated up to time step T, and $\bar{N}$ is the number of sites in its boundary. [Notice that $N = \sum_{t=0}^T n(t)$.] Figure 5 shows the behavior of $\log N$ and $\log \bar{N}$ as a function of $\log T$ for the cellular automata of Figs. 2 and 3. Unique values of D and $\bar{D}$ are indeed found in all cases. Rules that give pat-

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[5] This quantity is referred to as the "growth rate dimension" in Ref. 20.
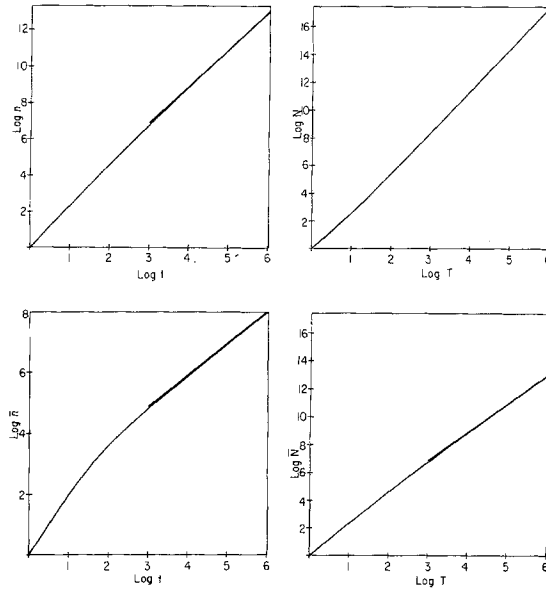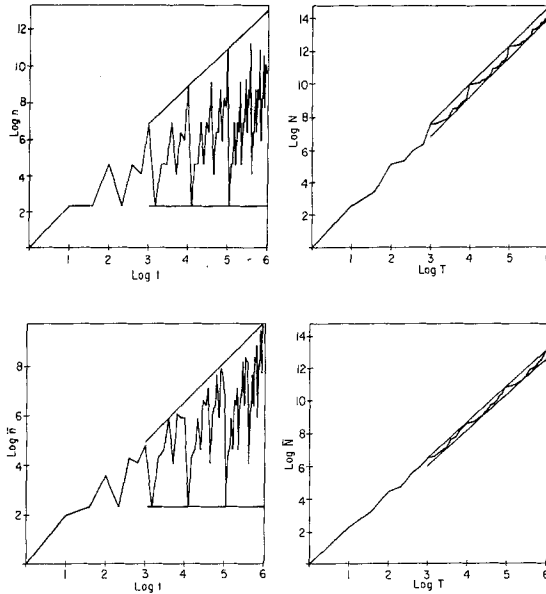
Fig. 5. Sizes of structures generated by the two-dimensional cellular automata of Fig. 2 growing from single nonzero initial sites as a function of time. (Although the sizes are defined only at integer times, their successive values are shown joined by straight lines.) $\bar{n}$ gives the number of sites on the boundaries of patterns obtained at time t. n gives the total number of sites contained within these boundaries. $\bar{N}$ is the number of sites in the boundary (surface) of the complete three-dimensional space-time structures illustrated in Fig. 3 up to time T, and N is the number of sites in their interior. The large-t limits of $\log n/\log t$ and so on give various growth dimensions for the structures. In cases (a), (b), and (c), structures with faceted boundaries are produced, and the growth dimensions have unique values. In cases (d), (e), and (f) the structures have dendritic boundaries, and the slopes of the bounding lines shown give upper (lim sup) and lower (lim inf) limits for the growth dimensions. In many of the cases shown, the numerical values of these upper and lower limits appear to coincide.

(e)

(f)
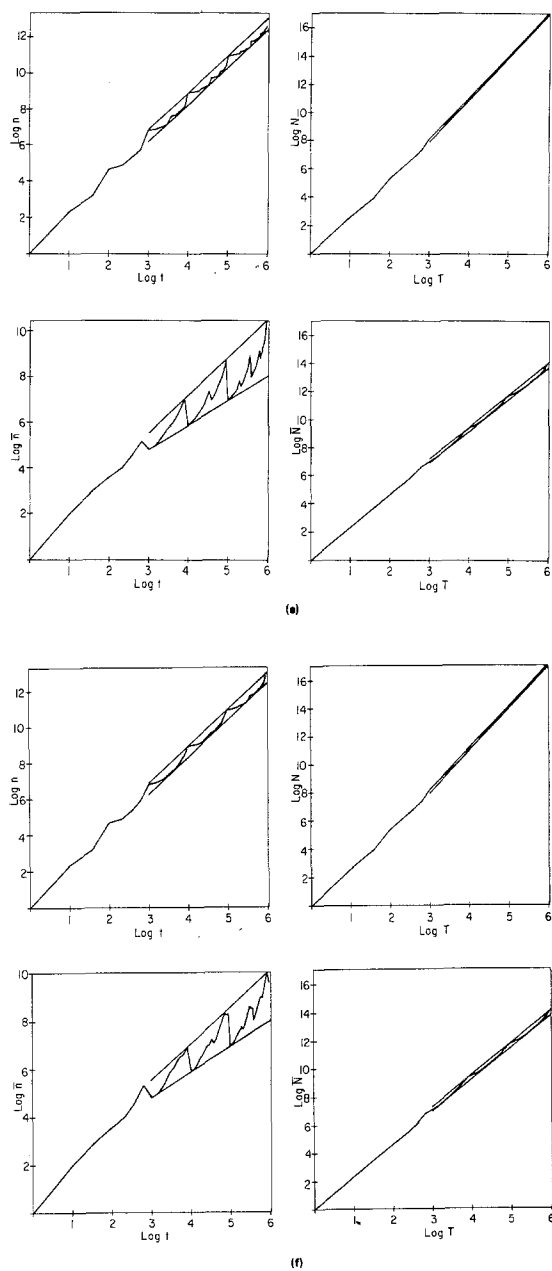
Fig. 5 (continued)

terns with faceted boundaries have $D = 3$, $\bar{D} = 2$. The additive rule of case (d) gives $D = 2.36 \pm 0.02$, $\bar{D} = 2.19 \pm 0.02$. Cases (e) and (f) both give $D = 3$, $\bar{D} = 2.27 \pm 0.02$.

Growth dimensions may be defined in general by considering the intersection of the complete space-time pattern, or its boundary, with various families of hyperplanes. With fixed-time hyperplanes one obtains the spatial growth dimensions D_x and $\bar{D}_x$. Temporal growth dimensions $D_t^{(x)}$ and $\bar{D}_t^{(x)}$ are obtained by considering sections through the space-time pattern in spatial direction x. (The section typically includes the site of the original seed.) The total growth dimension may evidently be obtained as an appropriate average over temporal growth dimensions in different directions. (The average must be taken over pattern sizes n, and so requires exponentiation of the growth dimensions.) The values of the temporal growth dimensions for the patterns of Figs. 2 and 3 depend on their internal structure. Cases (a), (c), (e), and (f) have $D_t = 2$; case (b) has $D_t = \log_2 3 \simeq 1.59$, and case (d) has $D_t = \log_2(1 + \sqrt{5}) \simeq 1.69$. The temporal growth dimensions $\bar{D}_t^{(x)}$ for the boundaries of the patterns are equal to one for the faceted boundary cases. These dimensions vary with direction in cases with dendritic boundaries. They are equal to one in directions of maximal growth, but are larger in other directions.

In general the values of growth dimensions associated with particular hyperplanes are bounded by the topological dimensions of those hyperplanes. Empirical studies indicate that among all (symmetric) two-dimensional cellular automata, patterns with the form of case (c), characterized by $D = 3$, $\bar{D} = 2$, $D_t = 2$ are the most commonly generated. Fractal boundaries are comparatively common, but their growth dimensions $\bar{D}$ are usually quite close to the minimal value of two. Fractal sections with $D_t < 2$ are also comparatively common for five-neighbor rules, but become less common for nine-neighbor rules.

