Two-Dimensional Description Logics for Context-Based Semantic Interoperability

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Abstract

Description Logics (DLs) provide a clear and broadly accepted paradigm for modeling and reasoning about terminological knowledge. However, it has been often noted, that although DLs are well-suited for representing a single, global viewpoint on an application domain, they offer no formal grounding for dealing with knowledge pertaining to multiple heterogeneous viewpoints — a scenario ever more often approached in practical applications, e.g. concerned with reasoning over distributed knowledge sources on the Semantic Web. In this paper, we study a natural extension of DLs, in the style of two-dimensional modal logics, which supports declarative modeling of viewpoints as *contexts*, in the sense of McCarthy, and their *semantic interoperability*. The formalism is based on two-dimensional semantics, where one dimension represents a usual object domain and the other a (possibly infinite) domain of viewpoints, addressed by additional modal operators and a metalanguage, on the syntactic level. We systematically introduce a number of expressive fragments of the proposed logic, study their computational complexity and connections to related formalisms.

Introduction

Description Logics (DLs) are popular knowledge representation formalisms, whose most prominent application is the design of ontologies — formal models of terminologies and instance data, representative of particular domains of interest (Baader et al. 2003) — used extensively on the Semantic Web and in biomedical applications. Under the standard Kripkean semantics, a DL ontology forces a unique, global view on the represented world, in which the ontology axioms are interpreted as universally true. This philosophy is well-suited as long as everyone can share the same conceptual perspective on the domain or there is no need for considering alternative viewpoints. Alas, this is hardly ever the case and very often, same domains are modeled differently depending on the intended use of an ontology. In practice, effective representation and reasoning about knowledge pertaining to such multiple heterogenous viewpoints becomes the primary objective for many applications, e.g. those concerned with reasoning over distributed knowledge sources on the Semantic Web (Guha, McCool, and Fikes 2004; Bao et al. 2010).

The challenges above resemble clearly the problems which once inspired J. McCarthy to introduce his theory of formalizing contexts in knowledge representation systems, as a way of granting them more generality (McCarthy 1987; Guha 1991). The gist of his proposal, motivating several existing logics of context (Buvač and Mason 1993; Buvač 1996; Nossum 2003), is to replace logical formulas φ, as the basic knowledge carriers, with assertions $ist(\boldsymbol{c}, \varphi)$. Assertions of this form state that φ is true in $\boldsymbol{c}$, where $\boldsymbol{c}$ denotes an abstract first-order entity called a *context*, which on its own can be described in a first-order language. For instance:

$$ist(\boldsymbol{c}, Heart(a)) \wedge \boldsymbol{HumanAnatomy}(\boldsymbol{c})$$

states that the object a is a heart in some context described as ***HumanAnatomy***. Based on this foundation, the theory allows for defining models of *semantic interoperability* within a possibly unbounded space of contexts, i.e. generic rules guiding the information flow between contexts, such as e.g.:

$$\forall xy\ \boldsymbol{HumanAnatomy}(x) \wedge \boldsymbol{Anatomy}(y) \rightarrow$$
$$\forall z(ist(x, Heart(z)) \rightarrow ist(y, HumanHeart(z)))$$

which ensures that in every ***Anatomy*** context, the interpretation of $HumanHeart$ includes also all those objects which are instances of $Heart$ in any ***HumanAnatomy*** context.

The formalism proposed in this paper incorporates these fundamental ideas of McCarthy's theory into the DL framework by considering contexts as abstract, first-class citizens, and offering an expressive formal apparatus for modeling their semantic interoperability. As a result, we harmonize and give a uniform formal treatment to two seemingly diverse aspects of the problem of reasoning with contexts in DL: 1) how to extend DLs to support the representation of inherently contextualized knowledge; 2) how to use knowledge from coexisting classical DL ontologies while respecting its context-specific scope. Our logic is essentially a two-dimensional DL, in the style of product-like combinations of DLs with modal logics (Wolter and Zakharyaschev 1999; Kurucz et al. 2003), similar to e.g. temporal DLs (Lutz, Wolter, and Zakharyaschev 2008; Artale, Lutz, and Toman 2007). In particular, we extend the standard DL semantics with a second dimension, representing a possibly infinite domain of contexts, and include additional modal operators along with a separate metalanguage in the syntax, for quantifying and expressing properties over the context entities.

In the following sections, we systematically introduce and motivate a number of expressive fragments of the logic, study their computational complexity and highlight the connections to some related formalisms.

Description Logics: preliminaries

A DL language $\mathcal{L}$ is specified by a vocabulary $\Sigma = (N_C, N_R, N_I)$, where N_C is a set of *concept names*, N_R a set of *role names*, N_I a set of *individual names*, and a number of operators for constructing complex concept descriptions (Baader et al. 2003). The semantics of $\mathcal{L}$ is given through *interpretations* of the form $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, where Δ is a non-empty *domain* of individuals, and $\cdot^{\mathcal{I}}$ is an *interpretation function*. The meaning of the vocabulary is fixed via mappings: $a^{\mathcal{I}} \in \Delta$ for every $a \in N_I$, $A^{\mathcal{I}} \subseteq \Delta$ for every $A \in N_C$ and $r^{\mathcal{I}} \subseteq \Delta \times \Delta$ for every $r \in N_R$, and $\top^{\mathcal{I}} = \Delta$. Then the function is inductively extended over complex expressions according to the fixed semantics of the constructors. Table 1 contains the list of concept constructors and their semantics which are considered in the rest of this paper: (1) top concept, (2) concept intersection, (3) existential role restriction, (4) complement, (5) nominal, where C, D are concepts, $r \in N_R$ and $a \in N_I$. We abbreviate $\neg\top$ with $\bot$, $\neg(\neg C \sqcap \neg D)$ with $C \sqcup D$ and $\neg\exists r.\neg C$ with $\forall r.C$.

	Syntax	Semantics
(1)	$\top$	Δ
(2)	$C \sqcap D$	$\{x \mid x \in C^{\mathcal{I}} \cap D^{\mathcal{I}}\}$
(3)	$\exists r.C$	$\{x \mid \exists y : (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
(4)	$\neg C$	$\{x \mid x \notin C^{\mathcal{I}}\}$
(5)	$\{a\}$	$\{a^{\mathcal{I}}\}$

Table 1: Concept constructors and their semantics.

A *knowledge base* (or an *ontology*) $\mathcal{K}$ is a finite set of axioms of three possible forms:

$$C \sqsubseteq D \mid C(a) \mid r(a, b) \qquad (\dagger)$$

where C, D are concepts, $a, b \in N_I$ and $r \in N_R$. We write $C \equiv D$, whenever $C \sqsubseteq D$ and $D \sqsubseteq C$ are both in $\mathcal{K}$. Typically, the formulas of the first type are denoted as *TBox axioms*, whereas the remaining two as *ABox axioms*. An interpretation $\mathcal{I}$ *satisfies* an axiom in either of the cases:

- $\mathcal{I} \models C \sqsubseteq D$ *iff* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,
- $\mathcal{I} \models C(a)$ *iff* $a^{\mathcal{I}} \in C^{\mathcal{I}}$,
- $\mathcal{I} \models r(a, b)$ *iff* $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$.

Finally, $\mathcal{I}$ is a *model* of $\mathcal{K}$ whenever it satisfies all its axioms. The computational complexity of reasoning in DLs varies depending on the expressiveness of the language. In the logic $\mathcal{EL}$, comprising only constructors (1-3), the central reasoning problem, *deciding concept subsumption* (i.e. verifying whether $\mathcal{K} \models C \sqsubseteq D$), is in PTIME (Baader, Brandt, and Lutz 2005). For $\mathcal{ALC}$ (1-4) and $\mathcal{ALCO}$ (1-5), the main reference problem of *deciding knowledge base satisfiability* (i.e. verifying whether $\mathcal{K}$ has a model) is EXPTIME-complete (Baader et al. 2003).

Interoperability systems

The DL semantics is extensional in its nature, in the sense that the meaning of an expression is its denotation in the object domain. Consequently, we define *semantic interoperability* also in a strictly extensional way. We say that it is the ability of a knowledge system to interpret expressions in different contexts via shared extensions, according to the declared constraints. For instance, the constraint $\alpha^{\mathcal{I}(\boldsymbol{c})} = \beta^{\mathcal{I}(\boldsymbol{d})}$ entails that the expression α has the same meaning in the context $\boldsymbol{c}$ as β in $\boldsymbol{d}$. A formal representation of the context-specific domain knowledge together with the interoperability constraints is denoted here as an *interoperability system*.

We introduce our framework in several steps. First, we demonstrate the basic interoperation mechanism in the simplest scenario involving a fixed number of explicitly named contexts. Next, we generalize the approach to account for a possibly infinite domain of contexts and include a lightweight metalanguage for describing them. Finally, we consider a few expressive extensions to the framework.

Simple interoperability systems

A *simple interoperation language* $SL_{\mathcal{L}}$ consists of a finite set M_I of individual context names, and an object language, which extends a DL $\mathcal{L}$ with special *context operators* applied to all constructs of $\mathcal{L}$.

Definition 1 ($SL_{\mathcal{L}}$-object language) *Let $\mathcal{L}$ be a DL with vocabulary $\Sigma = (N_C, N_R, N_I)$. Then the* object language *of $SL_{\mathcal{L}}$ is the smallest language containing $\mathcal{L}$ and closed under the constructors of $\mathcal{L}$ and the operators $\langle \boldsymbol{c} \rangle$, for $\boldsymbol{c} \in M_I$:*

$$\langle \boldsymbol{c} \rangle C \mid \langle \boldsymbol{c} \rangle r \mid \langle \boldsymbol{c} \rangle a$$

where C is a concept of the object language, $r \in N_R$ and $a \in N_I$. The resulting expressions are a concept, a role and an individual name of the object language, respectively.

Intuitively, the operator $\langle \boldsymbol{c} \rangle$ 'imports' the meaning of the bounded expression from the context denoted by name $\boldsymbol{c}$, to the context of occurrence. Formally, the semantics of $SL_{\mathcal{L}}$ is defined via extended interpretations.

Definition 2 ($SL_{\mathcal{L}}$-interpretations) *An $SL_{\mathcal{L}}$-*interpretation *is a tuple $\mathfrak{M} = (\mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(i)}\}_{i \in \mathfrak{C}})$, where:*

- *$\mathfrak{C}$ is a non-empty domain of contexts,*
- *$\cdot^{\mathcal{J}} : M_I \mapsto \mathfrak{C}$,*
- *$(\Delta, \cdot^{\mathcal{I}(i)})$, for every $i \in \mathfrak{C}$, is an interpretation of the object language, such that for every $\boldsymbol{c} \in M_I$ and expression α, $(\langle \boldsymbol{c} \rangle \alpha)^{\mathcal{I}(i)} = \alpha^{\mathcal{I}(\boldsymbol{c}^{\mathcal{J}})}$.*

Finally, we define the notion of Simple Interoperability System (SIS) and its $SL_{\mathcal{L}}$-model.

Definition 3 (Simple Interoperability System) *A* Simple Interoperability System *in $SL_{\mathcal{L}}$ is a finite set of formulas:*

$$\boldsymbol{c} : \varphi$$

where $\boldsymbol{c} \in M_I$ and φ is an axiom of the object language, in any of the forms $(\dagger)$.

Definition 4 ($SL_{\mathcal{L}}$-models) *An $SL_{\mathcal{L}}$-interpretation $\mathfrak{M} = (\mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(i)}\}_{i \in \mathfrak{C}})$ is a* model *of a SIS $\mathcal{O}$* iff *for every $\boldsymbol{c} : \varphi \in \mathcal{O}$, $(\Delta, \cdot^{\mathcal{I}(\boldsymbol{c}^{\mathcal{J}})})$ satisfies φ.*

Formulas $\boldsymbol{c} : \varphi$, corresponding to McCarthy's $ist(\boldsymbol{c}, \varphi)$, have a straightforward reading: φ holds (or is an axiom) in the context denoted by name $\boldsymbol{c}$. A SIS can be also viewed as a collection of ontologies $\{\mathcal{O}_{\boldsymbol{c}}\}_{\boldsymbol{c} \in M_I}$ in $SL_{\mathcal{L}}$, where each $\mathcal{O}_{\boldsymbol{c}} = \{\varphi \mid \boldsymbol{c} : \varphi \in \mathcal{O}\}$ represents the knowledge relevant to one context, related to others by means of operators $\langle\cdot\rangle$, e.g.:

$\mathcal{O}_{\boldsymbol{c}}$:	$Patient \sqsubseteq \exists hasPart.\langle \boldsymbol{d} \rangle HumanHeart$
$\mathcal{O}_{\boldsymbol{d}}$:	$HumanHeart \sqsubseteq Heart$
	$Heart \sqsubseteq Organ$

It is easy to observe, that $SL_{\mathcal{L}}$ can serve as a language for integrating a set of standard DL ontologies $\{\mathcal{O}_{\boldsymbol{c}}\}_{\boldsymbol{c} \in M_I}$ in $\mathcal{L}$, which supports simple, logic-based mappings aligning the semantics of concepts, roles and individual names used in the ontologies:

$$\langle \boldsymbol{c} \rangle C \sqsubseteq \langle \boldsymbol{d} \rangle D \ \mid \ \langle \boldsymbol{c} \rangle r \sqsubseteq \langle \boldsymbol{d} \rangle s \ \mid \ \langle \boldsymbol{c} \rangle \{a\} \sqsubseteq \langle \boldsymbol{d} \rangle \{b\} \qquad (\ddagger)$$

where $\boldsymbol{c}, \boldsymbol{d} \in M_I$, C, D are concepts of $\mathcal{L}$, $r, s \in N_R$ and $a, b \in N_I$. For instance, two ontologies with partly overlapping information, e.g:

$\mathcal{O}_{\boldsymbol{c}}$:	$Staff \sqsubseteq \exists isEmployed.Company$
	$Staff(J.Smith)$
$\mathcal{O}_{\boldsymbol{d}}$:	$Employee \sqsubseteq \exists employedIn.\top$
	$Employee(JohnSmith)$

might be integrated by means of constraints:

$$\langle \boldsymbol{c} \rangle Staff \equiv \langle \boldsymbol{d} \rangle Employee$$
$$\langle \boldsymbol{c} \rangle isEmployed \sqsubseteq \langle \boldsymbol{d} \rangle employedIn$$
$$\langle \boldsymbol{c} \rangle \{J.Smith\} \equiv \langle \boldsymbol{d} \rangle \{JohnSmith\}$$

Not surprisingly, the language $SL_{\mathcal{L}}$ bares some obvious similarities with other known formalisms for connecting/integrating ontologies, such as $\mathcal{E}$-connections (Kutz et al. 2003), Distributed DLs (DDLs) (Borgida and Serafini 2003) and Package-based DLs (P-DLs) (Bao et al. 2009). In particular, mappings (‡) have exactly the same function as *bridge rules* in DDLs, i.e. lifting information from one context to another. The major difference from the first two approaches is that integration in $SL_{\mathcal{L}}$ is achieved by interpreting the aligned elements of the language directly over the same domain objects, without involving intermediary link relations such as $\mathcal{E}$-connections or directional semantic mappings (DDLs). This renders our integration mechanism in principle stronger. In the case of P-DLs, it is not difficult to show that $SL_{\mathcal{L}}$, although based on a more natural semantics, can be mapped on the corresponding P-DL $\mathcal{LP}$. Analogically to P-DLs, reasoning in $SL_{\mathcal{L}}$ is polynomially reducible to reasoning in $\mathcal{L}$, which guarantees the same worst case complexity.

Theorem 1 *The complexity of reasoning in $SL_{\mathcal{L}}$ is the same as in $\mathcal{L}$.*

The full proof, along others from this paper, can be found in the appendix.

Abstract interoperability systems

The expressive power of $SL_{\mathcal{L}}$ is strongly limited by restricting the representation to a fixed number of contexts. In this section, we dispose of this constraint and permit an unbounded space of context entities, thus shifting towards a full-fetched two-dimensional semantics. This natural generalization stems from the introduction of a quantification mechanism over the context domain, often advocated by the continuators of McCarthy (Guha 1991; Buvač 1996) as a mean of constructing more abstract and generic interoperability constraints. On the philosophical side, this step might be interpreted as a manifestation of the Open World Assumption on the level of contexts (or knowledge sources), which in some open-ended environments such as the Web can be often justified.

An *abstract interoperation language* $AL_{\mathcal{L}}$ consists of a metalanguage, supporting atomic concept assertions and taxonomies of concept names, and an object language, equipped with generalized context operators over concepts. To distinguish between the atoms of the two languages, we use a ***bold font*** for writing the former and a *regular font* for the latter.

Definition 5 ($AL_{\mathcal{L}}$-metalanguage) *The* metalanguage *of $AL_{\mathcal{L}}$ consists of a set M_C of concept names, the top concept $\top$, and a set M_I of individual names. The axioms of the metalanguage are formulas:*

$$\boldsymbol{A} \sqsubseteq \boldsymbol{B} \ \mid \ \boldsymbol{A}(\boldsymbol{c})$$

where $\boldsymbol{A}, \boldsymbol{B}$ are concepts and $\boldsymbol{c} \in M_I$.

Definition 6 ($AL_{\mathcal{L}}$-object language) *Let $\mathcal{L}$ be a DL language. Then the* object language *of $AL_{\mathcal{L}}$ is the smallest language containing $\mathcal{L}$, and closed under the constructors of $\mathcal{L}$ and two concept-forming operators:*

$$\langle \boldsymbol{A} \rangle C \ \mid \ [\boldsymbol{A}]C$$

where $\boldsymbol{A}$ is a concept of the metalanguage and C a concept of the object language.

Informally, the concept $\langle \boldsymbol{A} \rangle C$ denotes all objects which are C in *some* context of type $\boldsymbol{A}$, whereas $[\boldsymbol{A}]C$ objects which are C in *all* such contexts. For instance, $\langle$***HumanAnatomy***$\rangle Heart$ refers to the concept $Heart$ in some ***HumanAnatomy*** context, which corresponds to McCarthy's: $\exists x(ist(x, Heart(y)) \wedge$ ***HumanAnatomy***$(x))$. The two context operators behave almost as the usual $\mathbf{S5}$ modalities, in particular preserving the duality $[\boldsymbol{A}] = \neg\langle \boldsymbol{A} \rangle\neg$, with the sole difference that an additional (metalanguage) condition is imposed on the accessed possible worlds.

Further, we define the notion of Abstract Interoperability System (AIS) in $AL_{\mathcal{L}}$.

Definition 7 (Abstract Interoperability System) *An* Abstract Interoperability System *in $AL_{\mathcal{L}}$ is a pair $\mathcal{K} = (\mathcal{C}, \mathcal{O})$, where $\mathcal{C}$ is a set of axioms of the metalanguage and $\mathcal{O}$ is a set of formulas:*

$$\boldsymbol{c} : \varphi \ \mid \ \boldsymbol{A} : \varphi$$

where φ is an axiom of the object language in any of the forms (†), $\boldsymbol{c} \in M_I$ and $\boldsymbol{A}$ is a concept of the metalanguage.

A formula $\boldsymbol{A} : \varphi$ states that the axiom φ must hold in all contexts of type $\boldsymbol{A}$. The semantics is given through the corresponding $AL_{\mathcal{L}}$-interpretations and $AL_{\mathcal{L}}$-models.

Definition 8 ($AL_{\mathcal{L}}$-interpretations) *An $AL_{\mathcal{L}}$-interpretation is a tuple $\mathfrak{M} = (\mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(i)}\}_{i \in \mathfrak{C}})$ where:*

- $\mathfrak{C}$ *is a non-empty domain of contexts,*
- $\cdot^{\mathcal{J}}$ *is an interpretation function of the metalanguage, which maps* $\boldsymbol{A}^{\mathcal{J}} \subseteq \mathfrak{C}$*, for every* $\boldsymbol{A} \in M_C$*,* $\top^{\mathcal{J}} = \mathfrak{C}$*, and* $\boldsymbol{c}^{\mathcal{J}} \in \mathfrak{C}$*, for every* $\boldsymbol{c} \in M_I$*,*
- $(\Delta, \cdot^{\mathcal{I}(i)})$*, for every* $i \in \mathfrak{C}$*, is an interpretation of the object language, such that for every* $\langle \boldsymbol{A} \rangle C$ *and* $[\boldsymbol{A}]C$*:*
  - $(\langle \boldsymbol{A} \rangle C)^{\mathcal{I}(i)} = \{x \mid \exists j \in \mathfrak{C}: \ j \in \boldsymbol{A}^{\mathcal{J}} \wedge x \in C^{\mathcal{I}(j)}\}$,
  - $([\boldsymbol{A}]C)^{\mathcal{I}(i)} = \{x \mid \forall j \in \mathfrak{C}: \ j \in \boldsymbol{A}^{\mathcal{J}} \rightarrow x \in C^{\mathcal{I}(j)}\}$.

