J. Math. Kyoto Univ. (JMKYAZ)
35-3 (1995) 357-376

Two-dimensional Finsler spaces whose geodesics constitute a family of special conic sections

By

Makoto MATSUMOTO

To find the Finsler spaces having a given family of curves as the geodesics is an interesting problem for geometricians and will be an important problem from the standpoint of applications [1]. A previous paper [5] of the present author gave the complete solutions of this problem in the two-dimensional case, and another paper [7] may be regarded as the first continuation of [5]. The present paper is the second continuation.

The purpose of the present paper is to give a geometrical development of the preceding two papers above. The given families of curves treated in this paper consist of semicircles, parabolas and hyperbolas respectively on the semiplane. Some preliminaries are necessary and, in particular, the formula (1.20) for the functions $G^i(x, y)$ will enable us to obtain them easily without finding the fundamental tensor.

§1. Preliminaries

Let us consider an n-dimensional Finsler space $F^n=(M^n, L(x, y))$ on an underlying smooth manifold M^n with the fundamental function $L(x, y)$. The fundamental tensor $g_{ij}(x, y)$, the angular metric tensor $h_{ij}(x, y)$ and the normalized supporting element $l_i(x, y)$ of F^n are defined respectively by

$$g_{ij}=h_{ij}+l_i l_j, \quad h_{ij}=LL_{(i)(j)}, \quad l_i=L_{(i)},$$

where $L_{(i)}=\partial L/\partial y^i$ and $L_{(i)(j)}=\partial L_{(i)}/\partial y^j$.

The geodesic, the extremal of the length integral $s=\int_{t_0}^{t} L(x, y)\mathrm{d}t$, $t\geqq t_0$, $y^i=\dot{x}^i=\mathrm{d}x^i/\mathrm{d}t$, along a curve $x^i=x^i(t)$, is given by the Euler equation

$$\text{(1.1)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}L_{(i)}-L_i=0,$$

where $L_i=\partial L/\partial x^i$ and $y^i=\dot{x}^i$. In terms of $F(x, y)=L^2(x, y)/2$, (1.1) is written in the well-known form

$$\text{(1.2)} \qquad \ddot{x}^i+2G^i(x, \dot{x})=h(t)\dot{x}^i,$$

where we put

Received in February 7, 1994

(1.3) $$2g_{ij}G^i(x, y) = y^r\left(\frac{\partial^2 F}{\partial x^r \partial y^j}\right) - \frac{\partial F}{\partial x^j} ,$$

and $h(t) = (\mathrm{d}^2 s/\mathrm{d}t^2)/(\mathrm{d}s/\mathrm{d}t)$.

We shall restrict our consideration to Finsler spaces of dimension two and use the notation (x, y) and (p, q) respectively, instead of (x^1, x^2) and (y^1, y^2) ([6]; 1.1 and 1.2 of [1]). Then, from the homogeneity of $L(x, y; p, q)$ in (p, q) we have

$$L_x = L_{xp}p + L_{xq}q , \quad L_y = L_{yp}p + L_{yq}q ,$$

and the *Weierstrass invariant*

$$W = \frac{L_{pp}}{q^2} = -\frac{L_{pq}}{pq} = \frac{L_{qq}}{p^2} .$$

Consequently the two equations of (1.1) reduce to the single equation

(1.4) $$L_{xq} - L_{yp} + (p\dot{q} - q\dot{p})W = 0 ,$$

which is called the *Weierstrass form* of geodesic equation.

Next we are concerned with the *associated fundamental function* $A(x, y, z)$, $z = y' = \mathrm{d}y/\mathrm{d}x$ defined as follows:

(1.5) $$A(x, y, z) = L(x, y; 1, z) , \quad L(x, y; p, q) = A\left(x, y, \frac{q}{p}\right)p .$$

Then it is easy to show that (1.4) is written in terms of $A(x, y, z)$ as

(1.6) $$A_{zz}y'' + A_{yz}y' + A_{xz} - A_y = 0 , \quad z = y' ,$$

which is called the *Rashevsky form* of geodesic equation.

We observe in (1.4) that $p\dot{q} - q\dot{p} = p^3 y''$. Hence from (1.2) we have another form of geodesic equation as follows:

(1.7) $$y'' = \frac{2}{p^3}(qG^1 - pG^2) .$$

Now we consider the functions $G^i(x, y)$ in detail. In a general F^n we have the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j)$, defined by $G^i{}_j = \partial G^i/\partial y^j$ and $G_j{}^i{}_k = \partial G^i{}_j/\partial y^k$, and obtain two kinds of covariant differentiations; for a Finslerian vector field $V^i(x, y)$, for instance, we have

$$V^i{}_{;j} = \frac{\partial V^i}{\partial x^j} - \frac{\partial V^i}{\partial y^r}G^r{}_j + V^r G_r{}^i{}_j , \quad V^i{}_{.j} = \frac{\partial V^i}{\partial y^j} ,$$

called the h-and v-covariant derivatives of V^i respectively.

$B\Gamma$ is L-metrical, $L_{;i} = 0$, which gives

(1.8) $$L_i = l_r G^r{}_i .$$

Further we get from (1.8)

(1.9) $$L_{i(j)}=\frac{1}{L}h_{rj}G^r{}_i+l_rG_i{}^r{}_j \ .$$

Next, from (1.3) we have $2g_{ij}G^iy^j=(\partial F/\partial x^r)y^r$, that is,

(1.10) $$2G^il_i=L_ry^r \ .$$

We shall return to the two-dimensional case. Since the matrix (h_{ij}) is of rank one, we get $\varepsilon=\pm 1$ and the vector field $m_i(x, y)$ satisfying

(1.11) $$h_{ij}=\varepsilon m_im_j \ .$$

It is noted that m_i is determined by (1.11) within its orientation. Since we have

(1.12) $$g_{ij}=l_il_j+\varepsilon m_im_j \ ,$$

we get easily

(1.13) $$l_il^i=1 \ , \quad l_im^i=m_il^i=0 \ , \quad m_im^i=\varepsilon \ ,$$

where the lowering and raising of indices are done by g_{ij} and its reciprocal g^{ij}. Therefore we obtain the orthonormal frame field (l^i, m^i), called the *Berwald frame*.