The rules for the two-dimensional cellular automata shown in Figs. 2 and 3 are completely invariant under all the rotation and reflection symmetry transformations on their neighborhoods. Figure 6 shows patterns generated by cellular automaton rules with lower symmetries. These patterns are often complicated both in their boundaries and internal structure. Even though the patterns grow from completely symmetric initial states consisting of single nonzero sites, they exhibit definite directionalities and vorticities as a consequence of asymmetries in the rules. Asymmetric patterns may be obtained with symmetrical rules from asymmetric initial states containing several nonzero sites. For example, some rules should support periodic structures that propagate in particular directions with time. Other rules should yield spiral patterns with definite vorticities. Structures of these kinds are expected to be simpler in many $k > 2$ rules than for
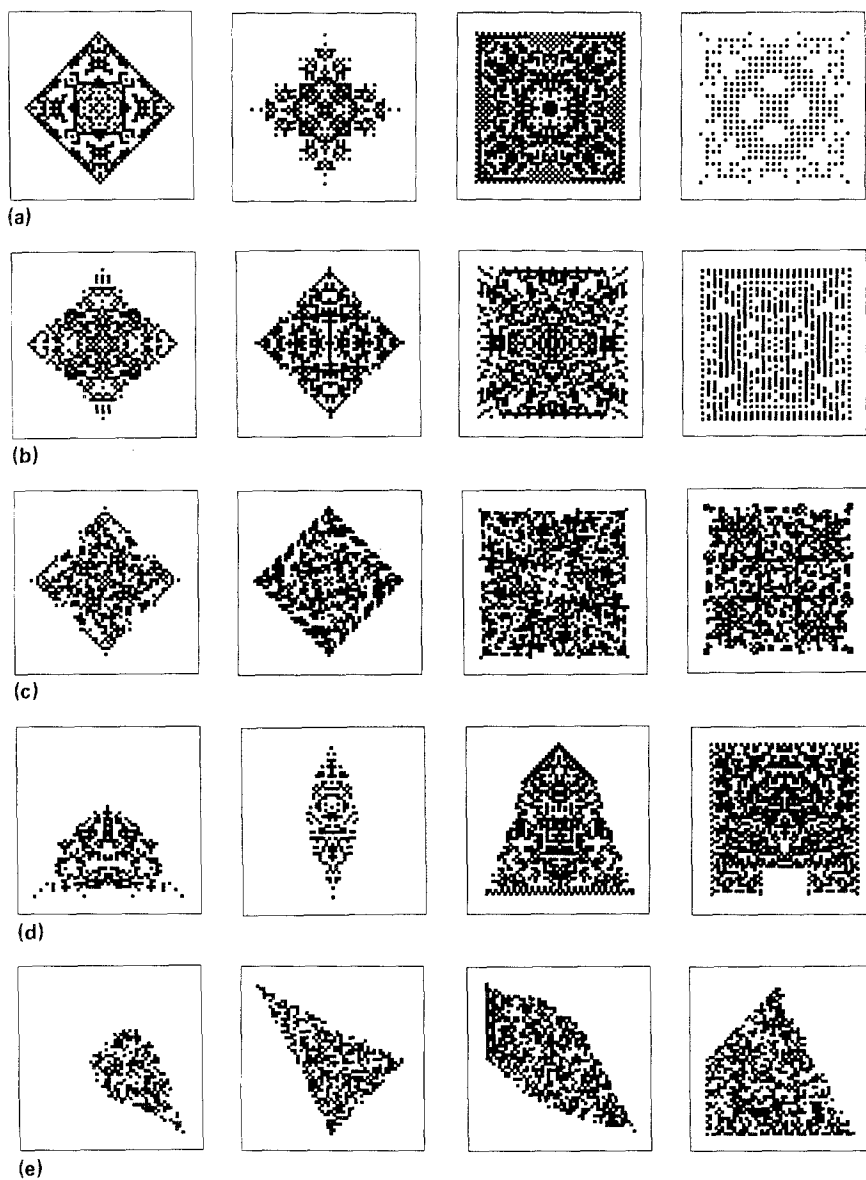
Fig. 6.   Examples of patterns generated by growth from single-site seeds for 24 time steps according to general nine-neighbor square rules, with symmetries: (a) all, (b) horizontal and vertical reflection, (c) rotation, (d) vertical reflection, (e) none.

$k = 2$ rules (cf. Ref. 24) just as in one-dimensional cellular automata. Notice that spiral patterns in two-dimensional cellular automata have total growth dimensions $D = \bar{D} = 2$.

Figure 7 shows the evolution of various two-dimensional cellular automata from initial states containing both single nonzero sites, and small regions with a few nonzero sites. In most cases, the overall patterns generated after a sufficiently long time are seen to be largely independent of the particular form of the initial state. In cases such as (c) and (e), features in the initial seed lead to specific dislocations in the final patterns. Nevertheless, deformations in the boundaries of the patterns usually occur only on length scales of order the size of the seed, and presumably become negligible in the infinite time limit. As a consequence, the growth dimensions for the resulting patterns are usually independent of the form of the initial seed (cf. Ref. 20 for additive rules).

There are nevertheless some cellular automaton rules for which slightly different seeds can lead to very different patterns. This phenomenon occurs when a cellular automaton whose configurations contain only certain blocks of site values satisfies an effective rule with special properties such as scale invariance. If the initial seed contains only these blocks, then the pattern generated follows the effective rule. However, if other blocks are present, a pattern of a different form may be generated. An example of this behavior for a one-dimensional cellular automaton is shown in Fig. 8. Patterns produced with one type of seed have temporal growth dimension $\log_2 3 \simeq 1.59$, while those with another type of seed have dimension 2.

Cellular automaton rules embody a finite maximum information propagation speed. This implies the existence of a "bounding surface" expanding at this finite speed. All nonzero sites generated by cellular automaton evolution from a localized seed must lie within this bounding surface. (The cellular automata considered here leave a background of zero sites invariant; such a background must be mapped to itself after at most k time steps with any cellular automaton rule.) Thus the pattern generated after t time steps by any cellular automaton is always bounded by the polytope (planar-faced surface) corresponding to the "unit cell" formed from the set of vectors specifying the displacements of sites in the neighborhood, magnified by a factor t in linear dimensions (cf. Ref. 14). Thus patterns generated by five-neighbor cellular automaton rules always lie within an expanding diamond-shaped region, while those with nine-neighbor rules may fill out a square region.
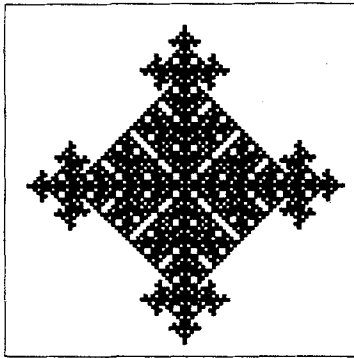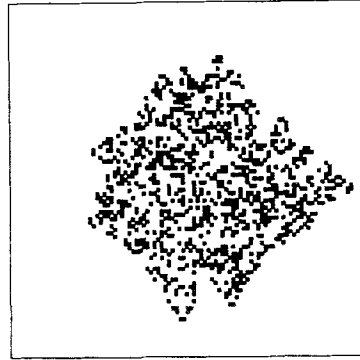
The actual minimal bounding surface for a particular cellular automaton rule often lies far inside the surface obtained by magnifying the unit cell. A sequence of better approximations to the bounding surface may be found as follows. First consider a set of sites representing the
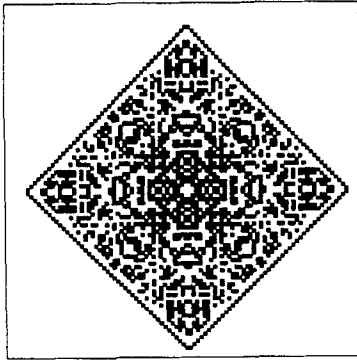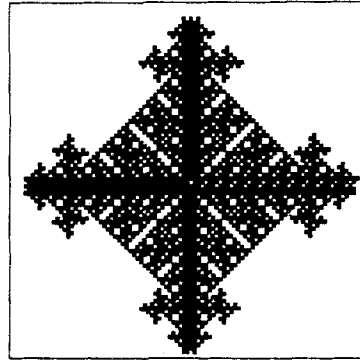
(a)
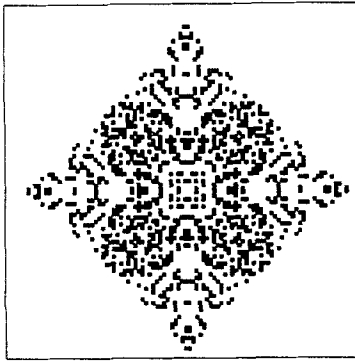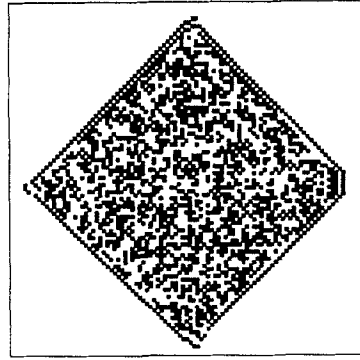
(b)

(c)

