TWO-DIMENSIONAL HAZARD ESTIMATION FOR LONGEVITY ANALYSIS.*

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Abstract

We investigate the practical aspects of some recent marker dependent hazard estimators when applied to mortality models. We analyse the development over time of Danish longevity from 1974-1998, whith mortality considered as a two-dimensional function of age and chronological time, respectively. A method of bootstrapping confidence bands of marker dependent estimator is applied to identify significant changes in the mortality patterns. Functionals of the hazard rate estimator, such as expected remaining lifetime and probability of survival, are also presented. We outline the usefulness of this methodology to analyse the age and time effect on longevity and its implications for life insurance risk management. The estimation techniques can be a starting point for more advanced prediction models.

Key Words: Longevity; Expected Remaining Lifetime; Kernel Hazard Estimation; Cross-validation.

1 Introduction

Longevity is a dynamic phenomenon. Welfare and scientific achievements have caused increasing values of life expectancy almost all over the world in recent decades (see Macdonald, Cairns, Gwilt and Miller, 1998 for a recent comparative study based on SMR, standard mortality ratios). This trend has been well established in the general population, but it may not be uniform across the age range because some causes of mortality have a stronger impact on specific age groups. The evolution of longevity over time may also be accelerating or decelerating. This development has immense importance for political planning and for pricing and reserving of life insurance products. It is therefore crucial for actuarial science to develop methods that are able to increase our insight into this question, especially as new medical advances come on to scene. Genetic knowledge, for example, may have a great impact on increasing life expectancy, thus we strongly believe that longevity prediction is of considerable importance for the future of life insurance companies. The method presented in this paper gives a visual impression of historical data. This is an important first step while searching for predictive methods of old age mortality.

Mortality analysis has a long tradition in actuarial science (see Cramer and Wold, 1935, Buus, 1960 and more recently Benjamin and Soliman, 1995). Conventional practice uses parametric graduation techniques to smooth out wild fluctuations when estimating probabilities of death for a given population. Graduation allows us to obtain a clear picture of the mortality curve, in other words the probability of death as a function of age. In the simple one-dimensional case in which it is assumed that mortality only depends on age, a significant amount of literature exists on nonparametric kernel estimators of the mortality function (see for example Ramlau-Hansen, 1983b, Gavin, Haberman and Verrall, 1995 and Nielsen 1998a). Renshaw, Haberman and Hatzopoulos (1996) developed a parametric model which incorporates both the effect of age on mortality and of underlying time trends on mortality rates so that the model captures the evolution of the mortality curve over time; and Ramlau-Hansen et al. (1987) presents an actuarial application of multidimensional mortality estimation of Danish diabetics based on a semiparametric kernel smoothing technique.

Here, we want to make use of the important recent developments in smoothing theory in the area of mathematical statistics. We work within the framework of counting processes, as they are suitable for statistical analysis in survival models. The model is described in the appendix, but for a complete presentation of counting processes in mortality analysis see Macdonald (1996). We take as our starting point the two-dimensional mortality estimator, which was defined theoretically in Nielsen and Linton (1995) and applied to Danish and Spanish mortality experience data in Felipe, Guillen and Nielsen (2001). This two-dimensional estimator considers mortality as a function of chronological time, t and age x. We apply the graduation principle of kernel hazard estimation as introduced by Nielsen and Linton (1995) to obtain an estimator of the force of mortality that we call $\alpha(t,x)$ (see Gerber, 1995, for standard actuarial definitions). Although this two-dimensional function is easily estimated, it can be hard to draw good conclusions from a two-dimensional graph. Felipe et al. (2001) looked at the relative quantity:

$$\frac{\alpha(t+\Delta t,x)}{\alpha(t,x)}$$

for fixed x, t and Δt. Based on this relative quantity they concluded that the development of mortality is indeed very different for different age groups, when studying the period from 1975 to 1993. The general tendency being that while mortality has been decreasing for children, teenagers and the elderly, it decreased a lot less for the working ages (between 30 and 60) or perhaps even increased for adults and specially for Danish women between 60 and 70. The most notable case is a 70 percent increase of mortality in 30-year-old Spanish males over those two decades. Felipe et al. (2001) also discussed the estimated level curves, where the mortality development is studied for one particular age x as a function of chronological time, $\alpha(\cdot, x)$.

In this paper we restrict ourselves to the Danish population data and update the data set to cover from 1974 until 1998. The data are also extended so that we have mortality information in the age interval from 0 to 98 years. The study of Felipe et al. (2001) did not consider mortality above age 90. This age group is of particular interest to us, since we are studying the development of expected future lifetime of people of old age. We also study the development of mortality as a function of chronological time and fixed age. Moreover, we study the evolution of the survival probability for pensioners of different age groups, the probability to survive at least until the 99th birthday. All the studied quantities are functions of the two-dimensional curve $\alpha(t, x)$ and we estimate them all by taking the corresponding function of our general kernel hazard estimator of $\alpha(t, x)$. The estimation of $\alpha(t, x)$ is therefore crucial to all our applications in this paper. Our non-parametric approach eliminates the need for a complete specification, arising when a generalized linear model framework is used (see Renshaw et al., 1996, p. 454). In this paper we also tackle two issues when using a kernel method: bandwidth choice and confidence interval estimation. We use cross-validation while selecting the degree of smoothing while estimating $\alpha(t, x)$. This method is based on the analogue to cross-validation for our type of data and has been introduced to similar counting process estimators in Ramlau-Hansen (1981), Nielsen (1990) and Nielsen and Linton (1995). We also introduce a general procedure based on bootstrapping for constructing confidence bands for all functionals of our estimator. See Efron (1979) for an early account of the bootstrap principle and Hall (1992) for a more recent monograph. Our method is an adjustment of the well known bootstrap principle to our counting process type of data. Details of our bootstrap method are given in appendix A.4.

The results in this paper deal with one of the core entities in insurance, mortality. We provide practitioners with a statistical methodology to explore the nature of longevity evolution in a given portfolio or subpopulation. Our discussion is not limited to estimated values, it is extended to the deviations from expected values, thereby quantifying risk in the proper sense. We apply these new techniques to describe mortality changes experienced in Denmark for both genders in the past twenty-five years. It is seen that Danish women over 80 have an increasingly longer future lifetime. Even for men we observe some longevity increasing trends, but these are not as significant as for women. These results are important and will have impact when actuaries calculate the needed reserves for the annuities of old-aged policy holders.

In section 2 we specify the marker dependent hazard model considering mortality as a function of age and chronological time. In section 3 we show how mortality curves can be analyzed based on the two-dimensional marker dependent hazard defined in the previous section. We consider risk profiles for people of age 30, 50, 70 and 90.