(1.13) shows that we have scalar fields $h(x, y; p, q)$ and $k(x, y; p, q)$ such that

(1.14) $$(m_1, m_2)=h(-l^2, l^1) \ , \quad (m^1, m^2)=k(-l_2, l_1) \ , \quad hk=\varepsilon \ .$$

Then (1.12) and (1.14) yield

(1.15) $$g\,(=\det(g_{ij}))=\varepsilon(l_1m_2-l_2m_1)^2=\varepsilon h^2 \ ,$$

(1.16) $$\frac{1}{h}(m^1, m^2)=\frac{1}{g}(-l_2, l_1) \ .$$

According to (1.15) ε is the *signature of the metric*.

We consider W. We have $h_{11}=LL_{pp}=LWq^2$ and $h_{11}=\varepsilon(m_1)^2=\varepsilon h^2(l^2)^2=\varepsilon h^2(q/L)^2$. Consequently we have

(1.17) $$L^3W=\varepsilon h^2=g \ .$$

Now we consider the expression of G^i in the Berwald frame [4]. (1.9) and (1.14) give

$$L_{xq}-L_{yp}=\frac{\varepsilon}{L}m_r(m_2G^r{}_1-m_1G^r{}_2)=\frac{\varepsilon}{L}m_rh(l^1G^r{}_1+l^2G^r{}_2)=\frac{\varepsilon}{L^2}m_rh(y^iG^r{}_i) \ ,$$

which implies $2m_rG^r=(L_{xq}-L_{yp})L^2/\varepsilon h$. Consequently this and (1.10) lead to

(1.18) $$2G^i=(L_ry^r)l^i+(L_{xq}-L_{yp})\frac{L^2}{h}m^i \ .$$

If we put

(1.19) $$L_0=L_r y^r\ ,\quad M=L_{xq}-L_{yp}\ ,$$

then it follows from (1.16) and (1.17) that (1.18) is written in the more convenient form

(1.20) $$2G^1=\frac{1}{L}\left(L_0 p-L_q\frac{M}{W}\right)\ ,\quad 2G^2=\frac{1}{L}\left(L_0 q+L_p\frac{M}{W}\right)\ .$$

This is also written in terms of $A(x, y, z)$ as follows:

(1.20′) $$\begin{cases} 2G^1=\dfrac{1}{A}\left(A_0 p-\dfrac{Mp^2A_z}{A_{zz}}\right)\ , \\ 2G^2=\dfrac{1}{A}\left\{A_0 q+\dfrac{Mp(Ap-A_z q)}{A_{zz}}\right\}\ , \end{cases}$$

where we put

$$A_0=A_x p+A_y q\ ,\quad M=A_{xz}+A_{yz}\frac{q}{p}-A_y\ .$$

Finally we shall summarize some essential points of projective change for the later use. Let us consider two Finsler spaces $F^n=(M^n, L(x, y))$ and $\bar{F}^n=(M^n, \bar{L}(x, y))$ on a common underlying manifold M^n. If any geodesic of F^n coincides with a geodesic of $\bar{F}^n$ as a set of points and vice versa, then the change $L\rightarrow\bar{L}$ of the metrics is called *projective* and F^n is said to be projective to $\bar{F}^n$ [1]. Then the geodesic equation (1.2) in an arbitrary parameter t shows that $\bar{F}^n$ is projective to F^n, if and only if there exists a scalar field $P(x, y)$, positively homogeneous of degree one in y, satisfying

(1.21) $$\bar{G}^i(x, y)=G^i(x, y)+P(x, y)y^i\ .$$

$P(x, y)$ is called the projective factor, or the *projective difference* of $\bar{L}$ from L.

In the two-dimensional case (1.21) can be easily verified from (1.7). In fact, if $\bar{F}^n$ is projective to F^n, then (1.7) gives $(\bar{G}^1-G^1)q=(\bar{G}^2-G^2)p$, which is written as $(\bar{G}^i-G^i)m_i=0$ from (1.14).

A Finsler space is called *projectively flat*, if it has a covering by coordinate neighborhoods in which it is projective to a locally Minkowski space. Then F^n is projectively flat, if and only if we have locally a function $P(x, y)$, positively homogeneous of degree one in y, satisfying

(1.22) $$G^i(x, y)=-P(x, y)y^i\ .$$

The notion of projective flatness coincides with the notion of "with rectilinear extremals", that is, any geodesic is locally represented by n linear equations $x^i=x^i{}_0+a^i t$ of a parameter t, or $n-1$ linearly independent linear equations $a^\alpha{}_i(x^i-x^i{}_0)=0$, $\alpha=1, \cdots, n-1$.

We shall deal with a *Randers change* $L\rightarrow\bar{L}=L+\beta$, where β is a differential one-form $\beta(x, dx)=b_i(x)dx^i$ [4]. A Randers change is projective, if and only if b_i is a locally gradient vector field, that is, $\partial b_i/\partial x^j-\partial b_j/\partial x^i=0$; such a change is called a *gradient Randers change*. In the case of dimension two it is easy to verify it from (1.4). In fact, we have from $\bar{L}=L+\beta$

$$\bar{L}_{xq}-\bar{L}_{yp}=L_{xq}-L_{yp}+\partial b_2/\partial x-\partial b_1/\partial y\ ,\quad \bar{W}=W\ .$$

Thus (1.4) shows that the geodesics coincide with each other if and only if $\partial b_2/\partial x-\partial b_1/\partial y=0$. Then we have locally a function $b(x^i)$ such that $b_i=\partial b/\partial x^i$.

For a gradient Randers change we have $\bar{M}=M$ from (1.19). Thus, applying (1.20) to $\bar{L}=L+\beta$ we get

$$2pP\bar{L}+2G^1b_0=b_{00}p-b_2\frac{M}{W}\ ,\quad 2qP\bar{L}+2G^2b_0=b_{00}q+b_1\frac{M}{W}\ ,$$

where $b_0=b_iy^i$ and $b_{00}=(\partial b_i/\partial x^j)y^iy^j$. Since these give $2(G^1q-G^2p)=-M/W$, we get

$$2P\bar{L}=b_{00}-2b_iG^i\ . \qquad (1.23)$$

Consequently the projective difference P of the gradient Randers change is given easily by (1.23) and such a change will be omitted in the following.

§2. From geodesics to the Finsler metrics

Let us consider a family of curves $\{C(a, b)\}$ on the (x, y)-plane R^2, given by the equation

$$y=f(x; a, b)\ , \qquad (2.1)$$

with two parameters (a, b). The first purpose of a previous paper [5] was to show *how to find the two-dimensional Finsler space* $F^2=(R^2, L(x, y; p, q))$ *whose geodesics are given by* (2.1). We shall give an outline of the method in the notation in [7].

