TWO-DIMENSIONAL EQUIVALENT STIFFNESS ANALYSIS OF SOIL-STRUCTURE INTERACTION PROBLEMS

by

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ABSTRACT

The finite element technique is a powerful method to study the dynamic response of a structure taking into account the effects of ground conditions. However, limitations of computer storage capacity and cost presently prevent its general application to three-dimensional problems. In this thesis it is shown that three-dimensional problems can be analyzed by applying appropriate modification factors to two-dimensional (plane strain) analyses.

Modification factors are first determined analytically by comparing the dynamic response of both strip and rectangular footings (uniform shear stress) for a range of input frequencies. It is found that for input frequencies which are less than the fundamental period of the soild layer the modification factor is essentially independent of the input frequency. This suggests that the modification factors could be obtained from static analyses. Modification factors based on static stiffness analyses for both uniform shear stress and uniform shear displacement (rigid foundation) conditions were obtained and were found to be in close agreement with those obtained from the dynamic analyses. Variation of the modification factor with both the depth of the layer and the ratio of the sides of the rectangular base are given in graphical form. These factors may be applied to finite element place strain analysis to predict the dynamic response of three-dimensional structures.

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CHAPTER 1

INTRODUCTION

In the design of some structures it may be important to consider the safety of the structure for dynamic loading as well as static loading. Dynamic loading is especially important for structures in seismic zones, machine foundations, missile facilities and some structures which must be resistant to explosions. These structures may be built on or under the ground surface.

An energy exchange takes place between the structure and the ground when the structure-ground system is subject to a dynamic loading. This energy exchange is called the soil-structure interaction phenomenon and it is important that it be properly taken into consideration.

Until recently, it was believed that a structure founded on soft ground suffered more earthquake damage than a similar structure founded on firm ground. However, during the Kanto earthquake more damage occurred to rigid structures on a firm ground. Now, it is recognized that it is not only the properties of the structure or the ground alone that governs the earthquake damage, but it is the properties of the soil-structure system that are important.

As an example to show the importance of the soil-structure interaction during earthquakes, the recorded acceleration response spectrum (5) at the base of a building is shown in Fig. 1. The maximum response does not occur at the resonant period of the building; instead, the response at that period is close to a minimum. The same tendencies may be found in other records (5, 6).

These tendencies may be attributed to the feedback of the motion of the structure into the ground and the response depends on the dynamical

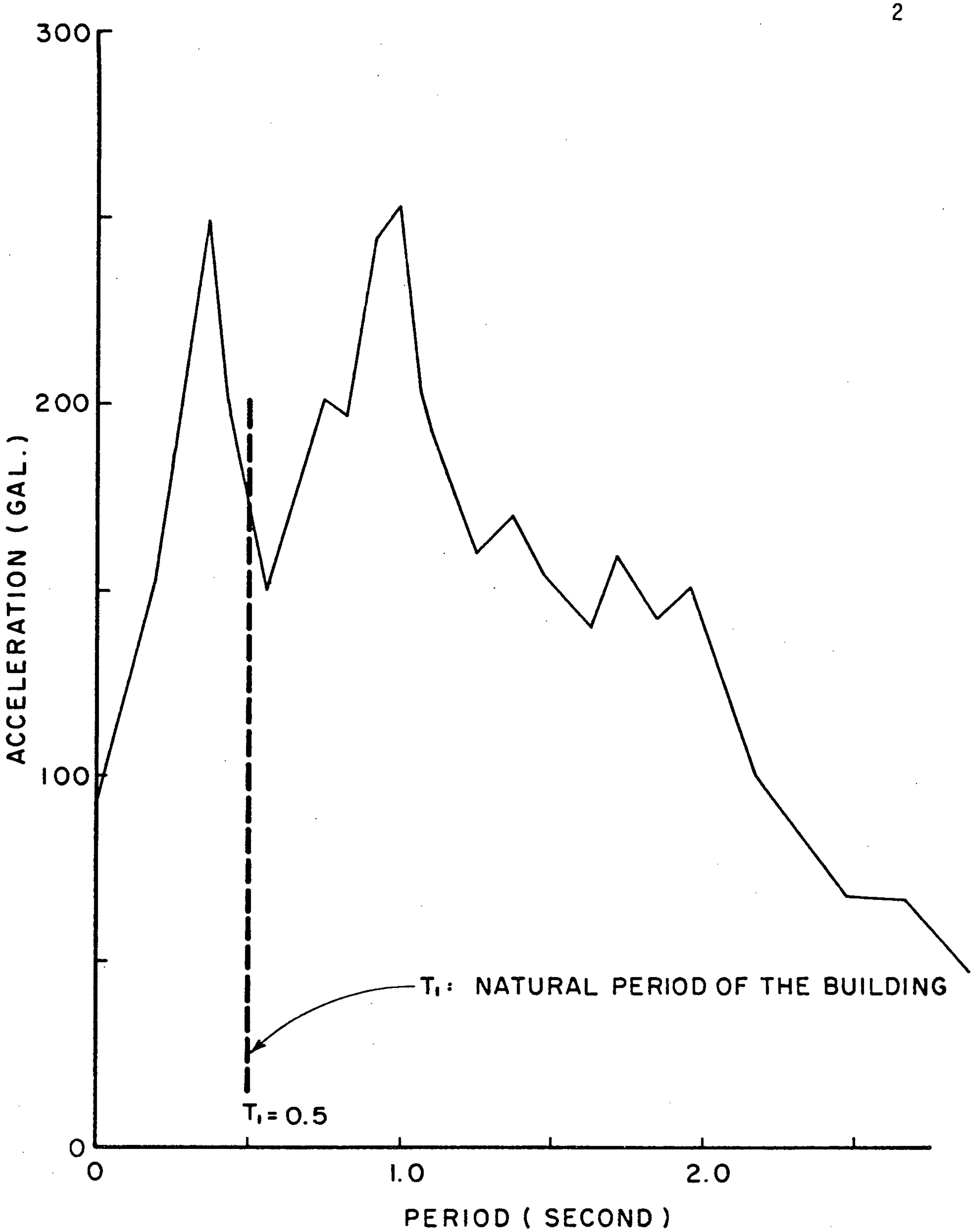


FIG. I ACCELERATION RESPONSE SPECTRUM AT THE BASE OF A BUILDING.

properties of the soil as well as those of the structure.

In the analysis of soil-structure systems, interaction between the ground and the structure may be taken into account by:

1. replacing the ground with some mass, spring, dash-pot combinations,
2. taking the ground as a continuous medium and applying wave propagation theory,
3. assuming that the ground is an assembly of a finite number of small elements and applying finite element technique.

The first approach involves important approximations in choosing a mass, spring, dash-pot system equivalent to a continuous medium. The second approach gives an exact solution under the idealized conditions of soil. However, mathematical difficulties make it almost impossible to consider complicated ground forms and unusual geometries. In the third approach, these complexities are treated without any mathematical difficulty.

In the investigation of soil structure systems, the finite element technique based on plane strain analysis has been used by several authors. However, when the ratio between the length and width of the foundation slab is small, the solution obtained using the plane strain analysis does not adequately represent the response of the system. A rigorous solution of this problem requires a three-dimensional analysis. However, storage limitations and economical reasons prevent the use of the three-dimensional finite element technique for such dynamic problems.

In this thesis, the problem of a structure with a rectangular rigid base resting upon a homogeneous elastic soil layer, which in turn rests upon a rigid base subject to a horizontal harmonic excitation, is investigated. A method is proposed whereby this problem can be reduced to an equivalent plane-strain problem by modifying the soil properties to account for the three-dimensional effect of the foundation.

CHAPTER 2

STEADY STATE RESPONSE OF A STRUCTURE OVER A SOIL LAYER FOR A HORIZONTAL EXCITATION AT THE BASE OF THE SOIL LAYER

A structure with a rigid rectangular base (2a x 2c) overlying a soil layer and subjected to a harmonic excitation at the base of the soil layer (bedrock) is shown in Fig. 2, where:

u_0 : Input harmonic displacement at the bedrock.

u_1 : Steady-state displacement response at the free surface of the soil layer alone.

u_2 : Steady-state displacement response at the base of the structure.

The processes involved in the computation of u_2 for a given u_0 are as follows:

1. Excitation at the bedrock gives the response u_1 at the free surface of the soil layer.
2. Displacement at the base of the structure is influenced by the motion of the structure, which gives the displacement response u_2 at the base. This response is transmitted into the structure and creates the force Q at the base of the structure.
3. The created force Q at the base of the structure gives the additional displacement u_2' at the contact surface between the structure and the soil layer.
4. Finally, this additional displacement u_2' and the displacement at the free surface of the soil layer u_1 are combined to give the displacement at the base of the structure u_2.

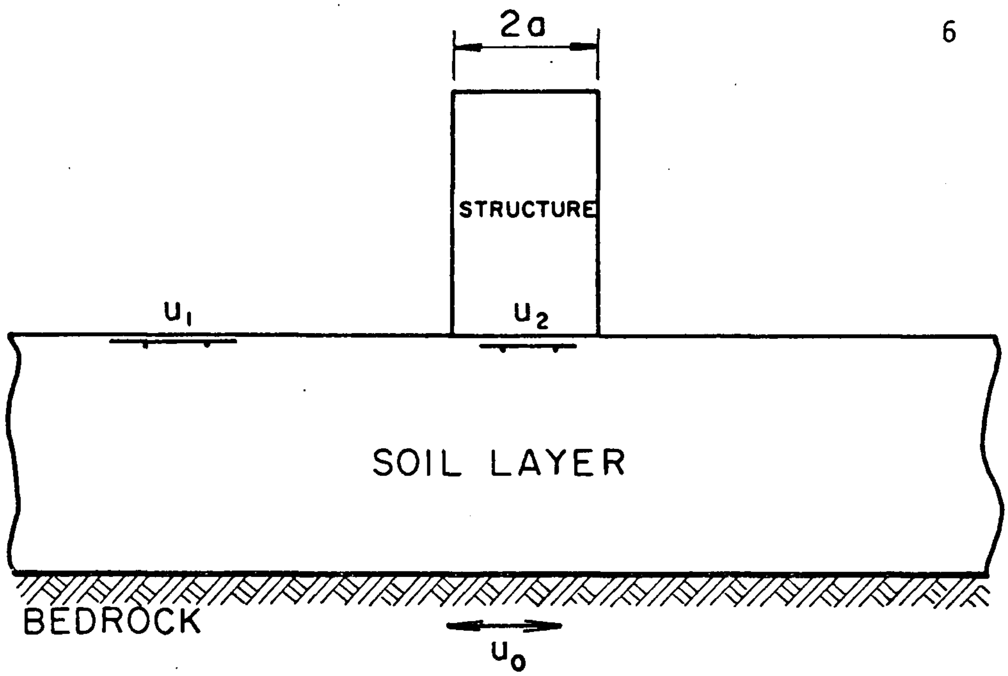


FIG. 2 STRUCTURE AND SOIL LAYER SYSTEM SUBJECTED TO A HARMONIC EXCITATION AT THE BEDROCK.

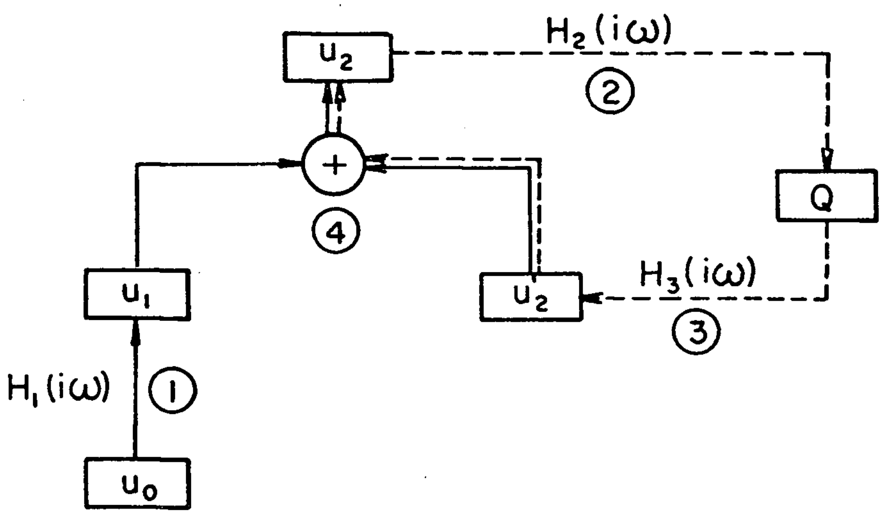


FIG. 3 STRUCTURE AND SOIL LAYER INTERACTION PROCESS FOR A HARMONIC EXCITATION AT THE BEDROCK.

The above process to compute the displacement response at the base for a given harmonic excitation is diagramatically illustrated in Fig. 3.

If the response of the structure and soil layer system is linear then the processes from 1 to 4 can be expressed mathematically using Fourier transformation and transfer functions as follows:

1. $$\bar{u}_1 = H_1(i\omega)\bar{u}_0 \tag{2:1}$$
2. $$\bar{Q} = H_2(i\omega)\bar{u}_2 \tag{2:2}$$
3. $$\bar{u}_2' = H_3(i\omega)\bar{Q} \tag{2:3}$$
4. $$\bar{u}_2 = \bar{u}_1 + \bar{u}_2' \tag{2:4}$$

where:

$\bar{u}_0$, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_2'$, $\bar{Q}$: Fourier transformations of u_0, u_1, u_2, u_2' and Q respectively.

$H_1(i\omega)$, $H_2(i\omega)$, $H_3(i\omega)$: Transfer functions in the processes 1, 2 and 3 respectively.

Substitution of the righthand side of Equation 2:2 into Equation 2:3 gives:

$$\bar{u}_2' = H_2(i\omega)H_3(i\omega)\bar{u}_2 \tag{2:5}$$

Substitution of the righthand side of Equations 2:1 and 2:5 into Equation 2:4 gives:

$$\bar{u}_2 = H_1(i\omega)\bar{u}_0 + H_2(i\omega)H_3(i\omega)\bar{u}_2 \tag{2:6}$$

Simplifying,

$$\bar{u}_2 = I(i\omega)_{3D}\bar{u}_0 \tag{2:7}$$

where:

$$I(i\omega)_{3D} = \frac{H_1(i\omega)}{1-H_2(i\omega)H_3(i\omega)} \tag{2:8}$$

The steady-state displacement response at the base of the structure can therefore be obtained by solving Equation 2:7.

If the structure is assumed to be infinitely long, then for the two-dimensional case Equations 2:1 to 2:4 become:

$$\overline{u}_1 = H_1(i\omega)\overline{u}_0 \quad (2:9)$$

$$\overline{Q}_{2D} = H_2'(i\omega)(\overline{u}_2)_{2D} \quad (2:10)$$

$$(\overline{u}_2')_{2D} = H_3'(i\omega)\overline{Q}_{2D} \quad (2:11)$$

$$(\overline{u}_2)_{2D} = \overline{u}_1 + (u_2')_{2D} \quad (2:12)$$

Substituting in the same manner as for the three-dimensional case,

$$(\overline{u}_2)_{2D} = I_{2D}(i\omega)\overline{u}_0 \quad (2:13)$$

where:

$$I_{2D}(i\omega) = \frac{H_1(i\omega)}{1-H_2'(i\omega)H_3'(i\omega)} \quad (2:14)$$

If the three-dimensional response is to be obtained from a two-dimensional analysis, then the two-dimensional analysis must be modified such that the displacement and shear force at the base of the structure will be the same as for a three-dimensional analysis, i.e.,

$$(\overline{u}_2)_{2D} = (\overline{u}_2)_{3D} \quad (2:15)$$

and

$$\overline{Q}_{2D} = Q/2c \quad (2:16)$$

From Equations 2:7 and 2:13:

$$I_{2D}(i\omega) = I_{3D}(i\omega) \quad (2:17)$$

hence:

$$\frac{H_1(i\omega)}{1-H_2'(i\omega)H_3'(i\omega)} = \frac{H_1(i\omega)}{1-H_2(i\omega)H_3(i\omega)} \quad (2:18)$$

From Equations 2:16, 2:2 and 2:10:

$$H_2'(i\omega) = \frac{1}{2c}H_2(i\omega) \quad (2:19)$$

If a plane strain analysis is applied for the soil-structure system with the rectangular base (2a x 2c), $H_1(i\omega)$ will not change but $H_2(i\omega)$ and

$H_3(i\omega)$ will be replaced by $\frac{H_2(i\omega)}{2c}$ and $H_3'(i\omega)$ respectively.

The term $H_3(i\omega)$ is the ground compliance when a soil layer is subject to a horizontal load Q over a rectangular area on the surface. Similarly, the term $H_3'(i\omega)$ is the ground compliance when a soil layer is subject to a horizontal load Q/2c over the per unit length of a strip. These terms will be evaluated in the following chapters.

CHAPTER 3

DYNAMIC GROUND COMPLIANCE FOR UNIFORM TANGENTIAL LOADING OVER A SOIL LAYER

In Chapter 2 it is mentioned that the response of a structure on a soil layer, subjected to harmonic excitation at bedrock, can be obtained if the transfer functions are known. The transfer function $H_1(i\omega)$ for the strip base of a structure is the same as that for the rectangular base since it is not related to the structure but only related to the dynamical character of the soil layer. The transfer function $H_2(i\omega)$ changes simply into $H_2(i\omega)/2c$ when plane strain analysis is applied to a structure with rectangular base ($2a \times 2c$) instead of three-dimensional analysis. However, the change of the transfer function $H_3(i\omega)$ is not easily determined. This transfer function, termed the dynamic ground compliance, will be considered in this chapter.

The following assumptions will be made:

1. Stress distribution under the base is assumed uniform.
2. Soil layer is assumed to be homogeneous, isotropic and linear elastic material.
3. Soil layer is assumed to be fixed at the rigid bedrock.

3.1 Analytical Solution, Plane Strain

A soil layer which is subject to a uniformly distributed dynamical shear loading over a strip is shown in Figure 4.

The equations of motion in terms of displacements are:

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u \qquad (3:1)$$

$$\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial y} + \mu \nabla^2 v \qquad (3:2)$$

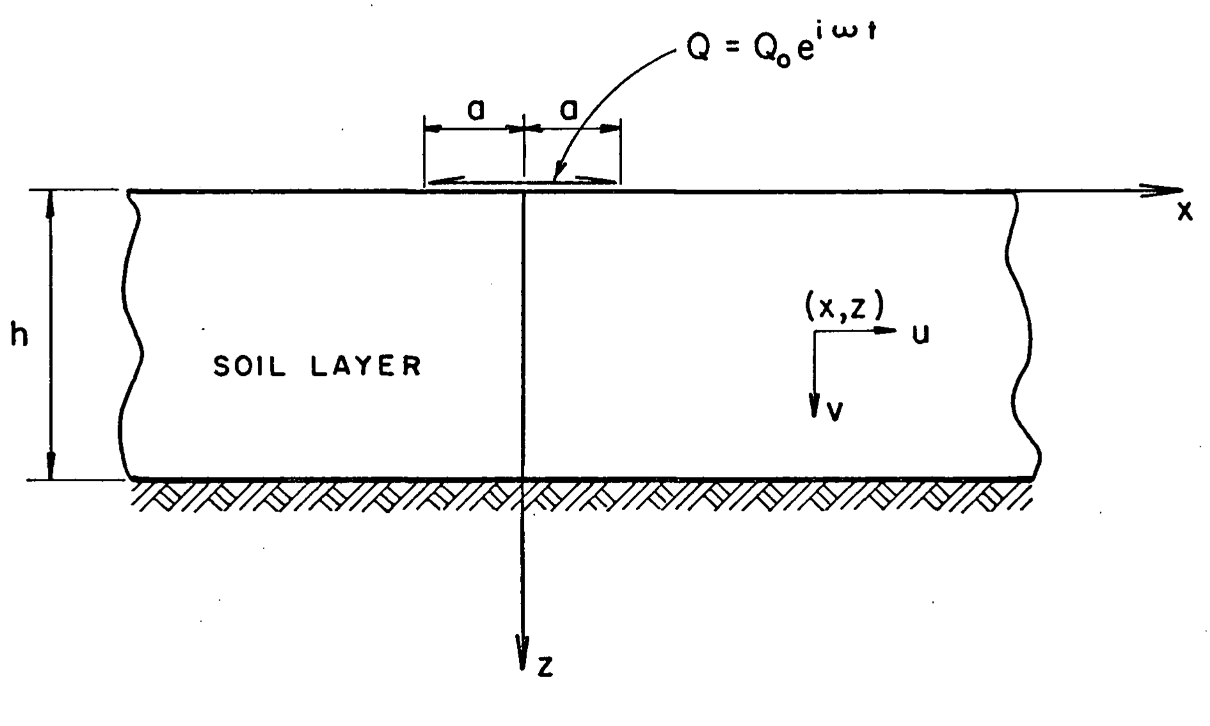


FIG. 4 SOIL LAYER SUBJECTED TO A HARMONIC UNIFORM HORIZONTAL STRIP LOAD ON THE SURFACE.

$$\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 w \tag{3:3}$$

where
$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Here λ and μ denote Lame's constants. ρ is the mass density of the soil layer. u,v,w are displacement components.