In section 4 we extend the analysis to the expected remaining lifetime. We consider the expected future lifetime for people of age 60, 70, 80 and 90. In section 5 we extend the analysis to the remaining survival time to age 99. We consider survival probabilities for people of age 60, 70, 80 and 90.

All technicalities and estimation techniques are left to the appendix. Appendix A.1 states the general model formulation based on counting processes, and in A.2 we define the version of the kernel hazard estimator of Nielsen and Linton (1995), which is of particular interest to our study. In A.3 we define the cross-validation principle that the amount of smoothing of our actual estimator is based on and, finally, in A.4 we introduce the new general principle of bootstrapping confidence bands for all functionals of our estimator of $\alpha(t, x)$.

2 Model and data

Let us assume that the future lifetime T_x of a person aged x is a random variable with a probability distribution function $G_x(z) = \Pr(T_x \leq z)$ for $z \geq 0$. The international actuarial community uses the following notation: ${}_zq_x = G_x(z)$, indicating the probability for a person aged x to die before age $x + z$. So, ${}_zp_x = 1 - G_x(z)$ denotes the probability that a person aged x will survive at least z years.

In this context, the force of mortality (or hazard) is defined by:

$$\mu_x = -\frac{d}{dz} \ln[1 - G_x(z)],$$

so, the probability of dying between age x and age $x + dz$ equals $\mu_x dz$ for small dz.

All these quantities are the basic principles for life insurance calculations. But the estimation of mortality probabilities and of the force of mortality has to be done from statistical data and, consequently, it makes use of statistical assumptions. Our method is general in two directions. Firstly, it needs fewer assumptions than the existing graduation models i.e. we do not restrict ourselves to e.g. the Makeham formula and, secondly, it takes into account that modern societies experience continuous innovations that cause structural changes in the patterns of mortality. One immediate consequence is that it is not reasonable to assume that $G_{\mathsf{x}}(\cdot)$ stays the same over a time period of more than two decades. So, if we introduce a time dimension, we may denote $G_{\mathsf{t,x}}(\cdot)$ the probability distribution function of survival time for age x in year t. We will use $\alpha(t,x)$ to denote the force of mortality in year t for age x, thus allowing for a chronological evolution of mortality as well as the usual dependence on the age component. Then, by definition, the time-dependent hazard is:

$$\alpha(t,x) = -\frac{d}{dz}\ln[1 - G_{\mathsf{t,x}}(z)]. \tag{1}$$

We propose the estimation of $\alpha(t,x)$, so that we will be able to analyse the changes in shape over the two axes simultaneously. This is useful for identifying age-specific evolution of mortality over time, or in other words, to be able to visualize the dynamics of the estimated current life table. Our method is nonparametric with smosthness as the only functional assumption α.

The data used in this paper were provided by Statistics Denmark, with detailed information on exposure counts and the number of occurrences observed in the Danish population each year from 1974 to 1998. Counts were given by gender and integer age intervals.

Table 1. Number of people (Exp.) and number of deaths (Occ.) by age group. Danish males.

	1974		1998	
Age group	Exp.	Occ.	Exp.	Occ.
0-9	394,202	639	344,604	244
10-19	384,983	259	295,522	112
20-29	418,037	449	377,733	289
30-39	323,385	491	416,226	579
40-49	280,712	1,182	378,116	1,288
50-59	282,655	2,976	354,812	2,727
60-69	240,667	6,394	224,045	5,046
70-79	131,743	8,747	156,835	8,707
80-89	41,424	6,013	61,184	8,062
90-99	3,730	1,159	6,463	1,863

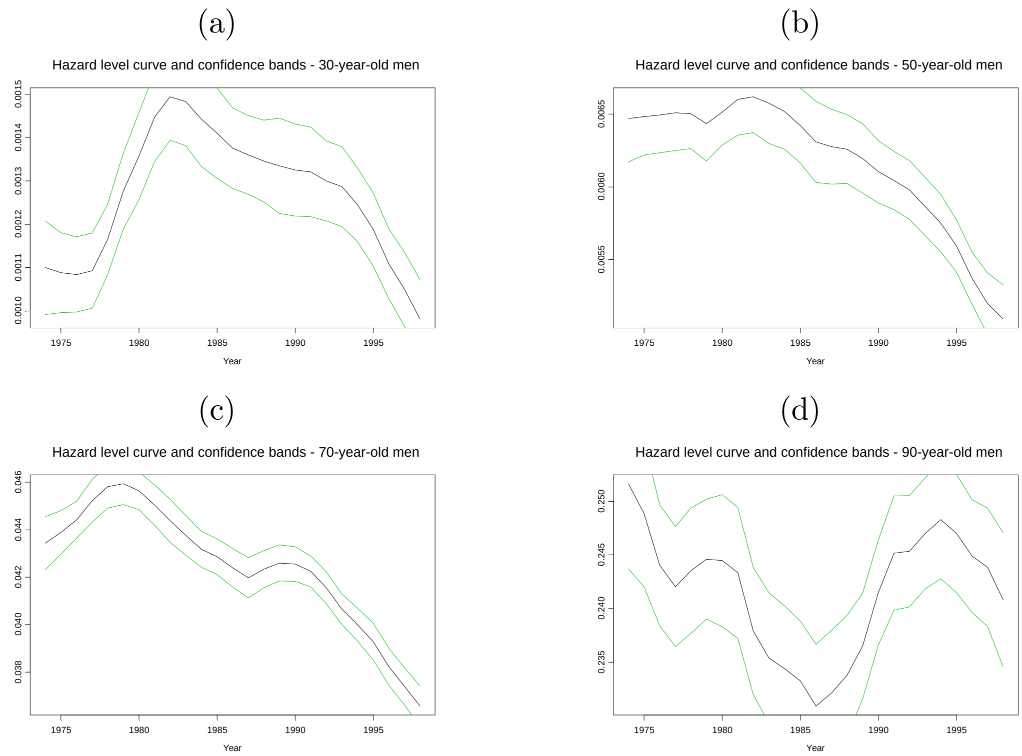
Table 1 shows the aggregated data in ten age groups, but the original information for each integer age and each year is used in the estimation. In columns we present the raw data in 1974 and 1998. The ratio between the number of occurrences and the number of exposures produces the raw mortality rates.

Since the model specification of the two-dimensional mortality is slightly technical, it is left to the appendix. The appendix relates the exact consequences of assuming an underlying two-dimensional true hazard $\alpha(t, x)$ to the analyses of our data. It specifies how $\alpha(t, x)$ can be estimated based on a given amount of smoothing and it introduces a model selection criteria that chooses the amount of smoothing that we apply in our actual study. Finally, a bootstrapping method is introduced. The method is designed to construct confidence bands for all well behaved functionals of the underlying two-dimensional force of mortality. The considered functionals in this paper are the risk profiles, the remaining life expectancies and various survival probabilities. In the rest of the paper we therefore consider the situation in which a two-dimensional estimator of the force mortality for Danish men and Danish women is constructed and bootstrapped confidence bands exist on all considered functionals of the underlying mortality. All details regarding these issues are fully described in the appendix.