Definition 9 ($AL_{\mathcal{L}}$**-models)** *An* $AL_{\mathcal{L}}$*-interpretation* $\mathfrak{M} = (\mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(i)}\}_{i \in \mathfrak{C}})$ *is a* model *of an AIS* $\mathcal{K} = (\mathcal{C}, \mathcal{O})$ iff*:*

- *for every* $\boldsymbol{A}(\boldsymbol{c}) \in \mathcal{C}$*,* $\boldsymbol{c}^{\mathcal{J}} \in \boldsymbol{A}^{\mathcal{J}}$*,*
- *for every* $\boldsymbol{A} \sqsubseteq \boldsymbol{B} \in \mathcal{C}$*,* $\boldsymbol{A}^{\mathcal{J}} \subseteq \boldsymbol{B}^{\mathcal{J}}$*,*
- *for every* $\boldsymbol{c} : \varphi \in \mathcal{O}$*,* $(\Delta, \cdot^{\mathcal{I}(\boldsymbol{c}^{\mathcal{J}})})$ *satisfies* φ*,*
- *for every* $\boldsymbol{A} : \varphi \in \mathcal{O}$ *and* $i \in \mathfrak{C}$*, if* $i \in \boldsymbol{A}^{\mathcal{J}}$ *then* $(\Delta, \cdot^{\mathcal{I}(i)})$ *satisfies* φ*.*

Application scenarios

Similarly to $SL_{\mathcal{L}}$, $AL_{\mathcal{L}}$ can be used both as a native language for constructing contextualized knowledge bases or as an external layer for imposing generic interoperability constraints over standard DL representations. However, unlike in $SL_{\mathcal{L}}$, the context operators in $AL_{\mathcal{L}}$ govern the semantic interoperation not only among a fixed number of explicitly introduced contexts, but rather within an entire space of possible contexts — some of which might be only logically entailed. Hence, the operators $\langle \cdot \rangle$ and $[\cdot]$ serve an analogical purpose to $\exists$ and $\forall$ in the object dimension: they restrict the set of possible (two-dimensional) models only to those in which certain entities — here contexts with specific object knowledge — are present. In the following paragraphs, we present a few sample applications of $AL_{\mathcal{L}}$.

Contextualized knowledge base. We model a piece of information presented on the disambiguation website of Wikipedia on querying for the term $Ring$[1].

$\boldsymbol{disambiguation} : Ring \sqsubseteq \langle \boldsymbol{Math} \rangle Ring \sqcup \langle \boldsymbol{People} \rangle Ring$
$\boldsymbol{Math} : Ring \sqsubseteq AlgebStruct \sqcup \langle \boldsymbol{Geometry} \rangle Annulus$
$\boldsymbol{People} : Ring \sqsubseteq \{nickRing\}$

Observe, that the named context ***disambiguation*** provides basic distinction on $Ring$ in some ***Math*** context and in some ***People*** context. This is further enhanced, by the distinction defined on the level of all ***Math*** contexts. There, $Ring$ denotes either $AlgebStruct$ or, in some further ***Geometry*** context, $Annulus$. In case of ***People*** context, $Ring$ actually denotes an individual $nickRing$.

Interoperation constraints for ontology alignment and reuse. Consider an infrastructure such as the NCBO BioPortal project[2], which gathers numerous published biohealth ontologies, and categorizes them via simple thematic tags ***Cell***, ***Health***, ***Anatomy***, etc., organized in a simple concept hierarchy. The intention of the project is to facilitate the reuse of the collected resources in new applications. We assume that each ontology name is interpreted as a distinct context name in $AL_{\mathcal{L}}$. Note, that the division between the metalanguage and the object language is already present in the architecture of the BioPortal, which can be immediately utilized, for example to state:

$\top : \langle \boldsymbol{HumanAnat.} \rangle Heart \sqsubseteq [\boldsymbol{Anat.}] HumanHeart$	(1)
$\boldsymbol{Anatomy} : Heart \sqsubseteq Organ$	(2)
$\boldsymbol{HumanAnatomy} \sqsubseteq \boldsymbol{Anatomy}$	(3)

where (1) fixes the translation from $Heart$ to $HumanHeart$ (cf. Introduction); (2) imposes an axiom $Heart \sqsubseteq Organ$ of an upper anatomy ontology over all ontologies tagged with ***Anatomy***, which due to the metalanguage taxonomy (3) carries over to all ontologies tagged with ***HumanAnatomy***.

More generally, $AL_{\mathcal{L}}$ provides logic-based explications of some interesting notions, relevant to the problem of ontology alignment and reuse, such as:

concept alignment: $\top : \langle \boldsymbol{A} \rangle C \sqsubseteq [\boldsymbol{B}] D$
every instance of C in any ontology of type $\boldsymbol{A}$ is D in every ontology of type $\boldsymbol{B}$

semantic importing: $\boldsymbol{c} : \langle \boldsymbol{A} \rangle C \sqsubseteq D$
every instance of C in any ontology of type $\boldsymbol{A}$ is D in ontology $\boldsymbol{c}$

upper ontology axiom: $\boldsymbol{A} : C \sqsubseteq D$
axiom $C \sqsubseteq D$ holds in every ontology of type $\boldsymbol{A}$

Ontology versioning management and change analysis. The context operators can be also interpreted as change operators, in the style of DL of Change (Artale, Lutz, and Toman 2007), for instance, for representing and studying dynamic aspects of ontology versioning, especially when evolutionary constraints apply to a whole collection of semantically interoperable ontologies. Some central issues arising in this setup are integrity (constraining the scope of changes allowed due to versioning), evolvability (ability of coordinating the evolution of ontologies) and formal analysis of differences between the versions (Huang and Stuckenschmidt 2005). In the examples below, we assume that contexts represent possible versions, while each metalanguage concept refers to all versions of the same ontology.

version-invariant concepts: $\top : \langle \boldsymbol{A} \rangle C \equiv [\boldsymbol{A}] C$
C is a version-invariant concept within the scope of versions of type $\boldsymbol{A}$

dynamic analysis: $\top : \langle \boldsymbol{A} \rangle C \sqcap \langle \boldsymbol{A} \rangle \neg C \sqsubseteq C^{\star}$
$C^{\star}$ retrieves all instances which are C in some versions of type $\boldsymbol{A}$ and $\neg C$ in some others

evolvability constraints: $\top : \langle \boldsymbol{A} \rangle C \sqsubseteq \langle \boldsymbol{B} \rangle D$
every instance of C in a version of type $\boldsymbol{A}$ has to evolve into D in some version of type $\boldsymbol{B}$

Complexity and expressiveness

Interestingly, reasoning in $AL_{\mathcal{L}}$ is not significantly harder than reasoning in the underlying DLs.

Theorem 2 *The complexity of reasoning in* $AL_{\mathcal{L}}$ *ranges as in Table 2.*

[1] See http://en.wikipedia.org/wiki/Ring.
[2] See http://bioportal.bioontology.org/.

$\mathcal{L}$	
$\mathcal{EL}$	PTIME
$\mathcal{ALC}$	EXPTIME-complete
$\mathcal{ALCO}$	NEXPTIME-complete

Table 2: Complexity of reasoning in $AL_{\mathcal{L}}$.

Only in the case of $\mathcal{L} = \mathcal{ALCO}$ we encounter a jump from EXPTIME to NEXPTIME-completeness. The interaction of nominals with the context operators enables encoding of the usual 2^n-tiling problem, known to be NEXPTIME-complete (Kurucz et al. 2003). The result holds already when the metalanguage is trivialized by setting $M_C = M_I = \emptyset$.

As the next results show, $AL_{\mathcal{L}}$ is closely related to the product-like combination of DL with modal logic $\mathbf{S5}$ — a formalism well-studied in the literature (Kurucz et al. 2003), also used as the foundation for the DL of Change (Artale, Lutz, and Toman 2007) and connected to the Probabilistic DL (Lutz and Schröder 2010).

Theorem 3 *If $M_C = M_I = \emptyset$ and only TBox axioms are allowed, then $AL_{\mathcal{L}}$ is a notational variant of $\mathbf{S5}_{\mathcal{L}}$ with global TBoxes.*

Proof. Observe, that only axioms $\top : C \sqsubseteq D$ are allowed, for arbitrary concepts C, D in $AL_{\mathcal{L}}$. Replace every $\langle\top\rangle$ with $\Diamond$, every $[\top]$ with $\Box$ and every $\top : C \sqsubseteq D \in \mathcal{O}$ with $C \sqsubseteq D$. It is easy to see that the semantics of $AL_{\mathcal{L}}$ coincides with that of $\mathbf{S5}_{\mathcal{L}}$. Note, that a TBox is considered global *iff* its every axiom is satisfied in all possible $\mathbf{S5}$-worlds. ❑

Theorem 4 *If $M_I = \emptyset$ and only TBox axioms are allowed, then reasoning in $AL_{\mathcal{L}}$ is polynomially reducible to reasoning in $\mathbf{S5}_{\mathcal{L}}$ with global TBoxes and concepts from M_C interpreted globally.*

Proof. First note, that a concept C is interpreted globally *iff* for every possible $\mathbf{S5}$-world w, $C^{\mathcal{I}(w)} = \Delta$ or $C^{\mathcal{I}(w)} = \emptyset$. Observe also, that only axioms $\boldsymbol{A} : C \sqsubseteq D$ are allowed, for $\boldsymbol{A} \in M_C$ and arbitrary concepts C, D in $AL_{\mathcal{L}}$. Translate every occurrence $\langle\boldsymbol{A}\rangle C$ to $\Diamond(A \sqcap C)$, every $[\boldsymbol{A}]D$ to $\Box(\neg A \sqcup C)$ and every $\boldsymbol{A} : C \sqsubseteq D \in \mathcal{O}$ to $A \sqcap C \sqsubseteq D$. Clearly, the resulting set of formulas is satisfiable in $\mathbf{S5}_{\mathcal{L}}$ *iff* the original one was in $AL_{\mathcal{L}}$. ❑

The corresponding $\mathbf{S5}_{\mathcal{L}}$ logics are obviously not full $\mathbf{S5} \times \mathcal{L}$ products, as we deliberately do not allow the roles of $\mathcal{L}$ to be interpreted rigidly across the context dimension, i.e. such that $r^{\mathcal{I}(c)} = r^{\mathcal{I}(d)}$ for every pair $c, d \in \mathfrak{C}$. Hence, in the landscape of combinations of modal logics (Kurucz et al. 2003), $AL_{\mathcal{L}}$ classifies as an 'approximation' of modal products, i.e. a combination considerably more expressive than fusion of logics, but weaker from those based on full product semantics. We also do not consider here context operators over roles, which allow for emulating such behavior. As it turns out, adding constructs $\langle\boldsymbol{A}\rangle r, [\boldsymbol{A}]r$ to $AL_{\mathcal{L}}$, with the expected semantics, immediately rises the lower complexity bounds to PSPACE-hard for $\mathcal{L} = \mathcal{EL}$ and 2EXPTIME-hard for $\mathcal{L} = \{\mathcal{ALC}, \mathcal{ALCO}\}$, which follows by immediate reductions from the corresponding variants of $\mathbf{S5}_{\mathcal{L}}$ with modalized roles in (Lutz and Schröder 2010; Artale, Lutz, and Toman 2007).

A formalism similar in the spirit to $AL_{\mathcal{L}}$, both in the formal design and in the underlying motivation, has been studied in (Klarman and Gutiérrez-Basulto 2010) as a Context DL $\mathcal{ALC}_{\mathcal{ALC}}$. There, however, the combination of DLs with the context operators is based on $(\mathbf{K_n})_{\mathcal{L}}$-frames, rather than $\mathbf{S5}_{\mathcal{L}}$. Consequently, $\mathcal{ALC}_{\mathcal{ALC}}$ seems less suitable for applications dealing with semantic interoperability between loosely coexisting DL representations, which are more natural to represent as possible worlds in a universal frame. Moreover, $(\mathbf{K_n})_{\mathcal{L}}$ exhibits much worse computational behavior, with 2EXPTIME-complete satisfiability problem already for $\mathcal{L} = \mathcal{ALC}$ with no rigid roles, and undecidable when rigid (or modalized) roles are included (Klarman and Gutiérrez-Basulto 2010).

Expressive metalanguages

For many applications, particularly relevant for the Semantic Web, a practical metalanguage for describing knowledge sources requires not only concept tags but also properties, e.g. for describing the provenance (authorship, date, place, relationships to other sources, etc.) (Bao et al. 2010). A natural way to support such requirements in the presented setting is to employ a standard DL in the role of the metalanguage.

Definition 10 (**$AL_{\mathcal{L}}^{\mathcal{M}}$-metalanguage**) *The* metalanguage *of $AL_{\mathcal{L}}^{\mathcal{M}}$ is a DL language $\mathcal{M}$ based on vocabulary $\Gamma = (M_C, M_R, M_I^\star)$, where M_C is a set of concept names, M_R a set of role names and $M_I^\star$ a set of individual names, with a designated subset $M_I \subseteq M_I^\star$. Axioms of the metalanguage are formulas of the form (†).*

Observe, that the context names M_I are here only a subset of all individual names $M_I^\star$ which might be used in context descriptions. Further, we also allow possibly complex concepts $\boldsymbol{C}$ of $\mathcal{M}$ inside the operators $\langle\boldsymbol{C}\rangle D$, $[\boldsymbol{C}]D$ and axioms $\boldsymbol{C} : \varphi$. Presence of roles in the metalanguage allows for effective reasoning with such information as:

$$\frac{\textit{\textbf{hasAuthor}}(\textit{\textbf{anatomy\_ont}}, \textit{\textbf{johnSmith}})}{\exists \textit{\textbf{maintainedBy}}.\textit{\textbf{University}}(\textit{\textbf{anatomy\_ont}})}$$

where $\textit{\textbf{anatomy\_ont}} \in M_I$, $\textit{\textbf{johnSmith}} \in M_I^\star$, $\textit{\textbf{hasAuthor}}, \textit{\textbf{maintainedBy}} \in M_R$, $\textit{\textbf{University}} \in M_C$.

To accommodate the interpretation of $\mathcal{M}$ in the semantics, without damaging its original architecture, we pose a new domain of the metalanguage Θ, with the set of context domain being a subset of it, and extend the interpretation function accordingly.

Definition 11 (**$AL_{\mathcal{L}}^{\mathcal{M}}$-interpretations**) *An $AL_{\mathcal{L}}^{\mathcal{M}}$-interpretation is a tuple $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c\in\mathfrak{C}})$, where:*

- *Θ is a non-empty metalanguage domain,*
- *$\mathfrak{C} \subseteq \Theta$ is a non-empty context domain,*
- *$\cdot^{\mathcal{J}}$ is an interpretation function which maps $\boldsymbol{A}^{\mathcal{J}} \subseteq \Theta$, for every $\boldsymbol{A} \in M_C$, $\boldsymbol{r}^{\mathcal{J}} \subseteq \Theta \times \Theta$, for every $\boldsymbol{r} \in M_R$, $\boldsymbol{c}^{\mathcal{J}} \in \Theta$, for every $\boldsymbol{c} \in M_I^\star$, with $\boldsymbol{c}^{\mathcal{J}} \in \mathfrak{C}$, whenever $\boldsymbol{c} \in M_I$,*
- *$(\Delta, \cdot^{\mathcal{I}(i)})$ as in Definition 8.*

$\mathcal{L}$ \ $\mathcal{M}$	$\mathcal{EL}$	$\mathcal{ALC}, \mathcal{ALCO}$
$\mathcal{EL}$	as in $AL_{\mathcal{L}}$	EXPTIME-hard
$\mathcal{ALC}$	as in $AL_{\mathcal{L}}$	NEXPTIME-complete
$\mathcal{ALCO}$	as in $AL_{\mathcal{L}}$	NEXPTIME-complete

Table 3: Complexity of reasoning in $\mathrm{AL}_{\mathcal{L}}^{\mathcal{M}}$.

The notions of AIS and $AL_{\mathcal{L}}^{\mathcal{M}}$-model remain exactly the same as in the case of $AL_{\mathcal{L}}^{\mathcal{M}}$ (Definition 7 and 9).

It turns out, that a shift from simple taxonomies to much more convenient $\mathcal{EL}$, as the metalanguage of AISs, does not entail a further increase in the complexity, which remains the same as in the corresponding $AL_{\mathcal{L}}$. Pushing the metalanguage envelope, however, has its limits. The use of $\mathcal{ALC}$ and $\mathcal{ALCO}$ in the same role, noticeably affects the complexity. The EXPTIME-hardness for $\mathcal{L} = \mathcal{EL}$, transfers directly from the lower bound of the involved metalanguages. The non-determinism involved in the other two cases can be interpreted by the need of guessing the interpretation of the metalanguage first, before finding the model of the object component of the combination.

Theorem 5 *The complexity of reasoning in $AL_{\mathcal{L}}^{\mathcal{M}}$ ranges as in Table 3.*

The lower bound of $AL_{\mathcal{ALC}}^{\mathcal{ALC}}$ is again obtained by an encoding of the $2^n \times 2^n$ tiling problem. For the upper bounds for $\mathcal{L} \in \{\mathcal{ALC}, \mathcal{ALCO}\}$ we devise a variant of a type elimination algorithm, whereas for $\mathcal{L} = \mathcal{EL}$ a completion algorithm in the style of (Baader, Brandt, and Lutz 2005). In most cases the results are robust enough to allow generalizations to more expressive DLs (see the appendix).

Conclusions

The problems of 1) representing inherently contextualized knowledge within the paradigm of DLs and 2) reasoning with multiple heterogenous, but semantically interoperating, DL representations, are both interesting and important issues, motivated by numerous practical application scenarios. It is our belief that these two challenges are in fact two sides of the same coin and, consequently, they should be approached within the same, unifying formal framework. In this paper, we have argued that two-dimensional DLs incorporating the principles of McCarthy's theory of contexts achieve this objective to a great extent, by providing sufficient syntactic and semantic means to support both functionalities. As our results show, such an extension of the standard DLs does not necessarily entail an increase in the computational complexity of reasoning, nor does it affect the generally adopted knowledge representation methodology of DLs. We therefore consider the approach a worthwhile subject to further research. In particular, we intend to investigate how certain basic notions, which are essential for practical use and maintenance of multi-context knowledge systems (e.g. inconsistency handling), can be meaningfully restated within the presented framework.

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Appendix

This appendix contains proofs of the technical results presented in the paper.

Reasoning in $SL_{\mathcal{L}}$

The following result covers the computational complexity of reasoning in $SL_{\mathcal{L}}$, for $\mathcal{L}$ ranging over a number of possible DLs including $\mathcal{EL}, \mathcal{ALC}$ and $\mathcal{ALCO}$.

Theorem 6 *The complexity of reasoning in $SL_{\mathcal{L}}$ is the same as in $\mathcal{L}$.*

Proof. Let $\mathcal{O}$ be a SIS in $SL_{\mathcal{L}}$. W.l.o.g. we consider $\mathcal{O}$ as a set of ontologies $\{\mathcal{O}_{\boldsymbol{c}}\}_{\boldsymbol{c}\in M_I}$ in $SL_{\mathcal{L}}$, where each $\mathcal{O}_{\boldsymbol{c}} = \{\varphi \mid \boldsymbol{c} : \varphi \in \mathcal{O}\}$. We claim that $\{\mathcal{O}_{\boldsymbol{c}}\}_{\boldsymbol{c}\in M_I}$ can be polynomially transformed into a single ontology in $\mathcal{L}$. First, we fix a set of vocabularies $\{\Sigma_{\boldsymbol{c}}\}_{\boldsymbol{c}\in M_I}$, where for every $\boldsymbol{c} \in M_I$, $\Sigma_{\boldsymbol{c}}$ consists of:

- all atoms (possibly including $\top$) occurring in $\mathcal{O}_{\boldsymbol{c}}$ which are not inside the scope of any operator $\langle \boldsymbol{d} \rangle$, for any $\boldsymbol{d} \in M_I$,
- all atoms (possibly including $\top$) in $\mathcal{O}_{\boldsymbol{d}}$, for every $\boldsymbol{d} \in M_I$, within the scope of context operators, for which the inner-most binding operator is $\langle \boldsymbol{c} \rangle$.