Since displacement components are independent of the longitudinal coordinate y, we have:

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \tag{3:4}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{3:5}$$

Defining two potentials Φ_1 (x,z,t) and Ψ_1 (x,z,t) as follows:

$$u\ (x,z,t) = \frac{\partial \Phi_1}{\partial x} + \frac{\partial \Psi_1}{\partial z} \tag{3:6}$$

$$w\ (x,z,t) = \frac{\partial \Phi_1}{\partial z} - \frac{\partial \Psi_1}{\partial x} \tag{3:7}$$

Equations 3:1 and 3:3 reduce to:

$$\frac{\rho}{\lambda + 2\mu} \frac{\partial^2 \Phi_1}{\partial t^2} = \nabla^2 \Phi_1 \tag{3:8}$$

$$\frac{\rho}{\mu} \frac{\partial^2 \Psi_1}{\partial t^2} = \nabla^2 \Psi_1 \tag{3:9}$$

Since the applied boundary stress is harmonic, the potentials should have the following form:

$$\begin{aligned} \Phi_1(x,z,t) &= \Phi(x,z)\ e^{i\omega t} \\ \Psi_1(x,z,t) &= \Psi(x,z)\ e^{i\omega t} \end{aligned} \tag{3:10}$$

In view of Equation 3:10, Equations 3:8 and 3:9 reduce to:

$$(\nabla^2 + p^2)\ \phi\ (x,z) = 0 \tag{3:11}$$

$$(\nabla^2 + q^2)\ \psi\ (x,z) = 0 \tag{3:12}$$

where:

$$p = \frac{\rho\omega^2}{\lambda+2\mu}, \qquad q = \frac{\rho\omega^2}{\mu} \tag{3:13}$$

The stress-strain relations (Hooke's Law) can be expressed as:

$$\sigma_{zz} = \lambda\Delta + 2\mu\frac{\partial w}{\partial z}, \quad \sigma_{xx} = \lambda\Delta + \mu\frac{\partial u}{\partial x}, \quad \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

The boundary conditions of the problem are:

$$u(x,h,t) = w(x,h,t) = 0 \tag{3:14}$$

$$\sigma_{zz}(x,o,t) = 0 \tag{3:15}$$

$$\tau_{xz}(x,o,t) = -\frac{2\tau_0 a}{\pi} e^{i\omega t}\int_0^\infty \frac{\text{Sin}a\zeta}{a\zeta}\text{Cos}x\zeta d\zeta = \begin{cases} 0 & |x|>a \\ -\tau_0 e^{i\omega t} & |x|<a \end{cases} \tag{3:16}$$

Taking Sine Fourier transform and Cosine Fourier transform in x of Equations 3:11 and 3:12 respectively, we get:

$$\frac{d^2\overline{\phi}}{dz^2} - (\zeta^2 - p^2)\ \overline{\phi} = 0 \tag{3:17}$$

$$\frac{d^2\overline{\psi}}{dz^2} - (\zeta^2 - q^2)\ \overline{\psi} = 0 \tag{3:18}$$

where:

$$\overline{\phi}\ (\zeta,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty \phi\ (x,z)\ \text{Sin}x\zeta dx$$

$$\overline{\psi}\ (\zeta,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty \psi\ (x,z)\ \text{Cos}x\zeta dx$$

From the general solution of Equations 3:17 and 3:18, the two potentials become:

$$\phi\ (x,z) = \int_0^\infty [A\ \text{Cosh}\alpha z + B\ \text{Sinh}\alpha z]\ \text{Sin}x\zeta d\zeta \tag{3:19}$$

$$\psi(x,z) = \int_0^\infty [C \operatorname{Cosh}\beta z + D \operatorname{Sinh}\beta z] \operatorname{Cos} x\zeta d\zeta \qquad (3:20)$$

where:

$$\alpha = (\zeta^2 - p^2)^{1/2}, \qquad \beta = (\zeta^2 - q^2)^{1/2} \qquad (3:21)$$

and A,B,C,D are arbitrary functions of ζ. Employing boundary conditions of Equations 3:14, 3:15 and 3:16, we get:

$$\begin{aligned}
(\zeta \operatorname{Cosh}\alpha h)A + (\zeta \operatorname{Sinh}\alpha h)B + (\beta \operatorname{Sinh}\beta h)C + (\beta \operatorname{Cosh}\beta h)D &= 0 \\
(\alpha \operatorname{Sinh}\alpha h)A + (\alpha \operatorname{Cosh}\alpha h)B + (\zeta \operatorname{Cosh}\beta h)C + (\zeta \operatorname{Sinh}\beta h)D &= 0 \\
(\beta^2 + \zeta^2)A + 2\zeta\beta D &= 0 \\
2\zeta\alpha B + (\beta^2 + \zeta^2)C &= -\frac{2\tau_0 a}{\pi\mu}\delta(a\zeta)
\end{aligned} \qquad (3:22)$$

Here: $\delta(a\zeta) = \dfrac{\operatorname{Sin} a\zeta}{a\zeta}$

From these equations, $A(\zeta)$ and $D(\zeta)$ are obtained as:

$$A(\zeta) = \frac{2\tau_0 a\ \delta(a\zeta)}{\pi\mu} \cdot \frac{2\zeta\beta(\alpha\beta \operatorname{Sinh}\beta h \operatorname{Cosh}\alpha h - \zeta^2 \operatorname{Cosh}\beta h \operatorname{Sinh}\alpha h)}{4\zeta^2\alpha\beta(\zeta^2+\beta^2)+\alpha\beta[4\zeta^4+(\zeta^2+\beta^2)^2]\operatorname{Cosh}\beta h \operatorname{Cosh}\alpha h - \zeta^2[(\zeta^2+\beta^2)^2+4\alpha^2\beta^2]\operatorname{Sinh}\beta h \operatorname{Sinh}\alpha h} \qquad (3:23)$$

$$D(\zeta) = \frac{2\tau_0 a\ \delta(a\zeta)}{\pi\mu} \cdot \frac{2\zeta\beta(\alpha\beta \operatorname{Sinh}\beta h \operatorname{Cosh}\alpha h - \zeta^2 \operatorname{Cosh}\beta h \operatorname{Sinh}\alpha h)(\beta^2+\zeta^2)}{4\zeta^2\alpha\beta(\zeta^2+\beta^2)+\alpha\beta[4\zeta^4+(\zeta^2+\beta^2)^2]\operatorname{Cosh}\beta h \operatorname{Cosh}\alpha h - \zeta^2[(\zeta^2+\beta^2)^2+4\alpha^2\beta^2]\operatorname{Sinh}\beta h \operatorname{Sinh}\alpha h} \qquad (3:24)$$

From Equations 3:6, 3:10, 3:19 and 3:20, displacement component u on the surface of the soil layer can be expressed as:

$$u(x,o,t) = e^{i\omega t} \int_0^\infty [\zeta A(\zeta) + \beta D(\zeta)] \operatorname{Cos} x\zeta d\zeta \qquad (3:25)$$

Substituting the righthand side of Equation 3:23 and Equation 3:24 for $A(\zeta)$

$D(\zeta)$, into the righthand side of Equation 3:25, finally we get:

$$u(x,o,t) = \frac{2\tau_0 ae^{i\omega t}}{\pi\mu} \cdot \int_0^\infty \frac{\beta(\beta^2-\zeta^2)(\alpha\beta \text{Sinh}\beta h \text{Cosh}\alpha h - \zeta^2 \text{Cosh}\beta h \text{Sinh}\alpha h)\delta(\beta\zeta) \cdot \text{Cos}x\zeta d\zeta}{4\zeta^2\alpha\beta(\beta^2+\zeta^2)+\alpha\beta[4\zeta^4+(\zeta^2+\beta^2)^2]\text{Cosh}\beta h \text{Cosh}\alpha h - \zeta^2[(\zeta^2+\beta^2)^2+4\alpha^2\beta^2]\text{Sinh}\beta h \text{Sinh}\alpha h} \tag{3:26}$$

The dynamical ground compliance of a soil layer for a tangential strip loading can be expressed by the ratio:

$$\frac{u(o,o,t)}{qe^{i\omega t}} \mu a$$

where q is the amplitude of the total dynamical force acting per unit length of the soil layer, i.e. $q = 2\tau_0 a$. In view of Equation 3:26, the dynamical ground compliance becomes:

$$\frac{u(o,o,t)}{qe^{i\omega t}} \mu a = \frac{a}{\pi} \int_0^\infty \frac{\beta(\beta^2-\zeta^2)(\alpha\beta \text{Sinh}\beta h \text{Cosh}\alpha h - \zeta^2 \text{Cosh}\beta h \text{Sinh}\alpha h)\delta(\beta\zeta) d\zeta}{4\zeta^2\alpha\beta(\beta^2+\zeta^2)+\alpha\beta[(\beta^2+\zeta^2)^2+4\zeta^4]\text{Cosh}\beta h \text{Cosh}\alpha h - \zeta^2[(\zeta^2+\beta^2)^2+4\alpha^2\beta^2]\text{Sinh}\beta h \text{Sinh}\alpha h} \tag{3:27}$$

Employing change of variable $\zeta \quad \frac{\omega}{C_2} \xi$ in Equation 3:27, we get:

$$\frac{u(o,o,t)}{qe^{i\omega t}} \mu a = \frac{a}{\pi a_0} \int_0^\infty - \frac{\sqrt{\xi^2-1}}{u} \frac{D(\xi)}{F(\xi)} E(\xi) \text{Sin}(a_0 \xi) d\xi \tag{3:28}$$

where:

$$D(\xi) = \{\xi^2 \text{Coth}(\sqrt{\xi^2-1}\, a_1) - \sqrt{\xi^2-n^2}\sqrt{\xi^2-1}\, \text{Coth}(\sqrt{\xi^2-n^2}\, a_1)\}\, \text{Coth}(\sqrt{\xi^2-1}\, a_1)$$

$$E(\xi) = \tanh(\sqrt{\xi^2-1}\, a_1)$$

$$F(\xi) = 4\xi^2(2\xi^2-1)\sqrt{\xi^2-n^2}\sqrt{\xi^2-1}\, \text{Cosech}(\sqrt{\xi^2-n^2}a_1)\text{cosech}(\sqrt{\xi^2-1}\, a_1)$$
$$- \{4\xi^4+(2\xi^2-1)^2\}\sqrt{\xi^2-n^2}\sqrt{\xi^2-1}\, \text{Coth}(\sqrt{\xi^2-n^2}\, a_1)\text{Coth}(\sqrt{\xi^2-1}\, a_1)$$

$$+ \xi^2 \{4(\xi^2-n^2)(\xi^2-1) + (2\xi^2-1)^2\}$$

$$n^2 = \frac{1-2\nu}{2(1-\nu)}, \quad c_2 = \sqrt{\frac{\mu}{\rho}}, \quad a_1 = \frac{\omega}{c}$$

The functions $\sqrt{\xi^2-1}$ and $\sqrt{\xi^2-n^2}$ have branch points $\xi = 1$, $\xi = n$, for $\xi > 0$, respectively. To stay on one branch of these functions,

$$\sqrt{\xi^2-1} = \begin{cases} i\sqrt{1-\xi^2} & 0 \leq \xi \leq 1 \\ \sqrt{\xi^2-1} & \xi \geq 1 \end{cases}$$

$$\sqrt{\xi^2-n^2} = \begin{cases} i\sqrt{n^2-\xi^2} & 0 \leq \xi \leq n \\ \sqrt{\xi^2-n^2} & \xi \geq n \end{cases}$$

Also, $\xi = 0$, $\xi = 1$ and $\xi = \xi_k$, $k = 1,2,\ldots,N$ are poles of the integrand where ξ_k, $k = 1,2,\ldots N$ are the finite number real zeros of $F(\xi)$. Since the integrand in Equation 3:28 have some singularities such as branch points and poles, the integral on the righthand side of Equation 3:28 must be numerically evaluated in the complex plane. From the integration in the complex plane:

$$f_1^{2D} = \frac{a}{\pi a_0} P \int_0^\infty \frac{\sqrt{\xi^2-1}}{\xi F(\xi)} D(\xi)E(\xi) \operatorname{Sin}(a_0\xi)\, d\xi \tag{3:29}$$

$$f_2^{2D} = \frac{a}{a_0} \sum_{k=1}^{N} \left[\frac{\sqrt{\xi^2-1}}{\xi dF(\xi)/d\xi} D(\xi)E(\xi) \operatorname{Sin}(a_0\xi)\right]_{\xi=\xi_k} \tag{3:30}$$

where:

$$\frac{\overline{u(o,o,t)}}{qe^{i\omega t}} \mu a = f_1^{2D} + i f_2^{2D} \tag{3:31}$$

and P denotes Cauchy principal value of the integral. The numerical study of $F(\xi) = 0$ shows that f_1^{2D} has three types of singularities as follows:

(i) $a_1 = \frac{2m-1}{2}\pi$, $m = 1,2,\ldots$

(ii) $a_1 = 2.7012, 7.455, \ldots$

(iii) $a_1 = 2.7207, 8.162, \ldots$

The second and third type singularities correspond to double real roots of $F(\xi) = 0$. The function $F(\xi)$ does not have any real root for $a_1 < \pi/2$, hence $f_2^{2D} = 0$ for $a_1 < \pi/2$. The flow chart for the computation of f_1^{2D} is given in Fig. A1. (Appendix 1).

3.2 Solution by the Finite Element Method, Plane Strain

In the finite element method the mass of each element is concentrated at the nodal points of each element. The equations of motion of nodal points in a soil layer subjected to forced excitations are:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{Q\} \tag{3:32}$$

where: $[M]$: mass matrix

$[C]$: damping matrix

$[K]$: stiffness matrix

$\{Q\}$: exciting force matrix

$\{\ddot{u}\}, \{\dot{u}\}, \{u\}$: acceleration, velocity and displacement matrices respectively.

Equation 3:32 represents a system of coupled equations. These coupled equations can be uncoupled by an orthogonal matrix $[\phi]$ to yield:

$$\{\ddot{\xi}\} + 2\zeta\,[\omega_n]\{\dot{\xi}\} + [\omega_n^2]\{\xi\} = [M^*]^{-1}\{Q^*\} \tag{3:33}$$

where: $\{\ddot{\xi}\}, \{\dot{\xi}\}, \{\xi\}$: acceleration, velocity and displacement in normal coordinates

ω_n : the natural frequencies

ζ : % of the critical damping

$[M^*] = [\phi]^T[M][\phi]$

$\{Q^*\} = [\phi]^T\{Q\}$

where the column vectors of $[\phi]$ are the mode shapes.

If the excitation is harmonic, the displacement in the Sth normal coordinate can be obtained from Equation 3:33 as:

$$\xi^S = \frac{\{1 - (\frac{\omega}{\omega_n^S})^2\} - i\{2\zeta \frac{\omega}{\omega_n^S}\}}{\{1 - (\frac{\omega}{\omega_n^S})^2\}^2 + \{2\zeta \frac{\omega}{\omega_n^S}\}^2} \frac{Q^{*S}}{M^{*S}}$$

$$= \xi_r^S + i\xi_i^S \tag{3:34}$$

where:

$$\xi_r^S = \frac{1 - (\frac{\omega}{\omega_n^S})^2}{\{1 - (\frac{\omega}{\omega_n^S})^2\}^2 + \{2\zeta \frac{\omega}{\omega_n^S}\}^2} \frac{Q^{*S}}{M^{*S}} \tag{3:35}$$

$$\xi_i = \frac{-2\zeta \frac{\omega}{\omega_n^S}}{\{1 - (\frac{\omega}{\omega_n^S})^2\}^2 + \{2\zeta \frac{\omega}{\omega_n^S}\}^2} \frac{Q^{*S}}{M^{*S}} \tag{3:36}$$

The horizontal displacement in the nodal coordinates at the center of the loaded area is:

$$u_c = \sum_{S=1}^{N} \phi_c^S \xi^S$$

$$= \sum_{S=1}^{N} (\phi_c^S \xi_r^S + i\phi_c^S \xi_i^S) \tag{3:37}$$

where: u_c : horizontal displacement in the nodal coordinates at the center of the loaded area

ϕ_c^S : horizontal displacement at the center of the loaded area in the Sth normal coordinate

N : calculated number of modes

From the definition, the ground compliance can be expressed as:

$$f_1 = \frac{\mu a}{\Sigma Q} \sum_{s=1}^{N} \phi_c^s \xi_r^s$$

$$f_2 = \frac{\mu a}{\Sigma Q} \sum_{s=1}^{N} \phi_c^s \xi_i^s \qquad (3:38)$$

It should be noted here that the imaginary part does not appear when the damping matrix is zero.