3 Mortality Profiles

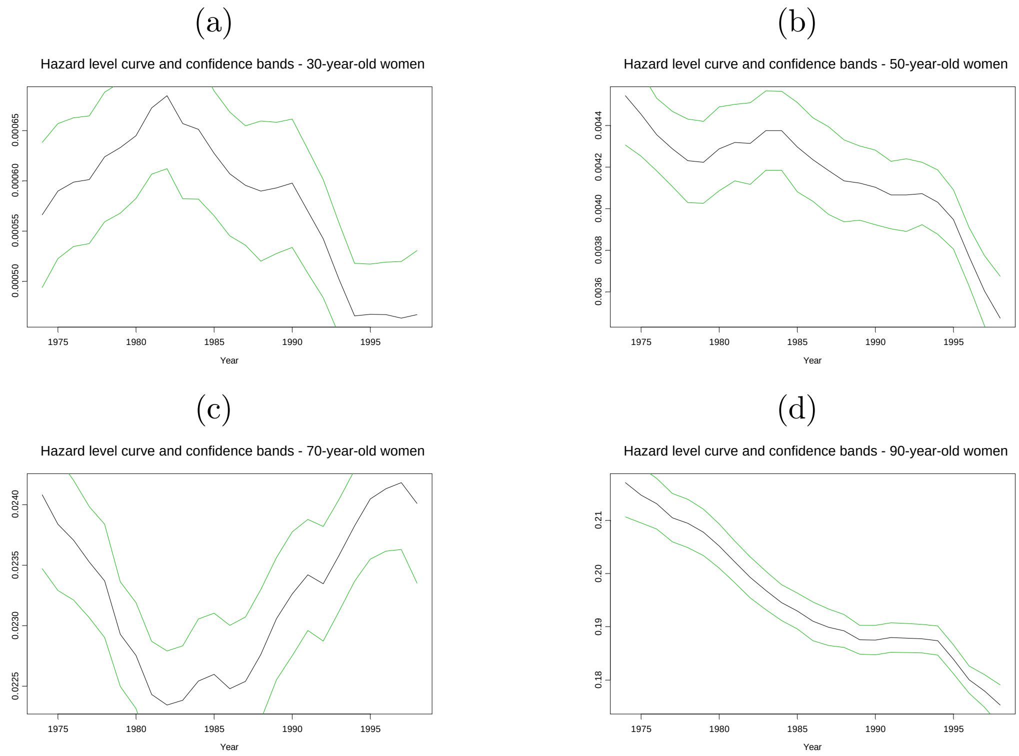
In this section we present the estimated mortality levels of (1) with fixed age and varying chronological time for Danish men and Danish women, together with their corresponding confidence bounds. We restrict this presentation to the eight curves resulting from considering both genders and ages of 30, 50, 70 and 90. The risk profiles provide a complete overview of the population's mortality evolution for the ages 30, 50, 70 and 90.

Firstly, we consider men, see Figure 1(a) to Figure 1(d), where the estimated force of mortality is plotted for a given age throughout the twenty-five year period, confidence bounds are also shown. The estimated hazard for the thirty-year-old men does not show any significant long-term effect. The curves regarding the fifty and seventy-year-old do have a decreasing shape, so the force of mortality has been lowering its level for these two ages. The decrease amounts to approximately 20 percent for the fifty-year-old group and 15 percent for the seventy-year-old group. Taking the confidence bands into consideration we conclude that the hazard level in 1998 is significantly lower than in 1974 for adult men of 50 and 70 years. For age 90, we see a somewhat unclear picture. The dominant peak in 1991-1996 has an explanation. In 1994 Denmark had a series of tropical nights. July had about twenty days with temperatures never going below 20 degrees Celsius (68^{o}F). These temperatures and a very high humidity, caused a high level in mortality for elderly people in Denmark. Throughout this paper we will see that 1994 has some special impact when regarding mortality or functionals of the mortality hazard. Due to our chosen smoothing procedure the peak shows in the interval 1992-1996.



Figures 1(a) to 1(d). Estimates of the force of mortality for Danish men from 1974 to 1998 and 95% confidence bounds. a) 30 years, b) 50 years, c) 70 years and d) 90 years.

Secondly, we considered women, see Figure 2(a) to Figure 2(d). We see that at age 30 the force of mortality reaches a maximum in 1983. From 1983 to 1998 the hazard curve decreases significantly. The shape of the curve is similar to the curve for males, but the general hazard level is about 50 percent lower. For the fifty-year-old women, we see a clear trend towards a lower mortality in the observation period of more than two decades. The decrease is approximately 22 percent. The hazard rate curve for women aged 70 shows a minimum overall level from 1979 to 1989. Though the graph looks dramatic, it is worth noting that the total variation of the hazard curve is only approximately 6 percent. Further studies of the hazard estimator in ages from 60-75 show that we actually find an increasing mortality in the range 62-69 years. For ages 70-74 the hazard does not show a clear trend, but from age 75 the hazard shifts downwards with respect to calender time indicating some longevity effect from this age. This result was also shown by Felipe et al. (2001). For women aged 90, we see a clear decrease at about 20 percent in the force of mortality estimates.



Figures 2(a) to 2(d). Estimates of the force of mortality for Danish women from 1974 to 1998 and 95% confidence bounds. (a) 30 years, (b) 50 years, (c) 70 years and (d) 90 years.

We have also investigated the curves covering all ages from age 0 to age 98. We were interested in any possible multiplicative or additive longevity effect. The variation in the shapes of the curves showed no systematic pattern to suggest a multiplicative or additive model.

4 Development of expected remaining lifetime for elderly people

For a given age and a given chronological time, the remaining lifetime is defined as:

$$\overset{\circ}{e}_{\mathsf{t,x}}= \int_0^\infty \exp\left\{-\int_0^{\mathsf{s}} \alpha\left(t+u, x+u\right) du\right\} ds.$$

Therefore the remaining lifetime is a function of the underlying two-dimensional hazard $\alpha(t,x)$. See, Gerber (1995) or Jordan (1967) for classical references to expected remaining lifetime formulation and estimation. Newman (1986) introduces some generalizations to life expentancy calculation, but he still assumes that the forces of mortality are (calendar) time invariant.