Next, we ensure disjointness of the vocabularies possibly by translating the atoms to fresh names. In particular for every $\top \in \Sigma_{\boldsymbol{c}}$ we replace it with a fresh concept name $\top_{\boldsymbol{c}}$. Further, we replace all the original atoms in $\{\mathcal{O}_{\boldsymbol{c}}\}_{\boldsymbol{c}\in M_I}$ with their translations and drop all context operators. It is easy to see that the resulting ontology $\mathcal{O}'$ is satisfiable in $\mathcal{L}$ *iff* $\mathcal{O}$ is satisfiable in $SL_{\mathcal{L}}$. Moreover, note that in case of $SL_{\mathcal{EL}}$ the subsumption problem is stated as: decide whether $\mathcal{O} \models \boldsymbol{c} : C \sqsubseteq D$. In order to reduce this problem to $\mathcal{L}$ we first transform $\mathcal{O}$ as described above and then restate the query $\boldsymbol{c} : C \sqsubseteq D$, by first transforming it to $\langle \boldsymbol{c} \rangle C \sqsubseteq \langle \boldsymbol{c} \rangle D$ and then replacing the atoms as in $\mathcal{O}$. Consequently, $\mathcal{O} \models \boldsymbol{c} : C \sqsubseteq D$ *iff* $\mathcal{O}' \models C' \sqsubseteq D'$, where $C' \sqsubseteq D'$ is the result of the transformation of the query.

What follows is that the complexity of reasoning in $SL_{\mathcal{L}}$ coincides with the complexity of reasoning in $\mathcal{L}$. ❑

NExpTime lower bounds

In this section we prove the NExpTime lower bound for the satisfiability problems in $AL_{\mathcal{ALCO}}$ and $AL^{\mathcal{ALC}}_{\mathcal{ALC}}$, respectively. All the remaining NExpTime lower bounds covered in the paper carry over directly from these two results or from the underlying DLs of known complexities (Baader et al. 2003). Resting on the close correspondence between our logics $AL^{\mathcal{M}}_{\mathcal{L}}$ and $AL_{\mathcal{L}}$ (Theorems 3 and 4) we sometimes refer to the elements of the context domain as **S5**-worlds rather than contexts.

Theorem 7 *Deciding concept satisfiability in $AL_{\mathcal{ALCO}}$ w.r.t. global TBoxes with only local roles is* NExpTime-*hard.*

Proof. The result is established by devising a polynomial reduction of the $2^n \times 2^n$ tiling problem, known to be NExpTime-complete (Marx 1999), to concept satisfiability in $AL^{\mathcal{ALC}}_{\mathcal{ALC}}$. An instance $\mathfrak{T} = (n, T)$ of the problem is defined as follows: given some $n \in \mathbb{N}$ in unary and a set of tiles $T = \{\tau_0, \ldots, \tau_m\}$, decide whether a $2^n \times 2^n$ grid can be tiled with T where the first cell in the grid is tiled with some $\tau_0 \in T$.

Let $\mathfrak{T} = (n, T)$ be an instance of the problem. In the consecutive steps, we define a TBox $\mathcal{T}_{\mathfrak{T}}$ and a concept $C_{\mathfrak{T}}$, such that there exists a tiling for $\mathfrak{T}$ *iff* $C_{\mathfrak{T}}$ is satisfiable w.r.t. $\mathcal{T}_{\mathfrak{T}}$.

The TBox consists of object axioms $\top : C \sqsubseteq D, \top : C \equiv D$ for $C, D \in \mathcal{ALC}$. For brevity we skip the qualification of the axioms "$\top$:" and write $C \sqsubseteq D$ and $C \equiv D$ instead.

First, the inclusions (1)-(10) enforce a 2^{2n}-long chain of individuals (*Grid*), uniquely identifiable by counting concepts X_i and Y_i, for $i \in (1, 2n)$. Notably, the Y-counter is shifted in the phase w.r.t. the X-counter by exactly 2^n, (i.e.: $X + 2^n = Y$), which further on is utilized for identifying the top-down neighbors in the tiling. Also, every 2^n-th individual, starting from the beginning of the chain, is made an instance of concept *RightEdge*, marking the right edge of the tiling (11):

$$\begin{aligned} &\mathit{StartGrid} \equiv \\ &\equiv \mathit{Grid} \sqcap \prod_{j=1}^{2n} \neg X_j \sqcap \prod_{j=1}^{n} \neg Y_j \sqcap Y_{n+1} \sqcap \prod_{j=n+2}^{2n} \neg Y_j, \end{aligned} \quad (1)$$

$$\mathit{EndGrid} \equiv \prod_{j=1}^{2n} X_j, \quad \mathit{Grid} \sqcap \neg \mathit{EndGrid} \sqsubseteq \exists s.\mathit{Grid}, \quad (2)$$

$$\neg X_i \sqcap \neg X_j \sqsubseteq \forall s.\neg X_i, \text{ for every } 1 \le j < i \le 2n, \quad (3)$$

$$X_i \sqcap \neg X_j \sqsubseteq \forall s.X_i, \text{ for every } 1 \le j < i \le 2n, \quad (4)$$

$$\neg X_j \sqcap X_{j-1} \sqcap \ldots \sqcap X_1 \sqsubseteq \forall s.X_j, \text{ for every } 1 \le j \le 2n, \quad (5)$$

$$X_j \sqcap X_{j-1} \sqcap \ldots \sqcap X_1 \sqsubseteq \forall s.\neg X_j, \text{ for every } 1 \le j \le 2n, \quad (6)$$

$$\neg Y_i \sqcap \neg Y_j \sqsubseteq \forall s.\neg Y_i, \text{ for every } 1 \le j < i \le 2n, \quad (7)$$

$$Y_i \sqcap \neg Y_j \sqsubseteq \forall s.Y_i, \text{ for every } 1 \le j < i \le 2n, \quad (8)$$

$$\neg Y_j \sqcap Y_{j-1} \sqcap \ldots \sqcap Y_1 \sqsubseteq \forall s.Y_j, \text{ for every } 1 \le j \le 2n, \quad (9)$$

$$Y_j \sqcap Y_{j-1} \sqcap \ldots \sqcap Y_1 \sqsubseteq \forall s.\neg Y_j, \text{ for every } 1 \le j \le 2n, \quad (10)$$

$$\mathit{RightEdge} \equiv \prod_{j=1}^{n} X_j. \quad (11)$$

Next, by (12)-(13), the values of the counting concepts are propagated globally across all **S5**-worlds:

$$X_i \sqsubseteq [\top] X_i, \quad \neg X_i \sqsubseteq [\top] \neg X_i, \text{ for every } 1 \le i \le 2n, \quad (12)$$

$$Y_i \sqsubseteq [\top] Y_i, \quad \neg Y_i \sqsubseteq [\top] \neg Y_i, \text{ for every } 1 \le i \le 2n. \quad (13)$$

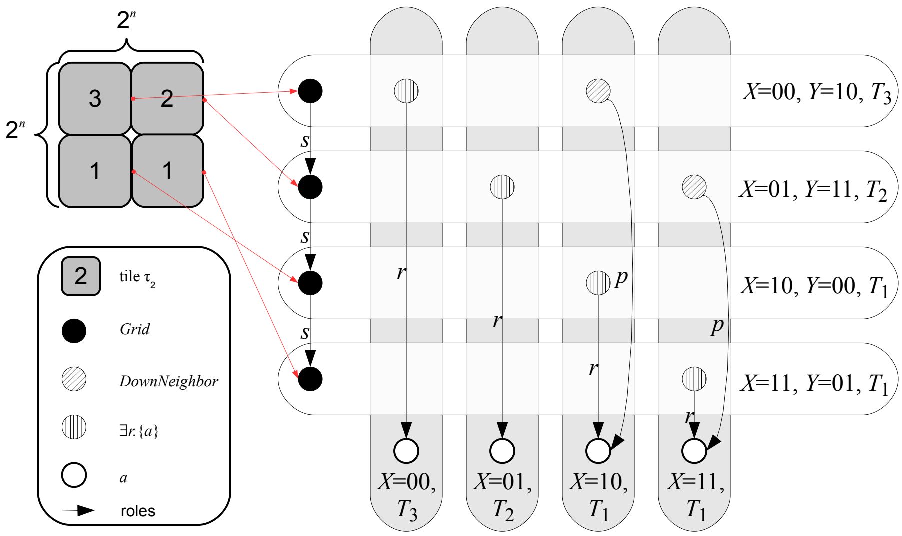

Figure 1: Encoding of a $2^n \times 2^n$ tiling in an $AL_{\mathcal{ALCO}}$-model.

Further, we impose the basic coloring constraints over all individuals (14), adjust the coloring of all the left-right neighbors: (15), and propagate the tile types over all **S5**-worlds (16):

$$\top \sqsubseteq (\bigsqcup_{\tau_i} T_i) \sqcap \bigsqcap_{\tau_i \neq \tau_j} \neg(T_i \sqcap T_j), \text{ for every } \tau_i, \tau_j \in T, \quad (14)$$

$$T_i \sqcap \neg \mathit{RightEdge} \sqsubseteq \forall s.(\bigsqcup_{\mathit{right}(\tau_i)=\mathit{left}(\tau_j)} T_j), \quad (15)$$
$$\text{for every } \tau_i, \tau_j \in T,$$

$$T_i \sqsubseteq [\top] T_i, \text{ for every } \tau_i \in T. \quad (16)$$

The key to the reduction is a suitable use of a single nominal $\{a\}$ (see Figure 1). By (17) every individual in the grid is linked to a via role r in some **S5**-world. There, due to (18)-(19), the value of the X-counter and the tile type assigned to the individual is forced upon a. Consequently, by assuming rigid individual names,[3] we generate 2^{2n} distinct **S5**-worlds:

$$\mathit{Grid} \sqsubseteq \langle\top\rangle \exists r.\{a\}, \quad (17)$$

$$X_i \sqsubseteq \forall r.X_i, \quad \neg X_i \sqsubseteq \forall r.\neg X_i, \quad (18)$$
$$\text{for every } 1 \leq i \leq 2n,$$

$$T_i \sqsubseteq \forall r.T_i, \text{ for every } \tau_i \in T. \quad (19)$$

Finally, in every **S5**-world, all individuals are linked to a via p (20). Whenever the value of the Y-counter on a grid-individual matches the value of the X-counter on a (21), the proper top-down coloring constraints are imposed (22):

$$\top \sqsubseteq \exists p.\{a\}, \quad (20)$$

$$\mathit{DownNeighbor} \equiv$$
$$\equiv \bigsqcap_{j=1}^{2n} ((Y_i \sqcap \exists p.X_i) \sqcup (\neg Y_i \sqcap \exists p.\neg X_i)), \quad (21)$$

$$T_i \sqcap \mathit{DownNeighbor} \sqsubseteq \forall p. \bigsqcup_{\mathit{down}(\tau_i)=\mathit{top}(\tau_j)} T_j, \quad (22)$$
$$\text{for every } \tau_i, \tau_j \in T.$$

The TBox $\mathcal{T}_{\mathfrak{T}}$ is defined as the union of the axioms (1)-(22). It is easy to see that the size of $\mathcal{T}_{\mathfrak{T}}$ is polynomial in the size of the instance $\mathfrak{T}$. Finally, we define the concept $C_{\mathfrak{T}} = \mathit{StartGrid} \sqcap T_0$ and claim that there is a tiling for $\mathfrak{T}$ *iff* $C_{\mathfrak{T}}$ is satisfiable w.r.t. globally interpreted $\mathcal{T}_{\mathfrak{T}}$.
($\Rightarrow$) Let τ be a tiling for $\mathfrak{T}$, i.e. a mapping from $2^n \times 2^n$ to T. Define an $AL_{\mathcal{ALCO}}$-model $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$ for $\mathcal{T}_{\mathfrak{T}}$ satisfying $C_{\mathfrak{T}}$ as follows. First, transform τ into $\pi : 2^{2n} \mapsto T$, such that for every $(x, y) \in 2^n \times 2^n$, $\tau(x, y) = \pi(y * 2^n + x)$. Then, fix $\Theta = \mathfrak{C} = \{c_i \mid i \in (0, 2^{2n})\}$ and $\Delta = \{d_i \mid i \in (0, 2^{2n})\}$ and ensure that the following interpretation constraints are satisfied:

- $a^{\mathcal{I}(c)} = d_0$ for $d_0 \in \Delta$ and every $c \in \mathfrak{C}$,
- for $c_0 \in \mathfrak{C}$:
  - $\mathit{Grid}^{\mathcal{I}(c_0)} = \Delta \setminus \{d_0\}$,
  - $\mathit{StartGrid}^{\mathcal{I}(c_0)} = \{d_1 \in \Delta\}$, $\mathit{EndGrid}^{\mathcal{I}(c_0)} = \{d_{2^{2n}} \in \Delta\}$,
  - $\mathit{RightEdge}^{\mathcal{I}(c_0)} = \{d_{2^n * i} \in \Delta\}$,
  - $s^{\mathcal{I}(c_0)} = \{\langle d_i, d_{i+1}\rangle \mid d_i, d_{i+1} \in \Delta, i \geq 1\}$,
- $\{d_i \mid \pi(i) = \tau_j\} \subseteq T_j^{\mathcal{I}(c)}$, for every $c \in \mathfrak{C}$ and $\tau_j \in T$,
- $d_0 \in T_j^{\mathcal{I}(c_i)}$ *iff* $\pi(i) = \tau_j$, for every $i \geq 1$ and $\tau_j \in T$,
  - $r^{\mathcal{I}(c_i)} = \{\langle d_i, d_0\rangle \mid d_i \in \Delta\}$ for $i \geq 1$,
  - $p^{\mathcal{I}(c)} = \{\langle d, d_0\rangle \mid d \in \Delta\}$ for every $c \in \mathfrak{C}$,
  - $\mathit{DownNeighbor}^{\mathcal{I}(c_i)} = \{d_{i-2^n} \in \Delta\}$, for every $c_i \in \mathfrak{C}$ and $i \geq 2^n + 1$.

The interpretations can be straightforwardly extended over the counting concepts X_i and Y_i so that $\mathfrak{M}$ is indeed a model for $\mathcal{T}_{\mathfrak{T}}$, where $d_1 \in (C_{\mathfrak{T}})^{\mathcal{I}(c_0)}$.

[3] Such assumption can be also made explicit by including axiom $\{a\} \sqsubseteq [\top]\{a\}$.

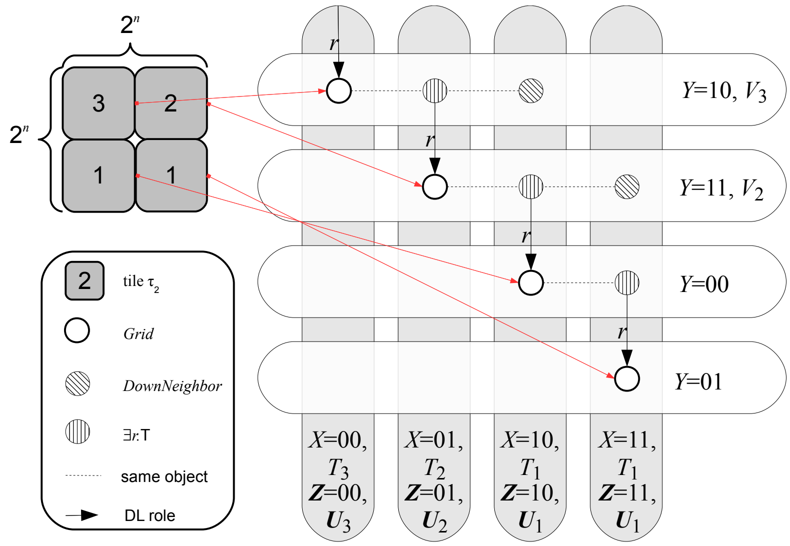

Figure 2: Encoding of a $2^n \times 2^n$ tiling in an $AL^{\mathcal{ALC}}_{\mathcal{ALC}}$-model.

($\Leftarrow$) Let $\mathfrak{M}$ be an $AL_{\mathcal{ALCO}}$-model of $\mathcal{T}_{\mathfrak{T}}$ satisfying $C_{\mathfrak{T}}$. Then, a tiling for $\mathfrak{T}$ can be retrieved from $\mathfrak{M}$ by mapping a chain of s-successors, which instantiate concept *Grid* in the **S5**-world in which $C_{\mathfrak{T}}$ is satisfied, on the $2^n \times 2^n$ grid, where the type of a tile in the grid is determined by the unique concept T_i satisfied by the individual in the chain. The coloring constraints have to be satisfied by the construction of the encoding. ❑

Theorem 8 *Deciding concept satisfiability in $AL^{\mathcal{ALC}}_{\mathcal{ALC}}$ w.r.t. global TBoxes with only local roles is* NEXPTIME*-hard.*

Proof. The result is established by reducing the $2^n \times 2^n$ tiling problem. Let $\mathfrak{T} = (n, T)$ be an instance of the problem. In the consecutive steps, we define a TBox $\mathcal{T}_{\mathfrak{T}}$ and a concept $C_{\mathfrak{T}}$, such that there exists a tiling for $\mathfrak{T}$ *iff* $C_{\mathfrak{T}}$ is satisfiable w.r.t. $\mathcal{T}_{\mathfrak{T}}$. Again, the encoding utilizes the possibility of constructing and constraining a "diagonal" in models, as depicted in Figure 2, representing the whole tiling in a linear projection.