Fig. 7. Examples of patterns generated by evolution of two-dimensional cellular automata from minimal seeds and small disordered regions. In most cases, growth is initiated by a seed consisting of a single nonzero site; for some of the rules shown, a square of four nonzero sites is required. The cellular automaton rules shown are nine-neighbor square outer totalistic, with codes (a) 143954, (b) 50224, (c) five-neighbor 750, (d) 15822, (e) 699054, (f) 191044, (g) 11202, (h) 93737, (i) 85507.
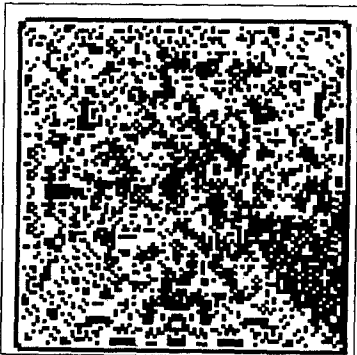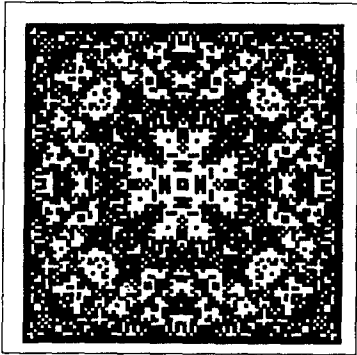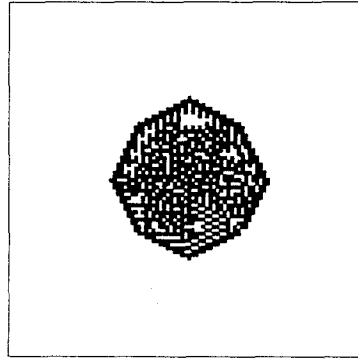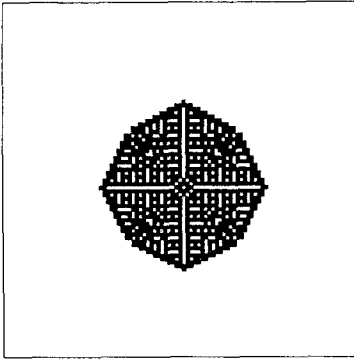
(d)
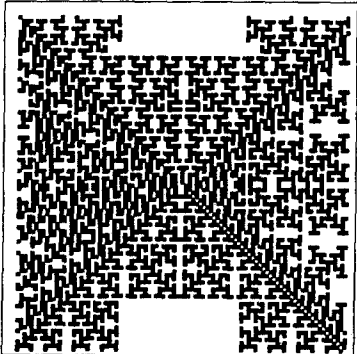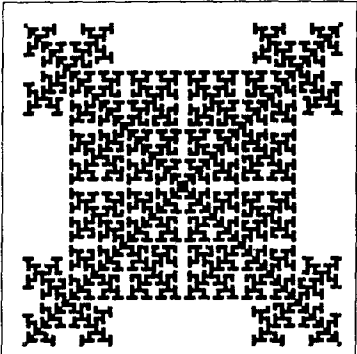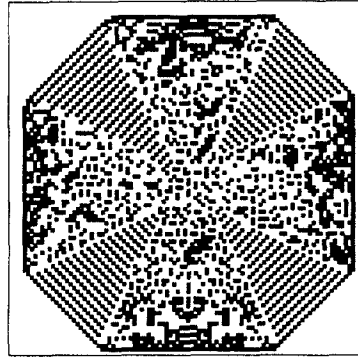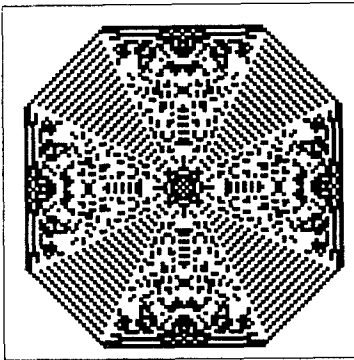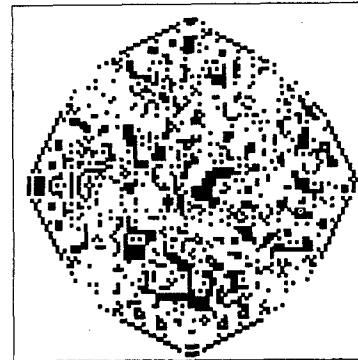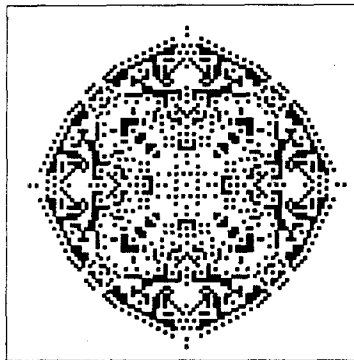
(e)
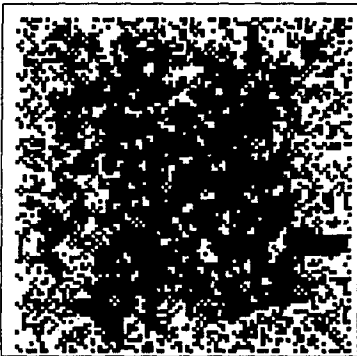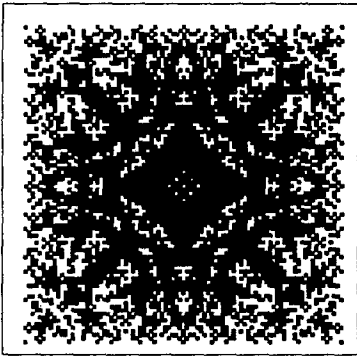
(f)

Fig. 7 (*continued*)

(g)

(h)

(i)

minimal
seed
t = 44

disordered
region
t = 44

Fig. 7 (*continued*)

Fig. 8. Example of a one-dimensional cellular automaton in which space-time patterns with different temporal growth dimensions are obtained with different initial seeds. The cellular automaton has $k = 2$, $r = 1$, and rule number 218. With an initial state containing only the blocks 00 and 10, it behaves like the additive rule 90, and yields a self-similar space-time pattern with fractal dimension $\log_2 3$. But when the initial state contains 10 and 11 blocks, it behaves like rule 128, and yields a uniform space-time pattern.
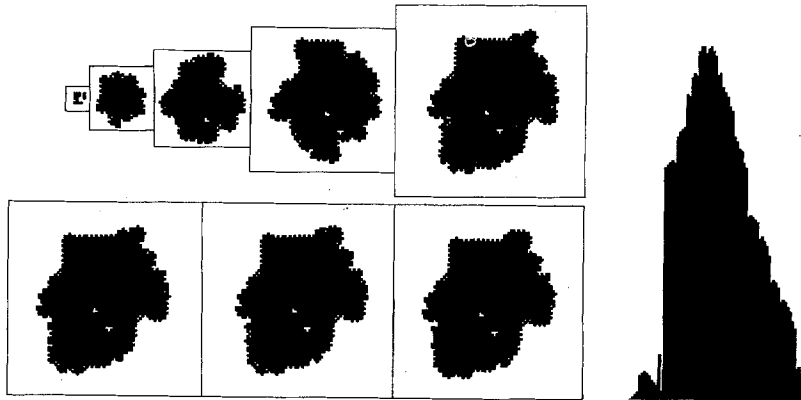
(a)
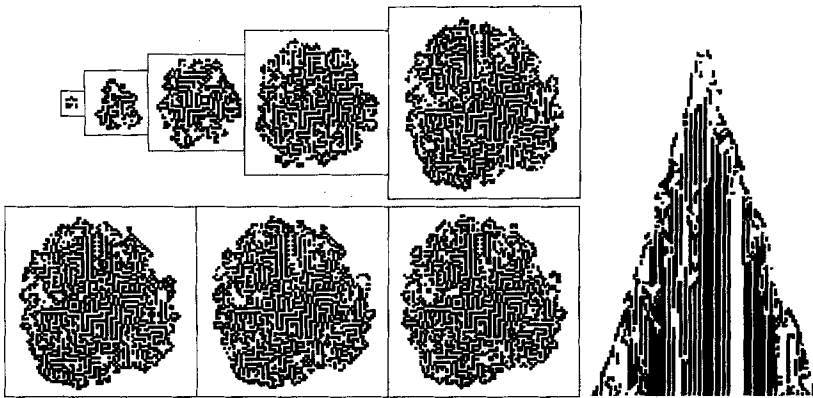
(b)

Fig. 9. Examples of two-dimensional cellular automata that exhibit slow diffusive growth from small disordered regions. The cellular automaton rules shown are nine-neighbor square outer totalistic, with codes (a) 256746, (b) 736, (c) 291552.

(c)

(section)
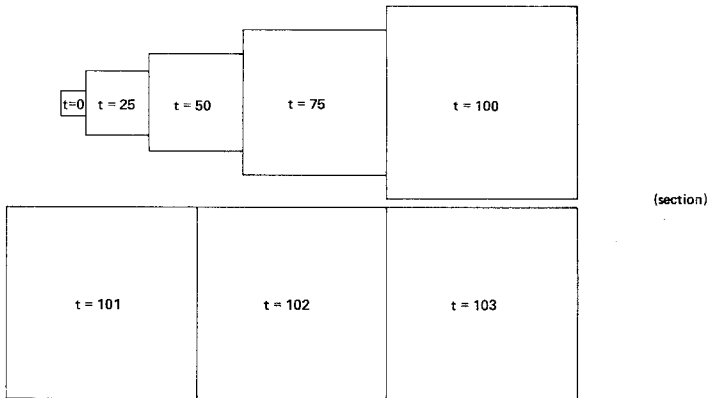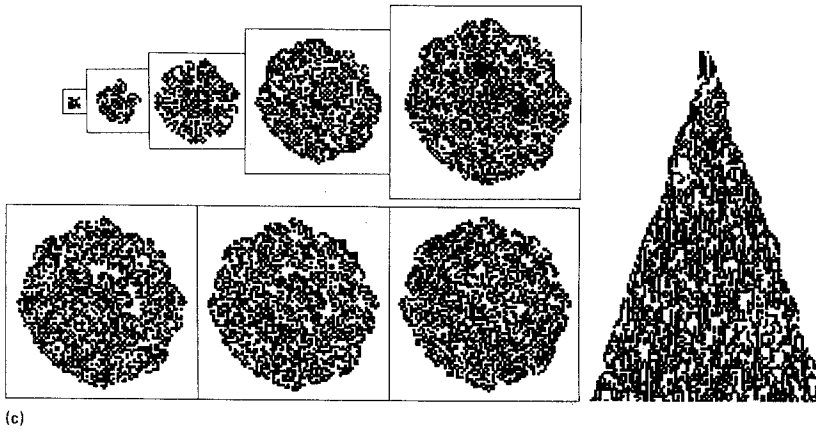
Fig. 9 (continued)

neighborhood for a cellular automaton rule. If the center site has value one at a particular time step, there could exist configurations for which all of the sites in the neighborhood would attain value one on the next time step. However, there may be some sites whose values cannot change from zero to one in a single time step with any configuration. Growth does not occur along directions corresponding to such sites. The polytope formed from sites in the neighborhood, excluding such sites, may be magnified by a factor t to yield a first approximation to the actual bounding surface for a cellular automaton rule. A better approximation is given by the polytope obtained after two time steps of cellular automaton evolution, magnified by a factor $t/2$.