To obtain mode shapes and natural frequencies, the existing computer program DYNAMIC was used. This program may be obtained from the Department of Civil Engineering Computer Program Library. The Flow chart for the above computations is shown in Fig. A2. (Appendix 1)

3.3 Analytical Solution for Three-Dimensional Case

Kobori (7, 8) obtained the dynamical ground compliance for this case and extended it for viscoelastic multi layers (9). According to his analysis, the displacement at the center of the load as shown in Fig. 5 is:

$$u(x=0,\ y=0,\ z=0,\ t) = \frac{Q}{\mu\pi^2 a_0 c} \int_0^{\infty} \int_0^{\frac{\pi}{2}} \left\{ \frac{\sin^2\phi}{\xi\sqrt{\xi^2-1}} - \frac{\sqrt{\xi^2-1}}{\xi F(\xi)} D(\xi)\cos^2\theta \right\}$$

$$E(\xi)S(a_0\xi,\theta)\,d\theta\,d\xi \qquad (3:39)$$

where:

$$F(\xi) = 4\xi^2(2\xi^2-1)\sqrt{\xi^2-n^2}\ \sqrt{\xi^2-1}\ \operatorname{cosech}(a_1\sqrt{\xi^2-n^2})\operatorname{cosech}(a_1\sqrt{\xi^2-1})$$

$$- \{4\xi^4 + (2\xi^2-1)^2\}\ \sqrt{\xi^2-n^2}\ \sqrt{\xi^2-1}\ \coth(a_1\sqrt{\xi^2-n^2})\coth(a_1\sqrt{\xi^2-1})$$

$$+ \xi^2\ \{4(\xi^2-n^2)(\xi^2-1) + (2\xi^2-1)^2\}$$

$$E(\xi) = \tanh(a_1\sqrt{\xi^2-1})$$

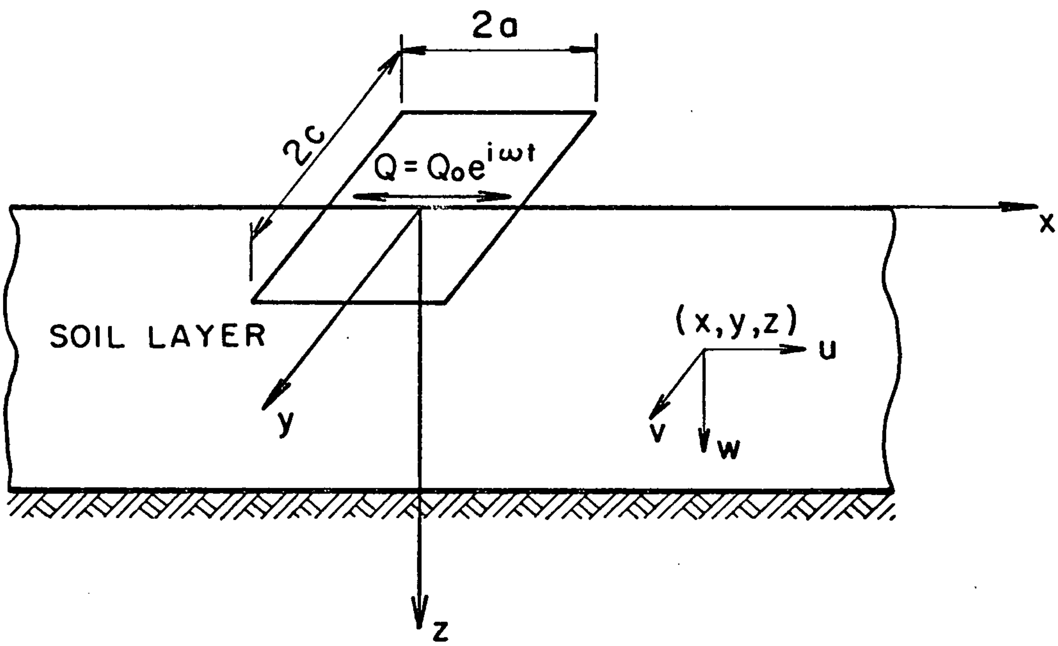


FIG. 5 SOIL LAYER SUBJECTED TO A HARMONIC UNIFORM HORIZONTAL RECTANGULAR LOAD ON THE SURFACE.

$$D(\xi) = \{\xi^2\coth(a_1\sqrt{\xi^2-1}) - \sqrt{\xi^2-n^2}\ \sqrt{\xi^2-1}\ \coth(a_1\sqrt{\xi^2-n^2}\}\ \coth(a_1\sqrt{\xi^2-1})$$

$$S(a_0\xi,\theta) = \frac{\sin(a_0\xi\cos\theta)\ \sin(\frac{c}{a}\ a_0\xi\sin\theta)}{\cos\theta\ \sin\theta}$$

Singularities in the integrand in Equation 3:39 are:

Branch points $\xi=n,\ \xi=1$

$\xi=0,\ \xi=1,\ \xi=\xi_{k(k=1,2,..N)},\ \xi=\xi'_{k(k=1,2,..,N')}$

where:

ξ_k : real roots for $F(\xi) = 0$

ξ'_k : real roots for $(E(\xi))^{-1} = 0$

Integrating Equation 3:39 in a complex plane, the ground compliance can be obtained as:

$$\frac{u^{3D}}{Q}\,a\mu = f_1^{3D} + if_2^{3D} \tag{3:40}$$

where:

$$f_1^{3D} = \frac{1}{\pi a_0\frac{c}{a}}\ P\int_0^\infty \left\{\frac{1}{\xi\sqrt{\xi^2-1}}\ S'(\xi) - \frac{\sqrt{\xi^2-1}}{\xi F(\xi)}\ D(\xi)\ S_2'(\xi)\right\} E(\xi)d\xi$$

$$f_2^{3D} = \frac{1}{\pi a_0\frac{c}{a}}\sum_{k=1}^{N'} \frac{\sqrt{\xi^2-1}}{\xi dF(\xi)/d\xi}\ D(\xi)E(\xi)\ S_2'(\xi)\Big|_{\xi=\xi_k}$$

$$+ \frac{1}{\pi a_0\frac{c}{a}}\sum_{k=1}^{N'} \frac{S_1'(\xi)}{a_1\xi^2}\ \Big|_{\xi=\xi'_k}$$

$$S_1'(\xi) = \int_0^{\frac{\pi}{2}} S(\xi,\theta)\ \sin^2\theta d\theta$$

$$S_2'(\xi) = \int_0^{\frac{\pi}{2}} S(\xi,\theta)\cos^2\theta d\theta$$

P : Cauchy's principal value of the integral.

The real part is the Cauchy's principal value of the integral. The imaginary part is the radius about the poles ξ_k and ξ_k'. Resonant frequencies are those where the equation $F(\xi) = 0$ has double roots. The equation $(E(\xi))^{-1} = 0$ does not govern resonant frequencies since this equation does not have double roots. These two equations do not have a real root for frequencies less than $a_1 = 1.57$.

CHAPTER 4

COMPARISON OF TWO-DIMENSIONAL AND THREE-DIMENSIONAL RESPONSE OF SOIL-STRUCTURES

The dynamic response of a structure located on a soil layer can be obtained for plane strain (two-dimensional) conditions using the finite element method of analysis. However, for the three-dimensional case the number of unknowns becomes very large and computer storage space and cost limit the size of the problem that can be solved.

For a harmonic base excitation it has been shown (Chapter 2) that both the two and three-dimensional responses depend essentially on the transfer function termed the ground compliance. The ground compliance for both the two and three-dimensional cases have been derived in Chapter 3. In this chapter these ground compliances will be compared and it will be shown that the three-dimensional response may be estimated from a two-dimensional plane strain analysis.

A comparison of the dynamic response of strip (two-dimensional) and rectangular (three-dimensional) loaded areas shows the following:

1. Resonant frequencies for a strip load and a rectangular load are identical since the functions $F(\xi)$ governing the resonant frequencies for them are the same (Equations 3:28 and 3:39).
2. Radiational dampings for a strip load and rectangular load do not appear for frequencies less than $a_1 = 1.57$, since the equations $F(\xi) = 0$ and $(E(\xi))^{-1} = 0$ governing the radiational damping do not have any real roots for these frequencies (Equations 3:31 and 3:40).
3. Radiational damping does not appear in a finite element

method as shown in Section 2, Chapter 3. However, material damping may be used to simulate the radiational damping.

Substituting u^{3D}/Q from Equation 3:40 into Equation 2:3, the transfer function $H_3(i\omega)$ is given by:

$$H_3(i\omega) = \frac{1}{a\mu}(f_1^{3D} + if_2^{3D}) \tag{4:1}$$

and the transfer function $I(i\omega)_{3D}$ from Equation 2:8 is given by:

$$I(i\omega)_{3D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu}(f_1^{3D} + if_2^{3D})\ H\ (\omega)} \tag{4:2}$$

If a plane strain analysis is used for the same system, then substituting u/q from Equation 3:31 into Equation 2:11, the transfer function $H_3(i\omega)$ can be expressed as:

$$H_3'(i\omega) = \frac{1}{a\mu}(f_1^{2D} + if_2^{2D}) \tag{4:3}$$

and the transfer function $I(i\omega)_{2D}$ from Equation 2:11 is given by:

$$I(i\omega)_{2D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu}\left(\frac{f_1^{2D}}{2c} + i\frac{f_2^{2D}}{2c}\right) H_2(\omega)} \tag{4:4}$$

The difference between the transfer functions $I(i\omega)_{3D}$ in Equation 4:2 and $I(i\omega)_{2D}$ in Equation 4:4 is the error which would arise by the use of a plane strain analysis for the structure with a rectangular base.

If the system has no radiational damping the transfer functions $I(i\omega)_{3D}$ and $I(i\omega)_{2D}$ will be:

$$I(\omega)_{3D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu} f_1^{3D} H_2(\omega)} \tag{4:5}$$

$$I(\omega)_{2D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu}\frac{f_1^{2D}}{2c} H_2(\omega)} \tag{4:6}$$

A modification factor α will be defined as:

$$\alpha = \frac{f_1^{2D}/2c}{f_1^{3D}} \tag{4:7}$$

Multiplying $I(\omega)_{2D}$ by $\frac{1}{\alpha}$ in Equation 4:6 gives:

$$I(\omega)'_{2D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu\alpha}\frac{f_1^{2D}}{c}H_2(\omega)} \tag{4:8}$$

This is identical with a three-dimensional transfer function $I(\omega)_{3D}$. Therefore, if a plane strain finite element analysis is modified to yield the transfer function $I(\omega)'_{2D}$, the three-dimensional effect is obtained for a harmonic motion at the bedrock.

Although it may be possible to give the response with three-dimensional radiational damping effects by choosing a suitable material damping, this thesis will not discuss such damping but only the modification factor α. Since radiational damping does not appear at frequencies less than the fundamental frequency in ground compliance ($a_1 = 1.57$), the precise considerations will be given for this range.

The ground compliances f_1^{2D} and f_1^{3D} for frequencies of a harmonic excitation less than $a_1 = 1.57$ are shown in Fig. 6, where f_1^{2D} is computed as shown in Appendix 3 and f_1^{3D} is cited from Reference 8. The modification factor α calculated from these ground compliances is shown in Fig. 7. The results show that the modification factor α for a square footing (2a = 2c) is almost constant with frequency (except near the resonant frequency). For a depth of soil, h, equal to the width of the footing, 2a, i.e. h/a = 2, $\alpha \simeq 1.25$ and for h/a = 4, $\alpha \approx 1.6$.

The fact that the modification factor α is essentially independent of frequency for a given geometry is important because it indicates that α

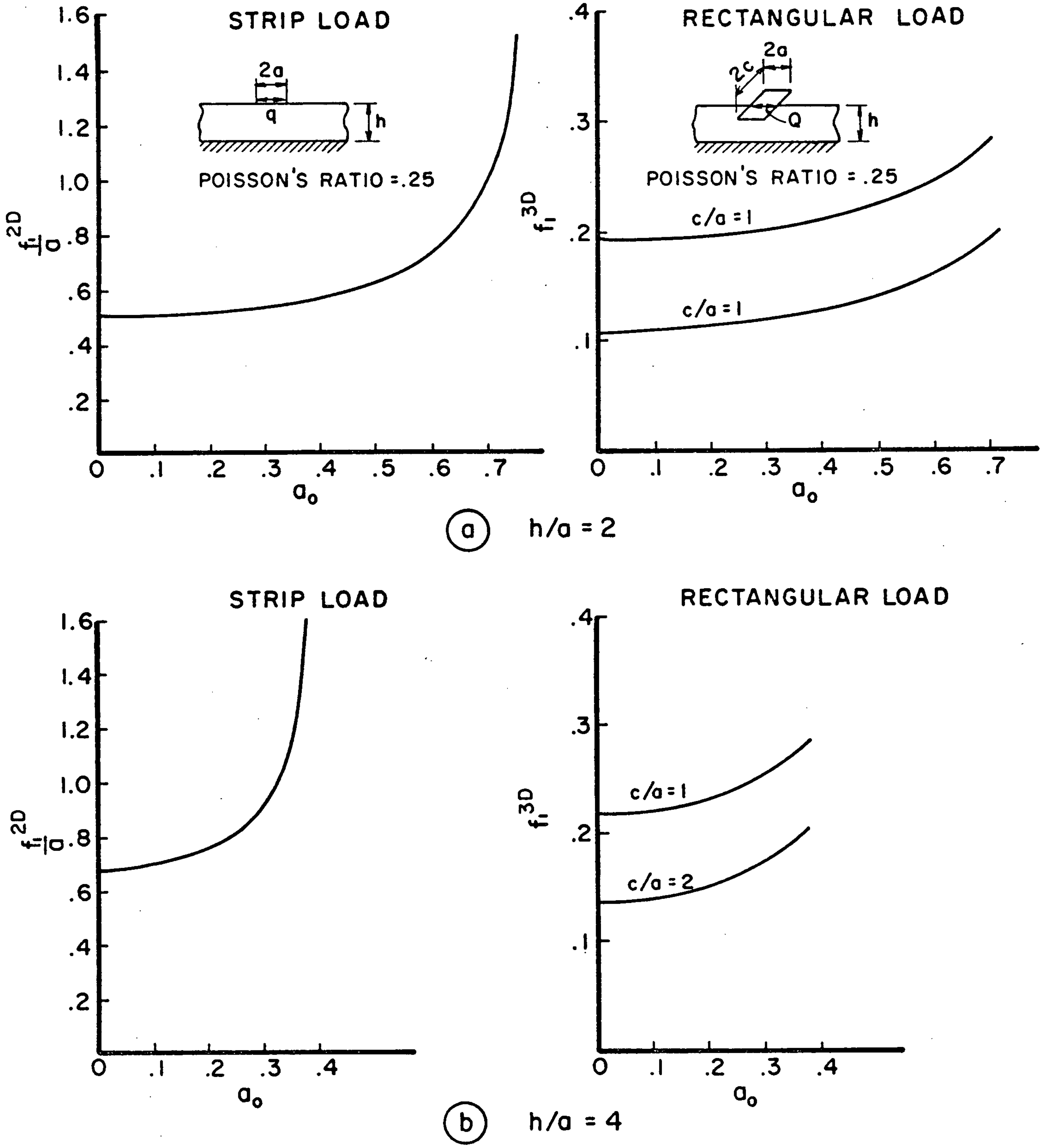
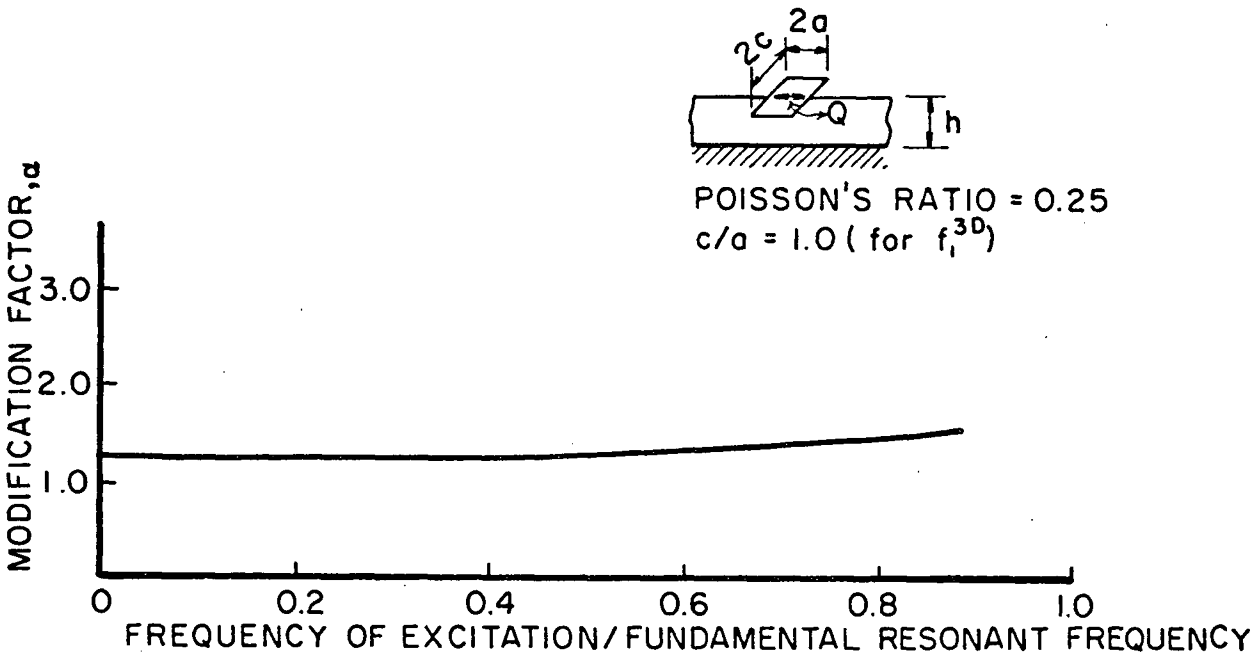
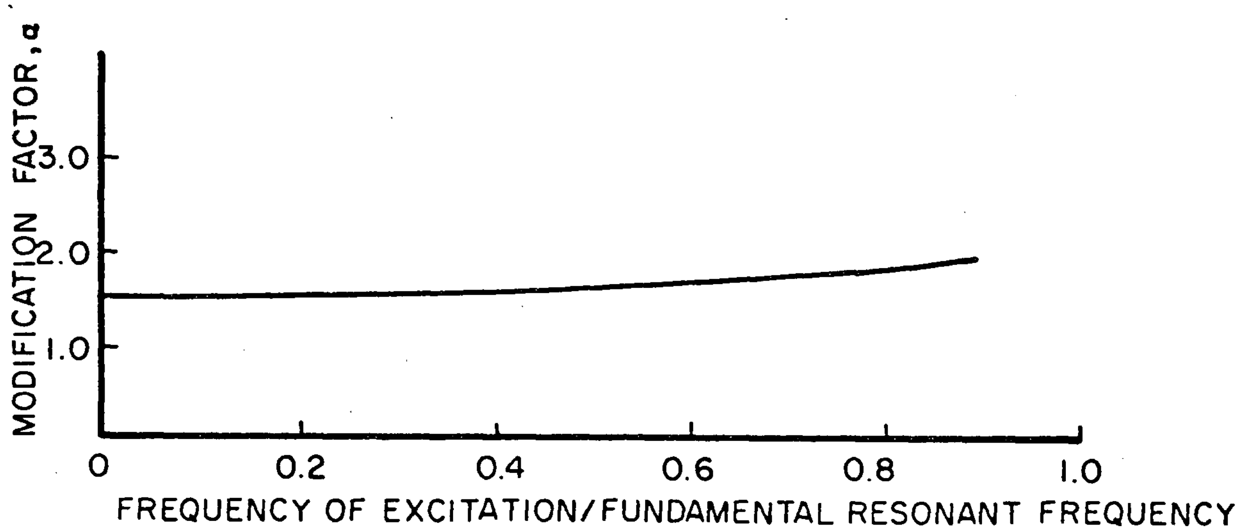


FIG.6 DYNAMIC GROUND COMPLIANCE OF A SOIL LAYER.



(a) $h/a = 2.0$



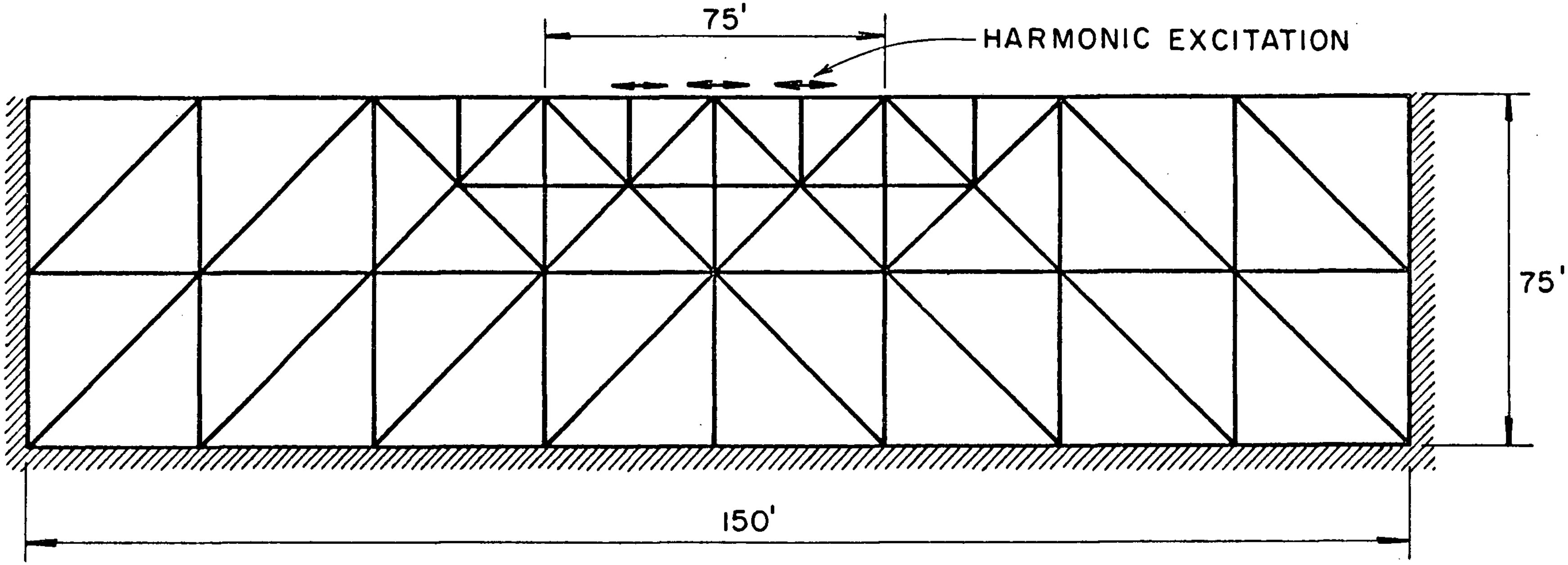
(b) $h/a = 4.0$

FIG. 7 MODIFICATION FACTOR, a.

could be obtained from a static analysis and this will be considered in Chapter 5.