We can therefore use our estimator of $\alpha(t,x)$ as an intermediate step while estimating the expected remaining lifetime and we can use our general bootstrapping method to evaluate the confidence bands. As our observation interval 1974-1998 only gave us a limited possibility for working in both the time and age direction at the same time, we used another estimator of the populations expected remaining lifetime at age x in year t.

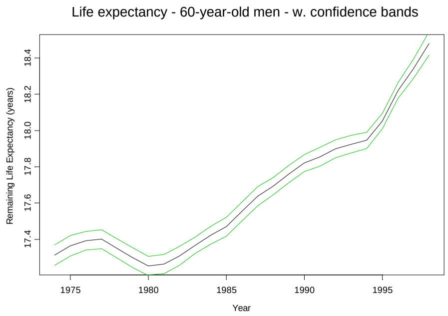
This estimator is defined as:

$$e_{\mathsf{t,x}} = \int_0^\infty \exp\left\{-\int_0^{\mathsf{s}} \alpha\left(t, x+u\right) du\right\} ds.$$

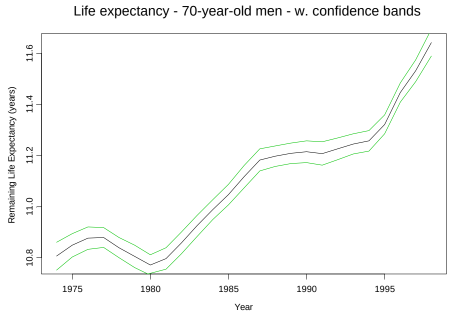
We estimate the expected remaining lifetime for the sample under exposure in every year from 1974 until 1998. Here, we restrict our presentation to the elderly, namely to the eight curves resulting from considering ages 60, 70, 80 and 90 and both genders, separately. The estimator is often referred to as the population's life expectancy.

Regarding men, we observe that there has been an increase in the expected remaining lifetime in the past two decades (Figure 3a to 3d). For men aged 60, the remaining life expectancy shifts from 17.3 years in 1974 to 18.5 years in 1998. This corresponds to an increase of 7 percent. The high mortality peak experienced in 1994 is shown on the graph. If this year is omitted from the data, a linear development in the remaining life expectancy from 1980 to 1998 shows up. During our observation period there seems to be a systematic increase with no sign of convergence towards a fixed level of remaining life expectancy for men aged 60. The 70-year-old men show a similar graph in Figure 3(b). Again, the 1994 effect is shown even more prominently in the graph. We see an increase from 10.8 years in 1974 to 11.6 years in 1998. As for age 60, it corresponds to a 7 percent increase. There is still no sign of convergence. For men aged 80, the same features appear in Figure 3(c). While the 1994 peak is significant, it seems to be explained primarily by the particular weather conditions. The increase in life expectancy for an 80-year-old man is slightly less than 7 percent. Finally, the remaining life expectancy for 90-year-old men does not show any significant trend.

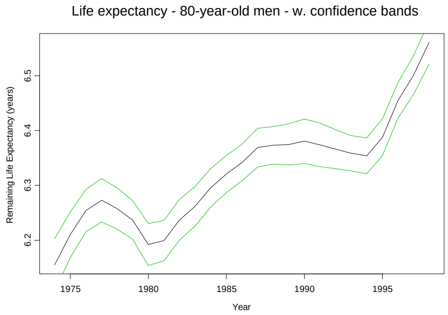
(a)



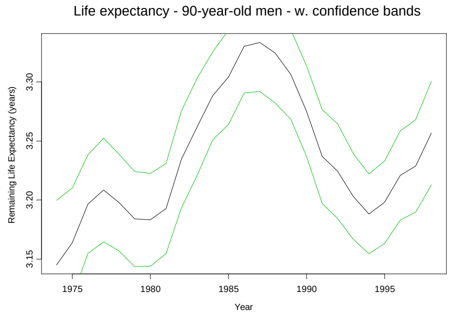
(b)



(c)



(d)



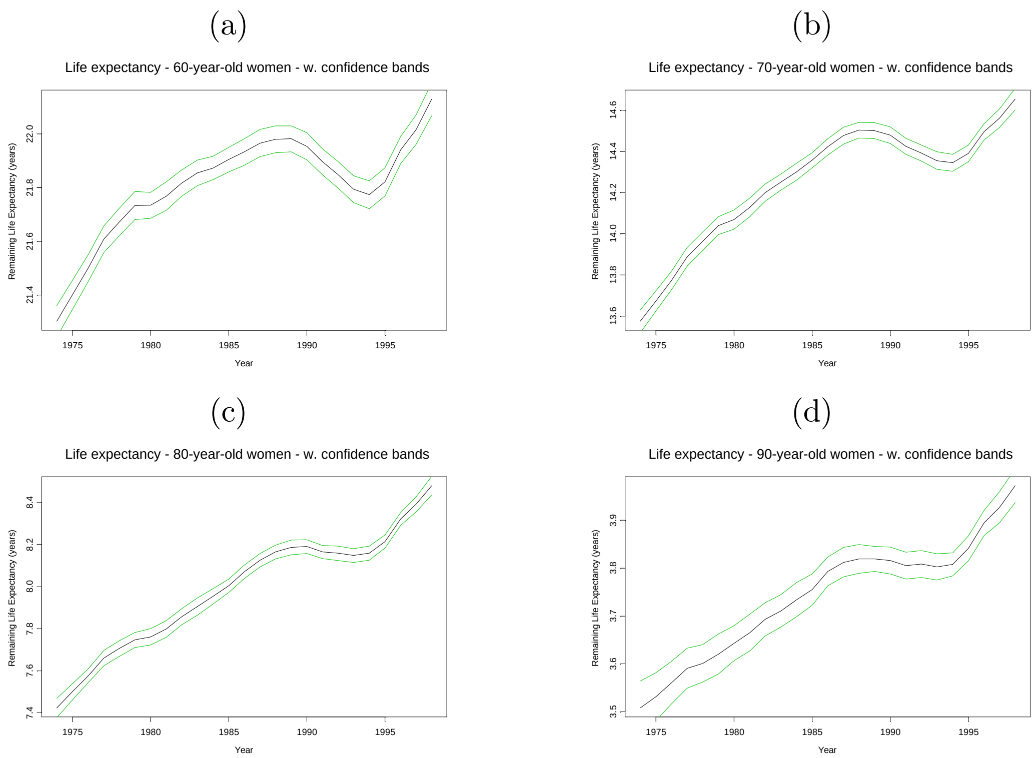
Figures 3(a) to 3(d). Estimates of the expected remaining lifetime for Danish men from 1974 to 1998 and 95% confidence bounds. (a) 60, (b) 70, (c) 80 and (d) 90 years old.