From (2.1) we get

$$z(=y')=f_x(x; a, b)\ , \qquad (2.2)$$

and these equations enable us to solve a and b as functions of x, y and z:

$$a=\alpha(x, y, z)\ ,\quad b=\beta(x, y, z)\ . \qquad (2.3)$$

Next, $y''=f_{xx}(x; a, b)$ and (2.3) give the following function $u(x, y, z)$:

$$z'=f_{xx}(x; \alpha, \beta)=u(x, y, z)\ , \qquad (2.4)$$

which is precisely the second order differential equation of y characterizing $\{C(a, b)\}$.

Now we are concerned with the Rashevsky form (1.6) of geodesic equation. Since we get $L_{qq}=A_{zz}/p$ from (1.5) and $A_{zz}=Wp^3$, (1.5) and (1.17) show

$$L^3W=A^3A_{zz}=g \ . \tag{2.5}$$

Thus it is suitable to call A_{zz} the *associated Weierstrass invariant.* We put $B=A_{zz}$. Then differentiation by z of (1.6), that is, $Bu+A_{yz}z+A_{xz}-A_y=0$, yields

$$B_x+B_yz+B_zu+Bu_z=0 \ . \tag{2.6}$$

The solution $B(x, y, z)$ of the first order quasilinear differential equation (2.6) is given as follows: Defining $U(x; a, b)$ and $V(x, y, z)$ by

$$U(x; a, b)=\exp \int u_z(x, f, f_x)\,\mathrm{d}x \ , \quad V(x, y, z)=U(x; \alpha, \beta) \ , \tag{2.7}$$

we obtain

$$B(x, y, z)=\frac{H(\alpha, \beta)}{V(x, y, z)} \ , \tag{2.8}$$

where H is an arbitrary non-zero function of two arguments.

From $A_{zz}=B$ we get A in the from

$$\begin{cases} A(x, y, z)=A^*(x, y, z)+C(x, y)+D(x, y)z \ , \\ A^*=\int \{\int B(x, y, z)dz\} dz=\int_{z0}^{z}(z-t)B(x, y, t)dt \ . \end{cases} \tag{2.9}$$

where C and D are arbitrary but must be chosen so that A may satisfy (1.6), that is,

$$C_y-D_x=A^*_{zz}u+A^*_{yz}z+A^*_{xz}-A^*_y \ . \tag{2.10}$$

It is easy to verify that the right-hand side of (2.10) does not depend on z.

Assume that a pair (C_0, D_0) has been chosen so as to satisfy (2.10). Then $(C-C_0)_y=(D-D_0)_x$, so that we have locally a function $E(x, y)$ satisfying $E_x=C-C_0$ and $E_y=D-D_0$. Thus (2.9) is written as

$$A=A^*+C_0+D_0z+E_x+E_yz \ .$$

Therefore (1.5) leads to the fundamental function

$$\begin{cases} L(x, y; p, q)=L_0(x, y; p, q)+e(x, y; p, q) \ , \\ L_0=A^*(x, y, p/q)p+C_0(x, y)p+D_0(x, y)q \ , \end{cases} \tag{2.11}$$

where e is the derived form

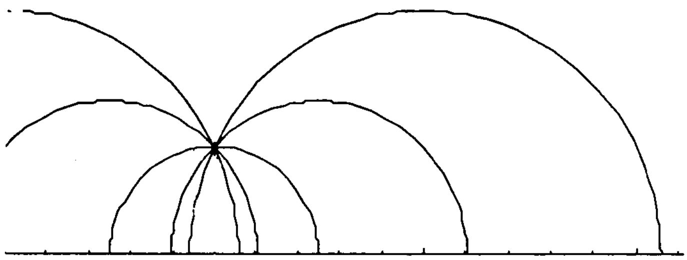
$$e(x, y; \mathrm{d}x, \mathrm{d}y)=E_x\mathrm{d}x+E_y\mathrm{d}y \ . \tag{2.12}$$

Since the change $L_0 \to L$ is a gradient change, (R^2, L) is projective to (R^2, L_0). Therefore the Finsler metric we found is uniquely determined depending on the choice of the functions H and E of two arguments. Further, for any choice of a function H we obtain a Finsler space (R^2, L) which is projective to each other, because each one has the same geodesics $\{C(a, b)\}$.

§3. Family of semicircles

We shall consider the family of semicircles $\{C(a, b)\}$ given by the equation

Fig. 1



(3.1) $$(x-a)^2+y^2=b^2\ ,\quad y, b>0\ ,$$

on the semiplane $R^2_+=\{(x, y), y>0\}$ having the centre $(a, 0)$ on the x-axis, and find the Finsler metrics $L(x, y; p, q)$ such that all geodesics of the two-dimensional Finsler space $(R^2_+, L(x, y; p, q))$ belong to $\{C(a, b)\}$.

First, from (3.1) we have

(3.2) $$x-a+yz=0\ ,\quad z=y'\ .$$

These equations give the functions $\alpha(x, y, z)$ and $\beta(x, y, z)$ as in the last section:

(3.3) $$a=x+yz=\alpha(x, y, z)\ ,\quad b=y\sqrt{1+z^2}=\beta(x, y, z)\ .$$

The function $u(x, y, z)$ is given by

(3.4) $$z'(=y'')=-\frac{1+z^2}{y}=u(x, y, z)\ ,$$

which yields the differential equation

(3.4′) $$yy''+1+(y')^2=0\ ,$$

characterizing $\{C(a, b)\}$.

Next we shall find the functions $U(x; a, b)$ and $V(x, y, z)$ defined by (2.7):

$$u_z=-\frac{2z}{y}\ ,\quad u_z(x, f, f_x)=\frac{-2(x-a)}{(x-a)^2-b^2}\ ,$$
$$U=\exp\int\frac{-2(x-a)}{(x-a)^2-b^2}dx=\frac{1}{|(x-a)^2-b^2|}\ ,\quad V=\frac{1}{y^2}\ .$$

Consequently we have from (2.8) and (2.9)

(3.5) $B(x, y, z)=y^2H(\alpha, \beta)$, $A^*(x, y, z)=y^2\int\{\int H(\alpha, \beta)dz\}dz$.

Thus the associated fundamental function $A(x, y, z)$ is written as

$$A=A^*(x, y, z)+C(x, y)+D(x, y)z \tag{3.6}$$

where C and D must be chosen so that they satisfy (2.10).