The analyses performed so far have been based on the assumption that the shear stress distribution at the contact between the structure and the soil layer is uniform. However, it is more likely that the displacements rather than the shear stresses would be uniform at the contact. The ground compliances (plane strain) for these two cases were obtained using a finite element dynamic analysis. The finite element mesh used is shown in Fig. 8. A flow chart for the computer program used is shown in Fig. 9. The ground compliance factors f_1^{2D} for the two assumptions are compared in Fig. 10 where it may be seen that the assumption of a uniform shear stress at the contact gives a higher (approximately 20%) ground compliance for the range of frequencies $0 < a_0 < 1$.

It would appear reasonable to assume that the ground compliance factors f_1^{3D} for the cases of uniform stress and uniform displacement would be similar to those for the two-dimensional case and hence the modification factor α obtained for the case of uniform stress would also apply for the case of a rigid foundation.



POISSON'S RATIO = 0.3
YOUNG'S MODULUS = 10^6 lbs./ft^2

FIG. 8 FINITE ELEMENT MESH.

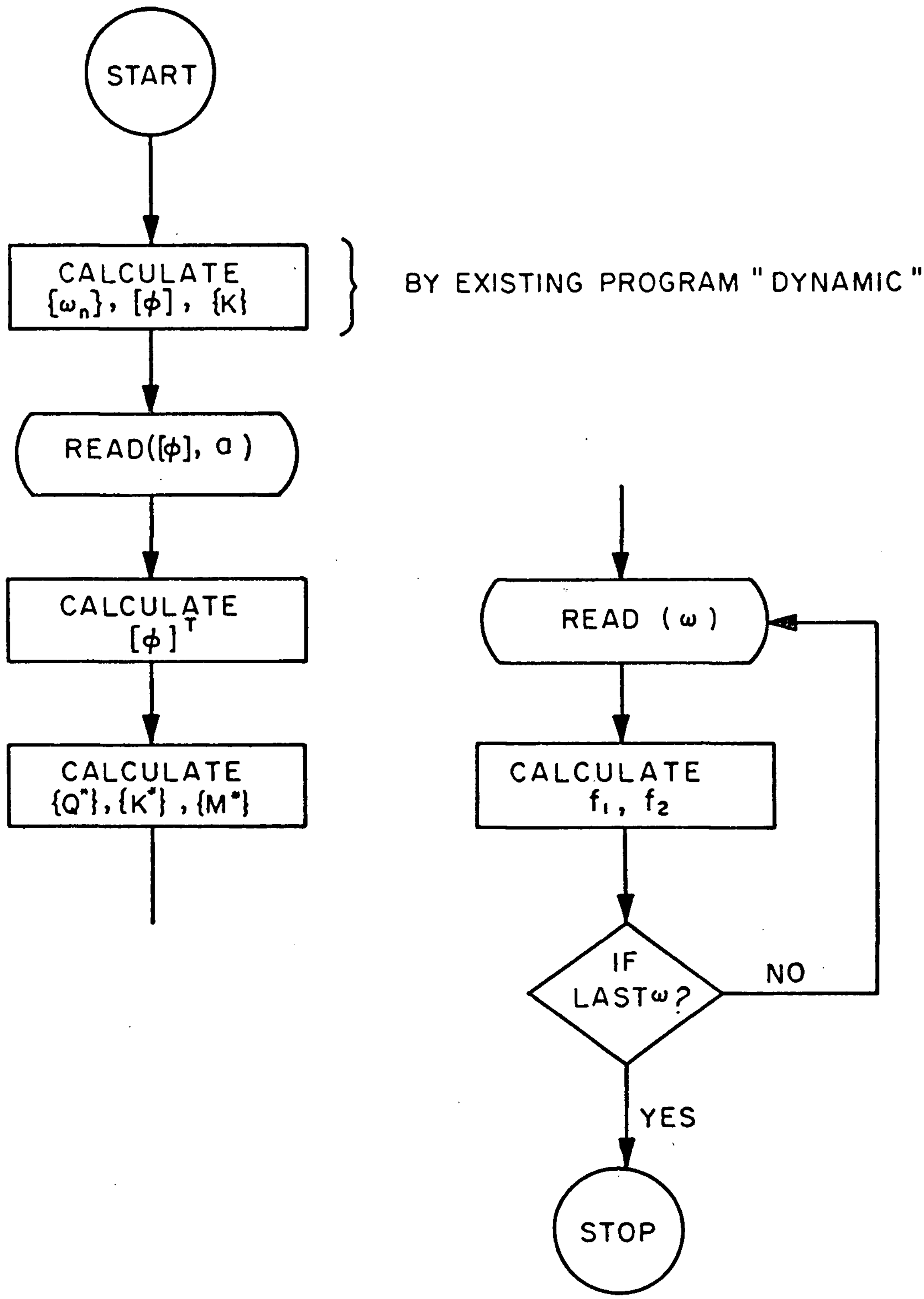


FIG.9 PROGRAM FOR COMPUTATION OF GROUND COMPLIANCE BY FINITE ELEMENT METHOD.

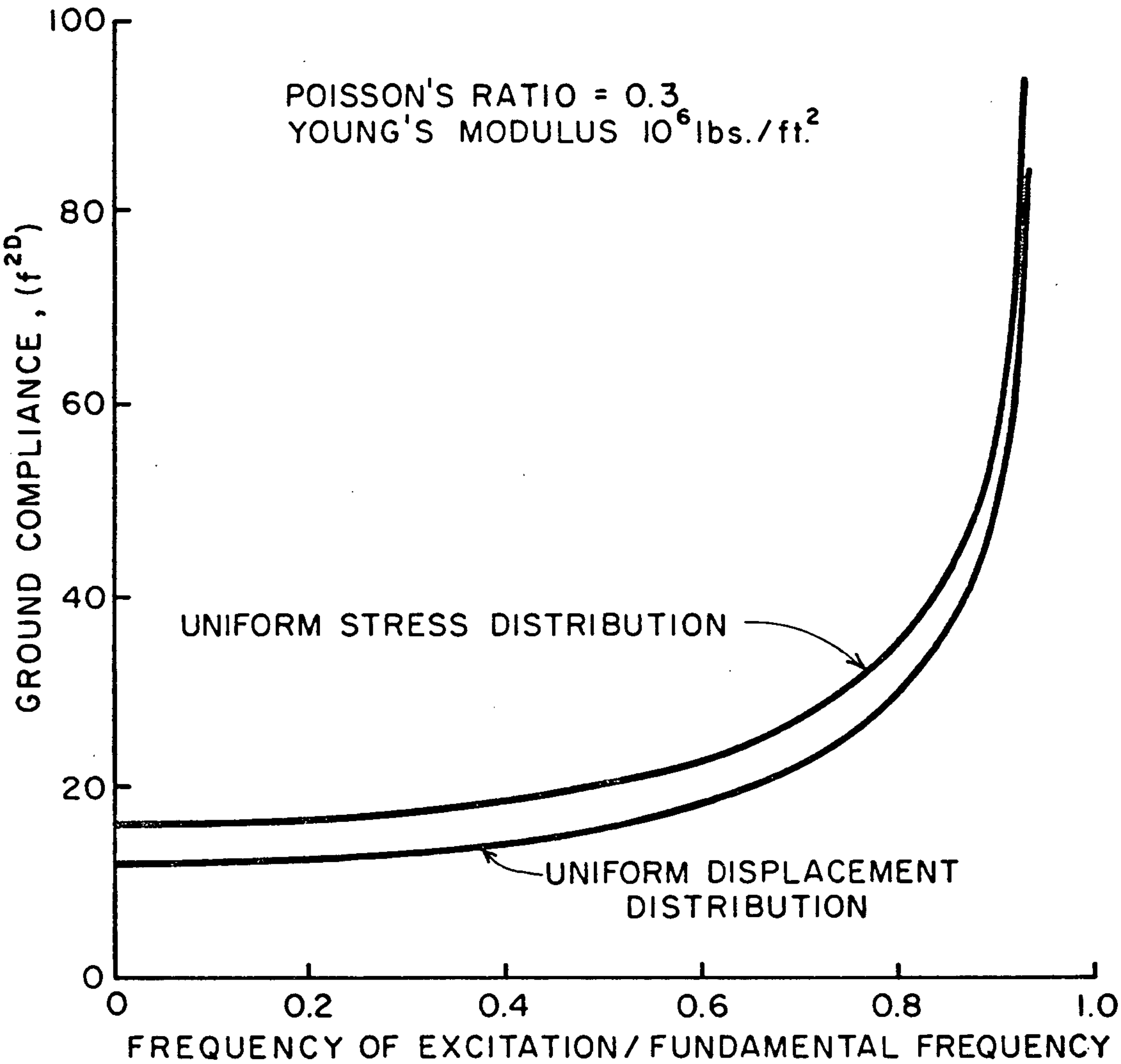


FIG. 10 COMPARISON OF THE GROUND COMPLIANCE FACTOR, f_1 FOR UNIFORM STRESS AND UNIFORM DISPLACEMENT.

CHAPTER 5

AN APPROXIMATE TWO-DIMENSIONAL SHEAR STIFFNESS ANALYSIS BASED ON THE STATIC SHEAR STIFFNESS OF THE SOIL LAYER

Chapter 4 indicates that the dynamic response of a three-dimensional structure involving soil-structure interaction may be estimated from a two-dimensional analysis if the appropriate modification factor is used. Since the modification factor α is almost constant for frequencies less than $a_1 = 1.57$, the constant value of α based on a static analysis may be used. In this chapter, the modification factor α based on a static shear stiffness will be evaluated.

5.1 Shear Stiffness of a Rigid Rectangular Foundation and Soil Layer System

As shown in Fig. 11, the displacement at the surface of a soil layer of depth h subjected to a surface load is approximately given as (10, 11):

$$u(x,y,o) = u_1(x,y,o) - u_2(x,y,h) \tag{5:1}$$

where u : displacement at the surface of a soil layer of depth h.

u_1 : displacement at the surface of a half space.

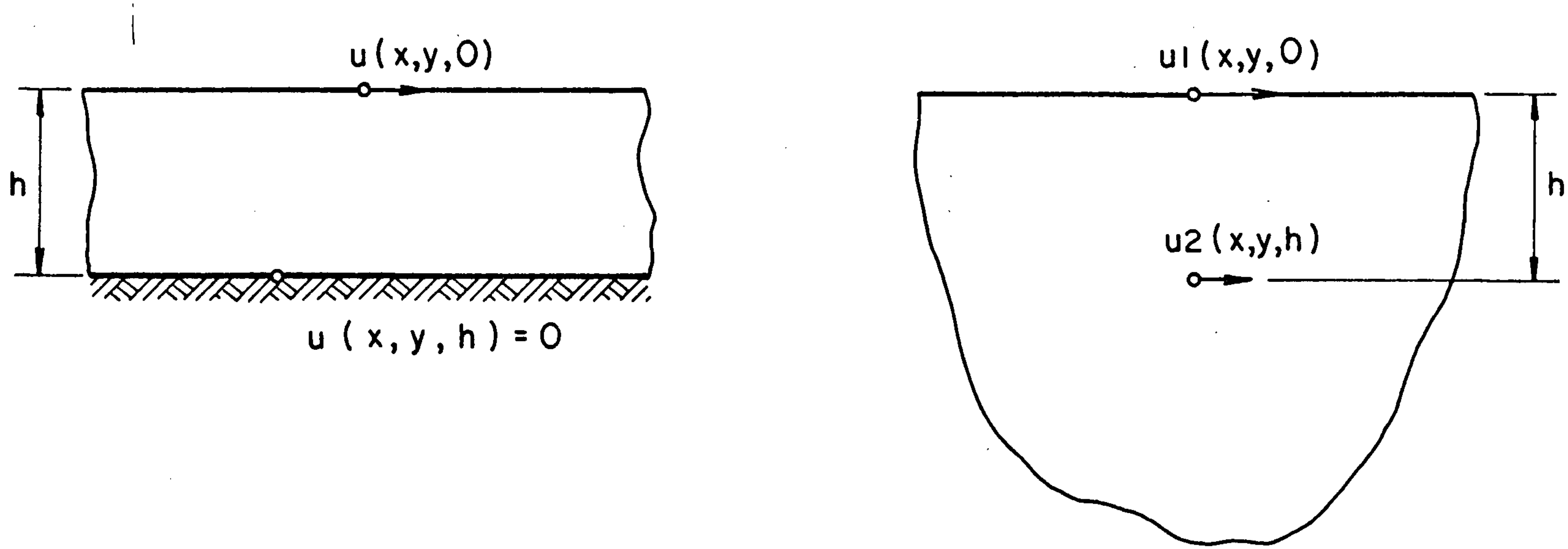
u_2 : displacement at the depth h in the half space.

The displacements u_1 and u_2 at any point in a horizontal plane for a uniform rectangular load can be calculated as follows.

If a horizontal point load is applied on the surface of a semi-infinite elastic body (Fig. 12), the displacement at the point (x,y,z) will be given by Cerruti's formula:

$$u(x,y,z) = \frac{Q}{4\pi G}\left\{\frac{x^2}{R^2} + 1 + \frac{1-2\nu}{R+z} R\left(1 - \frac{x^2}{R(R+z)}\right)\right\} \tag{5:2}$$

where Q : applied horizontal point load.



a) SOIL LAYER

b) HALF SPACE

FIG. 11 DISPLACEMENT IN SOIL LAYER AND HALF SPACE.

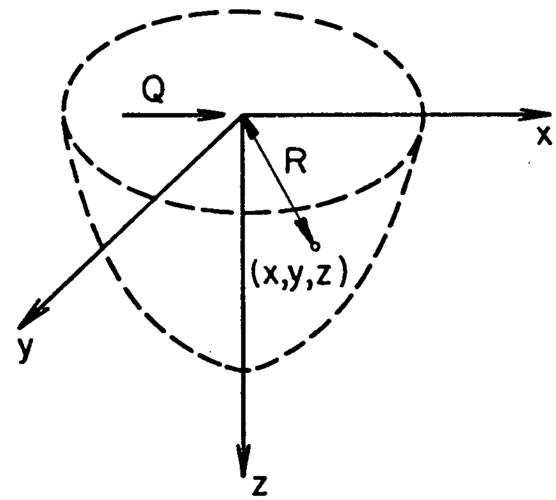


FIG.12 HALF SPACE SUBJECTED TO A CONCENTRATED LOAD

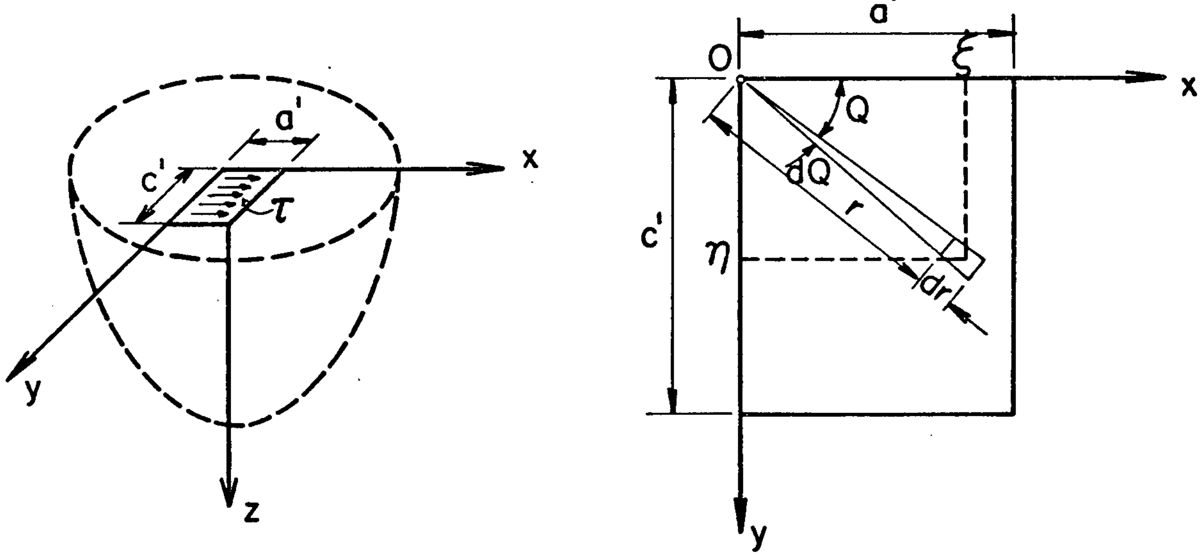


FIG. 13 UNIFORM SHEAR LOAD OVER A RECTANGULAR AREA ON A HALF SHEAR.

G,ν : shear modulus and poisson's ratio of soil.

$R = \sqrt{x^2 + y^2 + z^2}$

Integration of Equation 5:2 over the rectangular area, as shown in Fig. 13, with z = 0 will lead to the displacement at the corner of the uniform rectangular shear load as:

$$u_1^{corner} = \frac{\tau}{2xG}\left\{\int_0^{\tan^{-1}c'/a'}\int_0^{a'/\cos\theta}(1-\nu\sin^2\theta)d\theta\, dr + \int_{\tan^{-1}G'/a'}^{\pi/2}\int_0^{c'/\sin\theta}(1-\nu\sin^2\theta)d\theta\, dr\right\}$$

$$= \frac{\tau a'}{2\pi G}F(\lambda) \tag{5:3}$$

where $F(\lambda) = (1-\nu)\log(\sqrt{1+\lambda^2}+\lambda) + \lambda\log\frac{\sqrt{1\ \lambda^2+1}}{\lambda}$

$\lambda = c'/a'$

This displacement at any point (x,y,o) within the loaded area may be determined by considering the loaded area to be comprised of four rectangular areas with the point (x,y) a common corner to the four areas. The displacement $u_1(x,y,o)$ is then obtained by adding the contributions from each of the four areas:

$$u_1(x,y,o) = \begin{cases} \frac{1}{2\pi G}[x\{F(\frac{y}{x}) - F(\frac{y-c'}{x})\} - (x-a')\{F(\frac{y}{x-a'}) - F(\frac{y-c'}{x-a'})\}] & x>a',\ y<c' \\ \frac{1}{2\pi G}[x\{F(\frac{y}{x}) + F(\frac{y-c'}{x})\} - (x-a')\{F(\frac{y}{a'-x}) - F(\frac{y-c'}{a'-x})\}] & x>a',\ y>c' \\ \frac{1}{2\pi G}[x\{F(\frac{y}{x}) - F(\frac{y-c'}{x})\} - (x-a')\{F(\frac{y}{a'-x}) - F(\frac{y-c'}{a'-x})\}] & o<x<a',\ y>c' \\ \frac{\tau a'}{2\pi G}(\frac{c'}{a'}) \qquad x = \frac{a'}{2}\ ,\quad y = \frac{c'}{2} \end{cases} \tag{5:4}$$

The integration of Equation 5:2 over the rectangular area is difficult for z = h. If the sides of a uniform rectangular load are small compared with the depth of a soil layer, the horizontal displacement at the depth h can be obtained from Equation 5:2 under the coordinates shown in Fig. 13 replacing an area load with a concentrated load at the center of the area as:

$$u_2(x,y,h) \quad = \quad \frac{\tau a'c'}{4\pi G}\,\frac{1}{R_1}\,\left[\frac{(x-a'/2)^2}{R_1^2} + 1 + \frac{1-2\nu}{R_1+h}\,R_1\left\{1 - \frac{(x-a'/2)}{R_1(R_1+h)}\right\}\right] \qquad (5:5)$$

where $\quad R_1 \quad = \quad \sqrt{(x-a'/2)^2 + (y-c'/2) + h^2}$

Now the displacements u_1 and u_2 are obtained from Equation 5:4 and Equation 5:5 respectively. Substituting these values into Equation 5:1, the horizontal displacement on the surface of a soil layer subjected to a uniform rectangular surface shear load can be calculated.