The actual bounding surfaces for five-neighbor two-dimensional

cellular automaton rules usually have their maximal diamond-shaped form. However, many nine-neighbor rules have a diamond-shaped form, rather than their maximal square form. Some nine-neighbor rules, such as those of Figs. 7(g) and 7(h) have octagonal bounding surfaces, while still others, such as those of Fig. 7(i) have dodecagonal bounding surfaces. The cellular automata rules with lower symmetries illustrated in Fig. 6 in many cases exhibit more complicated boundaries, with lower symmetries.

Patterns that maintain regular boundaries with time typically fill out their bounding surface at all times. Dendritic patterns, however, usually expand with the bounding surface only along a few axes. In other directions, they meet the bounding surface only at specific times, typically of the form 2^j. At other times, they lie within the bounding surface.

Dendritic boundaries seem to be associated with cellular automaton rules that exhibit "growth inhibition" (cf. Ref. 14). Growth inhibition occurs if there exist some a_i for which $\phi(a_1,..., 0,..., a_n) = 1$, but $\phi(a_1,..., 1,..., a_n) = 0$, or vice versa. Such behavior appears to be common in physical and other systems.

Figures 9 and 10 show examples of two-dimensional cellular automata that exhibit the comparatively rare phenomenon of slow, diffusive, growth from simple seeds. Figure 11 gives a one-dimensional cellular automaton with essentially analogous behavior.

The phenomenon is most easily discussed in the one-dimensional case. The pattern shown in Fig. 11 is such that it expands by one site at a particular time step only if the site on the boundary has value one. If the boundary site has one of its other three possible nonzero values, then on average, no expansion occurs. The cellular automaton rule is such that the boundary sites have values one through four with roughly equal frequencies. Thus the pattern expands on average at a speed of about 1/4 sites per time step (on each side).

The origin of diffusive growth is similar in the two-dimensional case. Growth occurs there only when some particular several-site structure appears on the boundary. For example, in the cellular automaton of Fig. 9(a), a linear interface propagates at maximal velocity. Deformations of the interface slow its propagation, and a maximally corrugated interface with a "battlement" form does not propagate at all. Since many boundary structures occur with roughly equal probabilities, the average growth rate is small. In the cases investigated, the growth rate is asymptotically constant, so that the growth dimensions have definite values. A remarkable feature is that the boundaries of the patterns produced do not follow the polytopic form suggested by the underlying lattice construction of the cellular automaton. Instead, in many cases, asymptotically circular patterns appear to be produced.
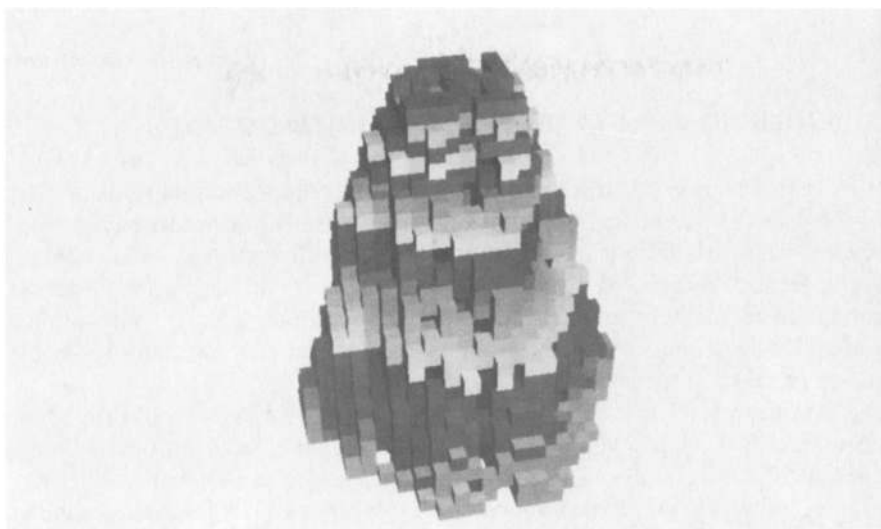
Fig. 10. View of three-dimensional structure formed from the configurations generated in the first 24 time steps of evolution according to the two-dimensional cellular automaton rule of Fig. 9(a).
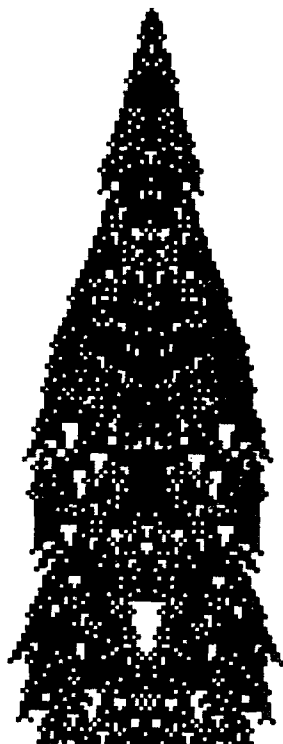
Fig. 11. Example of a one-dimensional cellular automaton that exhibits slow growth. The rule shown is totalistic $k = 5$, $r = 1$, with code 985707700. All nonzero sites are shown black. The initial state contains a single site with value 3. Growth occurs when a site with value 1 appears on the boundary.

3. EVOLUTION FROM DISORDERED INITIAL STATES

In this section, we discuss the evolution of cellular automata from dis-
ordered initial states, in which each site is randomly chosen to have value
zero or one (usually with probability 1/2). Such disordered configurations
are typical members of the set of all possible configurations. Patterns
generated from them are thus typical of those obtained with any initial
state. The presence of structure in these patterns is an indication of
self-organization in the cellular automaton.

As mentioned in Section 1, four qualitative classes of behavior have
been identified in the evolution of one-dimensional cellular automata from
disordered initial states. Examples of these classes are shown in Fig. 12.
Figure 13 shows the evolution of some typical two-dimensional cellular
automata from disordered initial states. The same four qualitative classes of
behavior may again be identified here. In fact, the space-time sections for
two-dimensional cellular automata have a striking qualitative similarity to
sections obtained from one-dimensional cellular automata, perhaps with
some probabilistic noise added.

Just as in one dimension, some two-dimensional cellular automata
evolve from almost all initial states to a unique homogeneous state, such as
the null configuration. The final state for such class 1 cellular automata is
usually reached after just a few time steps, but in some rare cases, there
may be a long transient.

Figures 13(a) and 14(a) give an example of a two-dimensional cellular
automaton with class-2 behavior. The disordered initial state evolves to a
collection of separated simple structures, each stable or oscillatory with a
small period. Each of these structures is a remnant of a particular feature in
the initial state. The cellular automaton rule acts as a "filter" which preser-
ves only certain features of the initial state. There is usually a simple pat-
tern to the set of features preserved, and to the set of persistent structures
produced. It should in fact be possible to devise cellular automaton rules
that recognize particular sets of features, and to use such class-2 cellular
automata for practical image processing tasks (cf. Ref. 25).

The patterns generated by evolution from several different disordered
configurations according to a particular cellular automaton rule are almost
always qualitatively similar. Yet in many cases the cellular automaton
evolution is unstable, in that small changes in the initial state lead to
increasing changes in the patterns generated with time. Figures 12 and 13
include difference patterns that illustrate the effect of changing the value of
a single site in the initial state. For class-2 cellular automata, such a change
affects only a finite region, and the difference pattern remains bounded with
time. Information propagates only a finite distance in class-2 cellular

automata, so that a particular region of the final state is determined from a bounded region in the initial state. For class-3 cellular automata, on the other hand, information generically propagates at a nonzero speed forever, and a small change in the initial state affects an ever-increasing region. The difference patterns for class-3 cellular automata thus grow without bound, usually at a constant rate.

The locally periodic patterns generated after many time steps by class-2 cellular automata such as in Fig 13(a) consist of many separated structures located at essentially arbitrary positions. Figure 13(b) shows another form of class-2 cellular automaton. There are four basic "phases." Two phases have vertical stripes, with either on even or odd sites. The other two phases have horizontal stripes. Regions that take on forms corresponding to one of these phases are invariant under the cellular automaton rule. Starting from a typical disordered state, each region in the cellular automaton lattice evolves toward a particular phase. At large times, the cellular automaton thus "crystallizes" into a patchwork of "domains." The domains consist of regions in particular phases. They are separated by domain walls. In the example of Fig. 13(b), these domain walls become essentially stationary after a finite time.

A change in a single initial site produces a difference pattern that ultimately spreads only along the domain walls. The spread continues only so long as each successive region on the domain wall contains only par-ticular arrangements of site values. The spread stops if a "pinning defect," corresponding to other arrangements of site values, is encountered. The arrangement of site values on the domain walls may in a first approximation be considered random. The difference pattern will thus spread forever only if the arrangements of site values necessary to support its propagation occur with a probability above the percolation threshold (e.g., Ref. 26), so that they form an infinite connected cluster with probability one.