We shall deal with (2.10) in this case. First A^* may be written in the form [5]

$$A^*=y^2\int_0^z(z-t)F(t)\,dt\ ,\quad F(t)=H(x+yt, y\sqrt{1+t^2})\ . \tag{3.7}$$

If we put

$$F_1(t)=H_\alpha(x+yt, y\sqrt{1+t^2})\ ,\quad F_2(t)=H_\beta(x+yt, y\sqrt{1+t^2})\ ,$$

then we have

$$A^*_y=2y\int_0^z(z-t)F(t)\,dt+y^2\int_0^z(z-t)\{F_1(t)t+F_2(t)\sqrt{1+t^2}\}\,dt\ ,$$

$$A^*_z=y^2\int_0^zF(t)\,dt\ ,\quad A^*_{xz}=y^2\int_0^zF_1(t)\,dt\ ,$$

$$A^*_{yz}=2y\int_0^zF(t)\,dt+y^2\int_0^2\{F_1(t)t+F_2(t)\sqrt{1+t^2}\}\,dt\ ,$$

and $A^*_{zz}=B=y^2H$. Thus (2.10) implies

$$\begin{aligned}C_y-D_x&=2y\int_0^zF(t)t\,dt+y^2\int_0^zF_1(t)\,dt\\&\quad+y^2\int_0^z\{F_1(t)t+F_2(t)\sqrt{1+t^2}\}t\,dt-y(1+z^2)H(\alpha, \beta)\ .\end{aligned} \tag{3.8}$$

Therefore we obtain

Theorem 1. *Every associated fundamental function $A(x, y, z)$ of a Finsler space $(R^2_+, L(x, y; p, q))$ having the semicircles (3.1) as the geodesics is given by*

$$A=A^*(x, y, z)+C(x, y)+D(x, y)z\ ,$$

where A^* is defined by (3.7), H is an arbitrary function of (α, β) given by (3.3) and functions (C, D) of (x, y) must be chosen so as to satisfy (3.8).

Example 1. In particular we first put $H(\alpha, \beta)=\alpha=x+yz$ in the result above. Then (3.7) and (3.8) yield

$$A^*=y^2\int_0^z(z-t)(x+yt)\,dt=\frac{y^2z^2}{6}(3x+yz)\ ,\quad C_y-D_x=-xy\ .$$

Thus, choosing $C=0$ and $D=x^2y/2$, we get

$$A(x, y, z)=\frac{yz}{2}\left(x^2+xyz+\frac{y^2z^2}{3}\right) .$$

Therefore it follows from (1.5) that the fundamental function $L(x, y; p, q)$ is written in the form

$$L(x, y; p, q)=\frac{yq}{p}\left(x^2p+xyq+\frac{y^2q^2}{3p}\right) , \tag{3.9}$$

where 1/2 was omitted by the homothetic change of metric.

To verify (3.9) we recall (1.4). For (3.9) we have

$$L_{xq}=2y\left(x+\frac{yq}{p}\right) , \quad L_{yp}=-\frac{2yq^2}{p^2}\left(x+\frac{yq}{p}\right) ,$$

$$L_{qq}=\frac{2y^2}{p}\left(x+\frac{yq}{p}\right) , \quad W=\frac{2y^2}{p^3}\left(x+\frac{yq}{p}\right) .$$

Hence (1.4) leads to (3.4′) immediately.

Further, using the formula (1.20), we have the functions G^i of (3.9) as follows:

$$2G^1L=-x^2p^3 , \quad 2G^2L=\frac{yq^2}{p^2}(p^2+q^2)\left(xp+\frac{yq}{3}\right)+x^2q^3 . \tag{3.10}$$

Example 2. Secondly we shall consider the metric where $H(\alpha, \beta)=\beta^n$, n being an arbitrary real number. Then we have from (3.7) and (3.8)

$$\begin{cases} A^*=y^{n+2}\{zI_n(z)-J_n(z)\} , \\ C_y-D_x=y^{n+1}\{(n+2)J_n(z)-(1+z^2)^{n/2+1}\} , \\ I_n(z)=\int_0^z(1+t^2)^{n/2}dt , \quad J_n=\int_0^z(1+t^2)^{n/2}tdt . \end{cases} \tag{3.11}$$

(i) We first treat the general case where $n\neq-2$: Then $J_n(z)=\{(1+z^2)^{n/2+1}-1\}/(n+2)$ and $C_y-D_x=-y^{n+1}$. Choosing $D=0$ and $C=-y^{n+2}/(n+2)$, we obtain

$$A(x, y, z)=y^{n+2}\left\{zI_n(z)-\frac{1}{n+2}(1+z^2)^{n/2+1}\right\} . \tag{3.12}$$

Making use of (1.20′) we obtain G^i of (3.12) as follows:

$$\begin{cases} 2G^1=K_np , \quad 2G^2=K_nq+\dfrac{p^2+q^2}{y} , \\ K_n=(n+2)\dfrac{q}{y}-\dfrac{p^2+q^2}{L}y^{n+1}I_n\left(\dfrac{q}{p}\right) . \end{cases} \tag{3.13}$$

We restrict our consideration to the case where n is an integer. Then we have the formula

$$(n+1)I_n(z)=nI_{n-2}(z)+z(1+z^2)^{n/2} .$$

Thus, for a positive integer n we get the reduction formula

$$(3.14)\qquad \begin{cases} I_n(z)=\dfrac{n}{n+1}I_{n-2}(z)+\dfrac{z(1+z^2)^{n/2}}{n+1} , & n\geqq 2 , \\ I_1(z)=\dfrac{1}{2}(z\sqrt{1+z^2}+\log|z+\sqrt{1+z^2}|) , & I_0(z)=z . \end{cases}$$

For a negative integer n we get

$$(3.15)\begin{cases} I_n(z)=\dfrac{n+3}{n+2}I_{n+2}(z)-\dfrac{z(1+z^2)^{n/2+1}}{n+2} , \quad n\leqq -4 , \\ I_{-3}(z)=\dfrac{z}{\sqrt{1+z^2}} , \quad I_{-2}(z)=\mathrm{Tan}^{-1}z , \quad I_{-1}(z)=\log|z+\sqrt{1+z^2}| . \end{cases}$$

As a consequence we are interested in the following cases:

(ia) $n=1$: We have from (3.12)

$$A(x,y,z)=y^3\left\{\frac{1}{6}\sqrt{1+z^2}(z^2-2)+\frac{z}{2}\log|z+\sqrt{1+z^2}|\right\} .$$

Therefore we obtain the fundamental function

$$(3.16)\qquad L(x,y;p,q)=y^3\sqrt{p^2+q^2}\left\{\left(\frac{q}{p}\right)^2-2\right\}+3q\log\left|\frac{q}{p}+\sqrt{1+\left(\frac{q}{p}\right)^2}\right| ,$$

where $\frac{1}{6}$ was omitted by the homothetic change.