The above solution is for a uniform shear stress over the loaded area. It was mentioned in earlier chapters that it is much more likely that the displacement rather than the shear stress will be uniform over the area. The solution for a uniform displacement will now be considered.

The shear stiffness of a rectangular rigid foundation on a soil layer is the total shear force on a rigid foundation to cause a unit horizontal displacement. Dividing a rigid foundation into 2m x 2n number of small elements with uniform load distributions as shown in Figure 14, this force will be calculated as follows:

The unit horizontal displacement of the element (i,j) in a rigid foundation subjected to a horizontal force can be expressed by superposing the influences due to each element as:

$$\sum_{k=1}^{2m} \sum_{\ell=1}^{2n} u_{i,j}^{k,\ell}\, \tau_{k,\ell} \quad = \quad 1 \qquad (5:6)$$

where $\quad \tau_{k,\ell} \quad :\quad$ uniform shear stress applied on the element (k,ℓ)

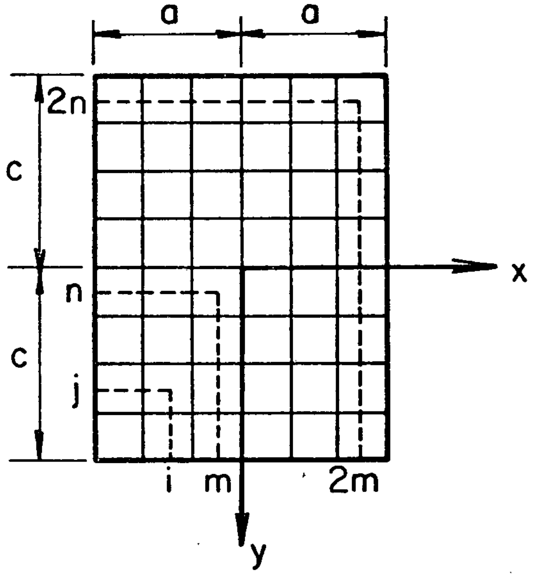


FIG. 14 RECTANGULAR AREA (2a x 2c) DIVIDED INTO (2n x 2m) ELEMENTS.

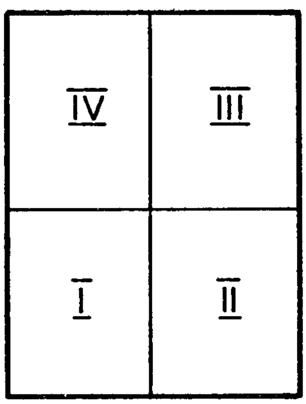


FIG. 15 RECTANGULAR AREA (2a x 2c) DIVIDED INTO FOUR ELEMENTS.

$u_{i,j}^{k,\ell}$: horizontal displacement of the element (i,j) due to a unit uniform shear stress on the element (k,ℓ).

Each quarter of the foundation area will be denoted as I, II, III and IV as shown in Fig. 15. Separating the influences of the element stress by the section I, II, III and IV, Equation 5:6 can be rewritten as:

$$\overset{\text{I}}{\sum_{k=1}^{m}\sum_{\ell=1}^{n} u_{i,j}^{k,\ell}\,\tau_{k,\ell}} + \overset{\text{II}}{\sum_{k=m+1}^{2m}\sum_{\ell=1}^{n} u_{i,j}^{k,\ell}\,\tau_{k,\ell}} + \overset{\text{III}}{\sum_{k=1}^{m}\sum_{\ell=n+1}^{2n} u_{i,j}^{k,\ell}\,\tau_{k,\ell}}$$

$$+ \overset{\text{IV}}{\sum_{k=m+1}^{2m}\sum_{\ell=n+1}^{2n} u_{i,j}^{k,\ell}\,\tau_{k,\ell}} = 1$$

or

$$\sum_{k=1}^{m}\sum_{\ell=1}^{n} \left(u_{i,j}^{k,\ell}\,\tau_{k,\ell} + u_{i,j}^{m+k,\ell}\,\tau_{m+k,\ell} + u_{i,j}^{m,n+\ell}\,\tau_{k,n+\ell} + u_{i,j}^{m+k,n+\ell}\,\tau_{m+k,n+\ell}\right) = 1 \tag{5:7}$$

Since the contact shear stress distribution under the rigid foundation is symmetrical with respect to x and y axes, the uniform shear stresses in the sections II, III and IV can be replaced with those in the section I.

$$\left.\begin{array}{ll} \text{II} & \tau_{2m-k+1,\ell} \\ \text{III} & \tau_{k,2n-\ell+1} \\ \text{IV} & \tau_{2m-k+1,2n-\ell+1} \end{array}\right\} = \tau_{k,\ell} \tag{5:8}$$

Substituting these relationships from Equation 5:8 into Equation 5:9 gives:

$$\sum_{k=1}^{m}\sum_{\ell=1}^{n}\left(u_{i,j}^{k,\ell} + u_{i,j}^{2m-k+1,\ell} + u_{i,j}^{k,2n-\ell+1} + u_{i,j}^{2m-k+1,2n-\ell+1}\right)\tau_{k,\ell} = 1 \qquad (5:9)$$

Applying Equation 5:9 to all elements in the section I gives:

$$\left.\begin{array}{l}
\sum_{k=1}^{m}\sum_{\ell=1}^{n}\left(u_{1,1}^{k,\ell} + u_{1,1}^{2m-k+1,\ell} + u_{1,1}^{k,2n-\ell+1} + u_{1,1}^{2m-k+1,2n-\ell+1}\right)\tau_{k,\ell} = 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{k=1}^{m}\sum_{\ell=1}^{n}\left(u_{i,j}^{k,\ell} + u_{i,j}^{2m-k+1,\ell} + u_{i,j}^{k,2n-\ell+1} + u_{i,j}^{2m-k+1,2n-\ell+1}\right)\tau_{k,\ell} = 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{k=1}^{m}\sum_{\ell=1}^{n}\left(u_{m,n}^{k,\ell} + u_{m,n}^{2m-k+1,\ell} + u_{m,n}^{k,2n-\ell+1} + u_{m,n}^{2m-k+1,2n-\ell+1}\right)\tau_{k,\ell} = 1
\end{array}\right\}\ m \times n \text{ equations} \qquad (5:10)$$

where the displacements can be obtained by Equations 5:1, 5:4 and 5:5 and the stresses can be obtained by solving Equation 5:10. Equation 5:10 has been put in a more compact form in Appendix 2.

The shear stiffness of the rectangular rigid foundation and soil layer system K_{3D} can be calculated as:

$$K_{3D} = \frac{4ac}{m \cdot n}\sum_{k=1}\sum_{\ell=1}\tau_{k,\ell} \qquad (5:11)$$

5.2 The Shear Stiffness of the Rigid Strip Foundation Soil Layer System

The rigid strip foundation area is divided into 2m number of elements as shown in Fig. 16 and a shear stress distribution on each element is assumed uniform. Then a unit horizontal displacement of the element (i) in a rigid foundation subjected to a horizontal force can be expressed by superposing the influences due to each element as:

$$\sum_{k=1}^{2m} u_i^k \tau_k = 1 \qquad (5:12)$$

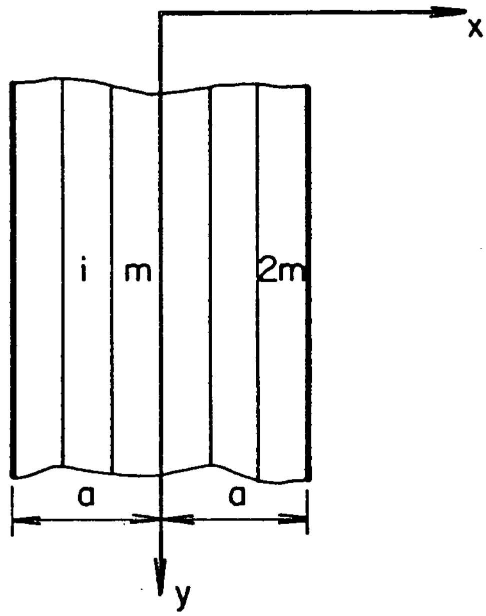


FIG.16 STRIP AREA DIVIDED INTO 2m ELEMENTS.

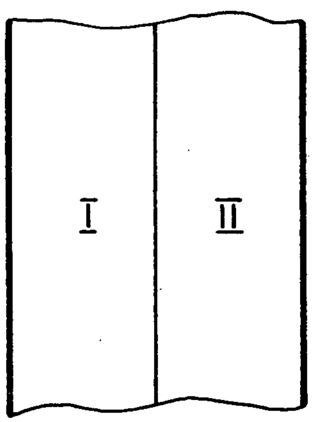


FIG.17 STRIP AREA DIVIDED INTO TWO SECTIONS.

where τ_k : uniform horizontal stress applied on the element (k).

u_i^k : horizontal displacement of the element (i) due to a unit uniform shear stress on the element (k).

Separating the influences of the element stress by the section I and II as shown in Fig. 17, Equation 5:8 can be rewritten as:

$$\overset{\text{I}}{\sum_{k=1}^{m} u_i^k \tau_k} + \overset{\text{II}}{\sum_{k=m+1}^{2m} u_i^k \tau_k} = 1 \tag{5:13}$$

or

$$\sum_{k=1}^{m} (u_i^k \tau_k + u_i^{m+k} \tau_{m+k}) = 1 \tag{5:14}$$

Since contact shear stress distribution under the rigid foundation is symmetrical with respect to y axis, the uniform shear stresses τ in section II can be replaced with those in section I.

$$\tau_{2m-k+1} = \tau_k \tag{5:15}$$

Substituting Equation 5:15 into Equation 5:14 gives:

$$\sum_{k=1}^{m} (u_i^k \tau_k + u_i^{2m-k+1} \tau_{2m-k+1}) = 1 \tag{5:16}$$

Applying Equation 5:16 to all elements in section I gives:

$$\left.\begin{array}{l} \sum_{k=1}^{m} (u_1^k + u_1^{2m-k+1})\, \tau_k = 1 \\ \cdots\cdots\cdots\cdots\cdots \\ \sum_{k=1}^{m} (u_i^k + u_i^{2m-k+1})\, \tau_k = 1 \\ \cdots\cdots\cdots\cdots\cdots \\ \sum_{k}^{m} (u_m^k + u_m^{2m-k+1})\, \tau_k = 1 \end{array}\right\} \text{m equations} \tag{5:17}$$

where the displacements can be obtained by Equation A1:5 (Appendix 3) and the stresses can be obtained by solving Equation 5:17.

The shear stiffness of a rigid strip foundation and soil layer system K_{2D} can be calculated as:

$$K_{2D} = \frac{2a}{m} \sum_{k=1}^{m} \tau_k \tag{5:18}$$

5.3 Calculation of Two and Three-Dimensional Stiffness for the Rigid Foundation

The static shear stiffnesses for the two and three-dimensional cases are given by Equations 5:18 and 5:11. The width of the footing is 2a in both cases but a unit length of footing is considered for the two-dimensional case while a length, 2c, is considered for the three-dimensional case. To compare stiffnesses then, the stiffness for the three-dimensional case must be divided by the length of the footing, 2c. Also, if both stiffnesses are divided by Young's modulus, E, a dimensionless stiffness is obtained.

The stiffness values obtained will depend on the number of area elements used. This is shown for the three-dimensional case in Fig. 18. It may be seen that if 30 x 30 elements are used, little error will occur. The stiffness values computed and shown in the next section were obtained using 30 area elements for the two-dimensional case and 30 x 30 area elements for the three-dimensional case.

5.4 Comparison of Two and Three-Dimensional Stiffness

Dimensionless stiffnesses for the three-dimensional case are shown in Table 1 for various ratios of c/h, h/a and Poisson's ratio, ν. Values shown range from 1.31 to 0.48. Dimensionless stiffnesses for the two-dimensional case are shown in Table 2 for various ratios of h/a and Poisson's

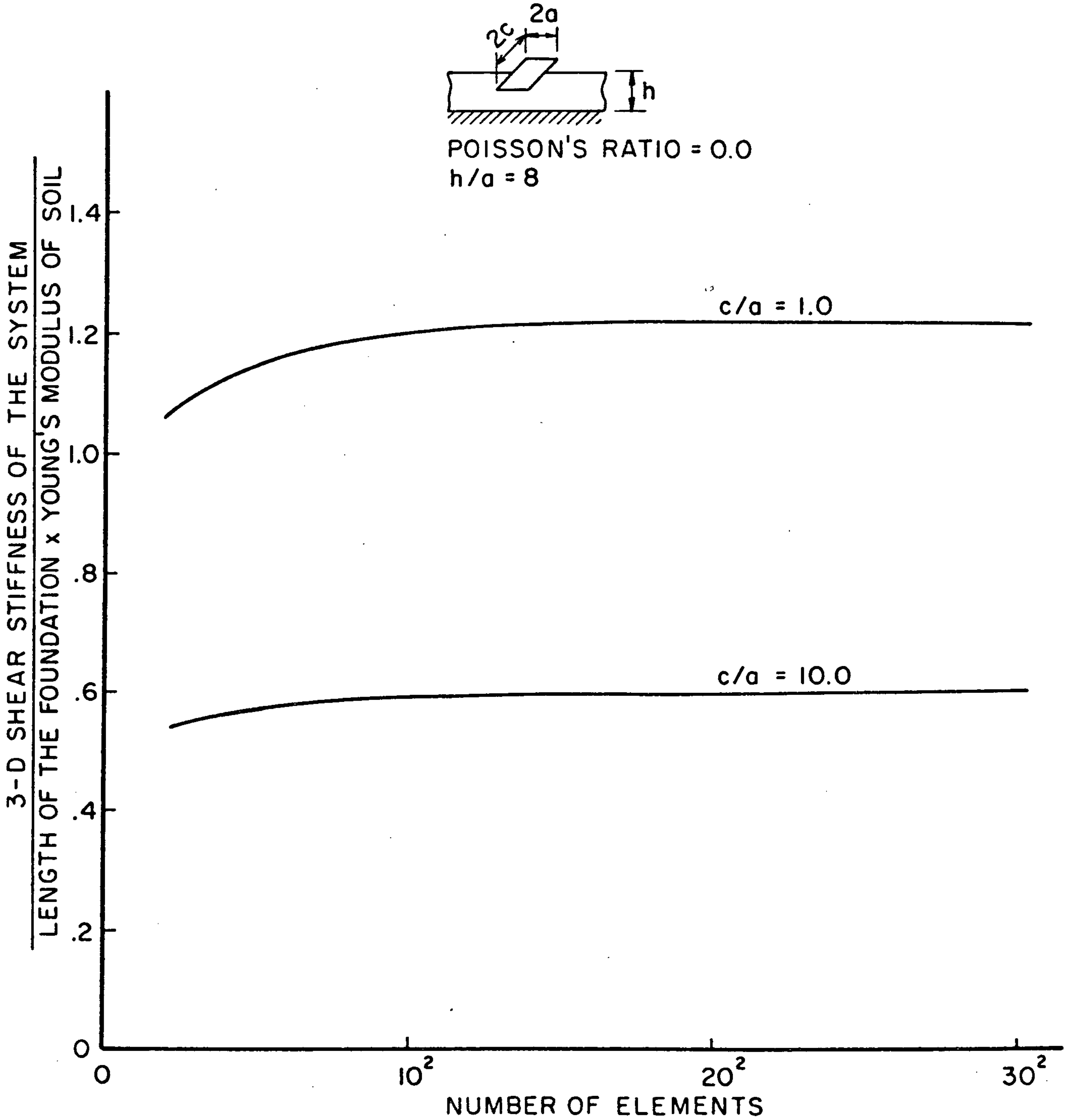
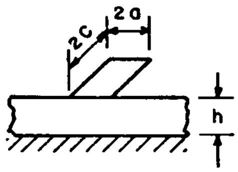


FIG. 18 EFFECTS OF NUMBER OF DIVISIONS OF RECTANGULAR AREA ON SHEAR STIFFNESS.

TABLE 1

3-D SHEAR STIFFNESS OF THE SOIL LAYER
———————————————————
YOUNG MODULUS x LENGTH OF THE FOUNDATION

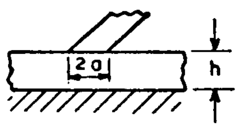


ν = Poisson's ratio

$\frac{c}{a}$	h/a = 4.			h/a = 8.			h/a = 16.		
	ν=0.	ν=.2	ν=.3	ν=0.	ν=.2	ν=.3	ν=0.	ν=.2	ν=.3
1.	1.314	1.211	1.180	1.222	1.128	1.100	1.179	1.090	1.065
3.	.898	.850	.842	.796	.754	.748	.746	.708	.702
5.	.798	.763	.761	.690	.661	.660	.634	.609	.608
10.	.712	.691	.695	.598	.581	.586	.535	.520	.524
20.	.690	.670	.665	.550	.538	.542	.489	.478	.483

TABLE 2

2-D SHEAR STIFFNESS OF THE SOIL LAYER
———————————————————
YOUNG MODULUS



ν =Poisson's ratio

h/a = 4.		h/a = 8.		h/a = 16.	
ν=0.	ν=0.3	ν=0.	ν=0.3	ν=0.	ν=0.3
.697	.683	.536	.538	.434	.443

ratio, ν, equal to 0.0 and 0.3. Values range from 0.70 to 0.44. As expected the three-dimensional stiffness is greater than the two-dimensional stiffness. However, as the length of the footing, 2c, becomes large compared with the width, 2a, the three-dimensional stiffness approaches the two-dimensional stiffness.

The dimensionless stiffness for a square and strip foundation are shown in Fig. 19. It may be seen that the square footing is considerably stiffer than the strip footing and that both decrease as the depth of the soil layer increases.

The dimensionless stiffness for the three-dimensional case is shown as a function of c/a and h/a in Fig. 20. It may be seen that the stiffness decreases with increasing values of both c/a and h/a.