When observing the behavior of data on Danish women aged 60 in Figure 4(a), we see an increase in remaining life expectancy from 21.3 in 1974 to 22.1 in 1998. This corresponds to an increase below 4 percent. The year 1994 shows a significant low level of life expectancy for this age group, and due to smoothing all the years 1992-1996 are influenced by this outlier. The confidence bands show that we are looking at a significant increase in life expectancy from the start in 1974 to the end in 1998.

The 70-year-old women give us some interesting information. Figure 4(b) depicts a one year absolute increase in life expectancy at this age, rising from 13.6 years in 1974 to 14.6 years in 1998. When compared to the 0.8 year increase for the 60-year-old women we conclude that the overall mortality in ages sixty to seventy has been increasing in the time period 1974-1998. We are not able to give an explanation for this result. Felipe et al. (2001) showed the same trend in their comparison between the mortality of Danish women in 1975 and in 1993.

The 80-year-old women life expectancy estimates in Figure 4(c) again show one year of increase (7.4 years in 1974 to 8.4 years in 1998). As this is the same increase as for the 70-year-old women, it implies that women aged 70-80 years do not contribute differently to the general development of life expectancy. Based on these graphs we can conclude that the changes in life expectancy mainly originate in the behavior of the female population over 80 year old.

The 90-year-old women show an increase in remaining life expectancy in Figure 4(d). While the total increase is only half a year, the relative increase is approximately 14 percent. As life expectancy for the 80 and 90-year-old women has significantly increased, this shall be taken into consideration when pricing for annuities.



Figures 4(a) to 4(d). Estimates of the expected remaining lifetime for Danish women from 1974 to 1998 and 95% confidence bounds. (a) 60, (b) 70, (c) 80 and (d) 90 years old.

5 Development of survival probability for elderly people

For a given age x and a year t, the probability of surviving to age 99 years equals:

$$S_{\mathsf{t,x}} = \exp\left\{-\int_0^{\mathsf{99-x}} \alpha\left(t+u, x+u\right) du\right\}.$$

It is clear that the survival function is a function of the underlying two-dimensional hazard $\alpha\left(t,x\right)$. We can therefore use our estimator of $\alpha\left(t,x\right)$ as an intermediate step while estimating the survival time, and we can use our general bootstrapping method to calculate the confidence bands.

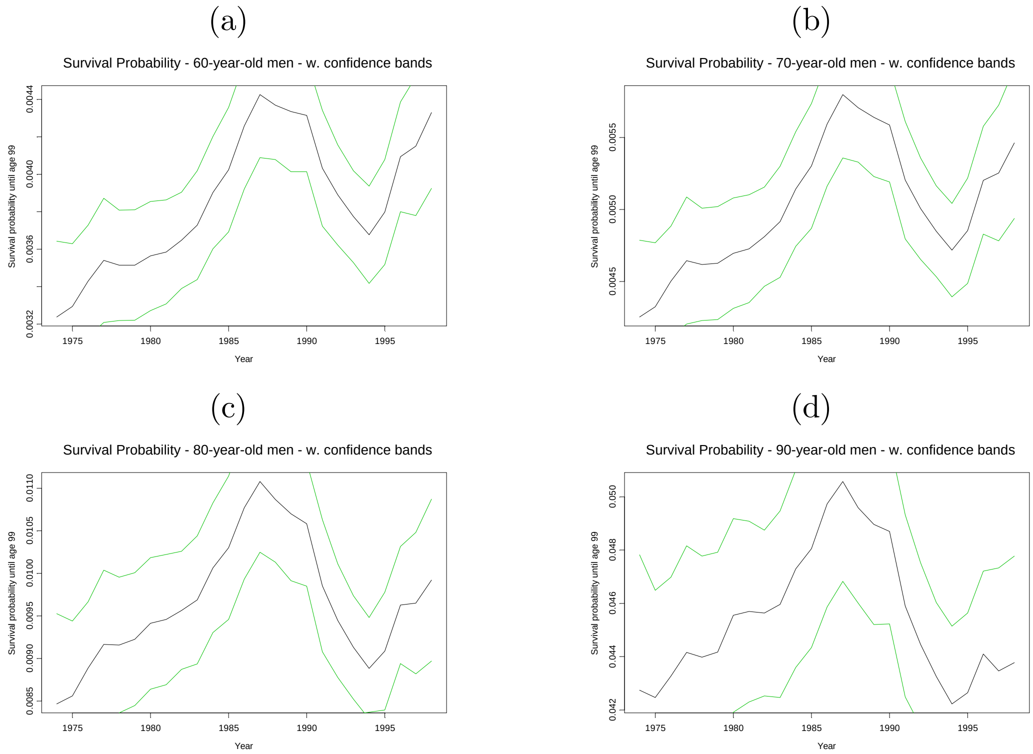
Following the same arguments as in the previous section we choose to use the following expression:

$$S_{\mathsf{t,x}} = \exp\left\{-\int_0^{99-\mathsf{x}} \alpha\left(t, x+u\right) du\right\}.$$

Again this estimator can be interpreted as the survival probablilty derived from the current moratlity pattern in the population.

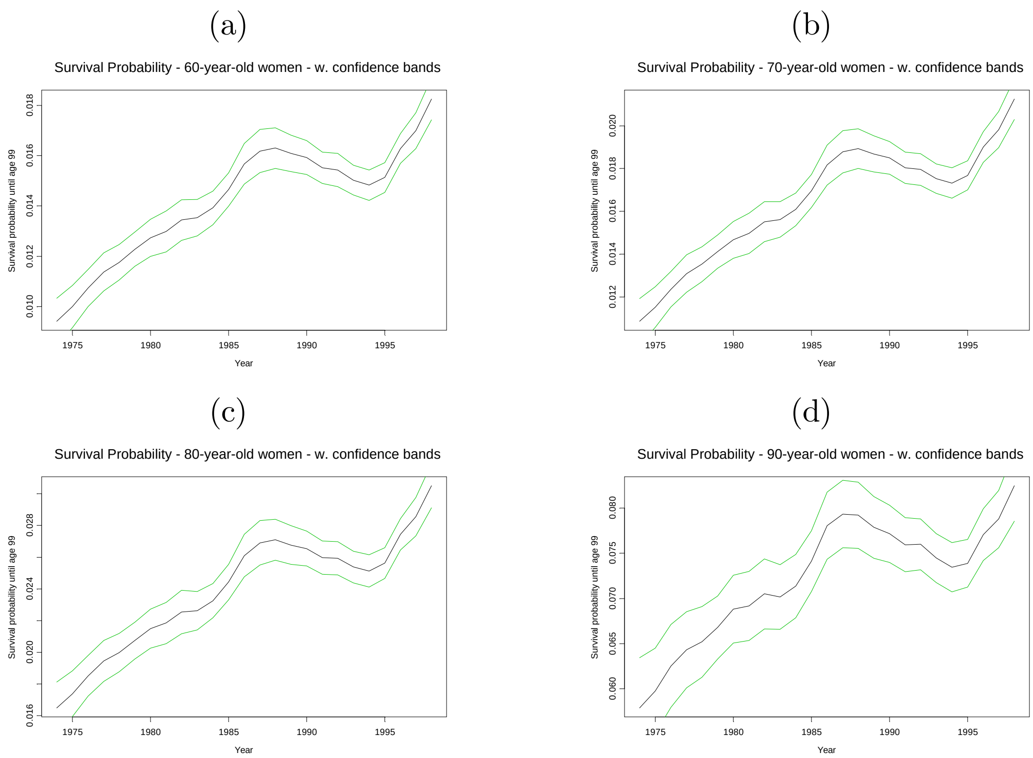
We present the evolution of the survival probability estimates from 1974 to 1998. The eight curves refers to the ages 60, 70, 80 and 90, and both genders separately. An overall increase of this survival probability is expected, but if the estimates indicate that the improvement is more significant for elderly people, then insurers should reconsider the way reserves for annuities have been calculated and update the estimated probability distribution of the net premiums (i.e. the expected present value of payments).