Phases in cellular automata may in general be described by "order parameters" that specify the spatially periodic patterns of sites corresponding to each phase. The size of domains generated by evolution from disordered initial states depends on the length of time before the domains become "frozen": slower relaxation leads to larger domains, as in annealing. A final state reached after any finite time can contain only finite size domains, and therefore cannot be a pure phase. States generated by two-dimensional cellular automata may contain "point" and "line" defects. Point defects are localized regions within domains. An example is the "L-shaped" region of zero sites in domains of the value one phase for the cellular automaton illustrated in Fig. 13(e). Line defects correspond to walls separating domains.
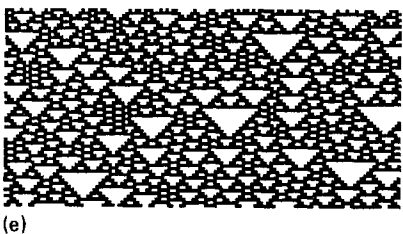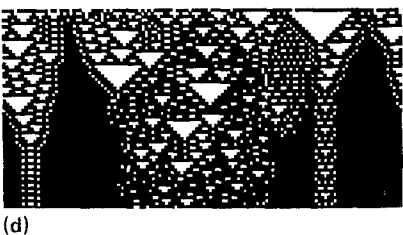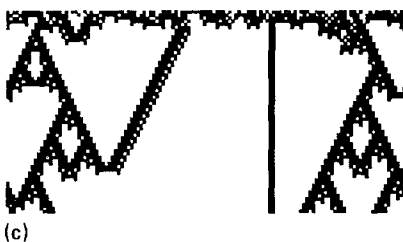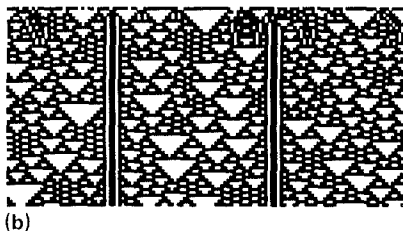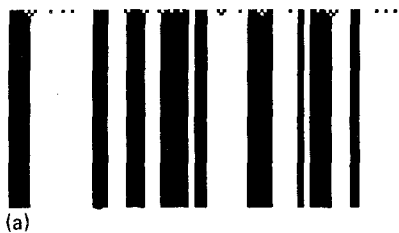
Fig. 12. Examples of the evolution of one-dimensional cellular automata from disordered initial states. The difference patterns on the right show site values that change when a single initial site value is changed. All nonzero sites are shown black. The cellular automaton rules shown are totalistic nearest neighbor ($r = 1$), with k possible values at each site: (a) $k = 2$, code 12, (b) $k = 5$, code 7530, (c) $k = 3$, code 681, (d) $k = 5$, code 3250, (e) $k = 2$, code 6, (f) $k = 3$, code 348, (g) $k = 3$, code 138, (h) $k = 3$, code 318, (i) $k = 3$, code 792.

(f)

(g)
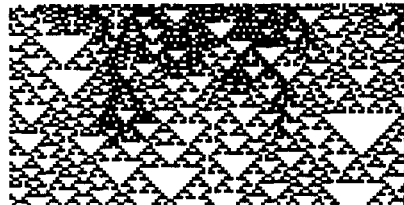
(h)

(i)

Fig. 12 (*continued*)

(a)

(b)

(c)
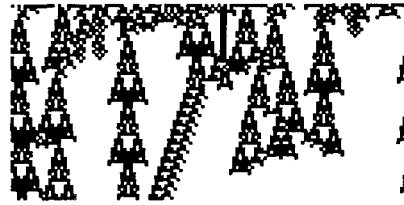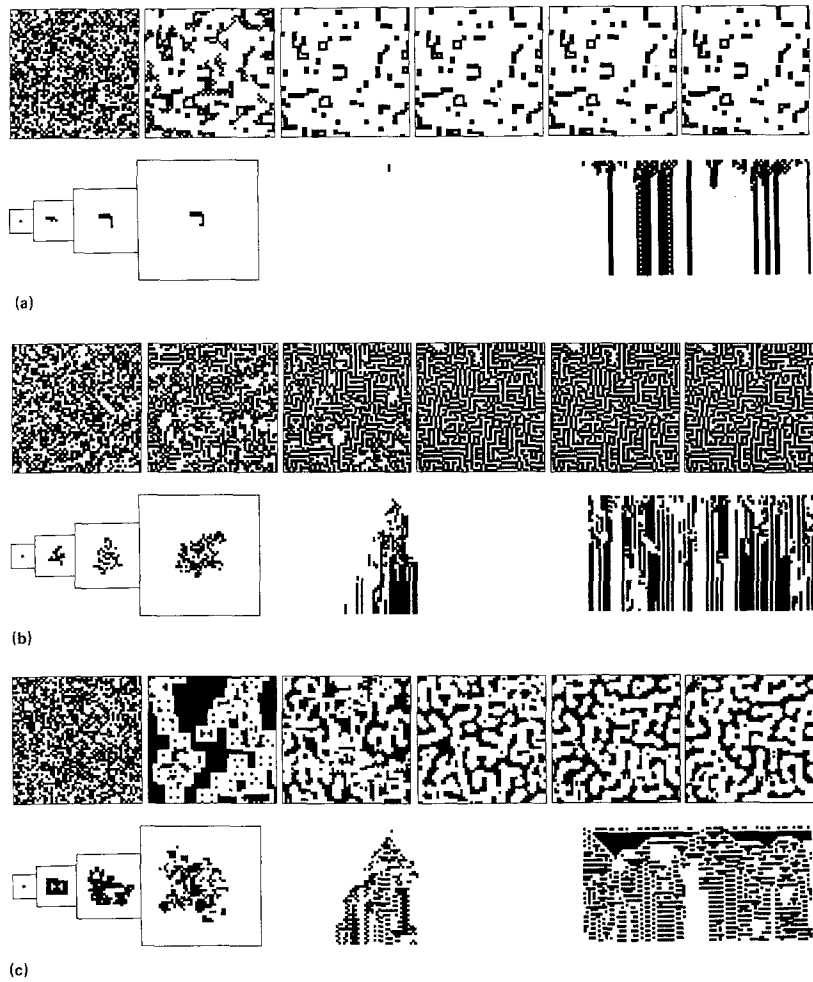
Fig. 13. Examples of the evolution of two-dimensional cellular automata from disordered initial states. The cellular automaton rules shown are totalistic five-neighbor square with codes: (a) 24, (d) 510, (e) 52; and outer totalistic nine-neighbor with codes: (b) 736, (c) 196623, (f) 152822, (g) 143954, (h) 3276, (i) 224 (the "Game of Life").

(d)

(e)

(f)

Fig. 13 (continued)

(g)

(h)

(i)

| t = 0 | t = 5 | t = 25 | t = 125 | t = 625 | t = 626 |

| t=0 | t = 5 | t = 10 | t = 20 | (section) | (section) |

difference pattern
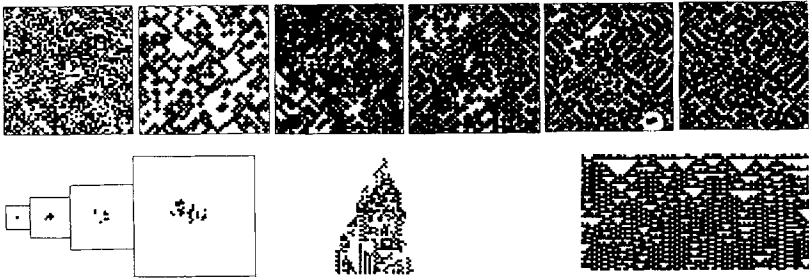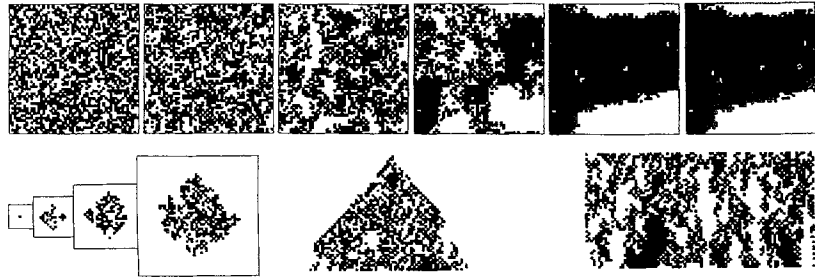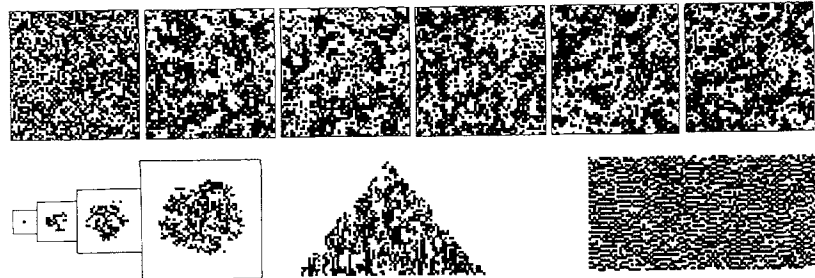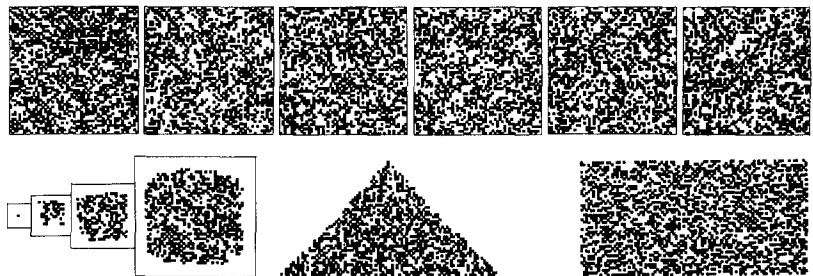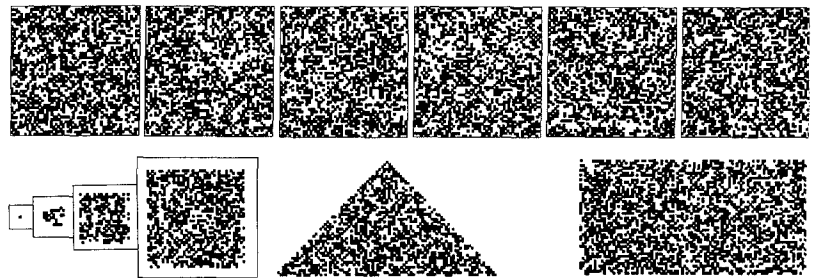
Fig. 13 (*continued*)

Fig. 14. View of three-dimensional structures formed by configurations generated in the first 24 times of evolution from disordered initial states (in a finite region) according to the cellular automaton rules of Figs. 13(a) and 13(i).