5.5 Modification Factor

The modification factor, α, was defined in Chapter 4, Equation 4:7 as $\alpha = \frac{f_1^{2D}/2c}{f_1^{3D}}$ and is the ratio of the stiffnesses, namely:

$$\alpha = \frac{K_{3D/2c}}{K_{2D}} \tag{5:19}$$

The variation of α with the geometry for Poisson's ratio equal to 0.3 are shown in Figs. 21, 22 and 23. It may be seen that the effect of the depth of soil is as follows:

1. The modification factor increases with the depth of a soil layer and will be infinite for an infinite depth since the displacement for a strip load on the surface of a semi-infinite body is infinite but not for a rectangular load. (Fig. 21)
2. The modification factor for a square foundation on a

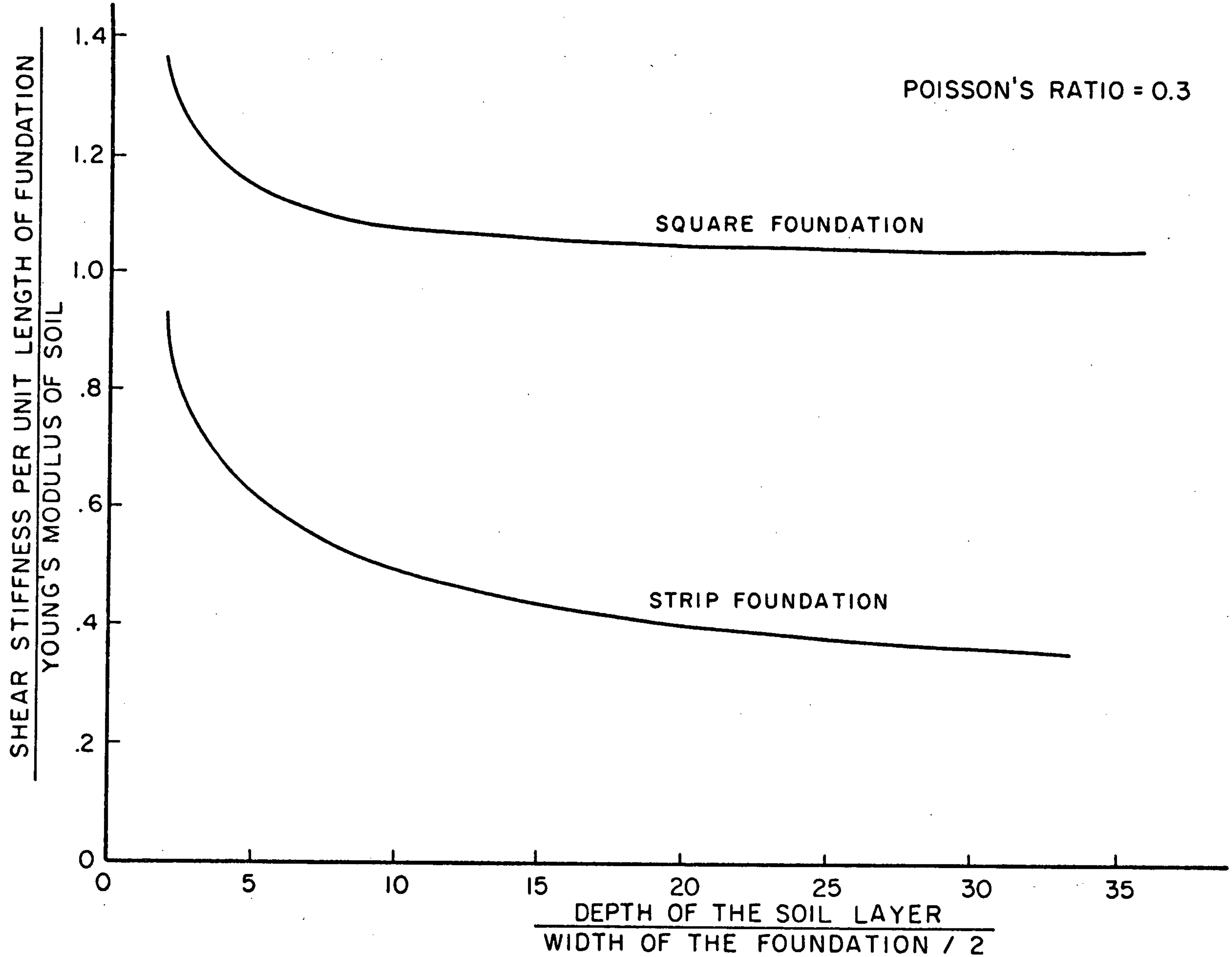


FIG. 19 SHEAR STIFFNESS OF THE SYSTEM PER UNIT LENGTH OF FOUNDATION.

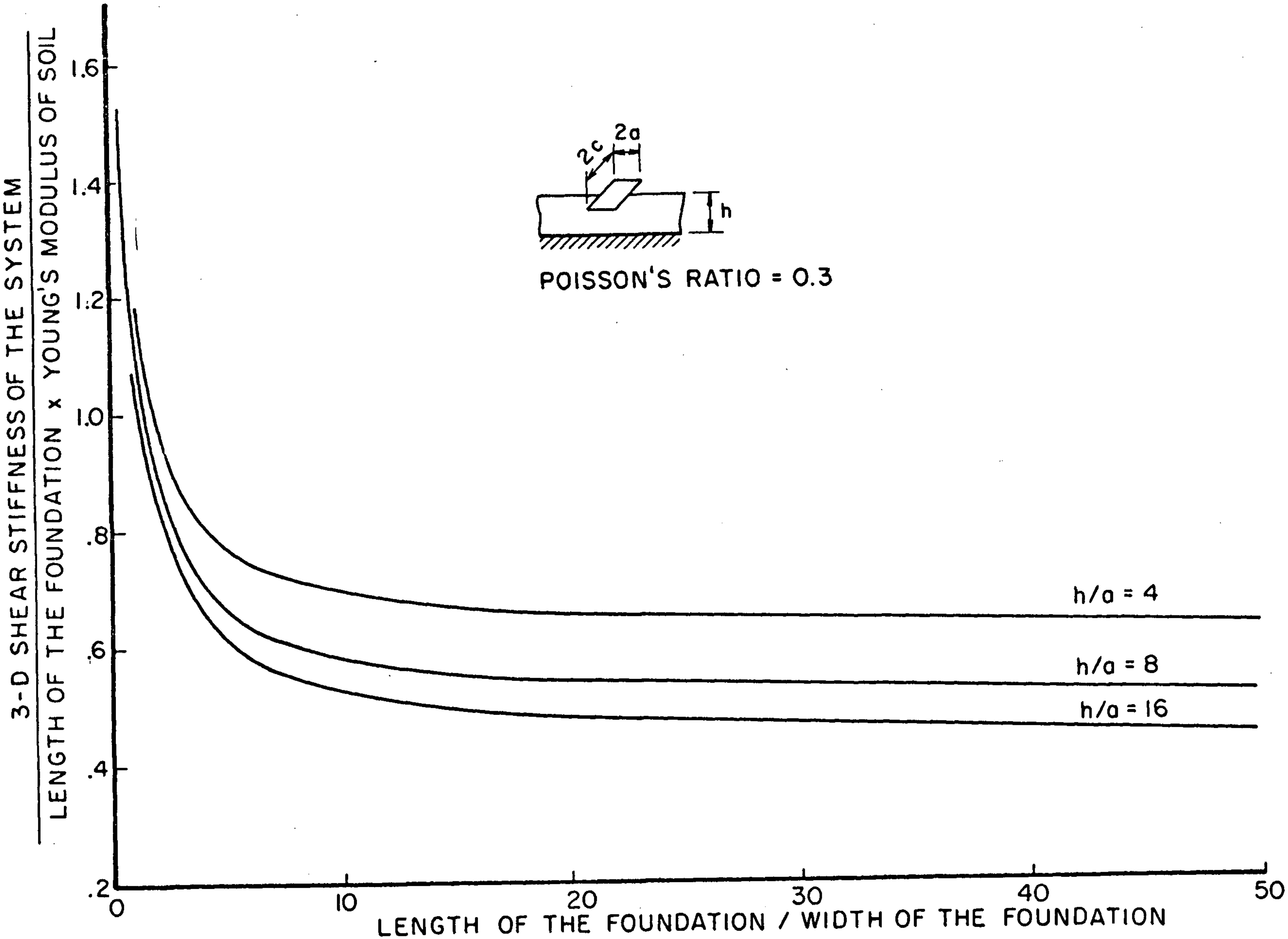


FIG.20 SHEAR STIFFNESS OF THE SYSTEM PER UNIT LENGTH OF RECTANGULAR FOUNDATION.

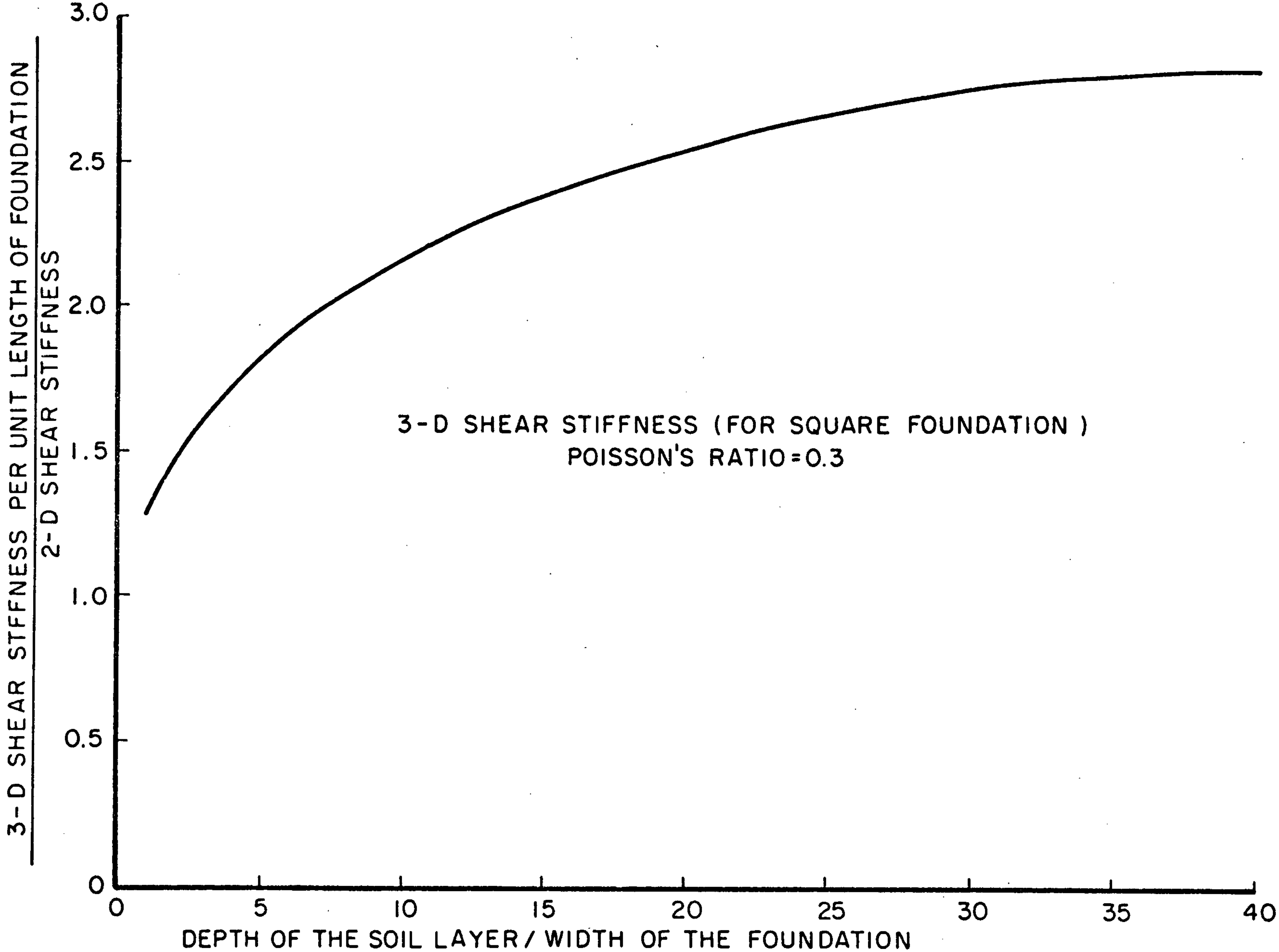


FIG. 21 SHEAR STIFFNESS RATIO.

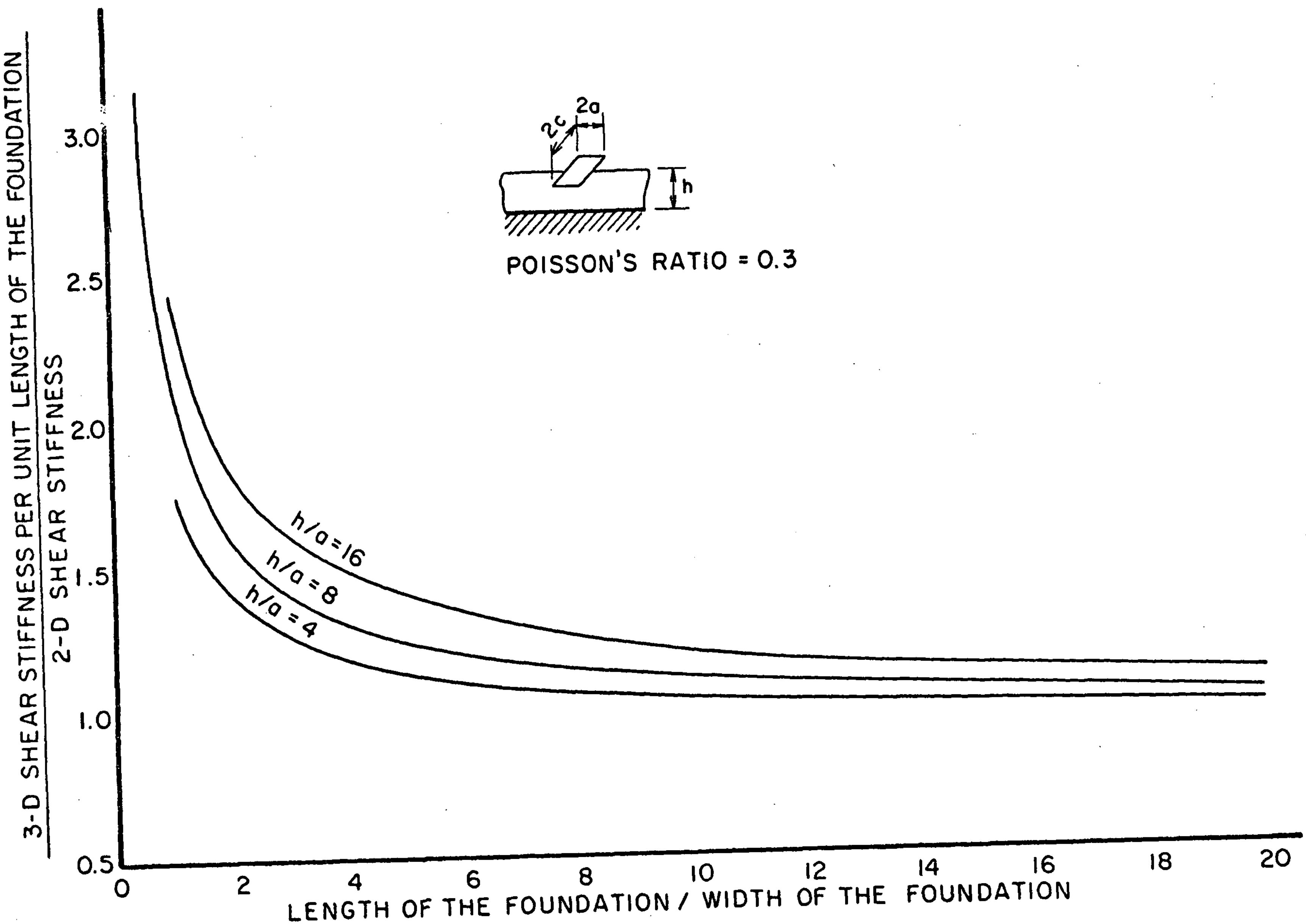


FIG.22 SHEAR STIFFNESS RATIO.

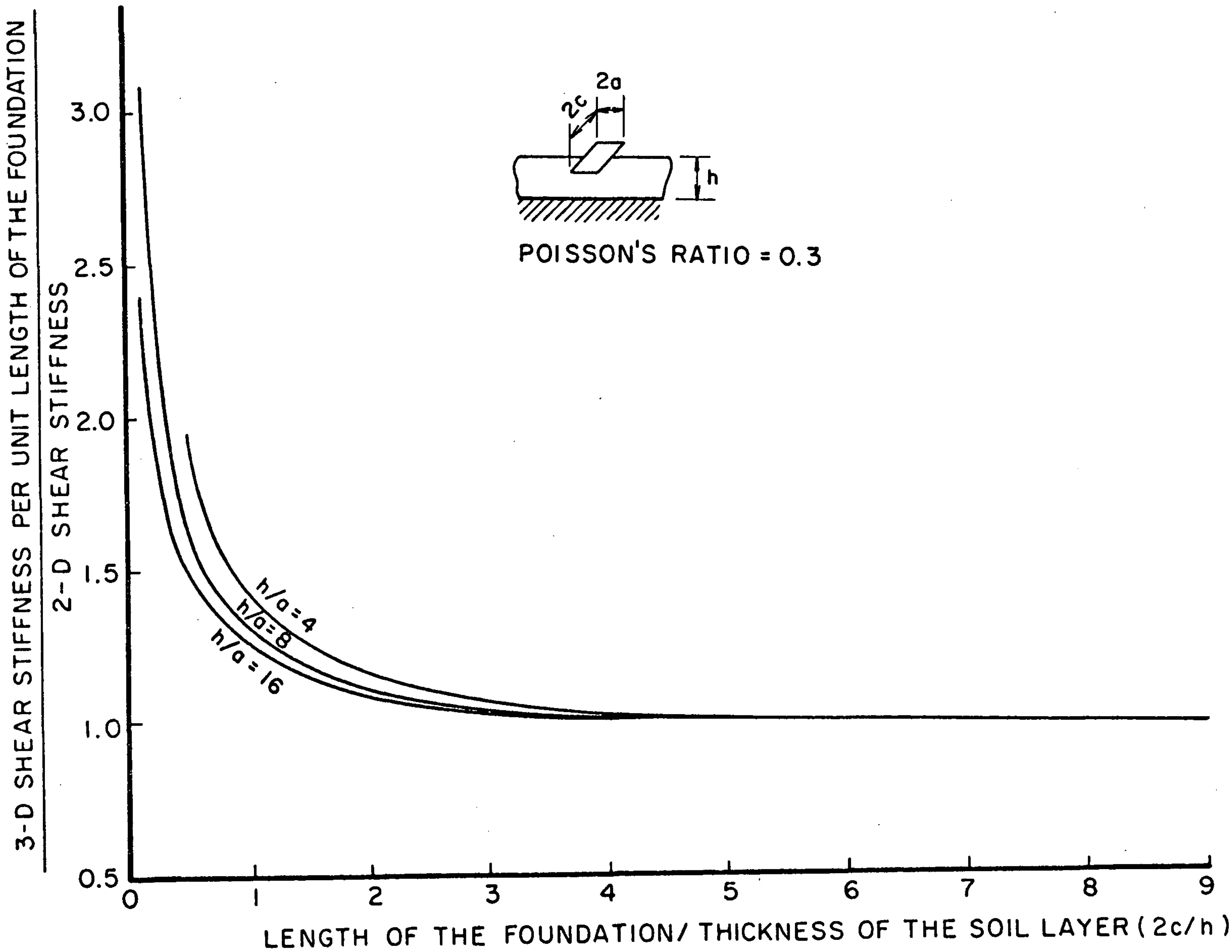


FIG.23 SHEAR STIFFNESS RATIO.

soil layer with the ratio $h/a \approx 1$ is about one (Fig. 21). The error caused by the use of a plane strain finite element method for this case will be negligible if radiational damping is ignored.

The effect of the ratio of the sides (c/a) of a rectangular foundation are as follows:

1. The modification factor decreases with the increase in sides ratio c/a and will be one for infinite sides ratio c/a. (Fig. 22, Fig. 23)
2. The modification factor is almost one at the ratio $2c/h = 5$. The error caused by the use of a plane strain finite element method for the ratio $2c/h \geq 5$ will be negligible if radiational damping is ignored. (Fig. 23)

It should be noted here that the above-mentioned results are based on the following assumptions.