(ib) $n=0$: Similarly we get

$$(3.17)\qquad Lx,y;p,q)=\frac{y^2}{p}(q^2-p^2) ,$$

which is of the Kropina type (**1.4** of [1]).

(ic) $n=-1$: We get as above

$$(3.18)\qquad L(x,y;p,q)=y\left(q\log\left|\frac{q}{p}+\sqrt{1+\left(\frac{q}{p}\right)^2}\right|-\sqrt{p^2+q^2}\right) .$$

(id) $n=-3$: We have $A=\sqrt{1+z^2}/y$ and the fundamental function is the simple

$$(3.19)\qquad L(x,y;p,q)=\frac{\sqrt{p^2+q^2}}{y} .$$

This is the well-known *Riemannian metric of constant curvature* -1 (§74 of [2]). Its G^i are given by

(3.20) $$\underset{r}{G}^1 = -\frac{pq}{y}\,, \quad \underset{r}{G}^2 = \frac{p^2-q^2}{2y}\,.$$

Since we obtain such a standard metric (3.19), we shall be concerned with the projective difference $P(x, y; p, q)$ of a general metric (3.12) from (3.19). From (3.13) and (3.20) we get

$$2(G^1 - G_r{}^1) = \left(K_n + \frac{2q}{y}\right)p\,, \quad 2(G^2 - G_r{}^2) = \left(K_n + \frac{2q}{y}\right)q\,.$$

Therefore we have the equation of the form (1.21):

(3.13′) $$G^i = \underset{r}{G}^i + Py^i\,, \quad P = \frac{(n+4)q}{2y} - \frac{p^2+q^2}{2L}y^{n+1}I_n\left(\frac{q}{p}\right)\,,$$

where L is the fundamental function determined by A of (3.12).

Similarly (3.10) is written in the form

(3.10′) $$G^i = \underset{r}{G}^i + Py^i\,, \quad P = \frac{x^2}{L}\left(q^2 - \frac{p^2}{2}\right) + \frac{yq^3}{Lp}\left(x + \frac{yq}{3p}\right)\,.$$

(ii) We deal with the exceptional case $n = -2$: From (3.11) we get

$$A^* = z\,\mathrm{Tan}^{-1}z - \frac{1}{2}\log(1+z^2)\,, \quad C_y - D_x = -\frac{1}{y}\,.$$

Hence, choosing $C=0$ and $D = x/y$, we obtain

(3.21) $$L(x, y; p, q) = q\,\mathrm{Tan}^{-1}\left(\frac{q}{p}\right) - \frac{p}{2}\log\left\{1 + \left(\frac{q}{p}\right)^2\right\} + \frac{xq}{y}\,.$$

The projective difference of this L from the Riemannian (3.19) is easily obtained as follows:

(3.22) $$\begin{cases} G^i = \underset{r}{G}^i + Py^i\,, \\ 2PL = \dfrac{1}{y}\left[p\left(q - \dfrac{xp}{y}\right) - (p^2-q^2)\,\mathrm{Tan}^{-1}\left(\dfrac{q}{p}\right) - pq\log\left\{1 + \left(\dfrac{q}{p}\right)^2\right\}\right]\,. \end{cases}$$

§4. Family of parabolas

We shall consider the family of parabolas $\{C(a, b)\}$ given by the equation

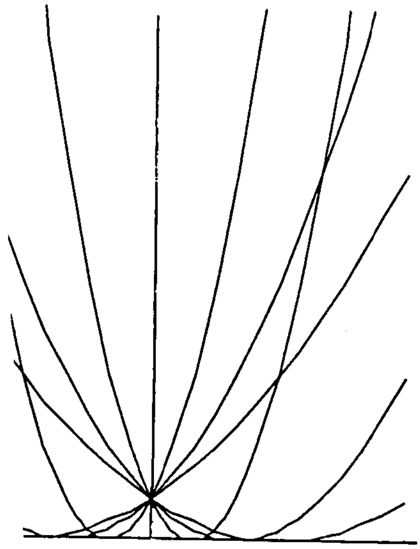
(4.1) $$by = (x-a)^2\,, \quad y, b > 0\,,$$

on the semiplane R^2_+ having the vertex $(a, 0)$ on the x-axis, and find the Finsler metrics similarly to the previous section. The family suggests us **Fig.20** of [2].

First (4.1) yields

$$z(=y') = \frac{2}{b}(x-a)\,, \quad z' = \frac{2}{b}\,.$$

Fig. 2



Consequently the functions $\alpha(x, y, z)$, $\beta(x, y, z)$ and $u(x, y, z)$ as in the preceding sections are given by

(4.2) $$a=x-\frac{2y}{z}=\alpha(x, y, z) , \quad b=\frac{4y}{z^2}=\beta(x, y, z) ,$$

(4.3) $$z'=\frac{z^2}{2y}=u(x, y, z) .$$

The latter yields the differential equation

(4.4) $$2yy''=(y')^2 ,$$

characterizing $\{C(a, b)\}$.

Next we get $u_z=z/y=2/(x-a)$ and

$$U(x; a, b)=\exp\int\frac{2dx}{x-a}=(x-a)^2 , \quad V(x, y, z)=\left(\frac{2y}{z}\right)^2 .$$

Thus (2.8) implies

$$B(x, y, z)=H(\alpha, \beta)\frac{z^2}{4y^2}=H(\alpha, \beta)\frac{1}{y\beta} .$$

On account of the arbitrariness of H, we may write B as $B=H(\alpha, \beta)/y$ and A^* of (2.9) is written in the form

$$A^*(x, y, z)=\frac{1}{y}\iint H(\alpha, \beta)(dz)^2 .$$

This is written, as in the last section, in the form

$$(4.5)\qquad A^*(x, y, z) = \frac{1}{y}\int_1^z (z-t)F(t)\,dt\ ,\quad F(t) = H\left(x - \frac{2y}{t}, \frac{4y}{t^2}\right)\ .$$

It is noted that $\int_1^z$ is applied in (4.5), instead of $\int_0^z$, because t is in the denominators in $F(t)$.