In the cellular automaton of Fig. 13(b), the domains become stationary after a few time steps. In the case of Fig. 13(e), however, the domains can continue to move forever, essentially by a diffusion process. Figure 12(d) shows a one-dimensional cellular automaton with domain walls that exhibit analogous behavior (cf. Refs. 27 and 17). In both cases, some domains become progressively larger with time, while others eventually disappear completely. The domain walls in Fig. 13(e) behave as if they carry a positive surface tension (cf. Ref. 28); the diffusion process responsible for their movement is biased to reduce the local curvature of the interface. A linear interface is stable under the cellular automaton rule of Fig. 13(e). In addition, the heights of any protrusions or intrusions cannot increase with time. In general, they decay, often quite slowly, until they are of height at most one. Deformations of height one, analogous to surface waves, do not decay further, and are governed by a one-dimensional cellular automaton rule (with $k = 2$, $r = 1$, and rule number 150). At large times, therefore, a domain must either shrink to zero size, or must have walls with continually decreasing curvatures.

Figure 13(c) shows a two-dimensional cellular automaton with structures analogous to domain walls that carry a negative surface tension. More and more convoluted patterns are obtained with time. The resulting labyrinthine state is strongly reminiscent of behavior observed with ferrofluids or magnetic bubbles.[29]

Figures 13(f), 13(g), and 13(h) are examples of two-dimensional cellular automata that exhibit class-3 behavior. Chaotic aperiodic patterns are obtained at all times. Moreover, t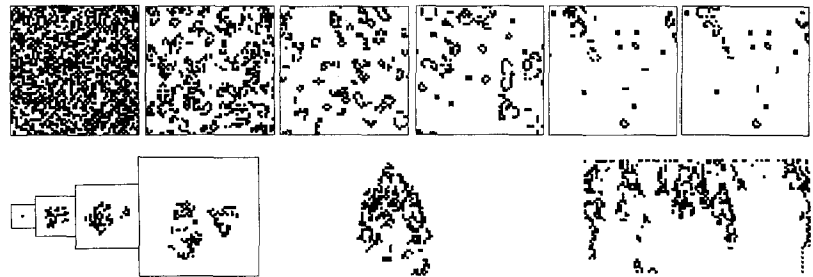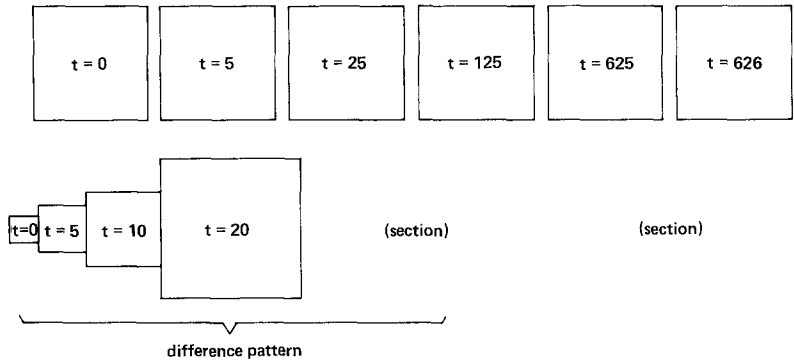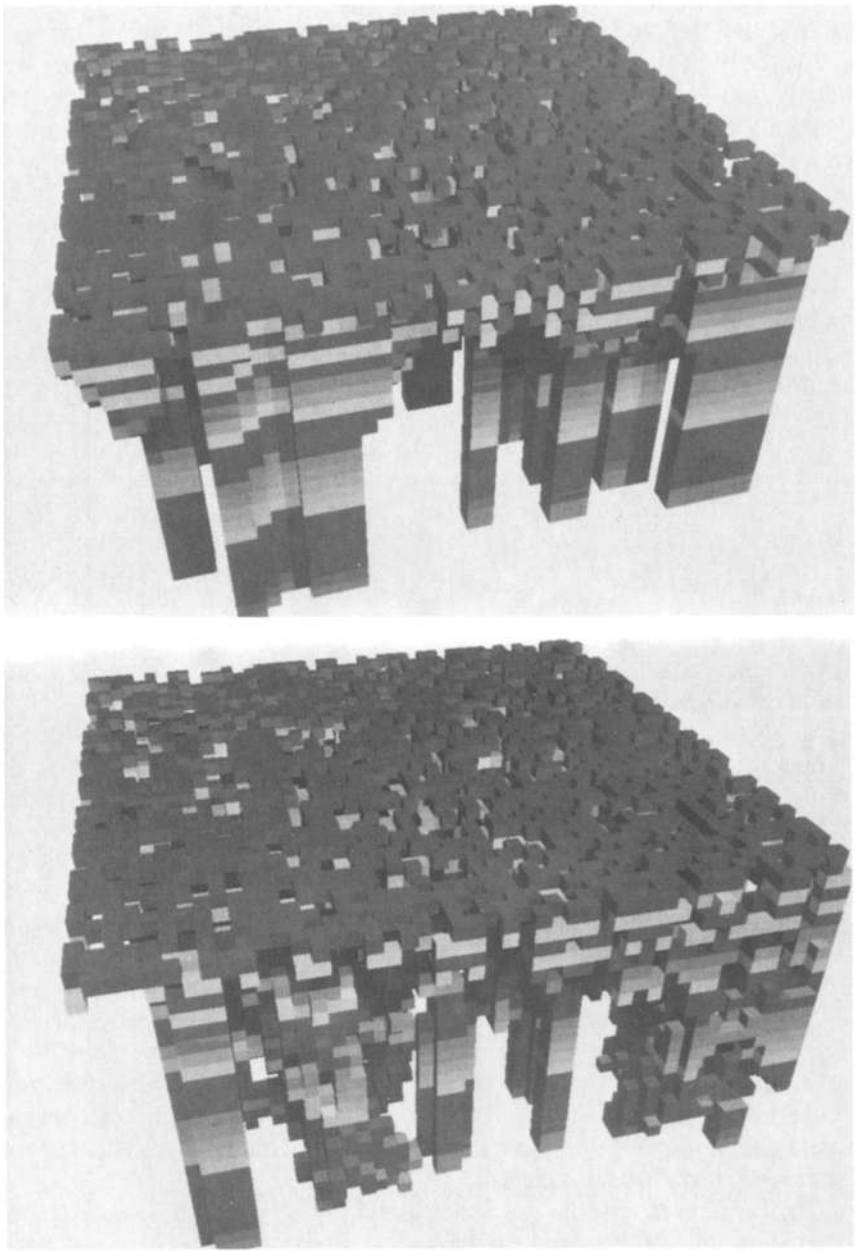he difference patterns resulting from changes in single initial site values expand at a fixed rate forever. A remarkable feature is that in almost all cases (Fig. 13(h) is an exception), the expansion occurs at the same speed in all directions, resulting in an asymptotically circular difference pattern. For some rules, the expansion occurs at maximal speed; but often the speed is about 0.8 times the maximum. When the difference patterns are not exactly circular, they tend to have rounded corners. And even with asymmetrical rules, circular difference patterns are often obtained. A rough analog of this behavior is found in asymmetric one-dimensional cellular automata which generate symmetrical difference patterns. Such behavior is found to become increasingly common as k and r increase, or as the number of independent parameters in the rule ϕ increases.

An argument based on the central limit theorem suggests an explanation for the appearance of circular difference patterns in two-dimensional class-3 cellular automata. Consider the set of sites corresponding to the neighborhood for a cellular automaton rule. For each site, compute the probability that the value of that site changes after one time step of cellular

automaton evolution when the value of the center site is changed, averaged over all possible arrangements of site values in the neighborhood. An approximation to the probability distribution of differences is then obtained as a multiple convolution of this kernel. (This approximation is effectively a linear one, analogous to Huygens' principle in optics.) The number of convolutions performed increases with time. If the number of neighborhood arrangements is sufficiently large, the kernel tends to be quite smooth. Convolutions of the kernel thus tend to a Gaussian form, independent of direction.

Some asymmetric class-3 cellular automata yield difference patterns that expand, say, in the horizontal direction, but contract in the vertical direction. At large times, such cellular automata produce patterns consisting of many independent horizontal lines, each behaving essentially as a one-dimensional class-3 cellular automaton.

Class-3 behavior is considerably the commonest among two-dimensional cellular automata, just as it is for one-dimensional cellular automata with large k and r. It appears that as the number of parameters or degrees of freedom in a cellular automaton rule increases, there is a higher probability for some degree of freedom to show chaotic behavior, leading to overall chaotic behavior.