1. The rigid foundation is divided into small equal area elements and the stress distribution on each element is assumed uniform.
2. The displacement of an element is assumed to be uniform and is equal to that at the centre of the element, although the displacement within an element is not uniform because of assumption 1.
3. The displacement at the surface of a soil layer is assumed to be the difference between the surface displacement and the displacement at a depth equal to the depth of soil layer (semi-infinite analysis).
4. The displacement in a semi-infinite body at the same depth

as that of a soil layer is calculated by replacing an area load on the element with a concentrated load at the centre of the element.

These assumptions lead to the following errors:

1. The error due to assumption 4 increases as the ratio of c/a increases. However, it reduces as the ratio h/a increases.
2. The error due to assumption 3 increases as h/a reduces.
3. The error due to assumption 2 gives a smaller stiffness under any conditions.

CHAPTER 6

APPLICATION OF THE EQUIVALENT SHEAR STIFFNESS ANALYSIS TO THE DYNAMIC RESPONSE OF A RECTANGULAR FOUNDATION OVER A SOIL LAYER

In this chapter a plane strain finite element analysis is modified to model the case of rigid rectangular base on a soil layer (Fig. 24). The results are compared with the analytical solution.

6.1 Transfer Function

The force Q created by a harmonic horizontal excitation u_2 under the rectangular rigid body is:

$$Q = - M \frac{d^2u_2}{dt^2} \tag{6:1}$$

where M : total mass of the rigid body on a soil layer.

Fourier transformation of Equation 6:1 gives:

$$\overline{Q} = M\omega^2\overline{u}_2 \tag{6:2}$$

where $\overline{Q}$, $\overline{u}_2$: Fourier transformations of Q and u_2 respectively.

ω : angular frequency of the excitation.

Therefore, the transfer function $H_2(\omega)$ is:

$$H_2(\omega) = M\omega^2 \tag{6:3}$$

Substitution of the transfer function $H_2(\omega)$ from Equation 6:3 into Equation 4:2 gives:

$$I(i\omega)_{3D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu}\,(f_1^{3D} + if_2^{3D})\; M\omega^2}$$

$$= \frac{H_1(\omega)}{1 - ba_0^2\;(f_1^{3D} + if_2^{3D})} \tag{6:4}$$

where $b = \dfrac{M}{\rho a^3}$

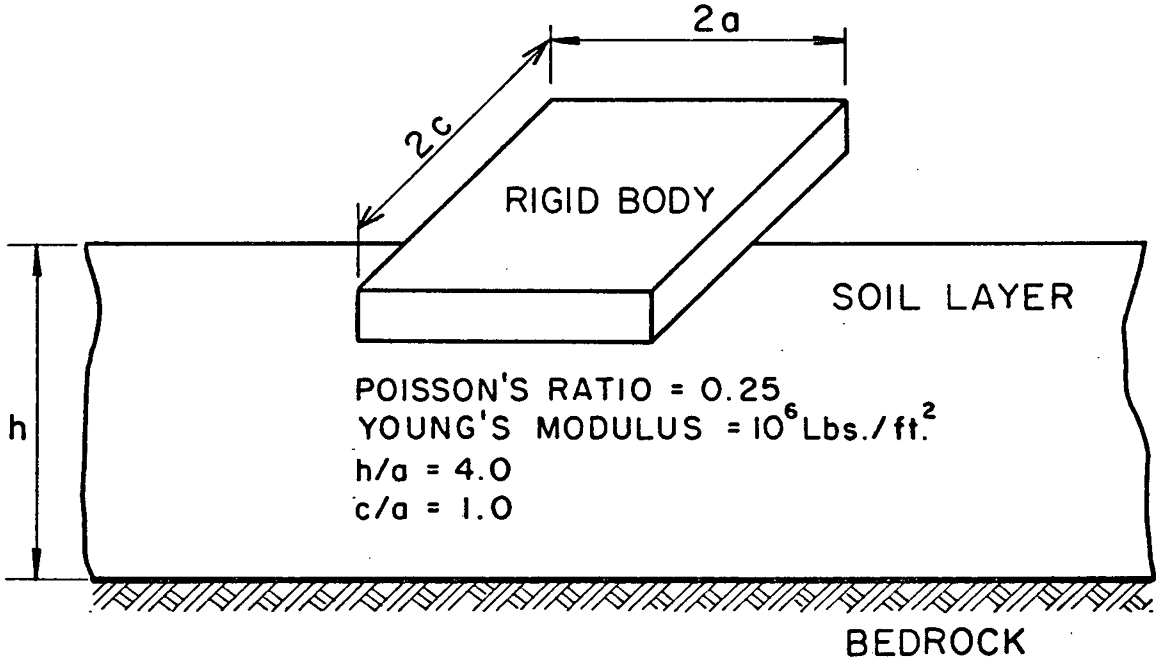


FIG. 24 RIGID BODY ON A SOIL LAYER.

$$a_0 = \frac{\omega a}{\sqrt{\frac{G}{\rho}}}$$

ρ : mass of a unit volume of soil.

The absolute value of the complex transfer function in Equation 6:4 is:

$$I(\omega)_{3D} = \frac{H_1(\omega)}{\sqrt{(1 - ba_0^2\ f_1^{3D})^2 + (ba_0^2\ f_2^{3D})^2}} \quad (6:5)$$

Ignoring f_2^{3D} gives:

$$I(\omega)_{3D} = \frac{H_1(\omega)}{1 - ba_0^2\ f_1^{3D}} \quad (6:6)$$

Also, the absolute transfer function $I(\omega)_{2D}$ by a plane strain analysis in Equation 4:4 is:

$$I(\omega)_{2D} = \frac{H_1(\omega)}{1 - ba_0^2\ (\frac{f^{2D}}{2c})} \quad (6:7)$$

Modification of the transfer function $I(\omega)_{2D}$ to get $I(\omega)_{3D}$ gives:

$$I(\omega)'_{2D} = \frac{H_1(\omega)}{1 - \frac{1}{\alpha}\ ba_0^2\ (\frac{f^{2D}}{2c})} \quad (6:8)$$

The transfer function $H_1(\omega)$ can be obtained by the methods discussed in reference 12. For this special case considered here, it can be given as:

$$H_1(\omega) = \sec a_1 \quad (6:9)$$

where $a = \frac{\omega h}{\sqrt{\frac{G}{\rho}}}$

For frequencies corresponding to $a_1 \leq \pi/2$, no radiational damping will occur (Chapter 3).

6.2 Fundamental Frequency of the System

The fundamental frequency is the lowest frequency at which the term $I(\omega)$ goes to infinity. This may occur when:

$$H_1(\omega) = \text{Sec } a_1 = \infty \tag{6:10}$$

or when $1 - ba_0^2 f_1^{3D} = 0$ (three-dimensional analysis)

or $1 - ba_0^2 f_1^{2D} = 0$ (plane strain analysis) (6:11)

or $1 - ba_0^2 \dfrac{f_1^{2D}}{2c} = 0$ (modified plane strain analysis)

From Equation 6:10, $H_1(\omega) = \infty$ when $a_1 = \pi/2$. Now $a_0 = \omega a/\sqrt{\frac{G}{\rho}}$ (Equation 6:4) and $a_1 = \omega h/\sqrt{\frac{G}{\rho}}$, hence the ratio $a_1/a_0 = h/a$. For $h/a = 4$, then $H_1(\omega) = \infty$ when $a_0 = \frac{1}{4}a_1$. Therefore resonance occurs when $a_0 = \frac{1}{4}a_1 = \pi/8 = 0.391$. Resonance may also occur for frequencies corresponding to a_0 less than 0.391 ($a_1 < \pi/2$) provided Equation 6:11 is satisfied and, since a_1 will be less than $\pi/2$ for these cases, no radiational damping will occur.

The fundamental frequencies for $a_1 < \pi/2$ or $a_0 < 0.391$ are calculated as follows:

1. f_1^{2D} and f_1^{3D} are obtained from Fig. 6b for values of a_0 from zero to 0.39.
2. The mass ratio, b, required to satisfy Equation 11 is then calculated.

The analytical relationships between a_0 and b are shown in Fig. 25 for both the plane strain and square footing cases. Modification factors based on static shear stiffness were calculated in Chapter 5. For a square footing with $h/a = 4$ the modification factor is 1.73 (Fig. 23). The plane strain result when modified for a square footing is also shown in Fig. 25 and it may be seen to be in close agreement with the analytical solution.

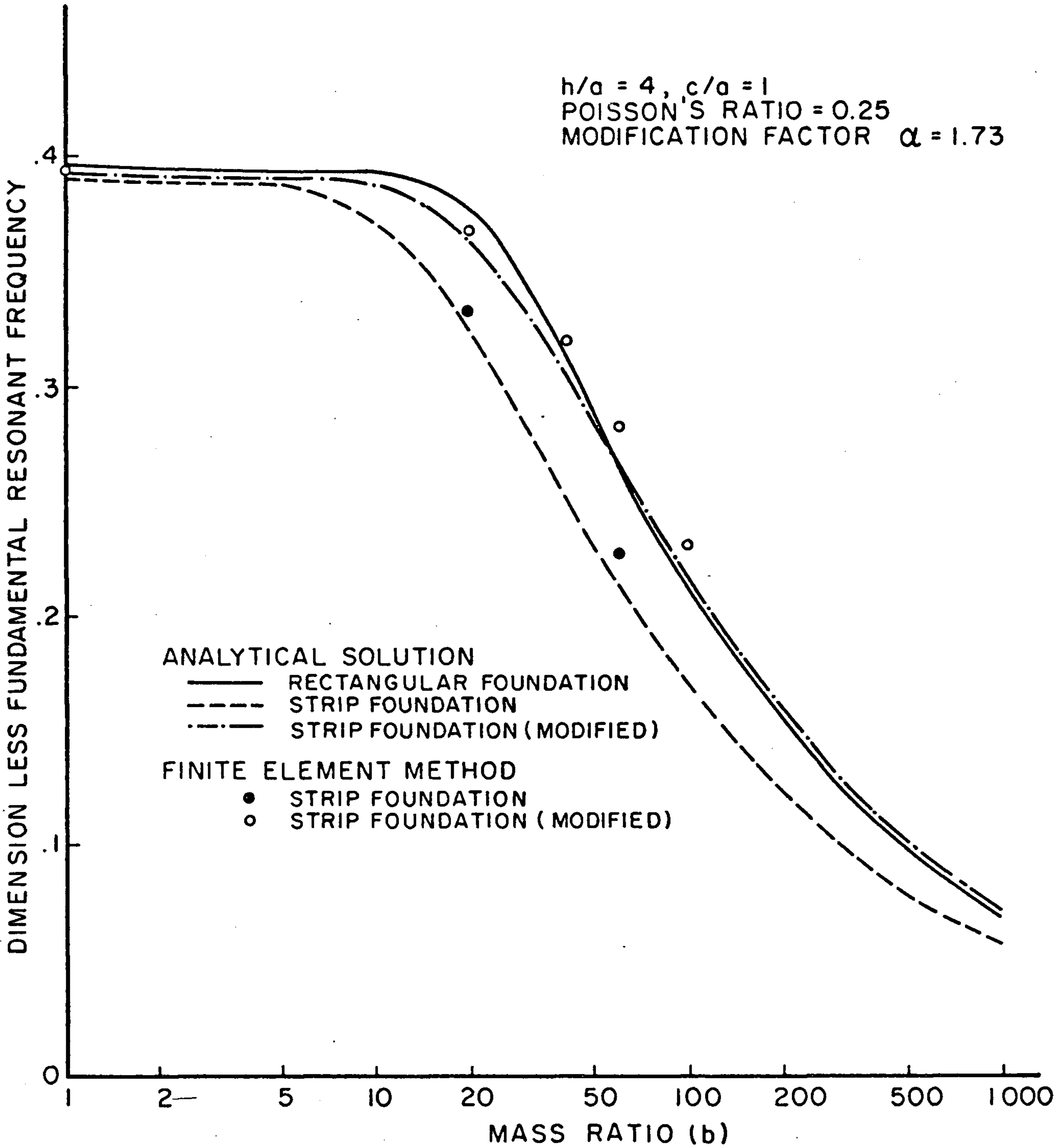


FIG.25 FUNDAMENTAL RESONANT FREQUENCY OF THE SYSTEM.

The fundamental frequencies were also obtained from a finite element analysis. The finite element mesh used is shown in Fig. 26. The results for the plane strain case and for the square footing (α = 1.73) are also shown in Fig. 25 and it is seen that they are very similar to the analytical results.

6.3 Discussion of the Results

The modification of a plane strain analysis based on the static modification factor gives the fundamental resonant frequency in shear close to, or the same as, that from a three-dimensional analysis as shown in Fig. 25. It is interesting that the results show no significant difference between the plane strain analysis and three-dimensional analysis for the mass ratio less than five.

The smaller ground compliance f_1 for the same frequency a_0 gives a larger resonant frequency of the system and the larger one gives a smaller resonant frequency under the constant mass ratio. The larger and smaller resonant frequencies obtained by a modified plane strain transfer function analysis than those by a three-dimensional one may be explained by $\frac{1}{\alpha}\frac{f_1^{2D}}{2C} > f_1^{3D}$ and $\frac{1}{\alpha}\frac{f_1^{2D}}{2C} < f_1^{3D}$ respectively. When the mass ratio is about 54 these two analyses give the same results for this particular case, where $\frac{1}{\alpha}\frac{f_1^{2D}}{2C}$ and f_1^{3D} will be the same values. The resonant frequencies from a finite element method are larger than those from a transfer function because the ground compliance f_1 for the uniform displacement distribution over the contact area under the foundation is smaller than that for the uniform stress distribution. Since the constant modification factor α is used for the modification on a finite element method, the same tendencies as mentioned above for the transfer function occur.

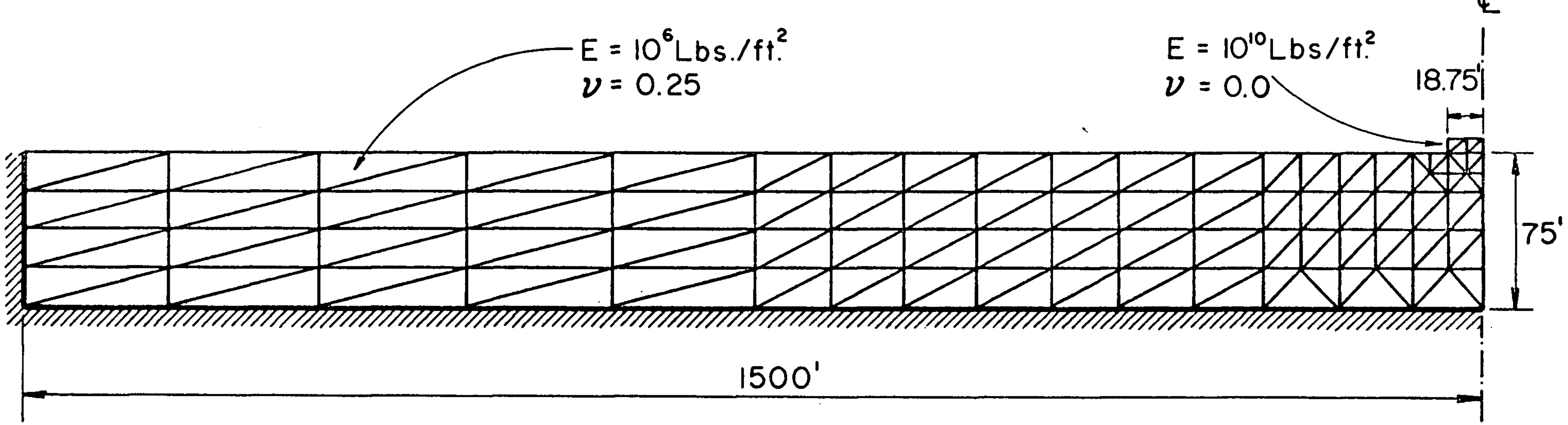


FIG.26 FINITE ELEMENT MESH FOR RIGID BODY AND SOIL LAYER SYSTEM.

It will be possible to get exact fundamental resonant frequencies for this system if a correct modification factor is used for each frequency a_0. Even a constant modification factor based on statical case will give a satisfactory result.

CHAPTER 7

CONCLUSIONS AND SUMMARY

If a structure with rectangular rigid base (2a x 2c) is attached on the surface of a homogeneous, isotropic and linear elastic soil layer, the transfer function between the input harmonic excitation on the bedrock and response at the base of the structure can be expressed with the ground compliance ignoring damping as:

$$I(\omega)_{3D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu} f_1^{3D} H_2(\omega)}$$

If a plane strain analysis is used for the same system the transfer function is:

$$I(\omega)_{2D} = \frac{H_1(\omega)}{1 - \frac{1}{a\mu} \frac{f_1^{2D}}{2c} H_2(\omega)}$$

If a plane strain analysis with the modification factor α is applied, the transfer function $I(\omega)_{3D}$ becomes:

$$I(\omega)'_{2D} = \frac{H_1(\omega)}{1 - \frac{1}{a\alpha\mu} \frac{f_1^{2D}}{2c} H_2(\omega)}$$

The following procedures will give this modification in a plane strain finite element method:

1. Find the modification factor α from $\alpha = \dfrac{f_1^{2D}/2c}{f_1^{3D}}$.
2. Multiply the mass and Young's modulus of soil by α.

For the frequencies of a harmonic excitation lower than $a_1 = 1.57$, the constant modification factor α based on the static shear stiffness can be used because no radiational damping occurs. For these frequencies of harmonic excitation, there will be no significant difference in the response of a structure between a plane strain and three-dimensional analysis for the following conditions:

1. Ratio 2c/h is larger than 5.
2. Ratio h/a is about one.
3. Mass ratio is smaller than certain ratio (i.e. = 5 for the ratio h/a = 4 and square foundation).

The modification factor based on the statical shear stiffness has the following tendencies:

1. Modification factor α is larger for larger ratio h/a.
2. Modification factor is smaller for larger ratio c/a.
3. Variation of Poisson's ratio of soil from 0 to 0.3 has negligible influence.

The calculated modification factor in this thesis can be applied to any shape of structure as long as the structure has a rectangular base and the same ratios h/a and c/a as those of calculated ones. Although this thesis considered the application of a plane strain finite element method to the structure with rectangular base on a soil layer only, it may be possible to apply this method for any case if the transfer functions $H_3(i\omega)_{2D}$ and $H_3(i\omega)_{3D}$ are known.