The inclusions (23)-(28) enforce a 2^{2n}-long chain of individuals, uniquely identifiable by counting concepts X_i, for $i \in (1, 2n)$. Moreover, every 2^n-th individual, starting from the beginning of the chain, is an instance of concept $RightEdge$, marking the right edge of the tiling, while the last 2^n individuals are instances of $BottomEdge$, marking the bottom of the tiling.

$$StartGrid \equiv \prod_{j=1}^{2n} \neg X_j, \quad EndGrid \equiv \prod_{j=1}^{2n} X_j, \qquad (23)$$
$$\neg EndGrid \sqsubseteq \langle \top \rangle \exists r.\top,$$

$$\neg X_i \sqcap \neg X_j \sqsubseteq [\top]\forall r.\neg X_i, \text{ for every } 1 \leq j < i \leq 2n, \qquad (24)$$

$$X_i \sqcap \neg X_j \sqsubseteq [\top]\forall r.X_i, \text{ for every } 1 \leq j < i \leq 2n, \qquad (25)$$

$$\neg X_j \sqcap X_{j-1} \sqcap \ldots \sqcap X_1 \sqsubseteq [\top]\forall r.X_j, \text{ for every } 1 \leq j \leq 2n, \qquad (26)$$

$$X_j \sqcap X_{j-1} \sqcap \ldots \sqcap X_1 \sqsubseteq [\top]\forall r.\neg X_j, \text{ for every } 1 \leq j \leq 2n, \qquad (27)$$

$$RightEdge \equiv \prod_{j=1}^{n} X_j, \quad BottomEdge \equiv \prod_{j=n+1}^{2n} X_j. \qquad (28)$$

The values of these counting concepts are then propagated over all the objects in the given context, by involving an interaction with concepts of the metalanguage $\mathbf{Z}_i$, for $i \in (1, 2n)$ (29).

$$\top \sqsubseteq [\mathbf{Z}_i]X_i, \quad \top \sqsubseteq [\neg \mathbf{Z}_i]\neg X_i, \text{ for every } 1 \leq i \leq 2n. \qquad (29)$$

Each individual is required to satisfy exactly one concept T_i, representing a tile type $\tau_i \in T$ (30). This type is then propagated to all individuals in the given world (31-32) and used to adjust the coloring of the left-right neighbors (33).

$$\top \sqsubseteq (\bigsqcup_{\tau_i} T_i) \sqcap \prod_{\tau_i \neq \tau_j} \neg(T_i \sqcap T_j), \text{ for every } \tau_i, \tau_j \in T, \qquad (30)$$

$$\top \sqsubseteq [\boldsymbol{U}_i]T_i, \text{ for every } \tau_i \in T. \qquad (31)$$

$$\top \sqsubseteq [\neg \boldsymbol{U}_i]\neg T_i, \text{ for every } \tau_i \in T, \qquad (32)$$

$$T_i \sqcap \neg RightEdge \sqsubseteq [\top]\forall r.(\bigsqcup_{right(\tau_i)=left(\tau_j)} T_j), \text{ for every } \tau_i, \tau_j \in T. \qquad (33)$$

For each individual we identify the counter of its down neighbor and encode this value rigidly across all **S5**-worlds by means of concepts Y_i (34-39). In the same manner, the tile type is propagated (40).

$$\neg X_i \sqcap \neg X_j \sqsubseteq \forall r.[\top]\neg Y_i, \text{ for every } n+1 \leq j < i \leq 2n, \qquad (34)$$

$$X_i \sqcap \neg X_j \sqsubseteq \forall r.[\top] Y_i, \text{ for every } n+1 \leq j < i \leq 2n, \quad (35)$$

$$\neg X_j \sqcap X_{j-1} \sqcap \ldots \sqcap X_{(n+1)} \sqsubseteq \forall r.[\top] Y_j, \text{ for every } n+1 \leq j \leq 2n, \quad (36)$$

$$X_j \sqcap X_{j-1} \sqcap \ldots \sqcap X_{(n+1)} \sqsubseteq \forall r.[\top] \neg Y_j, \text{ for every } n+1 \leq j \leq 2n, \quad (37)$$

$$X_i \sqsubseteq \forall r.[\top] Y_i, \text{ for every } 1 \leq i \leq n, \quad (38)$$

$$\neg X_i \sqsubseteq \forall r.[\top] \neg Y_i, \text{ for every } 1 \leq i \leq n, \quad (39)$$

$$\neg \mathit{BottomEdge} \sqcap T_i \sqsubseteq \forall r.[\top] V_i, \text{ for every } \tau_i \in T. \quad (40)$$

Finally, the up-down coloring constraints are enforced whenever the value of Y_i's agrees with the X_i-counter. (41-42).

$$\mathit{DownNeighbor} \equiv \bigsqcap_{1 \leq i \leq 2n} ((X_i \sqcap Y_i) \sqcup (\neg X_i \sqcap \neg Y_i)), \quad (41)$$

$$\mathit{DownNeighbor} \sqcap V_i \sqsubseteq \bigsqcap_{\mathit{down}(\tau_i) \neq \mathit{up}(\tau_j)} \neg T_j, \text{ for every } \tau_i \in T. \quad (42)$$

The TBox $\mathcal{T}_{\mathfrak{T}}$ is defined as the union of the axioms (23)-(42). It is easy to see that the size of $\mathcal{T}_{\mathfrak{T}}$ is polynomial in the size of the instance $\mathfrak{T}$. Finally, we define the concept $C_{\mathfrak{T}} = \exists r.(\mathit{StartGrid} \sqcap T_0)$ and claim that there is a tiling for $\mathfrak{T}$ *iff* $C_{\mathfrak{T}}$ is satisfiable w.r.t. $\mathcal{T}_{\mathfrak{T}}$.

($\Rightarrow$) Let τ be a tiling for $\mathfrak{T}$, i.e. a mapping from $2^n \times 2^n$ to T. Define an $AL^{\mathcal{ALC}}_{\mathcal{ALC}}$-model $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$ for $\mathcal{T}_{\mathfrak{T}}$ satisfying $C_{\mathfrak{T}}$ as follows. First, transform τ into $\pi : 2^{2n} \mapsto T$, such that for every $(x, y) \in 2^n \times 2^n$, $\tau(x, y) = \pi(y * 2^n + x)$. Then, fix $\Theta = \mathfrak{C} = \{c_i \mid i \in (1, 2^{2n})\}$ and $\Delta = \{d_i \mid i \in (0, 2^{2n})\}$ and ensure that the following interpretation constraints are satisfied:

- $r^{\mathcal{I}(c_i)} = \{(d_{i-1}, d_i) \mid d_{i-1}, d_i \in \Delta\}$,
- $\mathit{StartGrid}^{\mathcal{I}(c_1)} = \{d_1 \in \Delta\}$, $\mathit{EndGrid}^{\mathcal{I}(c_{2^{2n}})} = \{d_{2^{2n}} \in \Delta\}$,
- for every $c_i \in \mathfrak{C}$, $\mathit{DownNeighbor}^{\mathcal{I}(c_i)} = \{d_{i-2^n} \in \Delta\}$,
- for every $\tau_j \in T$ and $i \in (1, 2^{2n})$, $T_j^{\mathcal{I}(c_i)} = \Delta$, if $\pi(i) = \tau_j$, and else $T_j^{\mathcal{I}(c_i)} = \emptyset$.

The interpretations can be straightforwardly extended over the remaining concepts so that $\mathfrak{M}$ is indeed a model for $\mathcal{T}_{\mathfrak{T}}$, where $d_0 \in (C_{\mathfrak{T}})^{\mathcal{I}(c_1)}$.

($\Leftarrow$) Let $\mathfrak{M}$ be an $AL^{\mathcal{ALC}}_{\mathcal{ALC}}$-model of $\mathcal{T}_{\mathfrak{T}}$ satisfying $C_{\mathfrak{T}}$. Then, a tiling for $\mathfrak{T}$ can be retrieved from $\mathfrak{M}$ by mapping the diagonal of the model on the $2^n \times 2^n$ grid, where the type of a tile in the grid is determined by the unique concept T_i satisfied by the individual in the chain. The coloring constraints have to be satisfied by the construction of the encoding. ❑

PTime upper bound

In this section we show that satisfiability in $AL^{\mathcal{EL}}_{\mathcal{EL}}$ PTIME. As a first step, we introduce an algorithm to decide *instance checking* in $AL^{\mathcal{EL}}_{\mathcal{EL}}$. Then, we show how to use the algorithm to decide $AL^{\mathcal{EL}}_{\mathcal{EL}}$ satisfiability. During the definition of the algorithm, we refer to elements of the context domain as worlds rather than contexts.

First, we consider *instance checking* which is the problem to decide, given an abstract interoperability model $\mathcal{K} = (\mathcal{C}, \mathcal{O})$ with $\mathcal{O} = (\mathcal{T}^{\mathcal{O}}, \mathcal{A}^{\mathcal{O}})$, an individual name $a \in N_I$ and a object concept C whether, $\mathfrak{M} \models C(a)$ for all models $\mathfrak{M} = (\mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$ of $\mathcal{K}$.

We assume object TBoxes $\mathcal{T}^{\mathcal{O}}$ to be in *normal form*. A *basic meta concept* is either $\mathbf{A} \in M_C$ or $\top$. A *basic object concept* is either $\top$, a concept name over N_C, $\langle \mathbf{A} \rangle B$ or $[\mathbf{A}]B$ with $\mathbf{A} \in M_C$ and $B \in N_C$. For an object TBox $\mathcal{T}^{\mathcal{O}}$ to be in normal form we require that every GCI is of one of the following forms

$$X_1 \sqcap \ldots \sqcap X_n \sqsubseteq X,$$
$$\exists r.X \sqsubseteq B,$$
$$X \sqsubseteq \exists r.B$$

where the X_i and X denote basic object concepts and B denotes a concept name over N_C. It can be shown that by introducing fresh concept names every object TBox $\mathcal{T}^{\mathcal{O}}$ can be converted into an object TBox $\mathcal{T'}^{\mathcal{O}}$ in normal form that preserves subsumption (Baader, Brandt, and Lutz 2005).

We use $N_C^{\mathcal{O}}$ (resp. $N_R^{\mathcal{O}}$) to denote the set of object concept names (resp. role names) that occur in $\mathcal{O}$; $\mathsf{BC}^{\mathcal{O}}$ (resp. $\mathcal{P}^{\mathcal{O}}_{\langle \mathbf{A} \rangle}$, $\mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]}$) to denote the set of basic object concepts (resp. concepts $\langle \mathbf{A} \rangle B$, $[\mathbf{A}]B$) that occur (possibly as a subconcept) in $\mathcal{O}$. We denote by $\mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}}$ the set of *basic context concepts* that occur (possibly as subconcept) in $\mathcal{O}$

We extend the sets $\mathcal{P}^{\mathcal{O}}_{\langle \mathbf{A} \rangle}$ and $\mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]}$ as follows:

$$\mathcal{P}^{\mathcal{O}}_{\langle \text{-} \rangle} := \bigcup_{\substack{\mathbf{A}, \mathbf{A}' \in \mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}} \\ \langle \mathbf{A} \rangle B \in \mathcal{P}^{\mathcal{O}}_{\langle \mathbf{A} \rangle} \\ \mathbf{A} \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} \mathbf{A}'}} \langle \mathbf{A}' \rangle B$$

$$\mathcal{P}^{\mathcal{O}}_{[\text{-}]} := \bigcup_{\substack{\mathbf{A}, \mathbf{A}' \in \mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}} \\ [\mathbf{A}] B \in \mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]} \\ \mathbf{A}' \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} \mathbf{A}}} [\mathbf{A}'] B$$

We define the set of *extended basic concepts* as follows:

$$\mathsf{EBC}^{\mathcal{O}} := \mathsf{BC}^{\mathcal{O}} \cup \mathcal{P}^{\mathcal{O}}_{\langle \text{-} \rangle} \cup \mathcal{P}^{\mathcal{O}}_{[\text{-}]}.$$

Worlds Next, we introduce the set W of *worlds* of the model to be constructed.

R1	if $X_1 \sqcap \ldots \sqcap X_n \sqsubseteq X \in \mathcal{T}^{\mathcal{O}}$, $X_1, \ldots, X_n \in Q(a,w)$, and $X \notin Q(a,w)$ then $Q(a,w) := Q(a,w) \cup \{X\}$	
R2	if $\langle\mathbf{A}\rangle B \in Q(a,w)$, $B \notin Q(a,w)$, and $w \neq \langle\mathbf{A}\rangle B$ then $Q(a,\langle\mathsf{A}\rangle B) := Q(a,\langle\mathbf{A}\rangle B) \cup \{B\}$	
R3	if $\langle\mathbf{A}\rangle B \in Q(a,w)$, $B \notin Q(a,w)$, and $w = \langle\mathbf{A}\rangle B$ then $Q(a,\varepsilon_{\langle\mathbf{A}\rangle B}) := Q(a,\varepsilon_{\langle\mathbf{A}\rangle B}) \cup \{B\}$	
R4	if $[\mathbf{A}]B \in Q(a,v)$, $B \notin Q(a,w)$, and w is an $\mathbf{A}$-world then $Q(a,w) := Q(a,w) \cup \{B\}$	
R5	if $B \in Q(a,w)$, w is a $\mathbf{A}$-world, $\langle\mathbf{A}\rangle B \in \mathcal{P}^{\mathcal{O}}_{\langle\mathbf{A}\rangle}$, and $\langle\mathbf{A}\rangle B \notin Q(a,v)$ then $Q(a,v) := Q(a,v) \cup \{\langle\mathbf{A}\rangle B\}$	
R6	if $\langle\mathbf{A}\rangle B \in Q(a,w)$, and $\langle\mathbf{A}'\rangle B \notin Q(a,w)$ with $\mathsf{A}' \in \mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}}$, $\mathbf{A} \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} \mathbf{A}'$ then $Q(a,w) := Q(a,w) \cup \{\langle\mathbf{A}'\rangle B\}$	
R7	if $B \in Q(a,[\mathbf{A}]) \cap Q(a,\varepsilon_{[\mathbf{A}]})$, $[\mathbf{A}]B \in \mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]}$, and $[\mathbf{A}]B \notin Q(a,w)$ then, $Q(a,w) := Q(a,w) \cup \{[\mathbf{A}]B\}$	
R8	if $[\mathbf{A}]B \in Q(a,w)$, and $[\mathbf{A}']B \notin Q(a,w)$, with $\mathbf{A}' \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} \mathbf{A}$ then $Q(a,w) := Q(a,w) \cup \{[\mathbf{A}']B\}$	
R9	if $X \in Q(a,w)$, $X \sqsubseteq \exists r.B \in \mathcal{T}^{\mathcal{O}}$ then $R(r,w) := R(r,w) \cup (a,(B,w))$, and $Q((B,w),w) := Q((B,w),w) \cup \{B\}$	
R10	if $(a,b) \in R(r,w)$, $B \in Q(b,w)$, $\exists r.B \sqsubseteq Y \in \mathcal{T}^{\mathcal{O}}$ and $B \notin Q(a,w)$ then $Q(a,w) := Q(a,w) \cup \{Y\}$	

Figure 3: Completion rules

$$W := \bigcup_{\mathbf{A}\in\mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}}} \mathcal{P}^{\mathcal{O}}_{\langle\mathbf{A}\rangle} \cup \bigcup_{\substack{\mathbf{A}\in\mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}}\\ [\mathbf{A}]B\in\mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]}}} [\mathbf{A}] \cup \varepsilon_{\mathcal{A}^{\mathcal{O}}} \cup$$

$$\cup \bigcup_{\substack{\mathbf{A}\in\mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}}\\ \langle\mathbf{A}\rangle B\in\mathcal{P}^{\mathcal{O}}_{\langle\mathbf{A}\rangle}}} \varepsilon_{\langle\mathbf{A}\rangle B} \cup \bigcup_{\substack{\mathbf{A}\in\mathsf{BC}^{\mathcal{O}}_{\mathfrak{C}}\\ [\mathbf{A}]B\in\mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]}}} \varepsilon_{[\mathbf{A}]}$$

Intuitively, the worlds of the form $\langle\mathbf{A}\rangle B$ serve as a witnesses of the concepts of the form $\langle\mathbf{A}\rangle B$; the worlds of the form $[\mathbf{A}]$ are used to collect concepts that a domain element has to satisfy in *all* $\mathbf{A}$-worlds.Then, we have two kind of worlds for special cases. The worlds of the form $\varepsilon_{\langle\mathbf{A}\rangle B}$ serve for witnessing $\langle\mathbf{A}\rangle B$ restrictions that occurs in the world $\langle\mathbf{A}\rangle B$, and the worlds $\varepsilon_{[\mathbf{A}]}$ witness that $[\mathbf{A}]B$ is not true in all $\mathbf{A}$-worlds, i.e., if B is true *accidentally* in all $\mathbf{A}$-worlds. The world $\varepsilon_{\mathcal{A}^{\mathcal{O}}}$ serves to realize the ABox $A^{\mathcal{O}}$. An $\mathbf{A}$-*world* is of either form $\langle\mathbf{A}\rangle B$, $[\mathbf{A}]$, $\varepsilon_{[\mathbf{A}]}, \varepsilon_{\langle\mathbf{A}\rangle B}$. In particular, $\varepsilon_{\mathcal{A}^{\mathcal{O}}}$ is a $\top$-world.

Quasistates Let $\Omega := \mathsf{Ind}(\mathcal{A}^{\mathcal{O}}) \cup N^{\mathcal{O}}_C \times W$. A *quasistate* Q for $\mathcal{K}$ is a mapping that associates with each $a \in \Omega$ and each $w \in W$ a subset $Q(a,w) \subseteq \mathsf{EBC}^{\mathcal{O}}$. A *role quasistate* R is a mapping that associates with each $r \in N_R$ and each $w \in W$ a binary relation $R(r,w) \subseteq \Omega \times \Omega$. A *quasimodel* $\mathfrak{M}$ for $\mathcal{K}$ is a mapping pair (Q,R) with Q a quasistate and R a role quasistate.

The algorithm starts with the quasimodel (Q,R) , where

- $Q(b,\varepsilon_{\mathcal{A}^{\mathcal{O}}}) = \{\top\} \cup \{B \mid B(b) \in \mathcal{A}^{\mathcal{O}}\}$ for all $b \in \mathsf{Ind}(\mathcal{A}^{\mathcal{O}})$;
- $Q(b,w) = \{\top\}$ for all $b \in \mathsf{Ind}(\mathcal{A}^{\mathcal{O}})$ and $w \in W \setminus \{\mathcal{A}^{\mathcal{O}}\}$;
- $R(r,w) = \emptyset$ for all $r \in N^{\mathcal{O}}_R$ and $w \in W$.

This quasimodel is then extended by applying the completion rules shown in Figure 3 until no more rules apply. In Figure 3, $A \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} B$ denotes that A subsumes B w.r.t. the context TBox $\mathcal{T}^{\mathfrak{C}}$. This can be computed in polynomial time (Baader, Brandt, and Lutz 2005).

The the data structure $Q(a,w)$ describes the memberships of the element a in the world w. $R(r,w)$ describes the role memberships at world w.

Lemma 1 *For all $A_0 \in N_C$ and $a_0 \in \mathsf{Ind}(\mathcal{A}^{\mathcal{O}})$, $\mathcal{K} \models A_0(a_0)$ iff $A_0 \in Q(a_0,\varepsilon_{\mathcal{A}^{\mathcal{O}}})$.*

Proof. ($\Rightarrow$) Assume that $A_0 \notin Q(a_0,\varepsilon_{\mathcal{A}^{\mathcal{O}}})$. We define and interpretation $\mathfrak{M} = (\mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \mathcal{I})$ such that $\mathfrak{M} \models \mathcal{K}$ but $a_0 \notin A_0^{\mathfrak{M}}$.

$$\begin{aligned} \Delta &:= \Omega; \\ \mathfrak{C} &:= W; \\ A^{\mathcal{I}(w)} &:= \{a \in \Delta \mid A \in Q(a,w)\}; \\ r^{\mathcal{I}(w)} &:= \{R(r,w)\}. \end{aligned}$$

We define a model $\mathcal{J}$ of the meta knowledge base $\mathcal{C}$.

$$\mathbf{A}^{\mathcal{J}} := \{c \in \mathfrak{C} \mid c \text{ is an } \mathbf{A}\text{-world}\}$$

Then, we do standard $\mathcal{EL}$ reasoning over $\mathcal{T}^{\mathfrak{C}}$.

We show that $\mathfrak{M}$ is a model for $\mathcal{K} = (\mathcal{C},\mathcal{O})$. Then, it only needs to be proved that is a model of $\mathcal{O}$. First, we prove that

$$X \in Q(a,w) \text{ iff } a \in X^{\mathcal{I}(w)}$$

The proof of is by case distinction to the possible forms of X.

- $X = \top$ By initialization of construction of $\mathfrak{M}$.
- $X = A \in N_C$ Direct by construction of $\mathfrak{M}$.
- $X = \langle\mathsf{A}\rangle B \in \mathcal{P}^{\mathcal{O}}_{\langle\mathbf{A}\rangle}$ ($\Rightarrow$) Assume $\langle\mathsf{A}\rangle B \in Q(a,w)$
  1. $B \in Q(a,w) \wedge w$ is an A-world. Then, by I.H. $a \in B^{\mathcal{I}(w)}$, and thus $a \in \langle\mathsf{A}\rangle B^{\mathcal{I}(w)}$.
  2. $B \notin Q(a,w)$, $w \neq \langle\mathsf{A}\rangle B$. Then, by rule **R2**, $B \in Q(a,\langle\mathbf{A}\rangle B)$: By I.H. $a \in B^{\mathcal{I}(\langle\mathbf{A}\rangle B)}$. Then, by semantics, $a \in (\langle\mathbf{A}\rangle B)^{\mathcal{I}(w)}$.

3. $B \notin Q(a,w)$, $w = \langle \mathsf{A} \rangle B$. Then, by rule **R3**, $B \in Q(a, \varepsilon_{\langle \mathbf{A} \rangle B})$: By I.H. $a \in A^{\mathcal{I}(\varepsilon_{\langle \mathbf{A} \rangle B})}$.