If we put

$$F_1(t) = H_\alpha\left(x - \frac{2y}{t}, \frac{4y}{t^2}\right)\ ,\quad F_2(t) = H_\beta\left(x - \frac{2y}{t}, \frac{4y}{t^2}\right)\ ,$$

then it is easy to show that the condition (2.10) is written as

$$(4.6)\qquad C_y - D_x = \frac{z^2}{2y^2}H(\alpha, \beta) - \frac{1}{y^2}\int_1^z F(t)\,t\,dt - \frac{1}{y}\int_1^z \left\{F_1(t) - \frac{4}{t}F_2(t)\right\}dt\ .$$

Therefore we obtain

Theorem 2. *Every associated fundamental function $A(x, y, z)$ of a Finsler space $(R^2_+, L(x, y; p, q))$ having the parabolas (4.1) as the geodesics is given by*

$$A = A^*(x, y, z) + C(x, y) + D(x, y)z\ ,$$

where A^* is defined by (4.5), H is an arbitrary function of (α, β) given by (4.2) and the functions (C, D) of (x, y) must be chosen so as to satisfy (4.6).

Example 3. In particular we consider the case where $H(\alpha, \beta) = \beta^n$ for a real number n. Then $F(t) = (4y/t^2)^n$ from (4.2), $F_1 = 0$ and $F_2 = n(4y/t^2)^{n-1}$. Hence (4.5) and (4.6) give

$$(4.7)\qquad A^* = 4^n y^{n-1}\left(z\int_1^z t^{-2n}dt - \int_1^z t^{1-2n}dt\right)\ ,$$

$$(4.8)\qquad C_y - D_x = 4^n y^{n-2}\left\{\frac{z^{2-2n}}{2} + (n-1)\int_1^z t^{1-2n}dt\right\}\ .$$

Thus our discussion must be divided into the following cases:

(i) We first deal with the general case where $n \neq \frac{1}{2}$, 1: Then the equations above are written as

$$A^* = \frac{4^n y^{n-1}}{2(n-1)(2n-1)}\{z^{2-2n} + 2(n-1)z - (2n-1)\}\ ,$$

$$C_y - D_x = \frac{1}{2}4^n y^{n-2}\ .$$

Thus, choosing $C = 4^n y^{n-1}/2(n-1)$ and $D = -4^n y^{n-1}/(2n-1)$, we have $A(x, y, z) = 4^n y^{n-1} z^{2-2n}/2(n-1)(2n-1)$. Therefore we obtain the fundamental function

$$L(x, y; p, q) = y^{n-1} q^{2-2n} p^{2n-1}, \quad n \neq \frac{1}{2}, 1, \tag{4.9}$$

where $4^n/2(n-1)(2n-1)$ was omitted.

The functions G^i of this metric are given easily by (1.20) as

$$G^1 = 0, \quad G^2 = -\frac{q^2}{4y}. \tag{4.10}$$

(ii) $n=\frac{1}{2}$: (4.7) and (4.8) give

$$A^* = 2y^{-\frac{1}{2}}(z \log|z| - z + 1), \quad C_y - D_x = y^{-3/2}.$$

Choosing $D = -C = 2y^{-\frac{1}{2}}$, we have $A(x, y, z) = 2y^{-\frac{1}{2}} z \log|z|$. Consequently we omit 2 and obtain

$$L(x, y; p, q) = y^{-\frac{1}{2}} q \log\left|\frac{q}{p}\right|. \tag{4.11}$$

(iii) $n=1$: Similarly we have

$$A^* = 4(z - \log|z| - 1), \quad C_y - D_x = \frac{2}{y}.$$

Choosing $C=4$ and $D = -2x/y - 4$, we have $A(x, y, z) = -4\log|z| - 2xz/y$. Therefore, omitting -2, we obtain the metric

$$L(x, y; p, q) = 2p \log\left|\frac{q}{p}\right| + \frac{xq}{y}. \tag{4.12}$$

Now we shall return to the general case with the Finsler metric (4.9). If we refer to the new coordinate system $(\bar{x}, \bar{y}) = (x, \sqrt{y})$, then we have $(p, q) = (\bar{p}, 2\bar{y}\bar{q})$ and the metric (4.9) can be written in the form

$$\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q}) = 4^{1-n}(\bar{q})^{2-2n}(\bar{p})^{2n-1}. \tag{4.9'}$$

Since $\bar{L}$ does not depend on $\bar{x}$ and $\bar{y}$, this is a specially simple metric, called a *locally Minkowski metric* [1], [3], and $(\bar{x}, \bar{y})$ is an *adapted* coordinate system to the structure. Further its *main scalar* I, one of the two essential scalar fields in the two-dimensional case, are constant as follows: Since (4.9′) is of the form (i) or (iv) of *Theorem* 3.5.3.2 of [1], we have directly as follows:

(i) $\varepsilon=1$, $I^2>4$, $(2-2n)(2n-1)<0$, $\dfrac{1}{\sqrt{I^2-4}}+1=2(2-2n)$,

(iv) $\varepsilon=-1$, $(2-2n)(2n-1)>0$, $\dfrac{I}{\sqrt{I^2+4}}+1=2(2-2n)$.

Therefore we have

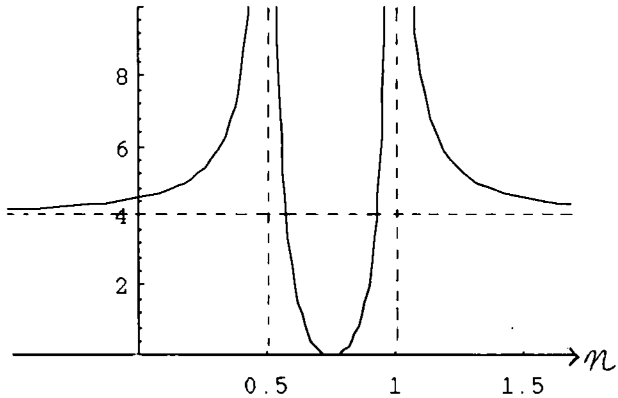
Proposition 1. *The Finsler space* $(R^2_+, L(x, y; p, q))$ *with a metric* (4.9) *is locally Minkowski, and has the signature* ε *and the constant main scalar* I *as follows*:

(1) $n<\frac{1}{2}, >1$: $\quad \varepsilon=1\ ,\quad I^2=\dfrac{(4n-3)^2}{2(n-1)(2n-1)}\ ,$

(2) $\frac{1}{2}<n<1$: $\quad \varepsilon=-1\ ,\quad I^2=-\dfrac{(4n-3)^2}{2(n-1)(2n-1)}\ .$

Remark. $\dfrac{(4n-3)^2}{2(n-1)(2n-1)}=4+\dfrac{1}{2(n-1)}-\dfrac{1}{2n-1}$. The graph of I^2 is shown in **Fig.3**.