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APPENDIX 1

Figure A1 Program for Computation of Ground Compliance by Analytical Method

Figure A2 Program for Computation of Ground Compliance by Finite Element Method

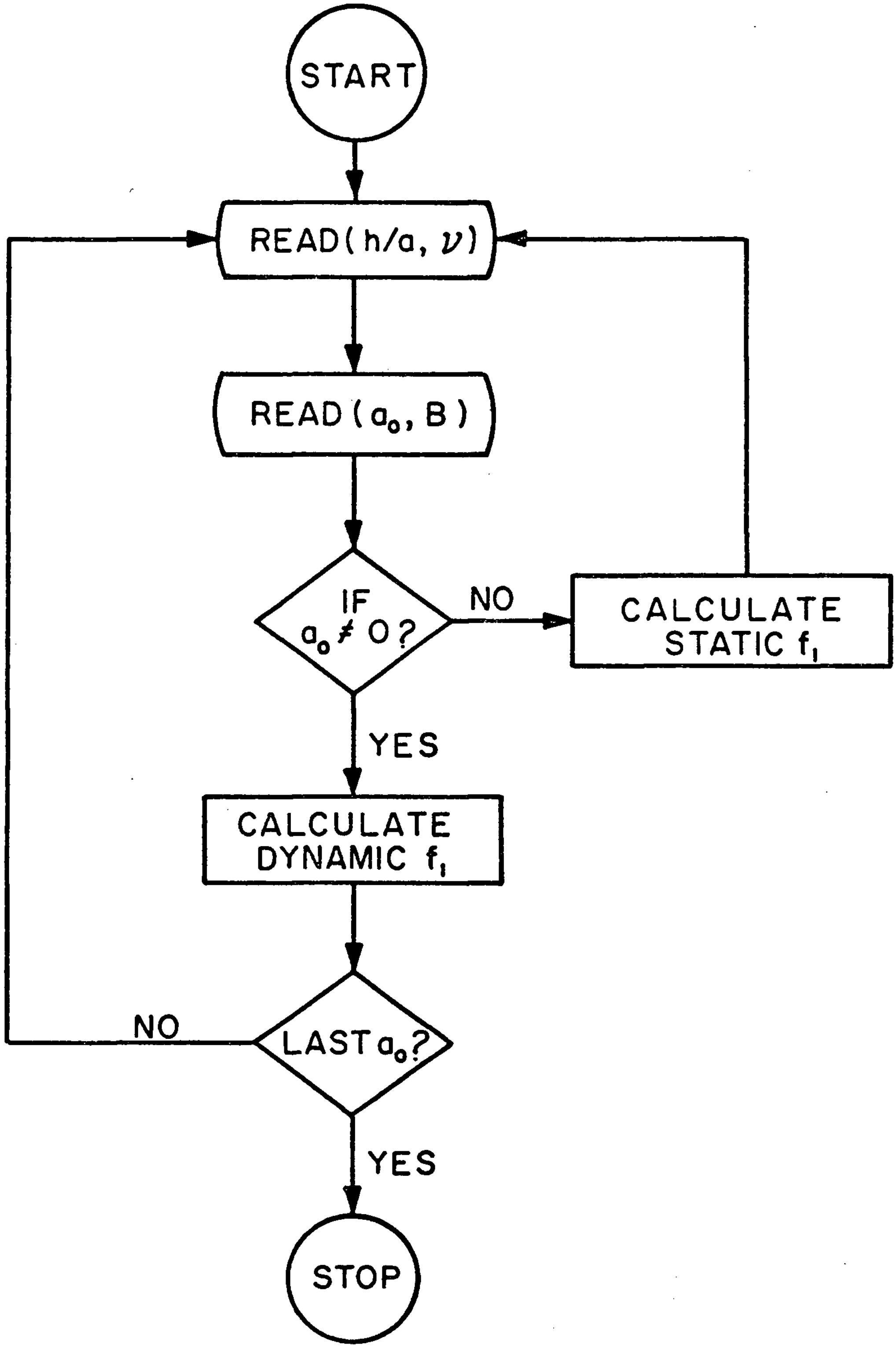

FIG. A1 PROGRAM FOR COMPUTATION OF GROUND COMPLIANCE BY ANALYTICAL METHOD.

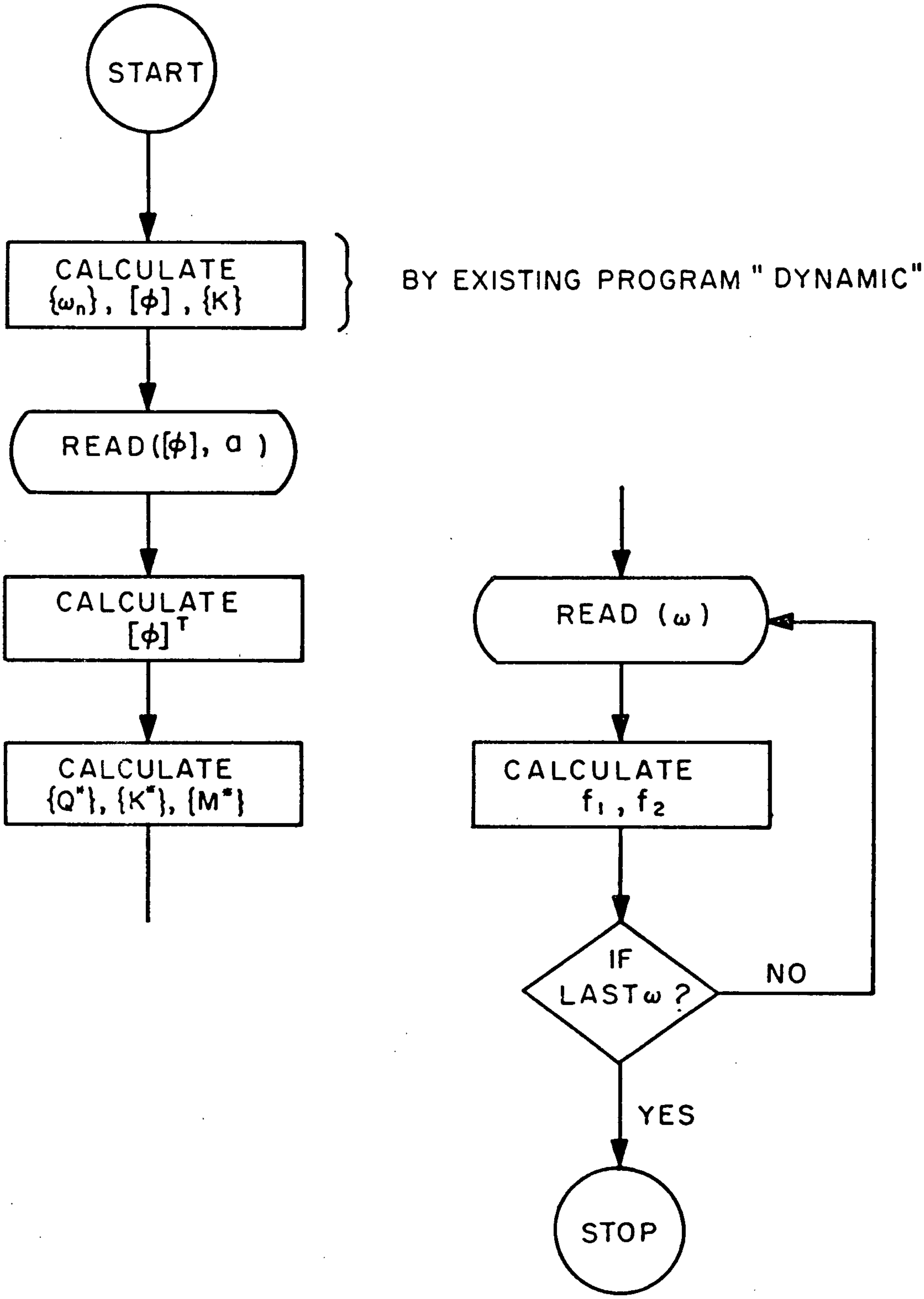

FIG. A2 PROGRAM FOR COMPUTATION OF GROUND COMPLIANCE BY FINITE ELEMENT METHOD.

APPENDIX 2

A COMPACT FORM FOR THE LINEAR SYSTEM OF EQUATIONS ARISING IN CHAPTER 5

The coefficients of unknowns in Equation 5:10 can be written in terms of $2m \times 2n$ displacements $u_{1,1}^{p,q}$ $p = 1,\cdots,2m$, $q = 1,\cdots,2n$, using the relation:

$$u_{i,j}^{p,q} = u_{1,1}^{|p-i|+1,|q-j|+1} \qquad \begin{array}{l} p = 1,\cdots,2m \\ q = 1,\cdots,2n \\ i = 1,\cdots, m \\ j = 1,\cdots, n \end{array} \qquad \text{(A2:1)}$$

as

$$\begin{aligned} u_{i,j}^{k,\ell} &= u_{1,1}^{|k-i|+1,|\ell-j|+1} \\ u_{i,j}^{k,2n+1-\ell} &= u_{1,1}^{|k-i|+1,2(n+1)-(\ell+j)} \\ u_{i,j}^{2m+1-k,\ell} &= u_{1,1}^{2(m+1)-(k+i),|\ell-j\ +1} \\ u_{i,j}^{2m+1-k,2n+1-\ell} &= u_{1,1}^{2(m+1)-(k+i),2(n+1)-(\ell+j)} \end{aligned} \qquad \begin{array}{l} i,k = 1,\cdots,m \\ j,\ell = 1,\cdots,n \end{array} \qquad \text{(A2:2)}$$

In view of Equation A2:2, Equation 5:10 becomes:

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} \Big[u_{1,1}^{|k-i|+1,|\ell-j|+1} + u_{1,1}^{|k-i|+1,2(n+1)-(\ell+j)} + u_{1,1}^{2(m+1)-(k+i),|\ell-j|+1} + u_{1,1}^{2(m+1)-(k+i),2(n+1)-(\ell+j)} \Big] \tau_{k,} = 1$$

$$\begin{array}{l} i = 1,\cdots,m \\ j = 1,\cdots,n \end{array} \qquad \text{(A2:3)}$$

This system of equations can be rewritten in matrix notation as:

$$[a]\{\tau\} = \{d\} \qquad \text{(A2:4)}$$

where:

$$a_{m(j-1)+i,m(i-1)+k} = u_{1,1}^{|k-i|+1,|\ell-j|+} + u_{1,1}^{|k-i|+1,2(n+1)-(\ell+j)} + u_{1,1}^{2(m+1)-(k+i),|\ell-j|+1} + u_{1,1}^{2(m+1)-(k+i),2(n+1)-(\ell+j)}$$

$$k,i = 1,\cdots,m \qquad \ell,j = 1,\cdots,n \tag{A2:5}$$

$$\tau_{m(i-1)+j} = \tau_{i,j} \qquad i = 1,\cdots,m \quad j = 1,\cdots,n \tag{A2:6}$$

$$d_r = 1 \qquad r = 1,\cdots,mn \tag{A2:7}$$

Similarly, the displacements u_i^k ($k = 1,2,\cdots,2m$) in Equation 5:16 can be expressed with the displacements u_1^k ($k = 1,2,\cdots,2m$) as:

$$u_i^k = u_1^{|k-i|+1} \tag{A2:8}$$

Using this relationship, Equation 5:17 can be written as:

$$\begin{aligned} &\sum_{k=1}^{m} (u_1^{|k-1|+1} + u_1^{2(m+1)-(k+1)})\tau_k = 1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\sum_{k=1}^{m} (u_1^{|k-i|+1} + u_1^{2(m+1)-(k+i)})\tau_k = 1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\sum_{k=1}^{m} (u_1^{|k-m|+1} + u_1^{2(m+1)-(k+m)})\tau_k = 1 \end{aligned} \tag{A2:9}$$

Therefore, if the displacements u_1^k ($k = 1,2,\cdots,2m$) are known, the stresses τ_k ($k = 1,2,\cdots,m$) can be obtained by solving Equation A2:9. The displacement u_1^k can be calculated by Equation A1:5.

Finally the shear stiffness is obtained by Equation 5:18. Computer programs for the computation of two-dimensional and three-dimensional shear stiffness of a soil layer is described in Fig. A3 and Fig. A4 respectively.

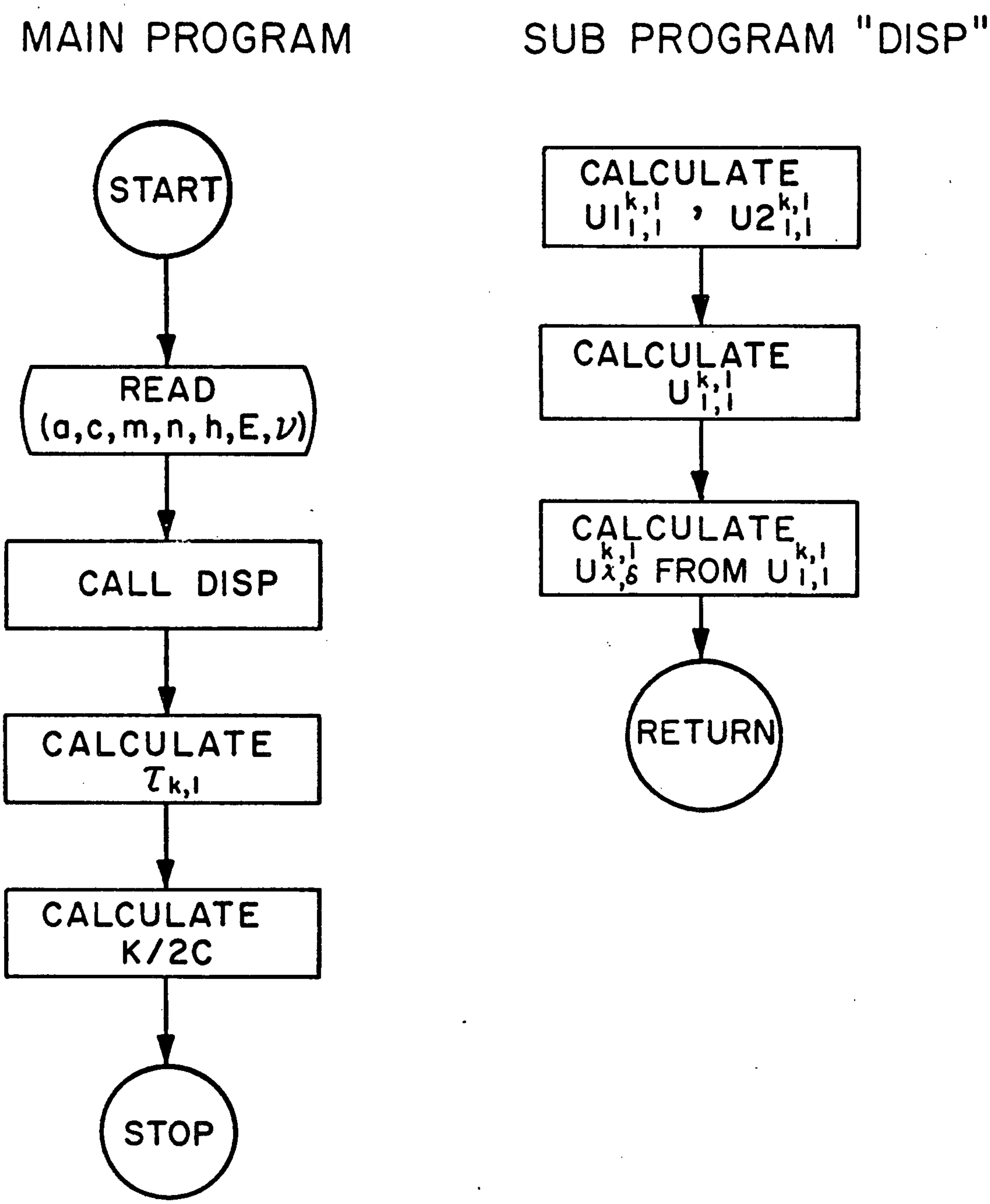

FIG. A3 PROGRAM FOR COMPUTATION OF SHEAR STIFFNESS OF SOIL LAYER AND RECTANGULAR RIGID FOUNDATION.

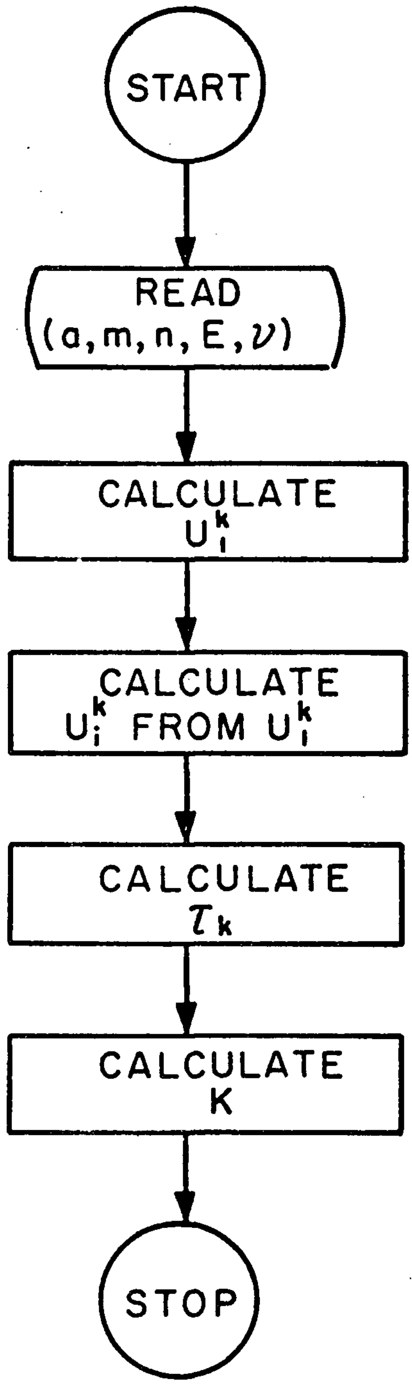

FIG. A4 PROGRAM FOR COMPUTATION OF SHEAR STIFFNESS OF SOIL LAYER AND RIGID STRIP FOUNDATION.

APPENDIX 3

TWO-DIMENSIONAL STATICAL GROUND COMPLIANCE

The static compliance can be obtained from the dynamic compliance (Chapter 3.1) when
New variable η will be denoted as:

$$\eta = \zeta h \tag{A1:1}$$

Expressing α, β, $\cosh\alpha h$, $\cosh\beta h$, $\sinh\alpha h$ and $\sinh\beta h$ with the new variable,

Maclaurins expansion for these gives:

$$\begin{aligned}
\alpha &= \zeta\,(1 - n^2\varepsilon - \tfrac{1}{2}n^4\varepsilon^2 + \cdots\cdots) \\
\beta &= \zeta\,(1 - \varepsilon - \tfrac{1}{2}\varepsilon^2 + \cdots\cdots) \\
\cosh h &= \cosh\eta - \varepsilon\eta n^2\sinh\eta + \tfrac{1}{2}\varepsilon^2\eta^2 n^4\cosh\eta - \cdots\cdots \\
\cosh\beta h &= \cosh\eta - \varepsilon\eta\sinh\eta + \tfrac{1}{2}\varepsilon^2\eta^2\cosh\eta - \cdots\cdots \\
\sinh\alpha h &= \sinh\eta - \varepsilon\eta n^2\cosh\eta + \tfrac{1}{2}\varepsilon^2\eta^2 n^4\sinh\eta - \cdots\cdots \\
\sinh\beta h &= \sinh\eta - \varepsilon\eta\cosh\eta + \tfrac{1}{2}\varepsilon^2\eta^2\sinh\eta - \cdots\cdots
\end{aligned} \tag{A1:2}$$

where

$$\varepsilon = \frac{1}{2}\,\frac{\rho}{\mu}\left(\frac{\omega}{\eta}\right)^2 \tag{A1:3}$$

$$\eta^2 = \left(\frac{H}{K}\right)^2 = \frac{1-2\nu}{2(1-\nu)} \tag{A1:4}$$

Substituting into the appropriate equations in Chapter 3.1 and ignoring ε terms of higher order when $\omega \to 0$ or $\varepsilon \to 0$ gives:

$$u(x,z=0,t) = \frac{2\tau h}{\pi\mu}\int_0^\infty \frac{\{(1+n^2)\sinh\eta\cosh\eta + (1-n^2)\eta\}\sin(\eta^a/h)\cos(\eta^x/h)}{2[1 + (1-n^2)\{(1+n^2)\sinh^2\eta + (1-n^2)\eta^2\}]\eta^2}\,d\eta \tag{A1:5}$$

Therefore, the ground compliance for a static case with $x = 0$ is:

$$\frac{u^{2D}}{q}\,a\mu = \frac{h}{\pi}\int_0^\infty \frac{\{(1+n^2)\sinh\eta\cosh\eta + (1-n^2)\eta\}\sin(\eta^a/h)}{2[1 + (1-n^2)\{(1+n^2)\sinh^2\eta + (1-n^2)\eta^2\}]\eta^2}\,d\eta \tag{A1:6}$$

where $q = 2a\tau$