Figure 12(i) shows an example of a class-4 one-dimensional cellular automaton. A characteristic feature of class-4 cellular automata is the existence of a complicated set of persistent structures, some of which propagate through space with time. Class-4 rules appear to occur with a frequency of a few per cent among all one-dimensional cellular automaton rules. Often one suspects that some degrees of freedom in a cellular automaton exhibit class-4 behavior, but they are masked by overall chaotic class-3 behavior.

Class-4 cellular automata appear to be much less common in two dimensions than in one dimension. Figures 13(i) and 13(b) show the evolution of a two-dimensional cellular automaton known as the "Game of Life".[8] Many persistent structures, some propagating, have been identified in this cellular automaton. It has in addition been shown that these structures can be combined to perform arbitrary information processing, so that the cellular automaton supports universal computation.[8] Starting from a disordered initial state, the density of propagating structures ("gliders") produced is about one per 2000 site region.

Except for a few simple variants on the Game of Life, no other definite class-4 two-dimensional cellular automata were found in a random sample of several thousand outer totalistic rules.[6] Some of the rules that appeared

[6] A few examples of class-4 behavior were however found among general rules. Requests for copies of the relevant rule tables should be directed to the authors.

to be of class 2 were found to have long transients, characteristic of class-4 behavior, but no propagating structures were seen. Other rules seemed to exhibit some class-4 features, but they were overwhelmed by dominant class-3 behavior.

4. GLOBAL PROPERTIES

Section 3 discussed the typical behavior of cellular automata evolving from particular initial states. This section considers the global properties of cellular automata, determined by evolution from all possible initial states. Studies of the global properties of one-dimensional cellular automata have been made using methods both from dynamical systems theory[7] and from computation theory.[12] Here these studies are generalized to the case of two-dimensional cellular automata. For those based on dynamical systems theory the generalization is quite straightforward; but in the computation theory approach substantial additional complications occur. Whereas the sets of configurations generated after any finite number of steps in the evolution of a one-dimensional cellular automata always correspond to regular formal languages,[12] the corresponding sets in two-dimensional cellular automata may be nonrecursive.[15]

Most cellular automaton rules are irreversible, so that several different initial states may evolve to the same final state. As a consequence, even starting from all possible initial states, only a subset of possible states may be generated with time. The properties of this set then determine the overall behavior of the cellular automaton, and the self-organization that occurs in it.

Entropy and dimension provide quantitative characterizations of the "sizes" of sets generated by cellular automaton evolution (e.g., Ref. 7). The spatial set entropy for a set of two-dimensional cellular automaton configurations is defined by considering a $X \times Y$ patch of sites. If the set contains all possible configurations, then all k^{XY} possible different arrangements of sites values must occur in the patch. In general $N(X, Y) \leqslant k^{XY}$ different arrangements will occur. Then the set entropy (or dimension) is defined as

$$s = \lim_{X,Y \to \infty} \frac{1}{XY} \log_k N(X, Y) \qquad (4.1)$$

If the cellular automaton mapping is surjective, so that all possible configurations occur, then this entropy is equal to one. In general it decreases with time in the evolution of the cellular automaton.

Spatial set entropy characterizes the set of configurations that can possibly be generated in the evolution of a cellular automaton, regardless

of their probabilities of occurrence. One may also define a spatial measure entropy in terms of the probabilities p_i for possible $X \times Y$ patches as

$$s_\mu = \lim_{X,Y \to \infty} \frac{-1}{XY} \sum_{i=1}^{k^{XY}} p_i \log_k p_i \tag{4.2}$$

The limiting value of s_μ at large times is typically nonzero for all but class-1 cellular automaton rules. Notice that in cases where domains with positive "surface tension" are formed, s_μ tends only very slowly to zero with time.

To find the spatial set entropy after, say, one time step in the evolution of a cellular automaton one must identify what configurations can be generated. In a one-dimensional cellular automaton, one can specify the set of configurations that can be generated in terms of rules that determine which sequences of site values can appear. These rules correspond to a regular formal grammar, and give the state transition graph for a finite state machine. The set of configurations that can be generated in a two-dimensional cellular automaton is more difficult to specify. In many circumstances in fact the occurrence of a particular patches of site values requires a global consistency that cannot be verified in general by any finite computation. As a consequence, many propositions concerning sets of configurations generated after even a finite number of steps in the evolution of two-(and higher-)dimensional cellular automata can be formally undecidable.

In a one-dimensional cellular automaton with a range-r rule, a particular sequence of X site values can be generated (reached) after one time step only if there exists some length $X + 2r$ sequence of initial site values that evolves to it. The locality of the cellular automaton rule ensures that in determining whether a length $X + 1$ sequence obtained by appending one new site can also be generated, it suffices to test only those length $X + 2r + 1$ predecessor configurations that differ in their last $2r + 1$ site values. In determining whether sequences of progressively greater lengths can be generated it suffices at each stage to record with which length $2r$ overlaps in the predecessor configuration a particular new site value can be appended. Since there are only k^{2r} possible sequences of site values in the overlaps, only a finite amount of information must be recorded, and a finite procedure can be given for determining whether any given sequence can be generated (cf. Ref. 12). Hence in particular there is a finite procedure to determine whether any given cellular automaton rule is surjective, so that all possible configurations can be reached in its evolution.[30]

In two-dimensional cellular automata there is no such simple iterative procedure for determining whether progressively larger patches of site values can be generated. An $X \times Y$ patch of site values is generated after

one step in the evolution of a two-dimensional cellular automaton with a range-r rule if there exists some $(X + 2r)(Y + 2r)$ patch of initial site values that evolves to it. Progressively larger patches can be generated if appropriate progressively large predecessor patches exist. The number of sites in the overlap between such progressively larger predecessor patches is not fixed, as in one-dimensional cellular automata, but instead grows essentially like the perimeter of the patch, $2r(X + Y + 2r)$. With this procedure, there is thus no upper bound on the amount of information that must be recorded to determine whether progressively larger patches can be generated. To find whether a patch of any particular size $X \times Y$ can be generated, it suffices to test all $k^{(X + 2r)(Y + 2r)}$ candidate predecessor patches. (As mentioned below, this is in fact an NP-complete problem, and therefore presumably cannot be solved in general in a time polynomial in the patch size.) However, questions concerning complete configurations can be answered only be considering arbitrarily large patches, and may require arbitrarily complex computations. As a consequence, there are global questions about configurations generated by two-dimensional cellular automata after a finite number of time steps that can posed, but cannot in general be answered by any finite computational process, and are therefore formally undecidable.[15]

Some examples of such undecidable questions about two-dimensional cellular automata are (i) whether a particular complete (but finitely specified) configuration can be generated after one time step from any initial configuration; (ii) whether a particular cellular automaton rule is surjective, so that all possible configurations can be generated; (iii) whether the set of complete configurations generated after say one time step has a nonempty intersection with some recursive formal language such as a regular language, whose words can be recognized by a finite computation; (iv) whether there exist configurations that have a particular period in time (and are thus invariant under some number of iterations of the cellular automaton rule).

It seems that global questions about the finite time behaviour of one-dimensional cellular automata are always decidable. Questions about their ultimate infinite time behavior may nevertheless be undecidable. To show this, one considers one-dimensional cellular automata whose evolution emulates that of a universal Turing machine. The successive arrangements of symbols on the Turing machine tape correspond to successive configurations of site values generated in the evolution of the cellular automaton. Undecidable questions such as halting for the Turing machines are then shown to be undecidable for the corresponding one-dimensional cellular automaton.[13]

In two-dimensional cellular automata, questions about global proper-

ties on infinite spatial scales can be undecidable even at finite times. This is proved[15] by considering the line-by-line construction of configurations. The rules used to obtain each successive line from the last can correspond to the rules for a universal Turing machine. The construction of the configuration can then be continued to infinity and completed only if this Turing machine does not halt with the input given, which is in general undecidable. Sets of configurations generated at finite times in the evolution of two-dimensional cellular automata can thus be nonrecursive.

Many global questions about two-dimensional cellular automata are closely analogous to geometrical questions associated with tilings of the plane. Consider for example the problem of finding configurations that remain invariant under a particular cellular automaton rule. All the neighborhoods in such configurations must be such that the values of their center sites are left unchanged by the cellular automaton rule. Each such neighborhood may be considered as a "tile." Complete invariant configurations are constructed from an array of tiles, with each adjacent pair of tiles subject to a consistency condition that the overlapping sites in the neighborhoods to which they correspond should agree. In a one-dimensional cellular automaton, the set of possible arrangements of tiles or configurations that satisfy the conditions can be enumerated immediately, and form a finite complement regular language (subshift of finite type).[12] In a two-dimensional cellular automaton, the problem of finding invariant configurations is equivalent to tiling the plane with a set of "dominoes" corresponding to the possible allowed neighborhoods, and subject to constraints that can be cast in the form of requiring adjacent pairs of edges to have complementary colours.[31] The problem of determining wether a particular set of dominoes can in fact be used to tile the plane is however known to be undecidable[32,33] (cf. Ref. 34). The problem of finding whether there exist invariant configurations under a particular two-dimensional cellular automaton rule is likewise undecidable.