Fig. 3



Since a Finsler space of dimension two is Riemannian if and only if $I=0$, Proposition 1 shows that $(R^2_+, L(x, y; p, q))$ under consideration is Riemannian, if and only if $n=3/4$; (4.9) reduces to

$$L^2(x, y; p, q)=\frac{pq}{\sqrt{y}}=2\bar{p}\bar{q}\ , \tag{4.13}$$

which is, of course, the *Lorentz metric*. Consequently we can state that all the Finsler spaces we consider in the present section are *projectively flat* [1], because they have the locally Minkowski spaces, given in Proposition 1, as the representatives. In fact, referring to $(\bar{x}, \bar{y})$, $\bar{G}^i$ of the metric (4.9′) vanish obviously. Further it is easy to show that the metrics (4.11) and (4.12) are written respectively in the form

$$\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q})=2\left(\bar{q}\log\left|\frac{2\bar{y}\bar{q}}{\bar{p}}\right|-\bar{q}\right)\ , \tag{4.11′}$$

$$\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q})=2\left(\bar{p}\log\left|\frac{2\bar{y}\bar{q}}{\bar{p}}\right|+\frac{\bar{x}\bar{q}}{\bar{y}}\right)\ , \tag{4.12′}$$

and $\bar{G}^i$ of these metrics are of the form (1.22) as follows:

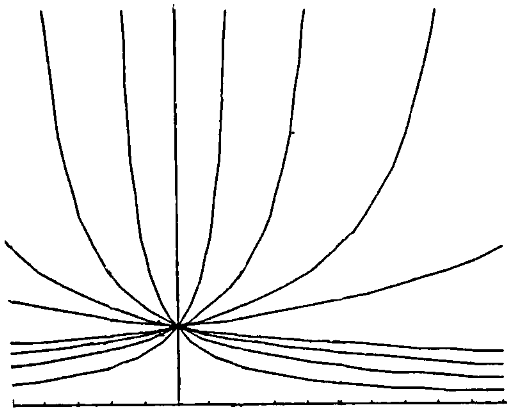
(4.11a) $$\bar{G}^i=-P\bar{y}^i, \quad P=-\frac{\bar{q}^2}{L\bar{y}}$$

(4.12a) $$\bar{G}^i=-P\bar{y}^i, \quad P=\frac{\bar{q}}{L\bar{y}}\left(\frac{\bar{x}\bar{q}}{\bar{y}}-2\bar{p}\right).$$

§5. Family of hyperbolas

Finally we shall consider the family of rectangular hyperbolas $\{C(a, b)\}$ given by the equation

Fig. 4



(5.1) $$(x-a)y=b, \quad y>0,$$

on the semiplane R^2_+ having the x-axis as one of the asymptotic lines, and find the Finsler metrics as in the preceding sections.

First (5.1) yields

$$z(=y')=-\frac{b}{(x-a)^2}, \quad z'=\frac{2b}{(x-a)^3}.$$

Consequently we get

(5.2) $$a=x+\frac{y}{z}=\alpha(x, y, z), \quad b=-\frac{y^2}{z}=\beta(x, y, z),$$

(5.3) $$z'=\frac{2z^2}{y}=u(x, y, z).$$

The latter gives the differential equation

(5.4) $$yy''=2(y')^2,$$

characterizing $\{C(a, b)\}$, which is quite similar to (4.4) in the form.

Next (5.3) gives $u_z = 4z/y = -4/(x-a)$ and

$$U(x;\, a,\, b) = \exp\int \frac{-4\mathrm{d}x}{x-a} = \frac{1}{(x-a)^4}\ , \quad V(x,\, y,\, z) = \left(\frac{z}{y}\right)^4 .$$

Thus (2.8) and (5.2) lead to

$$B(x,\, y,\, z) = H(\alpha,\, \beta)\left(\frac{y}{z}\right)^4 = H(\alpha,\, \beta)\left(\frac{\beta}{y}\right)^4 .$$

From the arbitrariness of H we may write $B = H(\alpha,\, \beta)/y^4$. Hence we have from (2.9)

$$\text{(5.5)} \qquad A^*(x,\, y,\, z) = \frac{1}{y^4}\int_1^z (z-t)F(t)\,\mathrm{d}t\ , \quad F(t) = H\left(x+\frac{y}{t},\, -\frac{y^2}{t}\right) .$$

Putting

$$F_1(t) = H_\alpha\left(x+\frac{y}{t},\, -\frac{y^2}{t}\right)\ , \quad F_2(t) = H_\beta\left(x+\frac{y}{t},\, -\frac{y^2}{t}\right)\ ,$$

the condition (2.10) for the functions $C(x,\, y)$ and $D(x,\, y)$ is written as follows:

$$\text{(5.6)} \qquad C_y - D_x = H(\alpha,\, \beta)\frac{2z^2}{y^5} - \frac{4}{y^5}\int_1^z F(t)\,t\,\mathrm{d}t + \frac{2}{y^4}\int_1^z \{F_1(t) - yF_2(t)\}\,\mathrm{d}t\ .$$

Theorem 3. *Every associated fundamental function* $A(x,\, y,\, z)$ *of a Finsler space* $(R^2_+,\, L(x,\, y;\, p,\, q))$ *having the hyperbolas* (5.1) *as the geodesics is given by*

$$A = A^*(x,\, y,\, z) + C(x,\, y) + D(x,\, y)z\ ,$$

where A^* *is defined by* (5.5), H *is an arbitrary function of* $(\alpha,\, \beta)$ *given by* (5.2) *and the functions* $(C,\, D)$ *of* $(x,\, y)$ *must be chosen so as to satisfy* (5.6).