   ($\Leftarrow$) Assume $a \in \langle \mathbf{A} \rangle B^{\mathcal{I}(w)}$. By the semantics, there is an $\mathbf{A}$-world v such that $a \in B^{\mathcal{I}(v)}$. I.H. $A \in Q(a,v)$. Then, by **S5**, $\langle \mathbf{A} \rangle B \in Q(a,v')$ for all worlds $v' \in W$. In particular, $\langle \mathbf{A} \rangle B \in Q(a,w)$.

- $X = \langle \mathsf{A} \rangle B \notin \mathcal{P}^{\mathcal{O}}_{\langle \mathbf{A} \rangle}$

  ($\Rightarrow$) Assume $\langle \mathsf{A} \rangle B \in Q(a,w)$ Then, by **S6** there is an $\mathbf{A}' \in M_C$ such that $\mathbf{A}' \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} \mathbf{A}$. By case before, $a \in \langle \mathbf{A}' \rangle B^{\mathcal{I}(w)}$. By semantics and $\mathbf{A}' \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} \mathbf{A}$, $a \in (\langle \mathbf{A} \rangle B)^{\mathcal{I}(w)}$.

  ($\Leftarrow$) Assume $a \in \langle \mathbf{A} \rangle B^{\mathcal{I}(w)}$. Then, by **R6** there is an $\mathbf{A}' \in M_C$ such that $\mathbf{A}' \sqsubseteq_{\mathcal{T}^{\mathfrak{C}}} \mathbf{A}$. By case before, $\langle \mathbf{A}' \rangle B \in Q(a,w)$. By **R6**, $\langle \mathbf{A}' \rangle B \in Q(a,w)$.

- $[\mathsf{A}]B \in \mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]}$.

  ($\Rightarrow$) Assume $[\mathsf{A}]B \in Q(a,w)$. By rule **R4**, $B \in Q(a,v)$ for all v that are an $\mathbf{A}$-world. Then, by I.H. $a \in B^{\mathcal{I}(v)}$. Thus, by semantics, $a \in ([\mathbf{A}]B)^{\mathcal{I}(w)}$.

  ($\Leftarrow$) Assume $a \in ([\mathbf{A}]B)^{\mathcal{I}(w)}$. Then $a \in B^{\mathcal{I}(v)}$ for all v that are an $\mathbf{A}$-world. Hence, $a \in B^{\mathcal{I}([\mathbf{A}])}$ and $a \in B^{\mathcal{I}(\varepsilon_{[\mathbf{A}]})}$. I.H. $B \in Q(a,[\mathbf{A}]) \cap Q(a, \varepsilon_{[\mathbf{A}]})$. Thus, by rule **R7**, $[\mathbf{A}]B \in Q(a,v)$ for all v, in particular w.

- $[\mathsf{A}]B \notin \mathcal{P}^{\mathcal{O}}_{[\mathbf{A}]}$.

  As in the case of $\langle \mathsf{A} \rangle B \notin \mathcal{P}^{\mathcal{O}}_{\langle \mathbf{A} \rangle}$ but using rule **R8**.

- $\exists r.A$

  ($\Rightarrow$) Assume $\exists r.A \in Q(a,w)$. Then, by rule **R9**, $(a,(A,w)) \in R(r,w)$ and $A \in Q((A,w),w)$. By I.H., $(A,w) \in A^{\mathcal{I}(w)}$. By construction, $(a,(A,w)) \in r^{\mathcal{I}(w)}$. Therefore, $a \in (\exists r.A)^{\mathcal{I}(w)}$.

  ($\Leftarrow$) Assume $a \in (\exists r.A)^{\mathcal{I}(w)}$. Then, there is an element (B,w) such that $(B,w) \in A^{\mathcal{I}(w)}$ and $(a,(B,w)) \in r^{\mathcal{I}(w)}$. By I.H. $A \in Q((B,w),w)$ and, by construction, $(a,(B,w)) \in R(r,w)$. Now, by **R10**, $\exists r.A \in Q(a,w)$.

We show that the interpretation $\mathfrak{M}$ is indeed a model for the TBox $\mathcal{T}^{\mathcal{O}}$. We make case distinction according with the type of GCIs

- $X_1 \sqcap \ldots \sqcap X_n \sqsubseteq X_n$. Let $a \in (X_1 \sqcap \ldots \sqcap X_n)^{\mathcal{I}(w)}$. Then, $a \in X_i^{\mathcal{I}(w)}, i \leq n$. By statement above, $X_i \in Q(a,w)$. By rule **R1**, $X \in Q(a,w)$. Again, by statement above, $a \in X^{\mathcal{I}(w)}$.
- $X \sqsubseteq \exists r.A$. Let $a \in X^{\mathcal{I}(w)}$. Then, $X \in Q(a,w)$. By rule **R9**, $A \in Q((A,w),w)$ and $(a,(A,w)) \in R(r,w)$. Then, by statement above, $(A,w) \in A^{\mathcal{I}(w)}$. By construction, $(a,(A,w)) \in r^{\mathcal{I}(w)}$. Therefore, $a \in (\exists r.A)^{\mathcal{I}}(w)$.
- $\exists r.A \sqsubseteq X$. Let $a \in (\exists r.A)^{\mathcal{I}(w)}$. By the semantics, there is a $b \in A^{\mathcal{I}(w)}$ such that $(a,b) \in r^{\mathcal{I}(w)}$. By statement above, $A \in Q(b,w)$. By definition of the interpretation, $(a,b) \in R(r,w)$. By rule, **R10**, $X \in Q(a,w)$. Therefore, $a \in X^{\mathcal{I}(w)}$.

Now, it remains to prove that $\mathcal{I}, \varepsilon_{\mathcal{A}^{\mathcal{O}}} \models \mathcal{A}^{\mathcal{O}}$.

- $A(a)$. By definition of the initial quasimodel, we have $A \in Q(a, \varepsilon_{\mathcal{A}^{\mathcal{O}}})$. By statement above, $a \in A^{\mathcal{I}(\varepsilon_{\mathcal{A}^{\mathcal{O}}})}$.
- $r(a,b)$ By construction of the model, $(a,b) \in r^{\mathcal{I}(\varepsilon(\mathcal{A}^{\mathcal{O}}))}$

($\Leftarrow$)

We prove this direction by using several invariants that hold during the completion algorithm.

Next we introduce the invariants for the point or the "point" $Q((A,w),v)$ with $C = \bigsqcap_{X \in Q((A,w),v)} X$. We make a case distinction according to the type of w, v world.

1. $w \in \{\langle \mathbf{A} \rangle B_2, \varepsilon_{\langle \mathbf{A} \rangle B_2}, [\mathbf{A}], \varepsilon_{[\mathbf{A}]}\}$ and $V \in \{[\mathbf{B}], \varepsilon_{[\mathbf{B}]}\}$. Then, $\mathcal{K} \models \langle \mathbf{A} \rangle A \sqsubseteq [\mathbf{B}]C$.
2. $w \in \{\langle \mathbf{A} \rangle \mathbf{B_2}, \varepsilon_{\langle \mathbf{A} \rangle \mathbf{B_2}}, [\mathbf{A}], \varepsilon_{[\mathbf{A}]}\}$ and $v \in \{\langle \mathbf{B} \rangle \mathbf{B_1}, \varepsilon_{\langle \mathbf{B} \rangle \mathbf{B_1}}\}$ such that $B_1 \notin C$. Then, $\mathcal{K} \models \langle \mathbf{A} \rangle A \sqsubseteq [\mathbf{B}]C$.
3. $w \in \{\langle \mathbf{A} \rangle \mathbf{B_2}, \varepsilon_{\langle \mathbf{A} \rangle \mathbf{B_2}}, [\mathbf{A}], \varepsilon_{[\mathbf{A}]}\}$ and $v \in \{\langle \mathbf{B} \rangle \mathbf{B_1}, \varepsilon_{\langle \mathbf{B} \rangle \mathbf{B_1}}\}$ such that $B_1 \in C$. Then, $\mathcal{K} \models \langle \mathbf{A} \rangle A \sqsubseteq \langle \mathbf{B} \rangle C$.
4. $w = v$ Then, $K \models A \sqsubseteq C$
5. $A \in Q(a, \varepsilon_{\mathcal{A}^{\mathcal{O}}})$, $a \in \mathsf{Ind}(\mathcal{A}^{\mathcal{O}})$. Then, $K \models A(a)$.
6. $A \in Q(a,v)$, v an $\mathbf{A}$-world, $a \in \mathsf{Ind}(\mathcal{A}^{\mathcal{O}})$. Then, $K \models \langle \mathbf{A} \rangle A(a)$.
7. $A \in Q(a,v)$, $v \in \{[\mathbf{A}], \varepsilon_{[\mathbf{A}]}\}$ $a \in \mathsf{Ind}(\mathcal{A}^{\mathcal{O}})$. Then, $K \models [\mathbf{A}]A(a)$.
8. An invariant for the structure R, $(a,b) \in R(r,v)$ implies $\exists X, \exists A.(X \in Q(a,v)) \wedge X \sqsubseteq \exists r.A \in \mathcal{T}^{\mathcal{O}} \wedge b = (A,v)$.

It is not hard to show by induction on the number of rule applications that the invariants hold. It is clear that the initial quasimodel satisfies these invariants, and they are preserved by each rule application. Then, Invariant 5 yields the desired result, i.e., for all $A_0 \in N_C$ and $a_0 \in \mathsf{Ind}(\mathcal{A}^{\mathcal{O}})$, $A_0 \in Q(a, \varepsilon_{\mathcal{A}^{\mathcal{O}}})$ implies $K \models A_0(a)$. ❑

It is readily checked that the cardinality of BC^{O} is linear on the size of $\mathcal{O}$. Since no rule removes elements of $Q(a,w)$ for some $a \in \Omega, w \in W$ or $R(r)$ for some $r \in N_R^{\mathcal{O}}$ the total number of rule applications is polynomial.

Theorem 9 *Instance checking in $AL^{\mathcal{EL}}_{\mathcal{EL}}$ can be decided in* PTIME

Note that the same algorithm can be used to decide subsumption w.r.t. global TBoxes. In this case the ABox is empty.

Corollary 1 *Given an object TBox $\mathcal{T}^{\mathcal{O}}$, and $A, B \in N_C$. $A \sqsubseteq_{\mathcal{T}^{\mathcal{O}}} B$ if and only if for each $a \in \Omega$, $w \in W$, $A \in Q(a,w)$ implies $B \in Q(a,w)$*

Corollary 2 *Subsumption w.r.t. to global TBoxes in $AL^{\mathcal{EL}}_{\mathcal{EL}}$ can be decided in* PTIME

Now, to check satisfiability in $AL^{\mathcal{EL}}_{\mathcal{EL}}$ we also need to check axioms $\mathbf{C} : \varphi$.

First note that, we can introduce a fresh concept name $\mathbf{A}'$ and include the axioms $\mathbf{A} \sqsubseteq \mathbf{C}$ and $\mathbf{C} \sqsubseteq \mathbf{A}$. Now, we need to check $\mathbf{A}' : \varphi$.

We show how to handle each different case:

- $\varphi = A \sqsubseteq B$ We use the subsumption algorithm restricting to check whether $A \in Q(a, w)$ implies $B \in Q(a, w)$ for the w that are $\mathbf{A}'$-worlds.
- $\varphi = B(a)$ We only need to check if $B \in Q(a, [\mathbf{A}'])$ or $B \in Q(a, \varepsilon_{[\mathbf{A}']})$.
- $\varphi = r(a, b)$ We only need to check if $(a, b) \in R(r, w)$ for the w that are $\mathbf{A}'$-worlds.

Theorem 10 *Satisfiability in $AL^{\mathcal{EL}}_{\mathcal{EL}}$ is in* PTIME.

NEXPTIME/EXPTIME upper bounds

In this section we derive some NEXPTIME and EXPTIME upper bounds which transfer directly to all decision problems of the same complexity discussed in the paper. In fact, the results obtained here apply to more expressive logics. Namely, we prove 1) the NEXPTIME upper bound for $AL^{\mathcal{L}}_{\mathcal{SHIO}}$ with object axioms of the form $\boldsymbol{C} : \varphi$, where φ is any boolean combination of DL formulas, possibly involving context operators, and where $\mathcal{L} \in \{\mathcal{SHIO}, \mathcal{EL}^{++}\}$; 2) the EXPTIME upper bound for $AL^{\mathcal{EL}^{++}}_{\mathcal{SHI}}$. Notably, the logic $\mathcal{EL}^{++}$ properly subsumes $\mathcal{EL}$ (Baader, Brandt, and Lutz 2008), $\mathcal{SHI}$ subsumes $\mathcal{ALC}$ and $\mathcal{SHIO}$ subsumes $\mathcal{ALCO}$ (Tobies 2001).

The decision procedures devised here are essentially variants of the type-based techniques, commonly used in proving the complexity results for the satisfiability problem in various modal logics and their combinations (Kurucz et al. 2003). First, we introduce a number of notational conventions and auxiliary results that should ease the layout of the target proofs. Whenever necessary we distinguish between the languages under consideration.

Consider an AIS $\mathcal{K} = (\mathcal{C}, \mathcal{O})$ in $AL^{\mathcal{M}}_{\mathcal{L}}$. We use the following notation to mark the sets of symbols of particular type occurring in $\mathcal{K}$:

- $con_c(\mathcal{K})$: the set of all metalanguage concepts, closed under negation ($\mathcal{M} = \mathcal{SHIO}$),
- $con_c(\mathcal{K})$: the set of all metalanguage concepts ($\mathcal{M} = \mathcal{EL}^{++}$),
- $con^{op}_c(\mathcal{K}) \subseteq con_c(\mathcal{K})$: the set of all metalanguage concepts occurring inside the context operators,
- $con_o(\mathcal{K})$: the set of all object language concepts, closed under negation,
- $rol_c(\mathcal{K})$: the set of all metalanguage roles,
- $rol^+_c(\mathcal{K}) \subseteq rol_c(\mathcal{K})$: the set of all transitive metalanguage roles ($\mathcal{M} = \mathcal{SHIO}$),
- $rol_o(\mathcal{K})$: the set of all object roles,
- $rol^+_o(\mathcal{K}) \subseteq rol_o(\mathcal{K})$: the set of all transitive object roles,
- $obj_o(\mathcal{K})$: the set of object individual names,
- $sub_o(\mathcal{K})$: the set of all object (sub)formulas, closed under negation.

By $\cdot^-$ we denote the inverse constructor for roles and assume that $(r^-)^- = r$ (resp. $(\boldsymbol{r}^-)^- = \boldsymbol{r}$). Let f be a set of $\mathcal{SHIO}$ formulas. Then by $\sqsubseteq^*_f$ we denote the reflexive-transitive closure of $\sqsubseteq$ on $\{r \sqsubseteq s, s^- \sqsubseteq r^- \mid r \sqsubseteq s \in f\}$ (resp. $\{\boldsymbol{r} \sqsubseteq \boldsymbol{s}, \boldsymbol{s}^- \sqsubseteq \boldsymbol{r}^- \mid \boldsymbol{r} \sqsubseteq \boldsymbol{s} \in f\}$). Without loss of generality we assume that neither $[\cdot]$, $\forall$ nor $\sqcup$ occur in $\mathcal{K}$. Further, in order to reduce the syntactic load in the considered cases, whenever possible we apply the following replacements of all the respective formulas with their equivalents:

$$
\begin{array}{llll}
a : C & \Rightarrow & \{a\} \sqsubseteq C, & (\mathcal{L} = \mathcal{SHIO}) \\
r(a, b) & \Rightarrow & \{a\} \sqsubseteq \exists r.\{b\}, & (\mathcal{L} = \mathcal{SHIO}) \\
\boldsymbol{a} : \boldsymbol{C} & \Rightarrow & \{\boldsymbol{a}\} \sqsubseteq \boldsymbol{C}, & \\
\boldsymbol{r}(\boldsymbol{a}, \boldsymbol{b}) & \Rightarrow & \{\boldsymbol{a}\} \sqsubseteq \exists \boldsymbol{r}.\{\boldsymbol{b}\}, & \\
\text{dom}(\boldsymbol{r}) \sqsubseteq \boldsymbol{C} & \Rightarrow & \exists \boldsymbol{r}.\top \sqsubseteq \boldsymbol{C}, & (\mathcal{M} = \mathcal{EL}^{++})
\end{array}
$$

An *object type* for $\mathcal{K}$ is a subset $t_o \subseteq con_o(\mathcal{K})$, where:

- $\neg\top \notin t_o$ and $\bot \notin t_o$,
- $C \in t_o$ iff $\neg C \notin t_o$, for all $C \in con_o(\mathcal{K})$,
- $C \sqcap D \in t_o$ iff $\{C, D\} \subseteq t_o$, for all $C \sqcap D \in con_o(\mathcal{K})$,
- $\{\exists s.C \mid \exists r.C \in t_c\} \subseteq t_c$, for every $s \in rol_c(\mathcal{K})$ such that $r \sqsubseteq^*_{\mathcal{C}} s$,
- $\{\neg\exists s.C \mid \neg\exists r.C \in t_c\} \subseteq t_c$, for every $\boldsymbol{s} \in rol_c(\mathcal{K})$ such that $s \sqsubseteq^*_{\mathcal{C}} r$.

The set of all object types for $\mathcal{K}$ is denoted by Π. An *object formula type* for $\mathcal{K}$ is a subset $f \subseteq sub_o(\mathcal{K})$, where:

- $\varphi \in f$ iff $\neg\varphi \notin f$, for all $\varphi \in sub_o(\mathcal{K})$,
- $\varphi \wedge \psi \in f$ iff $\{\varphi, \psi\} \subseteq f$, for all $\varphi \wedge \psi \in sub_o(\mathcal{K})$,

The set of all object formula types for $\mathcal{K}$ is denoted by Φ. A *context type* for $\mathcal{K}$ is a subset $t_c \subseteq con_c(\mathcal{K})$, where:

- $\neg\top \notin t_c$ and $\bot \notin t_c$,
- $\boldsymbol{C} \in t_c$ iff $\neg\boldsymbol{C} \notin t_c$, for all $\boldsymbol{C} \in con_c(\mathcal{K})$, ($\mathcal{M} = \mathcal{SHIO}$)
- $\boldsymbol{C} \sqcap \boldsymbol{D} \in t_c$ iff $\{\boldsymbol{C}, \boldsymbol{D}\} \subseteq t_c$, for all $\boldsymbol{C} \sqcap \boldsymbol{D} \in con_c(\mathcal{K})$,
- $\{\exists \boldsymbol{s}.\boldsymbol{C} \mid \exists \boldsymbol{r}.\boldsymbol{C} \in t_c\} \subseteq t_c$, for every $\boldsymbol{s} \in rol_c(\mathcal{K})$ such that $\boldsymbol{r} \sqsubseteq^*_{\mathcal{C}} \boldsymbol{s}$,
- $\{\neg\exists \boldsymbol{s}.\boldsymbol{C} \mid \neg\exists \boldsymbol{r}.\boldsymbol{C} \in t_c\} \subseteq t_c$, for every $\boldsymbol{s} \in rol_c(\mathcal{K})$ such that $\boldsymbol{s} \sqsubseteq^*_{\mathcal{C}} \boldsymbol{r}$.

The set of all context types for $\mathcal{K}$ is denoted by Ξ.

The following two definitions introduce the notions of *matching object role-successor* and *matching* **S5**-*successor*, used in the proofs for reconstructing the role relationships and accessibility relations between individuals in the object dimension.

Definition 12 (matching object role-successor) *Let t_o, t'_o be two object types for $\mathcal{K}$. For any $r \in rol_o(\mathcal{K})$, t'_o is a* matching r-successor *for t_o under $f \subseteq$* sub$_o(\mathcal{K})$ iff *the following conditions are satisfied:*

- *$\{\neg C \mid \neg\exists r.C \in t_o\} \subseteq t'_o$ and $\{\neg C \mid \neg\exists r^-.C \in t'_o\} \subseteq t_o$,*
- *if $\{r, r^-\} \cap rol^+_o(\mathcal{K}) \neq \emptyset$ then $\{\neg\exists r.C \in t_o\} \subseteq t'_o$ and $\{\neg\exists r^-.C \in t'_o\} \subseteq t_o$,*
- *t'_o is a matching s-successor for every $s \in rol_o(\mathcal{K})$ such that $r \sqsubseteq^*_f s$.*

Definition 13 (matching S5-successor) *For any object type t_o for $\mathcal{K}$, let $\mathbf{m}(t_o)$ denote the set of all object concepts containing context operators in t_o, i.e.: $\mathbf{m}(t_o) = \{\langle \boldsymbol{C} \rangle D, \neg\langle \boldsymbol{C} \rangle D \in t_o \mid \boldsymbol{C} \in con_c(\mathcal{K}), D \in con_o(\mathcal{K})\}$. Then, two object types t_o, t'_o for $\mathcal{K}$ are* matching **S5**-successors iff $\mathbf{m}(t_o) = \mathbf{m}(t'_o)$.