If any infinite sequence can be constructed from some set of dominoes in one dimension, then it is clear that a spatially periodic sequence can be found. Hence if there are to be any configurations with a particular temporal period in a one-dimensional cellular automaton, then there must be spatially periodic configurations with this temporal period. (The maximum necessary spatial period for configurations with temporal period p is k^{2rp+1} [12]: the existence of such spatially periodic configurations can be viewed as a consequence of the pumping lemma (e.g., Ref. 11) for regular languages.) In two dimensions, however, there are sets of dominoes for which a tiling of the plane is possible, but the tiling cannot be spatially periodic.[32,33,35] In the examples known, it appears that the basic arrangement of tiles is always self-similar, so that it is almost periodic. In

the simplest known examples, six square dominoes[33] or just two
irregularly shaped dominoes[35] are required for this phenomenon to occur.
(The simplest known example in three dimensions involves seven
polyhedral "dominoes".[36])

The problem of whether a set of dominoes can tile a finite, say, $X \times X$
region of the plane is clearly decidable, but is NP complete.[37] The
analogous problem of determining whether a particular patch can occur in
an invariant configuration for a two-dimensional cellular automaton, or
can in fact be generated by one time step of evolution from any initial state,
is thus also NP complete. These problems can presumably be solved only
by computations whose complication increases faster than a polynomial in
X, and are essentially equivalent to explicit testing of all $O(k^{X^2})$ possible
cases.

In addition to considering configurations of site values generated at a
particular step in the evolution of a cellular automaton, one may also dis-
cuss sequences of site values obtained with time. In general one may con-
sider the number of possible arrangements $N(\mathbf{v}_1,..., \mathbf{v}_p)$ of site values in a
space-time volume consisting of a parallelepiped with generator vectors $\mathbf{v}_i$.
The set entropy may than be defined as the exponential rate of increase of
N as the lengths of certain generators are taken to infinity (cf. Refs. 38 and
39):

$$s = \lim_{\alpha_1 \to \infty} \cdots \lim_{\alpha_p \to \infty} \frac{1}{\alpha_1 \cdots \alpha_{p'}} \log_k N(\alpha_1 \mathbf{v}_1, \alpha_2 \mathbf{v}_2,..., \alpha_p \mathbf{v}_p) \qquad (4.3)$$

where the α_i are scalar parameters, and $p' \leqslant p \leqslant d$. These entropies are in
fact functions of the unit p forms obtained as the exterior products of the
generator vectors $\mathbf{v}_i$ considered as one-forms in space-time. Certain con-
vergence properties of the limits in Eq. (4.3) can be proved from the fact
that the number of arrangements $N(V)$ of site values in a volume V is sub-
multiplicative, so that $N(V_1 \cup V_2) \leqslant N(V_1) N(V_2)$. A measure-theoretical
analog of the set entropy (4.3) may be defined in correspondence with
Eq. (4.2).

The spatial entropy (4.1) for two-dimensional cellular automata is
obtained from the general definition (4.3) by choosing $p = 3$, $p' = 2$ and
taking $\mathbf{v}_1$ and $\mathbf{v}_2$ to be orthogonal purely spacelike vectors along the two
lattice directions. The generator vector $\mathbf{v}_3$ is taken to be in the positive time
direction, but the number of arrangements N is independent of α_3 since a
complete configuration at one time step determines all future con-
figurations.

For a d-dimensional cellular automaton, there are critical values of p
and p' such that entropies corresponding to higher or lower-dimensional
parallelepipeds are zero or infinity. Entropies with exactly those critical

values may be nonzero and bounded by quantities that depend on the cellular automaton neighborhood size.

Entropies are essentially determined by the correlations between values of sites at different space-time points. These correlations depend on the propagation of information in the cellular automaton. The difference patterns discussed in Section 3 provide measures of such information propagation. They can be considered analogs of Green's functions (cf. Ref. 4) which describe the change produced at some space-time point x' in a cellular automaton as a consequence changes at another point x. The set-theoretical Green's function is defined to be nonzero whenever a change at x in any configuration could lead to a change at x'. In the measure-theoretical Green's function the possible configurations are weighted with their probabilities. The maximum rate of information propagation is determined by the slope of the space-time ("light") cone within which the Green's function is nonzero. The slope corresponding to propagation in a particular spatial direction in say a two-dimensional cellular automaton gives the Lyapunov exponent in that direction for the cellular automaton evolution.[7,40] In most cases it appears that the space-time structure corresponding to the set of sites on which the Green's function is nonzero tends to a fixed form after rescaling at large times, so that the structure has a unique growth dimension, and the Lyapunov exponents have definite values. Exceptions may occur in rules where difference patterns spread along domain boundaries, typically producing asymptotically self-similar structures analogous to percolation clusters (e.g., Ref. 26).

The Green's functions describe not only how a change at some time affects site values at later times, but also how the value of a particular site is affected by the previous values of other sites. The backward light cone of a site contains all the sites whose values can affect it. (Notice that the backward light cone for a bijective rule in general has little relation with the forward light cone for the inverse rule.[41]) The values of all sites in a volume V are thus determined by the values of sites on a surface S that "absorbs" (covers) all the backward light cones of points in V. The number of possible configurations in V is then bounded from above by the number of possible configurations of the set of sites within one cellular automaton neighborhood of the surface S. The entropy associated with the volume V is then not greater than the entropy associated with the volume around S. By choosing various "absorbing surfaces" S, whose sizes are determined by the rates of information propagation in different directions, one can derive various inequalities between entropies.

Many entropies can be defined for cellular automata using Eq. (4.3). One significant class is those that are invariant under continuous invertible

transformations on the space of cellular automaton configurations. Such entropies can be used to identify topologically inequivalent cellular automaton rules. For one-dimensional cellular automata, an invariant entropy may be defined by taking $p = 2$, $p' = 1$ in Eq. (4.3), and choosing v_1 in the positive time direction, and v_2 in the space direction. The entropy may be generalized by taking the v_i to be an arbitrary pair of orthogonal spacetime vectors (with v_1 having a positive time component).[38] The most direct generalization of these invariant entropies to two-dimensional cellular automata would have $p = 3$, $p' = 1$, and take the v_i to be an orthogonal triple of space-time vectors with v_1 having a positive time component. If v_1 were chosen purely timelike, then this entropy would have no dependence on spatial direction, and would correspond to the standard invariant entropy defined for the cellular automaton mapping. In general however, there is no upper bound on its value, and it is apparently infinite for most cellular automata that have positive Lyapunov exponents in more than one spatial direction. A finite entropy can nevertheless be constructed by choosing $p' = 2$. This entropy depends on the spatial (or in general space-time) vector $v_1 \times v_2$. To obtain an invariant entropy, one must perform some average over this vector (accounting for the fact that the entropy is a homogeneous function of degree one in the length of the vector). One possibility is to form the integral of the quantity (4.3) over those values of the vector for which the quantity is less than some constant (say, 1).

5. DISCUSSION

This paper has presented an exploratory study of two-dimensional cellular automata. Much remains to be done, but a few conclusions can be already be given.

A first approach to the study of cellular automaton behavior is statistical: one considers the average properties of evolution from typical initial configurations. Statistical studies of one-dimensional cellular automata have suggested that four basic qualitative classes of behavior can be identified. This paper has given analogs of these classes in two-dimensional cellular automata. One expects that the qualitative classification will also apply in three- and higher-dimensional systems.

Entropies and Lyapunov exponents are statistical quantitites that measure the information content and rate of information transmission in cellular automata. Their definitions for one-dimensional cellular automata are closest to those used in smooth dynamical systems. But rather direct generalizations can nevertheless be found for two- and higher-dimensional cellular automata.

Beyond statistical properties, one may consider geometrical aspects of patterns generated by cellular automaton evolution. Even though the basic construction of a cellular automaton is discrete, its "macroscopic" behavior at large times and on large spatial scales may be a close approximation to that of a continuous system. In particular domains of correlated sites may be formed, with boundaries that at a large scale seem to show continuous motions and deformations. While some such phenomena do occur in one dimension, they are most significant in two and higher dimensions. Often their motion appears to be determined by attributes such as curvature, that have no analog in one dimension.

The structures generated by two- and higher-dimensional cellular automata evolving from simple seeds show many geometrical phenomena. The most significant is probably the formation of dendritic patterns, characterized by noninteger growth dimensions.

Statistical measurements provide one method for comparing cellular automaton models with experimental data. Geometrical properties provide another. The geometry of patterns formed by cellular automata may be compared directly with the geometry of patterns generated by natural systems.

Topology is another aspect of cellular automaton patterns. When domains or regions containing many correlated sites exist, one may approximate them as continuous structures, and consider their topology. For example, domains produced by cellular automaton evolution may exhibit topological defects that are stable under the cellular automaton rule. In two-dimensional cellular automata, only point and line defects occur. But in three dimensions, knotted line defects (e.g., Ref. 42) and other complicated topological forms are possible. The topology of the structures supported by a cellular automaton rule may be compared directly with the topology of structures that arise in natural systems (cf. Ref. 43).

Geometry and topology provide essentially local descriptions of the behavior of cellular automata. Computation theory potentially provides a more global characterization. One may classify the behavior and properties of cellular automata in terms of the nature of the computations required to reproduce them. Even in one dimension, there are cellular automata that can perform arbitrary computations, so that at least some of their properties can be reproduced only by direct simulation or observation, and their limiting behavior is formally undecidable. The range of properties for which undecidability can occur in much larger in two dimensions than in one dimension. In particular, properties that involve a limit of infinite spatial size, even at finite times, can be undecidable. As higher-dimensional cellular automata are considered, the degree of undecidability that can be encountered in studies of particular properties increases.

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