Example 4. Similarly to Example 3, we are concerned with the space $(R^2_+,\, L(x,\, y;\, p,\, q))$ where $H(\alpha,\, \beta) = \beta^n = (-y^2/z)^n$ for a real number n. Then (5.5) and (5.6) are written in the form

$$\text{(5.7)} \qquad \begin{cases} A^* = (-1)^n y^{2n-4}\left(z\int_1^z t^{-n}\mathrm{d}t - \int_1^z t^{1-n}\mathrm{d}t\right)\ , \\ C_y - D_x = (-1)^n 2y^{2n-5}\left\{z^{2-n} + (n-2)\int_1^z t^{1-n}\mathrm{d}t\right\}\ . \end{cases}$$

To compute the integrals in these equations, we distinguish into three cases as follows:

(i) $n \neq 1,\, 2$: In this general case we get from (5.7)

$$A^*=\frac{(-1)^n y^{2n-4}}{(n-1)(n-2)}\{z^{2-n}+(n-2)z+(n-1)\}\ ,\quad C_y-D_x=(-1)^n 2y^{2n-5}\ .$$

Choosing $C=(-1)^n y^{2n-4}/(n-2)$ and $D=(-1)^{n+1}y^{2n-4}/(n-1)$, we have $A=(-1)^n y^{2n-4}z^{2-n}/(n-1)(n-2)$. Therefore, by the homothetic change of metric, we obtain the fundamental function

(5.8) $$L(x, y; p, q)=y^{2n-4}q^{2-n}p^{n-1}\ ,\quad n\neq 1, 2\ .$$

(ii) $n=1$: (5.7) gives

$$A^*=-\frac{1}{y^2}(z\log|z|-z+1)\ ,\quad C_y-D_x=-\frac{2}{y^3}\ .$$

If we choose $C=-D=1/y^2$, then we obtain

(5.9) $$L(x, y; p, q)=\frac{q}{y^2}\log\left|\frac{q}{p}\right|\ .$$

(iii) $n=2$: It follows from (5.7) that

$$A^*=z-1-\log|z|\ ,\quad C_y-D_x=\frac{2}{y}\ .$$

We can choose $C=1$ and $D=-1-2x/y$, and we obtain

(5.10) $$L(x, y; p, q)=p\log\left|\frac{q}{p}\right|+\frac{2xq}{y}\ .$$

Now we return to the metric (5.8). Since it is of quite a similar form to (4.9), we get also the result similar to Proposition 1 as follows:

Proposition 2. *The Finsler space* $(R^2_+, L(x, y; p, q))$ *with the metric* (5.8) *are locally Minkowski, and have the signature* ε *and the main scalar* I *as follows*:

(1) $n<1, >2$: $\quad\varepsilon=1,\quad I^2=\dfrac{(2n-3)^2}{(n-1)(n-2)}$.

(2) $1<n<2$: $\quad\varepsilon=-1,\quad I^2=-\dfrac{(2n-3)^2}{(n-1)(n-2)}$.

Remark. $\dfrac{(2n-3)^2}{(n-1)(n-2)}=4-\dfrac{1}{n-1}+\dfrac{1}{n-2}$.

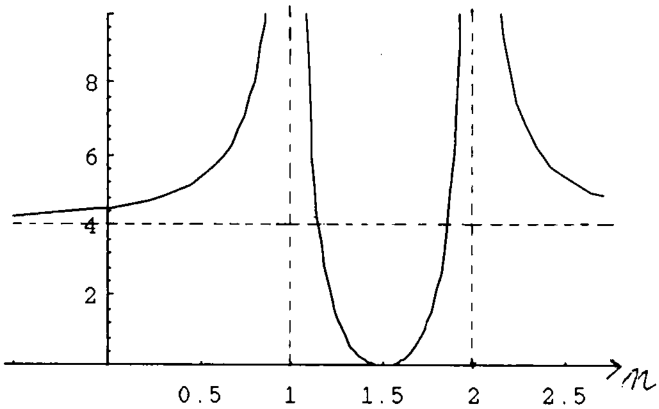
The graph of I^2 is shown in **Fig.5**.

Therefore for $n=3/2$ we get the *Lorentz metric*

(5.11) $$L^2(x, y; p, q)=\frac{pq}{y^2}\ .$$

Those special metrics (4.13) and (5.11) suggest us the consideration of the Lorentz metrics

Fig. 5



(5.12) $$L^2(x, y; p, q) = y^{2n}pq ,$$

for arbitrary real numbers n. On account of (1.4) we get the equation of their geodesics: $yy'' + 2n(y')^2 = 0$ which is equivalent to $(y^{2n}y')' = 0$. Thus we obtain their finite equations

(5.13) $$y^{2n+1} = c(x-a) , \quad n \neq -\frac{1}{2} ,$$

where a and c are arbitrary constants. If we put $n = -1/4$, then we get (4.1), and if $n = -1$, then we have (5.1).

Similarly to the last section, we have the new coordinate system $(\bar{x}, \bar{y}) = (x, 1/y)$ *adapted* to the local-Minkowski structure of (5.8). From $(\bar{p}, \bar{q}) = (p, -q/y^2)$ we get the expression of (5.8) in the form

(5.8′) $$\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q}) = (\bar{p})^{n-1}(-\bar{q})^{2-n} .$$

(5.9) and (5.10) are also written as

(5.9′) $$\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q}) = \bar{q}\left(1 - \log\left|\frac{\bar{q}}{\bar{y}^2\bar{p}}\right|\right) ,$$

(5.10′) $$\bar{L}(\bar{x}, \bar{y}; \bar{p}, \bar{q}) = \bar{p}\log\left|\frac{\bar{q}}{\bar{y}^2\bar{p}}\right| - \frac{2\bar{x}\bar{q}}{\bar{y}} .$$

From these expressions it follows that their $\bar{G}^i$ are written in the form (1.22) as follows:

(5.9a) $$\bar{G}^i = -P\bar{y}^i , \quad P = -\frac{\bar{q}^2}{\bar{L}\bar{y}} ,$$

(5.10a) $$\bar{G}^i = -P\bar{y}^i , \quad P = \frac{\bar{p}}{\bar{L}\bar{y}^2}(2\bar{y}\bar{p} - \bar{x}\bar{q}) .$$

15, ZENBU-CHO, SHIMOGAMO,
SAKYO-KU, KYOTO,
606 - JAPAN

References

[1] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwer Acad. Publishers, Dordrecht/Boston/London, 1993.

[2] W. Blaschke, Vorlesungen über Differentialgeometrie I (Dritte Auflage), Springer, 1930 (Chelsea 1967).

[3] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha Press, Saikawa 3-23-2, Ōtsu, Japan, 1986.

[4] M. Matsumoto, Projectively flat Finsler spaces of dimension two and an example of projective change, Proc. 5th Natl. Semin. Finsler and Lagrange spaces, Soc. Sti. Mat. Romania, Univ. Brasov, 1988, 233-239.

[5] M. Matsumoto, The inverse problem of variation calculus in two-dimensional Finsler space, J. Math. Kyoto Univ., 29(1989), 489-496.

[6] M. Matsumoto, Geodesics of two-dimensional Finsler spaces, Mathl. Comput. Modelling 20, No.4/5, 1-23, 1994.

[7] M. Matsumoto, Every path space of dimension two is projective to a Finsler space, Open systems & information dynamics, 3-3 (1995), 1-13.