When looking at the results for Danish men in Figures 5(a) to 5(d), the graphs show no significant tendency. The survival probability, $S_{\mathsf{t,x}}$ increases in the interval 1974-1987. From 1987 $S_{\mathsf{t,x}}$ decreases, reaching a minimum in 1994. Due to the bandwidth choice, the 1994 effect influences the estimates in the period from 1992 to 1996, but there is no evidence to see that in 1997 and 1998 higher survival probabilities are higher than in 1987. Note that the confidence bands of the survival probability are wide. Therefore, we are not able to derive any definite conclusion on male survival probability.



Figures 5(a) to 5(d). Estimates of the probability to survive at least until 99 years old for Danish men from 1974 to 1998 and 95% confidence bounds. (a) 60, (b) 70, (c) 80 and (d) 90 years old.

When analyzing Danish women, we see a much clearer trend in Figure 6(a) to 6(d). All the graphs show a significant increase in the survival probability. As we have plenty of exposure (the number of surviving elderly women is larger than for men), the confidence bands are quite narrow, giving us a good estimation of the survival probability. As expected, again we notice that 1994 was a special year with some excess mortality due to extreme weather conditions. Therefore 1994 shows a local minimum of $S_{\mathsf{t,x}}$. The overall picture does not give us any sign of convergence in $S_{\mathsf{t,x}}$ over the observation period 1974-1998.



Figures 6(a) to 6(d). Estimates of the probability to survive at least untill 99 years for Danish women from 1974 to 1998 and 95% confidence bounds. (a) 60, (b) 70, (c) 80 and (d) 90 years old.

These results indicate how important it is for insurers to use these methods in order to keep track of their risk. For instance, the increasing longevity of women has an immediate consequence on reserving. One could even be interested in testing whether the portfolio longevity behavior is similar to the population one.

As a general comment regarding confidence bands let us emphasise that in all our studies it turns out that most conclusions that can be arrived by looking at the graphs cannot be rejected on the basis of confidence. This was perhaps expected due to the quite substantial size of our data set, but it is nevertheless good to have a scientifically based confirmation of this expectation. The new methodology of confidence bands is even more important when studying data from different insurance populations, where the amount of data is often smaller than in a study of a nation-wide mortality development.

6 Discussion

Longevity or life expectancy at birth is a standard indicator of welfare and general socio-economic conditions of a population. For the life insurance industry, innovations affecting the longevity of the elderly have an enormous impact because they have a direct effect on the premium that should be charged for a number of their products. Therefore there is a need for methods to examine changes in longevity and to test their significance. Since annuities were probably calculated using an underestimated longevity, reserves should be modified accordingly. Just two decades ago, the probability of reaching the age of 100 was estimated to be so small that it was usually neglected in practical actuarial calculations. It seems that longevity expectations are so different nowadays that actuaries should start considering higher survival probabilities. In order to solve these questions, actuaries have to obtain precise estimations of the force of mortality for advanced ages, thus incorporating the possibility of improvements or even structural shocks. A structural change is likely to happen if medical research achieves revolutionary remedies for diseases that are among the main causes of mortality. The methods described in this paper are aimed at providing a statistical tool to visualise observed trends of mortality. This type of visualisation seems crucial as a starting point for the development of better predictions of future mortality. Since portfolio information is usually based on a smaller number of exposures than the usual frequencies in country mortality databases, one problem arising when inspecting portfolio mortality information is that portfolio raw mortality rates are very erratic. Our smoothing method is particularily suitable to eliminate these fluctuations and to provide the practitioner with a useful tool to figure out the shape of the estimated force of mortality and the form of all the other quantities that are derived from it (future life expectancy or probability of survival). We have also presented how to obtain confidence bounds. This latter result is fundamental to improve the practical usefulness of the methods described in this paper. In particular if the data set at hand is small. By comparing confidence bounds, one could for example evaluate whether longevity in a portfolio is significantly higher than the longevity in the population.

Another interesting result that can be derived from the proposed confidence bound calculation methodology is that using these bounds, conclusions can be drawn about the effective differences in life expectancy levels for different geographical regions or different subportfolios. Our proposed method seems to be very efficient for this purpose since it is designed to give valuable results for both big and small data sets.

Appendix

All technicalities and estimation techniques are left to this appendix. In § A.1 we give the very general model formulation based on counting processes. In § A.2. we define the version of the kernel hazard estimator of Nielsen and Linton (1995) that is of particular interest to our study. In § A.3 we define the cross-validation principle that the amount of smoothing of our actual estimator is based on, and in § A.4 we introduce the new general principle of bootstrapping confidence bands for all functionals of our estimator of $\alpha\,(t,x)$.