The analogous definition of matching role-successor applies to metalanguage roles in $\mathcal{M} = \mathcal{SHIO}$:

Definition 14 (matching metalanguage role-successor) *Let t_c, t'_c be two context types for $\mathcal{K}$. For any $\boldsymbol{r} \in rol_c(\mathcal{K})$, t'_c is a* matching $\boldsymbol{r}$-successor *for t_c under $\mathcal{C}$* iff *the following conditions are satisfied:*

- $\{\neg \boldsymbol{C} \mid \neg\exists \boldsymbol{r}.\boldsymbol{C} \in t_c\} \subseteq t'_c$ *and* $\{\neg \boldsymbol{C} \mid \neg\exists \boldsymbol{r}^-.\boldsymbol{C} \in t'_c\} \subseteq t_c$,
- *if* $\{\boldsymbol{r}, \boldsymbol{r}^-\} \cap rol_c^+(\mathcal{K}) \neq \emptyset$ *then* $\{\neg\exists \boldsymbol{r}.\boldsymbol{C} \in t_c\} \subseteq t'_c$ *and* $\{\neg\exists \boldsymbol{r}^-.\boldsymbol{C} \in t'_c\} \subseteq t_c$,
- t'_c *is a matching $\boldsymbol{s}$-successor for every $\boldsymbol{s} \in rol_c(\mathcal{K})$ such that $\boldsymbol{r} \sqsubseteq^*_{\mathcal{C}} \boldsymbol{s}$.*

Definition 15 ($\mathcal{C}$-admissibility) *Let S be a set of context types for $\mathcal{K} = (\mathcal{C}, \mathcal{O})$. We say that S is $\mathcal{C}$-*admissible iff *there exists a model $(S, \cdot^{\mathcal{J}})$ for $\mathcal{C}$, such that for every $t_c \in S$ and $\boldsymbol{C} \in con_c(\mathcal{K})$, $t_c \in \boldsymbol{C}^{\mathcal{J}}$* iff *$\boldsymbol{C} \in t_c$.*

Theorem 11 ($\mathcal{C}$-admissibility) *Let $S_\times$ be a multiset of context types for $\mathcal{K} = (\mathcal{C}, \mathcal{O})$, where $\mathcal{C}$ is a knowledge base in $\mathcal{L} \in \{\mathcal{SHIO}, \mathcal{EL}^{++}\}$, such that:*

- *S is the underlying set of elements of $S_\times$,*
- *for every $\boldsymbol{a} \in obj_c(\mathcal{K})$ and $t_c, t'_c \in S_\times$, if $\{\boldsymbol{a}\} \in t_c \cap t'_c$ then $t_c = t'_c$.*

Then the following two conditions are equivalent:

1. *There exists a model $(S_\times, \cdot^{\mathcal{J}})$ for $\mathcal{C}$, such that for every $t_c \in S_\times$ and $\boldsymbol{C} \in con_c(\mathcal{K})$, $t_c \in \boldsymbol{C}^{\mathcal{J}}$* iff *$\boldsymbol{C} \in t_c$.*
2. *S is $\mathcal{C}$-admissible.*

Proof. Intuitively, since neither of $\mathcal{L} \in \{\mathcal{SHIO}, \mathcal{EL}^{++}\}$ involves cardinality restrictions, it is straightforward to turn a model for $\mathcal{C}$ implied by condition (1) into a model implied by (2) (from Definition 15), and vice versa. This can be done simply by collapsing (resp. duplicating) individuals which realize the same type in the model. Formally, we demonstrate this by establishing a direct correspondence between both type of models. Let $\pi : S_\times \mapsto S$ be a surjective mapping, such that for every $t_c \in S_\times$, $\pi(t_c) = t_c$. Then $(S_\times, \cdot^{\mathcal{J}_\times})$ is a model implied by (1) *iff* $(S, \cdot^{\mathcal{J}})$ is a model implied by (2), provided that for every for every $t_c, t'_c \in S_\times$ the following conditions are satisfied:

- $t_c \in \boldsymbol{C}^{\mathcal{J}_\times}$ *iff* $\pi(t_c) \in \boldsymbol{C}^{\mathcal{J}}$,
- $\langle t_c, t'_c \rangle \in \boldsymbol{r}^{\mathcal{J}_\times}$ *iff* $\langle \pi(t_c), \tau(t'_c) \rangle \in \boldsymbol{r}^{\mathcal{J}}$.

By structural induction over constructs of $\mathcal{L}$ it is easy to find out that the models are bisimilar, and thus satisfy exactly the same formulas from $\mathcal{C}$. ❑

As a consequence of Definition 15 and Theorem 11, satisfiability of $\mathcal{C}$ can be reduced to the problem of finding a $\mathcal{C}$-admissible set of context types. The following theorems provide effectively verifiable, language-specific conditions for deciding whether a given set of context types is $\mathcal{C}$-admissible.

Theorem 12 (Deciding $\mathcal{C}$-admissibility in $\mathcal{SHIO}$) *Let S be a set of context types for $\mathcal{K} = (\mathcal{C}, \mathcal{O})$, where $\mathcal{C}$ is expressed in $\mathcal{SHIO}$. Then, S is $\mathcal{C}$-admissible* iff *the following conditions are satisfied:*

1. *for every $\boldsymbol{C} \sqsubseteq \boldsymbol{D} \in \mathcal{C}$ and $t_c \in S$, if $\boldsymbol{C} \in t_c$ then $\boldsymbol{D} \in t_c$,*
2. *for every $\boldsymbol{a} \in obj_c(\mathcal{K})$ there is a unique $t_c \in S$ such that $\{\boldsymbol{a}\} \in t_c$,*
3. *for every $\exists \boldsymbol{s}.\boldsymbol{C} \in con_c(\mathcal{K})$ and $t_c \in S$ with $\exists \boldsymbol{s}.\boldsymbol{C} \in t_c$, there is $t'_c \in S$, such that $\boldsymbol{C} \in t'_c$ and t'_c is a matching $\boldsymbol{s}$-successor for t_c under $\mathcal{C}$.*

The conditions can be effectively verified in a time at most exponential in the size of $\mathcal{K}$.

Proof. First, we construct a $\mathcal{SHIO}$-model $(S, \cdot^{\mathcal{J}})$ for $\mathcal{C}$ implied by $\mathcal{C}$-admissibility of S, as follows. For every $t_c, t'_c \in S$:

- $\boldsymbol{a}^{\mathcal{J}} = t_c$ *iff* $\{\boldsymbol{a}\} \in t_c$, for every $\boldsymbol{a} \in obj_c(\mathcal{K})$,
- $t_c \in \boldsymbol{C}^{\mathcal{J}}$ *iff* $\boldsymbol{C} \in t_c$, for every $\boldsymbol{C} \in con_c(\mathcal{K})$,
- $\langle t_c, t'_c \rangle \in \boldsymbol{s}^{\mathcal{J}}$ *iff* t'_c is a matching $\boldsymbol{s}$-successor for t_c under $\mathcal{C}$, for every $\boldsymbol{s} \in rol_c(\mathcal{K})$.

($\Leftarrow$) We show that $(S, \cdot^{\mathcal{J}})$ is indeed a model for $\mathcal{C}$. Observe that the respective conditions in the theorem guarantee that:

1. all GCIs are satisfied,
2. all individual names (and so the nominals) are given unique interpretations,
3. all individuals satisfying existential restrictions obtain proper role successors, and moreover, by Def. 14, it is ensured that:
   - role names and their inverses are interpreted as relations which are inverses of each other,
   - transitive roles are interpreted as transitive relations,
   - the role hierarchies entailed by $\mathcal{C}$ are respected,
   - for every $t_c \in S$, and $\boldsymbol{r} \in rol_c(\mathcal{K})$ all concepts of the form $\neg\exists \boldsymbol{r}.\boldsymbol{C} \in t_c$ are satisfied in the model.

The first two points are clear by the construction of the model and the conditions 1 and 2 in the theorem. The third one follows from the construction of the model, definition of context type (DCT) and of matching metalanguage role successor (Def. 14). We proceed by induction. Consider any $t_c, t'_c \in S$, such that t'_c is a matching $\boldsymbol{s}$-successor for t_c under $\mathcal{C}$ for some $\boldsymbol{s} \in rol_c(\mathcal{K})$ at the top level of the role hierarchy. Then $\langle t_c, t'_c \rangle \in \boldsymbol{s}^{\mathcal{J}}$ and, by Def. 14, t_c has to be a matching $\boldsymbol{s}^-$-successor for t'_c under $\mathcal{C}$, and thus $\langle t'_c, t_c \rangle \in (\boldsymbol{s}^-)^{\mathcal{J}}$. In both cases all concepts of the form $\neg\exists \boldsymbol{s}.\boldsymbol{C} \in t_c$ and $\neg\exists \boldsymbol{s}^-.\boldsymbol{C} \in t'_c$ need to be satisfied. Also, by the construction of the model, it is ensured that for all $t_c \in S$ all concepts $\exists \boldsymbol{s}.\boldsymbol{C} \in t_c$ are satisfied as well. Further, suppose $\boldsymbol{s}$ is a transitive role. Then for every t''_c which is a matching $\boldsymbol{s}$-successor for t'_c under $\mathcal{C}$, t''_c has to be also a matching $\boldsymbol{s}$-successor for t_c under $\mathcal{C}$ and so $\langle t_c, t''_c \rangle \in \boldsymbol{s}^{\mathcal{J}}$, which inductively extends over the whole interpretation of $\boldsymbol{s}$, rendering it a transitive relation. In such case, Def. 14 guarantees that the model satisfies all $\neg\exists \boldsymbol{s}.\boldsymbol{C} \in t_c$ and $\neg\exists \boldsymbol{s}^-.\boldsymbol{C} \in t''_c$.

Now, suppose that for some role $\boldsymbol{r}$ there is $\boldsymbol{s} \sqsubseteq^*_{\mathcal{C}} \boldsymbol{r}$ and let t'_c be a matching $\boldsymbol{s}$-successor for t_c under $\mathcal{C}$, for some $t_c, t'_c \in S$. Then by Def. 14, t'_c must be also a matching $\boldsymbol{r}$-successor for t_c under $\mathcal{C}$, and so by the construction of the model $\langle t_c, t'_c \rangle \in \boldsymbol{r}^{\mathcal{J}}$ and $\langle t'_c, t_c \rangle \in (\boldsymbol{r}^-)^{\mathcal{J}}$, which fulfills the semantics of the role inclusion. Finally, suppose $\boldsymbol{s}$ is a transitive role and $\langle t_c, t'_c\rangle, \langle t'_c, t''_c\rangle \in \boldsymbol{s}^{\mathcal{J}}$, for some $t_c, t'_c, t''_c \in S$. Since, as argued above, t''_c must be also a matching $\boldsymbol{s}$-successor for t_c under $\mathcal{C}$, it follows that $\langle t_c, t''_c\rangle \in \boldsymbol{s}^{\mathcal{J}}$. But then, by (DCT), for every concept of the form $\neg\exists \boldsymbol{r}.\boldsymbol{C} \in t_o$, there already is $\neg\exists \boldsymbol{s}.\boldsymbol{C} \in t_o$, and consequently, by transitivity of $\boldsymbol{s}$, also $\neg\exists \boldsymbol{s}.\boldsymbol{C} \in t'_o$. Therefore, it is also the case that t''_c is a matching $\boldsymbol{r}$-successor for t_c under $\mathcal{C}$ and $\langle t_c, t''_c\rangle \in \boldsymbol{r}^{\mathcal{J}}$. Clearly, all concepts of the form $\neg\exists \boldsymbol{r}.\boldsymbol{C} \in t_c$ and $\neg\exists \boldsymbol{r}^-.\boldsymbol{C} \in t''_c$ are satisfied in the model. By induction, the argument carries over to all roles in the hierarchy.

($\Rightarrow$) We demonstrate that $(S, \cdot^{\mathcal{J}})$, constructed as above, satisfies the conditions stated in the theorem. The first two are immediate. For the third one, suppose that for some $\exists \boldsymbol{s}.\boldsymbol{C} \in con_c(\mathcal{K})$ and $t_c \in S$ there is $\exists \boldsymbol{s}.\boldsymbol{C} \in t_c$. Clearly, by the semantics, there must be a $t'_c \in S$, such that $\langle t_c, t'_c\rangle \in \boldsymbol{s}^{\mathcal{J}}$ and $t'_c \in \boldsymbol{C}^{\mathcal{J}}$, and thus with $\boldsymbol{C} \in t'_c$. We show that in such case t'_c is a matching $\boldsymbol{s}$-successor for t_c under $\mathcal{C}$, i.e. that the three conditions in Def. 14 are satisfied. The first one is obvious. For the second one, suppose that $\boldsymbol{s}$ is transitive and some $\neg\exists \boldsymbol{s}.\boldsymbol{D}$ is satisfied in t_c. Then it must be the case that either t'_c has no $\boldsymbol{s}$-successors in the model (in such case $\neg\exists \boldsymbol{s}.\boldsymbol{D}$ is vacuously satisfied in t'_c) or it has some $\boldsymbol{s}$-successors. In the latter case, by transitivity of $\boldsymbol{s}$, such successors have to satisfy all $\boldsymbol{D}$ such that $t_c \in (\neg\exists \boldsymbol{s}.\boldsymbol{D})^{\mathcal{J}}$. It follows that all such $\neg\exists \boldsymbol{s}.\boldsymbol{D}$ have to be satisfied also in t'_c, and so the condition holds. Finally, by induction over the role hierarchy, t'_c is clearly a matching $\boldsymbol{r}$-successor for t_c under $\mathcal{C}$, for all $\boldsymbol{r}$ such that $\boldsymbol{s} \sqsubseteq^*_{\mathcal{C}} \boldsymbol{r}$.

Observe that the size of $\sqsubseteq^*_{\mathcal{C}}$ is at most polynomial in $\ell(\mathcal{K})$, while $|S| \leq 2^{\ell(\mathcal{K})}$ and $|t_c| \leq \ell(\mathcal{K})$ for every $t_c \in S$ (see also the proof of Lemma 2). Thus, deciding the conditions specified in the theorem cannot take more than a polynomial time in the size of S and, exponential in $\ell(\mathcal{K})$. ❑

In order to formulate a similar claim for $\mathcal{M} = \mathcal{EL}^{++}$ we require some additional notation. We write $\mathcal{C} \vdash \boldsymbol{r} \sqsubseteq \boldsymbol{s}$ *iff* $\boldsymbol{r} = \boldsymbol{s}$ or $\mathcal{C}$ contains role inclusions $\boldsymbol{r}_1 \sqsubseteq \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{n-1} \sqsubseteq \boldsymbol{r}_n$ with $\boldsymbol{r}_1 = \boldsymbol{r}$ and $\boldsymbol{r}_n = \boldsymbol{s}$. Further, we write $\mathcal{C} \vdash \mathrm{ran}(\boldsymbol{r}) \sqsubseteq \boldsymbol{C}$ if there is a role name $\boldsymbol{s}$ with $\mathcal{C} \vdash \boldsymbol{r} \sqsubseteq \boldsymbol{s}$ and $\mathrm{ran}(\boldsymbol{s}) \sqsubseteq \boldsymbol{C} \in \mathcal{C}$.

Let $X \subseteq con_c(\mathcal{K})$. Then by $X^{\sqsubseteq}_{\mathcal{C}}$ we denote the closure of X under subsumption in $\mathcal{C}$ w.r.t. $con_c(\mathcal{K})$, i.e.:

- $X \subseteq X^{\sqsubseteq}_{\mathcal{C}}$,
- if $\boldsymbol{C}, \boldsymbol{D} \in X^{\sqsubseteq}_{\mathcal{C}}$ then $\boldsymbol{C} \sqcap \boldsymbol{D} \in X^{\sqsubseteq}_{\mathcal{C}}$, for every $\boldsymbol{C} \sqcap \boldsymbol{D} \in con_c(\mathcal{K})$,
- for every $\boldsymbol{C} \in X^{\sqsubseteq}_{\mathcal{C}}$ and $\boldsymbol{D} \in con_c(\mathcal{K})$, if $\mathcal{C} \models \boldsymbol{C} \sqsubseteq \boldsymbol{D}$ then $\boldsymbol{D} \in X^{\sqsubseteq}_{\mathcal{C}}$.

Since the subsumption problem in $\mathcal{EL}^{++}$ is tractable (Baader, Brandt, and Lutz 2008), it is clear that $X^{\sqsubseteq}_{\mathcal{C}}$ can be computed in a polynomial time. By an abuse of notation we write $\boldsymbol{C}^{\sqsubseteq}_{\mathcal{C}}$, whenever $X = \{\boldsymbol{C}\}$ for any $\boldsymbol{C}$.

Algorithm 1 computes a $\mathcal{C}$-admissible set $S_{\mathcal{C},\Omega}$ of context types for $\mathcal{K}$, provided such set exists at all, and its subset $U_{\mathcal{C},\Omega}$. The subscript $\Omega \subseteq con_c(\mathcal{K})$ denotes an extra set of concepts which must be also satisfied in the model corresponding to $S_{\mathcal{C},\Omega}$. This parameter and the set $U_{\mathcal{C},\Omega}$ are necessary later on, when satisfiability of the whole knowledge base $\mathcal{K}$ is considered. In the special case, for $\Omega = \emptyset$, $S_{\mathcal{C},\Omega}$ corresponds exactly to the canonical model of $\mathcal{C}$. That is, for every $\mathcal{EL}^{++}$ concept $\boldsymbol{C}$, there exists $Y \in S_{\mathcal{C},\Omega}$ with $\boldsymbol{C} \in Y$ *iff* $\boldsymbol{C}$ is satisfied in every model of $\mathcal{C}$, provided such models exist. This dramatically reduces the search space for $\mathcal{C}$-admissible sets of context types for $\mathcal{M} = \mathcal{EL}^{++}$, which paves the way to the EXPTIME upper bound for $AL^{\mathcal{EL}^{++}}_{\mathcal{SHI}}$.

Algorithm 1 Computation of a set of context types for $\mathcal{K}$ in $\mathcal{EL}^{++}$.

Input: (context) ontology $\mathcal{C}$, a set of concepts $\Omega \subseteq con_c(\mathcal{K})$
Output: two sets of context types $S_{\mathcal{C},\Omega}$ and $U_{\mathcal{C},\Omega}$

$S := \emptyset$, $U := \emptyset$, *Marked* $:= \emptyset$
if $obj_c(\mathcal{K}) = \emptyset$ and $\Omega = \emptyset$ **then**
  add $\top^{\sqsubseteq}_{\mathcal{C}}$ to S
else
  for all $\boldsymbol{a} \in obj_c(\mathcal{K})$ **do**
    add $\{\boldsymbol{a}\}^{\sqsubseteq}_{\mathcal{C}}$ to S
  end for
  for all $\boldsymbol{C} \in \Omega$ **do**
    add $\boldsymbol{C}^{\sqsubseteq}_{\mathcal{C}}$ to S and to U
  end for
end if
while applicable **do**
  for all $Y \in S$ and $\exists \boldsymbol{s}.\boldsymbol{C} \in Y$ **do**
    if $\exists \boldsymbol{s}.\boldsymbol{C} \notin$ *Marked* **then**
      add $(\{\boldsymbol{C}\} \cup \{\boldsymbol{D} \mid \mathcal{C} \vdash \mathrm{ran}(\boldsymbol{s}) \sqsubseteq \boldsymbol{D}\})^{\sqsubseteq}_{\mathcal{C}}$ to S and add $\exists \boldsymbol{s}.\boldsymbol{C}$ to *Marked*
    end if
  end for
  for all $\boldsymbol{a} \in obj_c(\mathcal{K})$ and $Y, Z \in S$ **do**
    if $\{\boldsymbol{a}\} \in Y \cap Z$ **then**
      replace Y and Z in S with $(Y \cup Z)^{\sqsubseteq}_{\mathcal{C}}$
    end if
    if $Y \in U$ **or** $Z \in U$ **then**
      remove Y, Z from U and add $(Y \cup Z)^{\sqsubseteq}_{\mathcal{C}}$ to U
    end if
  end for
end while
if $\bot \notin Y$ for every $Y \in S$ **then**
  $S_{\mathcal{C},\Omega} := S$ and $U_{\mathcal{C},\Omega}$
else
  $S_{\mathcal{C},\Omega} := \emptyset$ and $U_{\mathcal{C},\Omega} = \emptyset$
end if

Theorem 13 (Semi-deciding $\mathcal{C}$-admissibility in $\mathcal{EL}^{++}$) *Let $\mathcal{K} = (\mathcal{C}, \mathcal{O})$ be a knowledge base, where $\mathcal{C}$ is expressed in $\mathcal{EL}^{++}$, and S be a set of context types for $\mathcal{K}$. Then S is $\mathcal{C}$-admissible if the following conditions are satisfied:*

1. *$S = S_{\mathcal{C},\Omega}$, where $S_{\mathcal{C},\Omega}$ is computed by Algorithm 1 for some $\Omega \subseteq \mathrm{con}_c(\mathcal{K})$,*
2. *$S_{\mathcal{C},\Omega}$ is non-empty.*

For a fixed Ω, the algorithm runs in a time polynomial in the size of $\mathcal{K}$.

Proof. Suppose $S_{\mathcal{C},\Omega}$ is non-empty. Then clearly, every element of $S_{\mathcal{C},\Omega}$ is a context type for $\mathcal{K}$. We construct an $\mathcal{EL}^{++}$-model $(S_{\mathcal{C},\Omega}, \cdot^{\mathcal{J}})$ for $\mathcal{C}$, implied by $\mathcal{C}$-admissibility of $S_{\mathcal{C},\Omega}$, as follows. For every $t_c, t'_c \in S$ fix:

- $\boldsymbol{a}^{\mathcal{J}} = t_c$ *iff* $\{\boldsymbol{a}\} \in t_c$, for every $\boldsymbol{a} \in obj_c(\mathcal{K})$,
- $t_c \in \boldsymbol{C}^{\mathcal{J}}$ *iff* $\boldsymbol{C} \in t_c$, for every $\boldsymbol{C} \in con_c(\mathcal{K})$,
- $\langle t_c, t'_c\rangle \in \boldsymbol{s}^{\mathcal{J}}$ *iff* $\{\boldsymbol{D} \mid \mathcal{C} \vdash \text{ran}(\boldsymbol{s}) \sqsubseteq \boldsymbol{D}\} \subseteq t'_c$.

Extend $\cdot^{\mathcal{J}}$ inductively over all roles by ensuring that for every $t_c, t'_c, t''_c \in S$:

- if $\langle t_c, t'_c\rangle \in \boldsymbol{r}$ and $\langle t'_c, t''_c\rangle \in \boldsymbol{s}$ then $\langle t_c, t''_c\rangle \in \boldsymbol{r} \circ \boldsymbol{s}$, for every $\boldsymbol{r} \circ \boldsymbol{s} \in rol_c(\mathcal{K})$,
- if $\langle t_c, t'_c\rangle \in (\boldsymbol{r}_1 \circ \ldots \circ \boldsymbol{r}_n)^{\mathcal{J}}$ then $\langle t_c, t'_c\rangle \in \boldsymbol{s}^{\mathcal{J}}$, for every $\boldsymbol{r}_1 \circ \ldots \circ \boldsymbol{r}_n \sqsubseteq \boldsymbol{s} \in \mathcal{C}$.

It is not hard to verify, that $(S_{\mathcal{C},\Omega}, \cdot^{\mathcal{J}})$ defined in this way is indeed a model for $\mathcal{C}$. In particular, by the construction of the model and definition of the algorithm, it is guaranteed that all GCIs are satisfied (by closure of the generated types under $\sqsubseteq$ in $\mathcal{C}$), individual names obtain unique interpretations (by merging types containing the same nominals) and that all individuals satisfying existential restrictions obtain proper successors. The only issue requiring more attention is the satisfaction of role ranges. Clearly, the ranges of roles included in existential restrictions are respected by the definition of the algorithm. For the inductive extension of $\cdot^{\mathcal{J}}$ over the remaining roles, we resort to the syntactic restriction permitting tractable reasoning in $\mathcal{EL}^{++}$, which has been identified in (Baader, Brandt, and Lutz 2008). The restriction states:

If $\boldsymbol{r}_1 \circ \ldots \circ \boldsymbol{r}_n \sqsubseteq \boldsymbol{s} \in \mathcal{C}$ with $n \geq 1$ and $\mathcal{C} \vdash \text{ran}(\boldsymbol{s}) \sqsubseteq \boldsymbol{D}$, then $\mathcal{C} \vdash \text{ran}(\boldsymbol{r}_n) \sqsubseteq \boldsymbol{D}$.

It immediately follows, that whenever $\langle t_c, t'_c\rangle \in \boldsymbol{s}^{\mathcal{J}}$ is included in $\cdot^{\mathcal{J}}$, for any $t_c, t'_c \in S_{\mathcal{C}}$, because of some role inclusion $\boldsymbol{r}_1 \circ \ldots \circ \boldsymbol{r}_n \sqsubseteq \boldsymbol{s} \in \mathcal{C}$, it is the case, that $\{\boldsymbol{D} \mid \mathcal{C} \vdash \text{ran}(\boldsymbol{s}) \sqsubseteq \boldsymbol{D}\} \subseteq t'_c$ if $\{\boldsymbol{D} \mid \mathcal{C} \vdash \text{ran}(\boldsymbol{r}_n) \sqsubseteq \boldsymbol{D}\} \subseteq t'_c$. But then, by induction, it is easy to see that the appropriate range restrictions are carried over from the roles occurring in some existential restrictions, which are sufficiently handled by the algorithm.

Observe, that the number of distinct concepts of the form $\exists \boldsymbol{r}.\boldsymbol{C}$ occurring in $S_{\mathcal{C},\Omega}$ is linearly bounded by the size of $\mathcal{C}$, and thus, computing $S_{\mathcal{C},\Omega}$ must terminate in a time polynomial in the size of $\mathcal{K}$. ❑

Theorem 14 (Satisfiability as $\mathcal{C}$-admissibility in $\mathcal{EL}^{++}$) *Let $\mathcal{K} = (\mathcal{C}, \mathcal{O})$ be a knowledge base, where $\mathcal{C}$ is expressed in $\mathcal{EL}^{++}$, and let $\Omega \subseteq con_c(\mathcal{K})$. Then $\mathcal{C}$ is satisfied in some model which also satisfies every $\boldsymbol{C} \in \Omega$* iff *$S_{\mathcal{C},\Omega}$, computed by Algorithm 1, is non-empty.*

Proof. ($\Rightarrow$) Let $(\mathfrak{C}, \cdot^{\mathcal{J}})$ be a model of $\mathcal{C}$ satisfying every $\boldsymbol{C} \in \Omega$. Define a mapping $\tau : \mathfrak{C} \mapsto \Xi$, such that for every $c \in \mathfrak{C}$ and $\boldsymbol{C} \in con_c(\mathcal{K})$, $\boldsymbol{C} \in \tau(c)$ *iff* $c \in \boldsymbol{C}^{\mathcal{J}}$. Now, let $S = \{\tau(c) \mid c \in \mathfrak{C}\}$. Observe, that the algorithm generating the context types from $S_{\mathcal{C},\Omega}$ is deterministic and enforces only the necessary consequences of $\mathcal{C}$ and the semantics of $\mathcal{EL}^{++}$. Hence, for every $t_c \in S_{\mathcal{C},\Omega}$ there must be some $t'_c \in S$, such that $t_c \subseteq t'_c$. Obviously, no $t_c \in S$ contains $\bot$. Thus, whenever $\mathcal{C}$ has a model satisfying all concepts from Ω, there has to exist a non-empty output from the algorithm.

($\Leftarrow$) Suppose $S_{\mathcal{C},\Omega}$ is non-empty. By the construction of $S_{\mathcal{C},\Omega}$, for every $\boldsymbol{C} \in \Omega$ there exists a type $t_c \in S_{\mathcal{C},\Omega}$, such that $\boldsymbol{C} \in t_c$. Thus, by Algorithm 1, Theorem 13 and Definition 15 there has to exist a model of $\mathcal{C}$ satisfying every $\boldsymbol{C} \in \Omega$. ❑

A *context structure* $\langle S, \mathfrak{S}\rangle$ for $\mathcal{K}$ is a pair consisting of a set $S \subseteq \Xi$ of context types for $\mathcal{K}$ and a non-empty set $\mathfrak{S}$ of tuples of the form $\langle t_c, f, \nu\rangle$, where $t_c \in S$, $f \subseteq sub_o(\mathcal{K})$ is an object formula type for $\mathcal{K}$, $\nu : obj_o(\mathcal{K}) \mapsto \Pi$ assigns unique object types to individual object names, and such that the following conditions are satisfied:

(CS1) for every $\boldsymbol{a} \in obj_c(\mathcal{K})$ there is a unique $t_c \in S$ such that $\{\boldsymbol{a}\} \in t_c$, and at most one $\langle t_c, f, \nu\rangle \in \mathfrak{S}$. If $\boldsymbol{a} : \varphi \in \mathcal{O}$, for any φ, then such $\langle t_c, f, \nu\rangle \in \mathfrak{S}$ must exist,

(CS2) S is $\mathcal{C}$-admissible,

(CS3) for every $\langle t_c, f, \nu\rangle \in \mathfrak{S}$ and $\boldsymbol{C} : \varphi \in \mathcal{O}$, if $\boldsymbol{C} \in t_c$ then $\varphi \in f$,

In the case of languages with full object formulas, the following requirement has to be also satisfied:

(CS4) for every $\langle t_c, f, \nu\rangle \in \mathfrak{S}$ it holds that:

- if $\boldsymbol{C} \in t_c$ and $\varphi \in f$ then $\langle \boldsymbol{C}\rangle\varphi \in f$, for every $\langle \boldsymbol{C}\rangle\varphi \in sub_o(\mathcal{K})$,
- for every $\langle \boldsymbol{C}\rangle\varphi \in f$ there is $\langle t'_c, f', \nu'\rangle \in \mathfrak{S}$, such that $\boldsymbol{C} \in t'_c$ and $\varphi \in f'$,
- for every $\neg\langle \boldsymbol{C}\rangle\neg\varphi \in f$ and $\langle t'_c, f', \nu'\rangle \in \mathfrak{S}$, if $\boldsymbol{C} \in t'_c$ then $\varphi \in f'$.

Intuitively, a context structure contains all the pieces necessary for reconstructing a single $AL^{\mathcal{M}}_{\mathcal{L}}$-interpretation. However, not all such interpretations might correspond to a genuine $AL^{\mathcal{M}}_{\mathcal{L}}$-model. To filter out the proper ones, some additional conditions need to be imposed. These are introduced in the notion of quasimodel candidate, and further, in the notions of quasimodel associated with specific logics under consideration.

Definition 16 (Quasimodel candidate) *A* quasimodel candidate $\mathfrak{Q}^S_{\mathfrak{S}}$ *for $\mathcal{K}$, where $\langle S, \mathfrak{S}\rangle$ is a context structure for $\mathcal{K}$, is a set of pairs $\langle k, t_o\rangle$, such that $k \in \mathfrak{S}$, $t_o \in \Pi$, satisfying the following conditions:*

(QC1) for every $k \in \mathfrak{S}$, with $k = \langle t_c, f, \nu\rangle$, and $a \in obj_o(\mathcal{K})$, $\langle k, \nu(a)\rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$.

For every $\langle k, t_o\rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$, with $k = \langle t_c, f, \nu\rangle$:

(QC2) if $\neg\langle \boldsymbol{C}\rangle\neg D \in t_o$ and $\boldsymbol{C} \in t_c$ then $D \in t_o$, for all $\neg\langle \boldsymbol{C}\rangle\neg D \in con_o(\mathcal{K})$,

(QC3) if $\boldsymbol{C} \in t_c$ and $D \in t_o$ then $\langle \boldsymbol{C}\rangle D \in t_o$, for all $\langle \boldsymbol{C}\rangle D \in con_o(\mathcal{K})$,

(QC4) for every $k' \in \mathfrak{S}$, there is some $\langle k', t'_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$ such that t_o, t'_o are matching **S5**-successors,

(QC5) if $\langle \boldsymbol{C} \rangle D \in t_o$ then there is $\langle k', t'_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$, such that $k' = \langle t'_c, f', \nu' \rangle$, $\boldsymbol{C} \in t'_c$, $D \in t'_o$ and t_o, t'_o are matching **S5**-successors. Moreover, if $t_o \neq t'_o$ then $k' \neq k$,

(QC6) for every $\exists r.C \in t_o$ there is $\langle k, t'_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$, such that $C \in t'_o$ and t'_o is a matching r-successor for t_o under f.

Lemma 2 (Quasimodel candidate space bound) *The size of a quasimodel candidate is exponentially bounded in the size of $\mathcal{K}$.*

Proof. By $\ell(\mathcal{K})$ we denote the size of $\mathcal{K}$, measured in the number of symbols used, and by $|X|$ — the number of elements of set X. We observe that the following (very liberally estimated) inequalities hold:

$$|con_c(\mathcal{K})| \leq 2\ell(\mathcal{K}), \quad |con_o(\mathcal{K})| \leq 2\ell(\mathcal{K}),$$
$$|sub_o(\mathcal{K})| \leq 2\ell(\mathcal{K}), \quad |obj_o(\mathcal{K})| \leq \ell(\mathcal{K})$$
$$|\Pi| \leq 2^{con_o(\mathcal{K})} \leq 2^{2\ell(\mathcal{K})}, \quad |\Xi| \leq 2^{con_c(\mathcal{K})} \leq 2^{2\ell(\mathcal{K})},$$
$$|\Phi| \leq 2^{sub_o(\mathcal{K})} \leq 2^{2\ell(\mathcal{K})}$$
$$|\Pi^{|obj_o(\mathcal{K})|}| = |\Pi|^{|obj_o(\mathcal{K})|} \leq 2^{2\ell(\mathcal{K})^2}$$
$$|\mathfrak{S}| \leq |\Xi| \cdot |\Phi| \cdot |\Pi^{|obj_o(\mathcal{K})|}| \leq 2^{2\ell(\mathcal{K})^2 + 4\ell(\mathcal{K})}$$
$$|\mathfrak{Q}^S_{\mathfrak{S}}| \leq |\mathfrak{S}| \cdot |\Pi| \leq 2^{2\ell(\mathcal{K})^2 + 6\ell(\mathcal{K})}$$

Since the maximum size of a single tuple in a quasimodel candidate is polynomial in $\ell(\mathcal{K})$ therefore the maximum size of a quasimodel is never greater than $2^{p(\ell(\mathcal{K}))}$, where p is a fixed polynomial. ❑

The structure of the proofs:

1. definition of a quasimodel
2. the quasimodel lemma
3. an algorithm with a specified time resource bound

Theorem 15 *Knowledge base satisfiability in $AL^{\mathcal{L}}_{\mathcal{SHIO}}$, for $\mathcal{L} \in \{\mathcal{SHIO}, \mathcal{EL}^{++}\}$, with full object formulas and only local roles is in* NExpTime.

Proof. We begin by defining the relevant notion of quasimodel.

Definition 17 (Quasimodel) *A quasimodel candidate $\mathfrak{Q}^S_{\mathfrak{S}}$ for $\mathcal{K}$, where $\mathcal{K}$ is a knowledge base in $AL^{\mathcal{L}}_{\mathcal{SHIO}}$, for $\mathcal{L} \in \{\mathcal{SHIO}, \mathcal{EL}^{++}\}$, with full object formulas and only local roles, is called a* quasimodel *for $\mathcal{K}$* iff *the following conditions are satisfied:*

(QM1) for every $\langle k, t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$, with $k = \langle t_c, f, \nu \rangle$, and $a \in obj_o(\mathcal{K})$, $\{a\} \in t_o$ *iff* $t_o = \nu(a)$.

For every $k \in \mathfrak{S}$ with $k = \langle t_c, f, \nu \rangle$:

(QM2) $C \sqsubseteq D \in f$ *iff* for every $\langle k, t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$ if $C \in t_o$ then $D \in t_o$, for every $C \sqsubseteq D \in sub_o(\mathcal{K})$.

(QM3) $\neg(C \sqsubseteq D) \in f$ *iff* there is $\langle k, t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$ such that $\{C, \neg D\} \subseteq t_o$, for every $\neg(C \sqsubseteq D) \in sub_o(\mathcal{K})$,

(QM4) $\neg(r \sqsubseteq s) \in f$ *iff* $r \not\sqsubseteq^*_f s$ and there is $\langle k, t_o \rangle, \langle k, t'_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$ such that t'_o is a matching r-successor for t_o under f, for every $\neg(r \sqsubseteq s) \in sub_o(\mathcal{K})$.

Next, we show the correspondence between quasimodels and models.

Lemma 3 (Quasimodel lemma) *A knowledge base $\mathcal{K}$ has an $AL^{\mathcal{L}}_{\mathcal{SHIO}}$-model, for $\mathcal{L} \in \{\mathcal{SHIO}, \mathcal{EL}^{++}\}$,* iff *there is a quasimodel for $\mathcal{K}$.*

Proof. ($\Rightarrow$) Let $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$ be a model for $\mathcal{K}$. First, we fix a mapping τ from Θ, $\mathfrak{C}$ and Δ to the corresponding context/object types. For every $c \in \Theta$ set $\tau(c) = t_c$, such that:

- $\boldsymbol{C} \in t_c$ *iff* $c \in \boldsymbol{C}^{\mathcal{J}}$, for every $\boldsymbol{C} \in con_c(\mathcal{K})$,

For every $\langle c, d \rangle \in \mathfrak{C} \times \Delta$ set $\tau(\langle c, d \rangle) = t_o$, such that:

- $C \in t_o$ *iff* $d \in C^{\mathcal{I}(c)}$, for every $C \in con_o(\mathcal{K})$.

Further, for every $c \in \mathfrak{C}$ set $\tau(c) = \langle t_c, f, \nu \rangle$, such that the following correspondences hold:

- $\boldsymbol{C} \in t_c$ *iff* $c \in \boldsymbol{C}^{\mathcal{J}}$, for every $\boldsymbol{C} \in con_c(\mathcal{K})$,
- $\varphi \in f$ *iff* $\mathfrak{M}, c \models \varphi$, for every $\varphi \in sub_o(\mathcal{K})$,
- $\nu(a) = \tau(\langle c, a^{\mathcal{I}(c)} \rangle)$ for every $a \in obj_o(\mathcal{K})$.

Fix $S = \{\tau(c) \mid c \in \Theta\}$ and $\mathfrak{S} = \{\tau(c) \mid c \in \mathfrak{C}\}$. By Theorem 11, $\langle S, \mathfrak{S} \rangle$ is a proper context structure satisfying all conditions (**CS**x). Next, define the quasimodel $\mathfrak{Q}^S_{\mathfrak{S}} = \{\langle \tau(c), \tau(\langle c, d \rangle) \rangle \mid \langle c, d \rangle \in \mathfrak{C} \times \Delta, \}$. It is easy to see, that all conditions (**QC1**)-(**QC5**) and (**QM1**)-(**QM3**) have to be satisfied. Since the notion of matching role successor is exactly the same for the object and metalanguage roles in $\mathcal{SHIO}$, the satisfaction of (**QC6**) can be demonstrated by the same argument as used in the proof of Theorem 12. Finally, for (**QM4**), observe that whenever $c \models \neg(r \sqsubseteq s)$ holds in $c \in \mathfrak{C}$, then there have to be $d, d' \in \Delta$, such that $\langle d, d' \rangle \in r^{\mathcal{J}}$ and $\langle d, d' \rangle \notin s^{\mathcal{J}}$ and so that the condition (**QM4**) has to be satisfied in $\mathfrak{Q}^S_{\mathfrak{S}}$ defined as above.