A1. The model

We observe n individuals $i = 1,..,n$. Let $N_i^{(n)}$ count observed deaths for the i'th individual in the time interval $[1974, 1998]$. We assume that $\mathsf{N}^{(n)} = (N_1^{(n)},..,N_n^{(n)})$ is an n-dimensional counting process with respect to an increasing, right continuous, complete filtration $\mathcal{F}_t^{(n)},\ t \in [1974, 1998]$, i.e. one that obeys les conditions habituelles (see Andersen et al. 1992, p. 60). We model the random intensity process $\lambda^{(n)} = (\lambda_1^{(n)},..,\lambda_n^{(n)})$ of $\mathsf{N}^{(n)}$ as depending on both chronological time, which is our time scale, and age represented as a covariate

$$\lambda_i^{(n)}(t) = \alpha\{t, X_i^{(n)}(t)\}Y_i^{(n)}(t),$$

where we have no restrictions on the functional form of $\alpha(\cdot)$. Y_i is a predictable process taking values in $\{0,1\}$, indicating (by the value 1) when the i'th individual is under risk, while the age $X_i(t)$ is a 1-dimensional, predictable, $CADLAG$, covariate process. We assume that $\mathcal{F}_t = \sigma\,(\mathsf{N}(s), \mathsf{X}(s), \mathsf{Y}(s); s \leq t)$, where $\mathsf{Y} = (Y_1, Y_2,.., Y_n)$ and $\mathsf{X} = (X_1, X_2,.., X_n)$. With these definitions, λ is predictable and the processes $M_i(t) = N_i(t) - \Lambda_i(t)$, $i = 1,\ldots,n$, with compensators $\Lambda_i(t) = \int_0^t \lambda_i(s)\,ds$, are square integrable local martingales on the considered time interval, see also Nielsen and Linton (1995).

A2. Estimating α

The estimator suggested by Nielsen and Linton (1995) and Felipe and Guillen

(1999) is:

$$\widehat{\alpha}(t,x) = \frac{\sum_{i=1}^{n} \int_{1974}^{1998} K_{b_1}(t-s)\, K_{b_2}\{x - X_i(s)\}\, dN_i(s)}{\sum_{i=1}^{n} \int_{1974}^{1998} K_{b_1}(t-s)\, K_{b_2}\{x - X_i(s)\}\, Y_i(s) ds} = \frac{O_{t,x}}{E_{t,x}}.$$

where

$$O_{t,x} = \sum_{i=1}^{n} \int_{1974}^{1998} K_{b_1}(t-s)\, K_{b_2}\{x - X_i(s)\}\, dN_i(s)$$

and

$$E_{t,x} = \sum_{i=1}^{n} \int_{1974}^{1998} K_{b_1}(t-s)\, K_{b_2}\{x - X_i(s)\}\, Y_i(s) ds$$

are respectively the smoothed occurrence and the smoothed exposure. Nielsen (1998b) pointed out that this estimator could be interpreted as a local constant marker dependent hazard estimator. Nielsen and Linton (1995) showed that this estimator is asymptotically normal with an asymptotic bias. This result parallels the standard type of results on asymptotic theory of smooth kernel estimators. While our life expectancy curve and our conditional survival function have not been analysed theoretically, then we do however, point out, that they can be analysed using the same principles as Nielsen and Linton (1995) did. These two latter estimators will also be asymptotically normal with bias. We use the Epanechnikov kernels;

$$K_{b_1}(t-s) = 0.75 I(|t-s| \le b_1) \left\{ 1 - \left(\frac{t-s}{b_1} \right)^2 \right\}$$

and

$$K_{b_2}(x - X_i(s)) = 0.75 I(|x - X_i(s)| \le b_2) \left\{ 1 - \left(\frac{x - X_i(s)}{b_2} \right)^2 \right\}.$$

We note that it is an important feature of this estimation procedure that the resulting estimator has the well known occurrence divided by exposure construction.

Most practitioners know different types of occurrence divided by exposure methods. For example the widespread method of piecewise constant mortalities has this construction. The local constant kernel estimator here corresponds to the occurrence exposure ratio. Let for a moment the kernel $K(x)$ equal $I(|x| < 1)$, then $O_{\mathrm{t,x}}$ corresponds to the observed number of failures with deaths in the chronological time interval $[t - b_1, t + b_1]$ and age in the interval $[x - b_2, x + b_2]$, and $E_{\mathrm{t,x}}$ corresponds to the exposure time observed in the area in the chronological time interval $[t - b_1, t + b_1]$ and age in the interval $[x - b_2, x + b_2]$. With the constant kernel, the estimator therefore is the traditional actuarial occurence exposure ratio. When we choose a smooth kernel, we therefore get a smooth adjustment of the traditional actuarial estimation method.

A3. Choice of bandwidth by cross-validation

In this section we consider the question of choosing the smoothing bandwidth automatically. More precisely, we use the marker depend hazard estimation equivalent to cross-validation. Little has been published on cross-validation for hazard models based on counting process theory, however, the working paper of Ramlau-Hansen (1981) contains a description of what to do in the one-dimensional case, and Nielsen (1990) gave an extensive theoretical investigation of this approach showing that cross-validation of one-dimensional kernel hazard estimation has all the same properties as what was known on the equivalent kernel density cross-validation estimator at the time. A practical application of the one-dimensional cross-validation approach was given in Andersen et al. (1993) and a practical application of a marker dependent hazard cross-validation was given in Nielsen and Linton (1995). We generalize the approach of Nielsen and Linton (1995) to our setting. There are many reasonable choices of criteria for selecting bandwidth. We work with a stochastic L_2 measure that is tractable from the point of view of mathematical analysis. Let $\widetilde{\alpha}_{\overline{b}}$ be any estimator of α considered in this paper that depends on a vector of bandwidths $\overline{b} = (b_1, b_2)$.

We want to minimize the quadratic term $Q_n(\overline{b})$:

$$
\begin{aligned}
Q_n(\overline{b}) &= \frac{1}{n}\sum_{i=1}^{n}\int_{1974}^{1998} \left[\widetilde{\alpha}_{\overline{b}}\{s, X_i(s)\} - \alpha\{s, X_i(s)\}\right]^2 Y_i(s)ds \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_{1974}^{1998} \widetilde{\alpha}_{\overline{b}}\left\{s, X_i(s)\right\}^2 Y_i(s)ds - \frac{2}{n}\sum_{i=1}^{n}\int_{1974}^{1998} \widetilde{\alpha}_{\overline{b}}\left\{s, X_i(s)\right\}\alpha\left\{s, X_i(s)\right\} Y_i(s)ds \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\int_{1974}^{1998} \alpha\left\{s, X_i(s)\right\}^2 Y_i(s)ds.
\end{aligned}
$$

The third term in $Q_n(\overline{b})$ does not depend on the bandwidth, so it is not computed. The first term depends only on the data and can be computed directly. Only the second term causes any problems, because it depends on the unknown α. The estimate $\frac{2}{n}\sum_{i=1}^{n}\int_{1974}^{1998} \widetilde{\alpha}_{\overline{b}}\left\{s, X_i(s)\right\} Y_i(s)dN_i(s)$ is biased due to the correlation between $\widetilde{\alpha}\left\{s, X_i(s)\right\}$ and $dN_i(s)$. This problem can be solved by replacing $\widetilde{\alpha}_{\overline{b}}\ \left\{s, X_i(s)\right\}$ by the leave-one-out version $\widetilde{\alpha}_{-i}\left\{s, X_i(s)\right\}$. Thus, we will choose $\overline{b}$ to minimize

$$
\widehat{Q}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n}\int_{1974}^{1998} \widetilde{\alpha}_{\overline{b}}\left\{s, X_i(s)\right\}^2 Y_i(s)ds - \frac{2}{n}\sum_{i=1}^{n}\int_{1974}^{1998} \widetilde{\alpha}_{\overline{b}-i}\left\{s, X_i(s)\right\} dN_i(s).
$$