($\Leftarrow$) Let $\mathfrak{Q}^S_{\mathfrak{S}}$ be a quasimodel for $\mathcal{K}$. In the following steps we define a model $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$ for $\mathcal{K}$. The interpretation of the context dimension follows immediately from the definition of the context structure. Since $\langle S, \mathfrak{S} \rangle$ is a context structure, then S must be a $\mathcal{C}$-admissible set of context types. But then, by Theorem 11, for any multiset $S_\times$ such that $S \subseteq S_\times$, there must be some interpretation function $\cdot^{\mathcal{J}}$ such that $(S_\times, \cdot^{\mathcal{J}})$ is a model for $\mathcal{C}$. Fix $\mathfrak{C} = \{k^{t_o} \mid \langle k, t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}\}$ and $\Theta = \{t^k_c \mid k = \langle t_c, f, \nu \rangle \in \mathfrak{C}\} \cup \{t_c \in S \mid \langle t_c, f, \nu \rangle \notin \mathfrak{C}$ for any $f, \nu\}$. Then clearly, $(\Theta, \cdot^{\mathcal{J}})$ is also a model for $\mathcal{C}$. The tuple $(\Theta, \mathfrak{C}, \cdot^{\mathcal{J}})$ is incorporated into $\mathfrak{M}$.

Now, consider the object dimension. For every $k \in \mathfrak{C}$, we fix the set of object types $T_k = \{t_o \mid \langle k, t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}\}$ realized in this context. A *run* ρ through $\mathfrak{Q}^S_{\mathfrak{S}}$ is a function which to every $k \in \mathfrak{C}$ assigns a single type from T_k, such that:

- for every $k, k' \in \mathfrak{C}$ it is the case that $\rho(k), \rho(k')$ are matching **S5**-successors,
- for every $k \in \mathfrak{C}$, if $\langle \boldsymbol{C} \rangle D \in \rho(k)$ then there is $k' \in \mathfrak{C}$, such that $k' = \langle t'_c, f', \nu' \rangle$, $\boldsymbol{C} \in t'_c$ and $D \in \rho(k')'$.

A set $\mathfrak{R}$ of runs through $\mathfrak{Q}^S_{\mathfrak{S}}$ is called *coherent iff* the following conditions are satisfied:

- for every $k \in \mathfrak{C}$ and $t_o \in T_k$, there is a $\rho \in \mathfrak{R}$ such that $\rho(k) = t_o$,
- for every $a \in obj_o(\mathcal{K})$ and $k \in \mathfrak{C}$, with $k = \langle t_o, f, \nu \rangle$, there is a unique $\rho \in \mathfrak{R}$, such that $\rho(k) = \nu(a)$,

Next, we define the interpretation of the object dimension as follows. First, fix the object domain as:

- $\Delta := \mathfrak{R}$

Then, for every $k \in \mathfrak{C}$, with $k = \langle t_c, f, \nu \rangle$, and $\rho, \rho' \in \Delta$ set the interpretation function as:

- $a^{\mathcal{I}(k)} = \rho$ *iff* $\nu(a) = \rho(k)$, for every $a \in obj_o(\mathcal{K})$,
- $\rho \in C^{\mathcal{I}(k)}$ *iff* $C \in \rho(k)$, for every $C \in con_o(\mathcal{K})$,
- $\langle \rho, \rho' \rangle \in r^{\mathcal{I}(k)}$ *iff* $\rho'(k)$ is a matching r-successor for $\rho(k)$ under f.

As all the conditions (**CS**x), (**QC**x) and (**QM**x) are satisfied by the assumption, it is not difficult to verify that $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$, defined as above, is indeed an $AL^{\mathcal{L}}_{\mathcal{SHIO}}$-model for $\mathcal{K}$, for $\mathcal{L} \in \{\mathcal{SHIO}, \mathcal{EL}^{++}\}$. Again for the interpretation of the roles (including inverses and transitive roles) and satisfaction of the role hierarchy we apply the same argument as used for the case of the context ontology in the proof of Theorem 12. ❑

By Lemmas 2 and 3, the simplest brute-force NEXPTIME algorithm for checking satisfiability of $\mathcal{K}$ first guesses a quasimodel and then checks whether all conditions (**CS**x), (**QC**x) and(**QM**x) are satisfied. Clearly, such a check can be accomplished in a polynomial time in the size of the quasimodel, and thus in at most an exponential time in the size of $\mathcal{K}$. ❑

Theorem 16 *Knowledge base satisfiability in* $AL^{\mathcal{EL}^{++}}_{\mathcal{SHI}}$*, with only local roles, is in* EXPTIME.

Proof.

Again, we start by defining the relevant notion of quasimodel.

Definition 18 (Quasimodel) *A quasimodel candidate* $\mathfrak{Q}^S_{\mathfrak{S}}$ *for* $\mathcal{K}$*, where* $\mathcal{K}$ *is a knowledge base in* $AL^{\mathcal{EL}^{++}}_{\mathcal{SHI}}$*, with only local roles, is called a* quasimodel iff *it satisfies the following conditions:*

(QM1) for every $k, k' \in \mathfrak{S}$, with $k = \langle t_c, f, \nu \rangle$ and $k' = \langle t'_c, f', \nu' \rangle$, and $t_o \in \Pi$, whenever $t_c = t'_c$ then $\langle k, t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$ *iff* $\langle k', t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$,

(QM2) for every $\langle k, t_o \rangle \in \mathfrak{Q}^S_{\mathfrak{S}}$, with $k = \langle t_c, f, \nu \rangle$, and $C \sqsubseteq D \in f$, if $C \in t_o$ then $D \in t_o$.

For every $k \in \mathfrak{S}$ with $k = \langle t_c, f, \nu \rangle$:

(QM3) if $a : C \in f$ then $C \in \nu(a)$,

(QM4) if $r(a, b) \in f$ then $\nu(b)$ is a matching r-successor for $\nu(a)$ under f,

(QM5) (**rigid object names**) for every $a \in obj_o(\mathcal{K})$ and $k' \in \mathfrak{S}$ with $k' = \langle t'_c, f', \nu' \rangle$, $\nu(a)$ and $\nu'(a)$ are matching **S5**-successors,

(QM6) (**rigid object names**) for every $a \in obj_o(\mathcal{K})$ and $\langle \boldsymbol{C} \rangle D \in \nu(a)$ there is $\langle t'_c, f', \nu' \rangle \in \mathfrak{S}$, such that $\boldsymbol{C} \in t'_c$, $D \in \nu'(a)$.

Next, we prove the corresponding quasimodel lemma.

Lemma 4 (Quasimodel lemma) *A knowledge base* $\mathcal{K}$ *has an* $AL^{\mathcal{EL}^{++}}_{\mathcal{SHI}}$ *-model* iff *there is a quasimodel for* $\mathcal{K}$*.*

Proof. The proof is slightly more involved than in the case of Lemma 3, as here we need to restrict the space of possible quasimodels only to those built over the minimal context structures, generated by Algorithm 1.

($\Rightarrow$) Let $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$ be a model for $\mathcal{K}$. As in the proof of Lemma 3, we first fix a mapping τ from $\mathfrak{C}$ and Δ to the corresponding context/object types. For every $c \in \Theta$ set $\tau(c) = t_c$, such that:

- $\boldsymbol{C} \in t_c$ *iff* $c \in \boldsymbol{C}^{\mathcal{J}}$, for every $\boldsymbol{C} \in con_c(\mathcal{K})$,

For every $\langle c, d \rangle \in \mathfrak{C} \times \Delta$ set $\tau(\langle c, d \rangle) = t_o$, such that:

- $C \in t_o$ *iff* $d \in C^{\mathcal{I}(c)}$, for every $C \in con_o(\mathcal{K})$.

Further, for every $c \in \mathfrak{C}$ set $\tau(c) = \langle t_c, f, \nu \rangle$, such that the following correspondences hold:

- $\boldsymbol{C} \in t_c$ *iff* $c \in \boldsymbol{C}^{\mathcal{J}}$, for every $\boldsymbol{C} \in con_c(\mathcal{K})$,
- $\varphi \in f$ *iff* $\mathfrak{M}, c \models \varphi$, for every $\varphi \in sub_o(\mathcal{K})$,
- $\nu(a) = \tau(\langle c, a^{\mathcal{I}(c)} \rangle)$ for every $a \in obj_o(\mathcal{K})$.

Define set $\Omega = \{\boldsymbol{C} \mid (\langle \boldsymbol{C} \rangle D)^{\mathcal{I}(c)} \neq \emptyset$ for any $\langle \boldsymbol{C} \rangle D \in con_o(\mathcal{K})$ and $c \in \mathfrak{C}\}$. The set contains all those metalanguage concepts whose satisfaction is enforced by means of object concepts containing context operators which are actually satisfied in the model. Then compute the set $S_{\mathcal{C},\Omega}$. Observe, that by Theorem 14, $S_{\mathcal{C},\Omega}$ has to be non-empty. Next, for every $t_c \in S_{\mathcal{C},\Omega}$, set $f(t_c) = \{\varphi \mid \boldsymbol{C} : \varphi \in \mathcal{O}, \boldsymbol{C} \in t_c\}$ and define the context structure $\langle S, \mathfrak{S} \rangle$ and the quasimodel $\mathfrak{Q}^S_{\mathfrak{S}}$ by applying the following steps.

1. Set $S = S_{\mathcal{C},\Omega}$, $\mathfrak{S} := \emptyset$, $\mathfrak{Q}^S_{\mathfrak{S}} := \emptyset$, and $T_{t_c} := \emptyset$, for every $t_c \in S_{\mathcal{C},\Omega}$.
2. For every $c \in \mathfrak{C}$ and every $t_c \in S_{\mathcal{C},\Omega}$:
   - let $\tau(c) = \langle t'_c, f', \nu \rangle$. If $t_c \subseteq t'_c$, then add $\langle t_c, f(t_c), \nu \rangle$ to $\mathfrak{S}$ and for all $d \in \Delta$ add $\tau(\langle c, d \rangle)$ to T_{t_c}.
3. For every $k \in \mathfrak{S}$, with $k = \langle t_c, f, \nu \rangle$, and $t_o \in T_{t_c}$, add $\langle k, t_o \rangle$ to $\mathfrak{Q}^S_{\mathfrak{S}}$.

It is not difficult to verify that all conditions (**CS**x), (**QC**x) and (**QM1**)-(**QM4**) have to be satisfied by $\langle S, \mathfrak{S} \rangle$ and $\mathfrak{Q}^S_{\mathfrak{S}}$. Conditions (**QM5**) and (**QM6**) are special variants of (**QC4**) and (**QC5**) and impose rigid name assumption on individual object names, i.e. the requirement that for every $a \in obj_o(\mathcal{K})$ and $c, c' \in \mathfrak{C}$, it is always the case that $a^{\mathcal{I}(c)} = a^{\mathcal{I}(c')}$. It is not hard to see that whenever an $AL^{\mathcal{EL}^{++}}_{\mathcal{SHI}}$ -model satisfies this constraint, (**QM5**) and (**QM6**) are also satisfied in the corresponding quasimodel.

($\Leftarrow$) Let $\mathfrak{Q}^S_{\mathfrak{S}}$ be a quasimodel for $\mathcal{K}$. In order to construct a model $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$ for $\mathcal{K}$ we proceed in a similar manner as in the proof of Lemma 3. In the case without the rigid name assumption, we first fix the interpretation of the context dimension $(\Theta, \mathfrak{C}, \cdot^{\mathcal{J}})$ and define runs through $\mathfrak{Q}^S_{\mathfrak{S}}$ as before, and impose only one coherency condition on sets of runs. We say that a set $\mathfrak{R}$ of runs through $\mathfrak{Q}^S_{\mathfrak{S}}$ is called *coherent iff* the following condition is satisfied:

- for every $k \in \mathfrak{C}$ and $t_o \in T_k$, there is a $\rho \in \mathfrak{R}$ such that $\rho(k) = t_o$.

Next, we define the interpretation of the object dimension as follows. First, fix the object domain as:

- $\Delta := \mathfrak{R}$

Then, for every $k \in \mathfrak{C}$, with $k = \langle t_c, f, \nu \rangle$, and $\rho, \rho' \in \Delta$ set the interpretation function as:

- $\rho \in C^{\mathcal{I}(k)}$ *iff* $C \in \rho(k)$, for every $C \in con_o(\mathcal{K})$,
- $\langle \rho, \rho' \rangle \in r^{\mathcal{I}(k)}$ *iff* $\rho'(k)$ is a matching r-successor for $\rho(k)$ under f.

Finally, we fix the interpretation of the individual object names. For every $k \in \mathfrak{C}$, with $k = \langle t_c, f, \nu \rangle$, and $a \in obj_o(\mathcal{K})$:

- $a^{\mathcal{I}(c)} = \rho(k)$, for some unique $\rho \in \mathfrak{R}$ such that $\nu(a) = \rho(k)$.

When the rigid name assumption applies, first pick any $k \in \mathfrak{S}$, with $k = \langle t_c, f, \nu \rangle$, and then remove all k' from $\mathfrak{S}$, with $k' = \langle t'_c, f', \nu' \rangle$, and $\langle k', t_o \rangle$ from $\mathfrak{Q}^S_{\mathfrak{S}}$, such that for some $a \in obj_o(\mathcal{K})$, $\nu'(a)$ and $\nu(a)$ are not matching **S5**-successors. By conditions (**QM5**) and (**QM6**) the resulting sets $\mathfrak{S}$ and ${\mathfrak{Q}^S_{\mathfrak{S}}}'$ must be still a context structure and a quasimodel, respectively. Then we formulate the coherency conditions as:

- for every $k \in \mathfrak{C}$ and $t_o \in T_k$, there is a $\rho \in \mathfrak{R}$ such that $\rho(k) = t_o$,
- (**rigid object names**) for every $a \in obj_o(\mathcal{K})$ there is a unique run $\rho_a \in \mathfrak{R}$ such that for every $k \in \mathfrak{C}$, with $k = \langle t_c, f, \nu \rangle$, it is the case that $\nu(a) = \rho_a(k)$.

The construction of the model remains the same as in the case without the assumption, with the only difference in assigning the interpretation to individual names. For every $k \in \mathfrak{C}$ and $a \in obj_o(\mathcal{K})$:

- $a^{\mathcal{I}(c)} = \rho_a(k)$.

In both cases, one can see by the construction that the resulting interpretation $\mathfrak{M} = (\Theta, \mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$, is an $AL^{\mathcal{EL}^{++}}_{\mathcal{SHI}}$-model for $\mathcal{K}$. ❑

The EXPTIME procedure, which we sketch here, finds a quasimodel for $\mathcal{K}$, whenever it exists. This, by Lemma 4, provides a decision for the satisfiability of $\mathcal{K}$. The procedure combines computations of Algorithm 1 with the type elimination technique. It involves two non-deterministic steps marked with (♣) and (♠) below, which, as shown later, can be reduced to a deterministic computation over exponentially many possible choice.

Let $f(t_c) = \{\varphi \mid \boldsymbol{C} : \varphi \in \mathcal{O} \text{ and } \boldsymbol{C} \in t_c\}$, for every $t_c \in \Xi$. Start by (♣) picking a subset $P \subseteq con^{op}_c(\mathcal{K})$, fixing $\Omega = P \cup \{\{\boldsymbol{a}\} \mid \boldsymbol{a} : \varphi \in \mathcal{O}\}$ and computing $S_{\mathcal{C},\Omega}$ and $U_{\mathcal{C},\Omega}$. Then fix a context structure $\langle S, \mathfrak{S} \rangle$ as follows:

- Set $S := S_{\mathcal{C},\Omega}$ and $\mathfrak{S}$,
- (♠) for every $t_c \in U_{\mathcal{C},\Omega}$ and $\boldsymbol{a} \in obj_c(\mathcal{K})$, such that $\{\boldsymbol{a}\} \in t_c$ add a single tuple $\langle t_c, f(t_c), \nu \rangle \in \mathfrak{S}$, for some unique mapping $\nu : obj_c(\mathcal{K}) \mapsto \Pi$;
- for every $t_c \in U_{\mathcal{C},\Omega}$, such that $\{\boldsymbol{a}\} \notin t_c$ for every $\boldsymbol{a} \in obj_c(\mathcal{K})$, add $\langle t_c, f(t_c), \nu \rangle \in \mathfrak{S}$, for every mapping $\nu : obj_c(\mathcal{K}) \mapsto \Pi$.

Then define a set $\mathfrak{Q}^S_{\mathfrak{S}} = \{\langle k, t_o \rangle \mid k \in \mathfrak{S}, t_o \in \Pi\}$ and proceed with elimination of elements of $\mathfrak{Q}^S_{\mathfrak{S}}$ and $\mathfrak{S}$:

- for (**QC1**): eliminate k from $\mathfrak{S}$, whenever it violates the condition. Subsequently, eliminate every $\langle k, t_o \rangle$ from $\mathfrak{Q}^S_{\mathfrak{S}}$,
- for (**QC2**)-(**QC6**): eliminate $\langle k, t_o \rangle$ from $\mathfrak{Q}^S_{\mathfrak{S}}$, whenever it violates any of the conditions,
- for (**QM1**): eliminate $\langle k, t_o \rangle$ from $\mathfrak{Q}^S_{\mathfrak{S}}$, whenever $\langle k, t_o \rangle$ gets eliminated,
- for (**QM2**): eliminate $\langle k, t_o \rangle$ from $\mathfrak{Q}^S_{\mathfrak{S}}$, whenever it violates the condition,
- for (**QM3**)-(**QM4**), (**rigid name assumption** (**QM3**)-(**QM4**)): eliminate k from $\mathfrak{S}$, whenever it violates any of the conditions. Subsequently, eliminate every $\langle k, t_o \rangle$ from $\mathfrak{Q}^S_{\mathfrak{S}}$.

Let $\mathfrak{Q}^S_{\mathfrak{S}}$ be the result of the elimination. If $\mathfrak{Q}^S_{\mathfrak{S}} \neq \emptyset$ and $\langle S, \mathfrak{S} \rangle$ is a context structure, then clearly $\mathfrak{Q}^S_{\mathfrak{S}}$ is a quasimodel. In such case, by Lemma 4, the algorithm returns "$\mathcal{K}$ *is satisfiable*". Else, the algorithm repeats the elimination procedure using a different subset $P \subseteq con^{op}_c(\mathcal{K})$ in step (♣) and/or a different set of mappings $\nu : obj_c(\mathcal{K}) \mapsto \Pi$ in step (♠). Note, that by Theorem 14, and the fact that only concepts in $con^{op}_c(\mathcal{K})$ might occur inside the context operators, it follows that if a quasimodel for $\mathcal{K}$ exists, there has to exist also a quasimodel based on a context structure corresponding to one of the pairs $S_{\mathcal{C},\Omega}, U_{\mathcal{C},\Omega}$ computed by Algorith 1. Thus, if for all such combinations the procedure fails to find a quasimodel, then evidently such quasimodel does not exist and, by Lemma 4, the procedure returns "$\mathcal{K}$ *is unsatisfiable*".

It is easy to see, that at the start of the elimination the exponential space-bound for quasimodel candidates, Lemma 2, applies also to $\mathfrak{Q}^S_{\mathfrak{S}}$. Consequently, a single run of the elimination procedure cannot take more than an exponential time in order to terminate. Further, observe that the non-deterministic steps (♣) and (♠) can be replaced by a deterministic enumeration of all possible choices. In both cases there are at most exponentially many of them. For (♣) it is $2^{|con^{op}_c(\mathcal{K})|}$, while for (♠) — $(|\Pi|^{|obj_o(\mathcal{K})|})^{|obj_c(\mathcal{K})|}$, which all together results in at most $2^{\ell\mathcal{K}} \cdot 2^{2\ell\mathcal{K}^3} = 2^{2\ell\mathcal{K}^3 + \ell\mathcal{K}}$ possible sets $\mathfrak{Q}^S_{\mathfrak{S}}$ to perform elimination on. Therefore, the algorithm has to return the correct answer in a time at most exponential in the size of $\mathcal{K}$. ❑