From Figure 7 it can be seen that a smoothing parameter vector of chronological time around 1-5 years and of age around 2-3 years seems appropriate for both genders. So, the best smoothing parameter is a bandwidth of three years in the chronological time direction and two years in the age direction. However it is clear that several choices of $\overline{b} = (b_1, b_2)$ result in approximately the same $Q_n(\overline{b})$. Moreover, bandwidth selection by cross validation is not too important when working on datasets with size like the Danish population. We also used the cross validation technique on smaller datasets, and in that situation $Q_n(\bar{b})$ showed a clearer minimum. Correct bandwidth selection is more important with smaller datasets than with larger datasets.

(a)   (b)

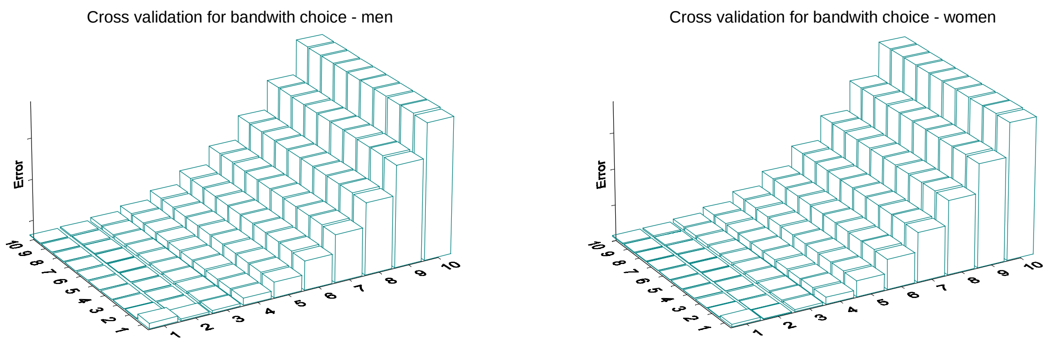


Figure 7(a) and (b). Cross validation results for different bandwidth choices (a) Men (b) Women.

A4. bootstrapping the confidence bands

In this section we give a general bootstrapping procedure intended to evaluate the confidence of any two-dimensional hazard estimator $\widehat{\alpha}(t, x)$ or functional of this hazard. We follow the original idea of Efron (1979). Let F be some distribution and suppose we intend to estimate a functional $H(F)$ of this distribution. The basic bootstrap answer to this question is to estimate $H(F)$ by $H(\widehat{F})$ where $\widehat{F}$ is an estimator of F. In most practical situations $H(\widehat{F})$ is so complicated that is has to be calculated by simulations. Therefore most bootstrap procedures involves a simulation step. While we omit a theoretical analyses of our bootstrap procedure, we do note, that a standard bootstrap analyses involves an expansion of $H(F)$ - $H(\widehat{F})$. Hall (1992) is still state-of-the-art when it comes to this type of bootstrap analyses.

Consider now a functional of the underlying counting process data

$$\Psi \left\{ (\Lambda_1, X_1, Y_1), .., (\Lambda_n, X_n, Y_n) \right\}, \tag{2}$$

where Ψ is the appropriate functional. Our bootstrap procedure estimates this functional by

$$\Psi \left\{ (\widehat{\Lambda}_1, X_1, \mathsf{Y}_1), .., (\widehat{\Lambda}_n, X_n, \mathsf{Y}_n) \right\}, \tag{3}$$

where the $\widehat{\Lambda}_i$'s are the integrated kernel hazard smoother taken as a function of the observed counting processes:

$$\widehat{\Lambda}_i(t) = \int_0^t \widehat{\alpha} \left\{ s, X_i(s) \right\} \mathsf{Y}_i(s) ds.$$

Note that we have chosen to condition on the exposure. Preliminary theoretical considerations has led to the conclusion that this gives the correct answer. For our purposes we note that the three estimators considered in this paper, the hazard, the conditional survival function and the expected lifetime, all can be represented as a functional of the underlying counting process data. In practice we are interested in the length of the confidence band of our estimators, namely the distance between the 95% quantile and the 5% quantile of the estimator. This distance is therefore also a functional of the underlying counting process data. These latter functionals are estimated by our bootstrap procedure and used en Section 3 to construct our confidence bands. We are not interested in representing the expected bias in our confidence bands.

Conditional on the observed covariate and exposure processes the correct confidence bands can be written as in the formula (2), see Theorem 1 in Nielsen and

Linton (1995) for the corresponding unconditional version. In practise we calculate our bootstrapped confidence band expression on the form (3) based on our grouped data using a general procedure that we illustrate below for the hazard case presented in Section 3. The calculations are performed as follows:

From the observed occurrence $O_{\mathsf{t,x}}$ and exposure $E_{\mathsf{t,x}}$ we calculate the estimator $\widehat{\alpha}(t,x)$ of $\alpha(t,x)$ using the techniques of the sections A2 and A3. The second step is to simulate n new set of occurrences. Each observation point in the occurrence matrix is drawn from the binomial distribution $O_{\mathsf{t,x}}^{\mathsf{k},*} = Bin(E_{\mathsf{t,x}}, \widehat{\alpha}(t,x))$. For each new simulated occurrence matrix $O_{\mathsf{t,x}}^{\mathsf{k},*}$ a new estimator $\widehat{\alpha}^{\mathsf{k},*}(t,x)$ is calculated based on this new occurence matrix and the original exposure $E_{\mathsf{t,x}}$. These $\widehat{\alpha}^{\mathsf{k},*}(t,x)'s$, $k = 1, ..., 200$ are stored. For a given age x and year t, all $\widehat{\alpha}^{\mathsf{k},*}(t,x)'s$ are ordered as $\widehat{\alpha}^{[1],*}(t,x), \ldots, \widehat{\alpha}^{[200],*}(t,x)$. The size of the 95% percent confidence bound can then be calculated as $\widehat{\alpha}^{[195],*}(t,x) - \widehat{\alpha}^{[25],*}(t,x)$. The bootstraped confidence bounds of the life expectancy and the conditional survival curves given in section 4 and 5 are calculated using the same principle as used in the hazard case.

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