Two-Dimensional Orthogonal Filter Banks and Wavelets with Linear Phase

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***Abstract*—Two-dimensional (2-D) compactly supported, orthogonal wavelets and filter banks having linear phase are presented. Two cases are discussed: wavelets with two-fold symmetry (centrosymmetric) and wavelets with four-fold symmetry that are symmetric (or anti-symmetric) about the vertical and horizontal axes. We show that imposing the requirement of linear phase in the case of order-factorable wavelets imposes a simple constraint on each of its polynomial order-1 factors. We thus obtain a simple and complete method of constructing orthogonal order-factorable wavelets with linear phase. This method is exemplified by design in the case of four-band separable sampling. An interesting result that is similar to the one well-known in the one-dimensional (1-D) case is obtained: Orthogonal order-factorable wavelets cannot be both continuous and have four-fold symmetry.**

***Index Terms*— Filter banks, linear phase, symmetry, 2-D wavelets, wavelets.**

I. Introduction

LINEAR PHASE is a desired quality in many applications of multirate filter banks and wavelets for several reasons:

- Applying filters with linear phase results in minimum phase distortion. This is important when the intraresolution level relation is to be exploited in multiresolution applications, as is done for instance in zero tree coding [1], where the correlation between adjacent resolution levels is used to improve the compression.
- It is commonly accepted that the human visual system is more tolerant to symmetric distortions than it is to asymmetric ones.
- Using symmetric filters allows one to use a symmetric periodical extension of finite length signals, which minimizes the discontinuity at the boundaries without enlarging the support size of the signal.

The latter argument holds in two dimensions (2-D) only for four-fold symmetric filters, i.e., filters with both horizontal and vertical axes of symmetry. The issue of paraunitary filter banks with linear phase in one dimension (1-D) has been dealt with by several authors [2]–[4]. Linear phase cannot be achieved in two-band orthogonal filter banks and wavelets, except for the Haar case, which corresponds to a discontinuous wavelet function and to filters with very poor frequency selectivity. Hence, there exist no continuous, orthogonal wavelets with linear phase, using 2-D quincunx sampling or 2-D separable four-band sampling. Yet, linear phase can be achieved in the general case of 2-D orthogonal wavelets, even when we use separable sampling. This point was recently highlighted by Kovačević and Vetterli [5], who also gave some design examples.

In this paper, we follow the rationale adopted in our earlier paper [6]. We investigate the space of 2-D orthogonal wavelets in the most complete, yet manageable way, and then impose a set of constraints to obtain desired wavelets. In [6], we dealt mainly with *degree-factorable* (2-D) orthogonal filter banks and wavelets and showed how to design wavelets with vanishing moments by solving a set of nonlinear equations. The polyphase matrix of an orthogonal degree-factorable M-band filter bank is a paraunitary polyphase (PUP) matrix and can be factored as a minimal product of McMillan degree-1 factors. In this paper, we further restrict the discussion to filter banks that are *order-factorable*, i.e., the polyphase matrix can be factored as a minimal product of polynomial order-1 factors. It is shown that this is a proper subset of degree-factorable filter banks.

We deal with the necessary and sufficient conditions for 2-D paraunitary filter banks and wavelets to have linear phase. We show that for order-factorable filter banks, similarly to the 1-D case (which is necessarily factorable in both senses), these conditions impose a simple linear constraint on each of the order-1 factors. Thus, we arrive at a simple and complete parameterization of 2-D order-factorable orthogonal filter banks and wavelets with linear phase. By adding a few extra constraints, we can obtain in a similar manner a parameterization of orthogonal filter banks with four-fold symmetry. All our results are given for general orthogonal (lossless) filter banks. Yet, only the design examples are limited to wavelets with a certain degree of regularity. We use all the notations and conventions given in [6]. In particular, the term *orthogonal wavelet* is used for wavelets that induce an orthogonal discrete wavelet transform; this does not necessarily mean that the wavelet functions are orthogonal [7], and in the polyphase matrix used here, each column (*not row*) corresponds to a different filter.

II. Factorable Filter Banks and Wavelets

Degree factorable or simply factorable orthogonal 2-D filter banks were suggested by Karlsson and Vetterli [8] and were further discussed in [6] and [9]. The polyphase matrix $\boldsymbol{H}(z_1, z_2)$ of degree factorable orthogonal filter banks is paraunitary and can be factored as

$$\boldsymbol{H}(z_1, z_2) = \boldsymbol{H}_I(z_1, z_2)\boldsymbol{H}_0 = \prod_{k=1}^{m+n} [I + (z_{i_k}^{-1} - 1)V_k V_k^T]\boldsymbol{H}_0 \qquad (1)$$

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where

- m McMillan degree in z_1;
- n McMillan degree in z_2;
- z_{i_k} either z_1 or z_2;
- V_k unit vectors of length M;
- $\boldsymbol{H}_0$ characteristic matrix of $\boldsymbol{H}(z_1, z_2)$.

Note that $\boldsymbol{H}(1,1) = \boldsymbol{H}_0$, and $\boldsymbol{H}_I(1,1) = I$. Wavelet functions correspond to filter banks that satisfy $\Sigma_m \ \boldsymbol{H}_0(m,n) = \delta_{n,0}\sqrt{M}$. In this case, the term *wavelet matrix* is used for the polyphase matrix, and the characteristic matrix is called a *Haar wavelet matrix* (HWM) of rank $M, \mathcal{H}^M$ [10].

Order-factorable filter banks are defined in a similar way

Definition 1: A 2-D M-band, orthogonal FIR filter bank of polynomial orders (s,r) in the variables z_1^{-1} and z_2^{-1} is called *order factorable* if the corresponding polyphase matrix can be factored as a product of $s+r$ order-1 factors

$$\boldsymbol{H}(z_1,z_2) = \boldsymbol{H}_I(z_1,z_2)\boldsymbol{H}_0 = \prod_{l=1}^{s+r} [I + (z_{i_l}^{-1} - 1)P_l]\boldsymbol{H}_0 \quad (2)$$

where

- z_{i_l} either z_1 or z_2:
- P_l symmetric projection matrix of rank R_l;
- $\boldsymbol{H}_0$ characteristic matrix of $\boldsymbol{H}(z_1, z_2)$.

Let us define a few notions to simplify the treatment of order-factorable filter banks. It is readily seen that if $\boldsymbol{H}(z_1,z_2)$ is an order-factorable PUP matrix of polynomial orders $(s,r)+$, it can be written as

$$\boldsymbol{H}(z_1,z_2) = [I + (z_i^{-1} - 1)P]\boldsymbol{H}'(z_1,z_2),$$

where z_i is either z_1 or z_2, and $\boldsymbol{H}'(z_1,z_2)$ is a PUP matrix of polynomial orders $(s-1,r)$ or $(s,r-1)$, respectively. Since P is a projection matrix, we have

$$\boldsymbol{H}'(z_1,z_2) = [I + (z_i - 1)P]\boldsymbol{H}(z_1,z_2).$$

Hence, if $z_i = z_1$ (and similarly for the z_2 case), the polynomial coefficients satisfy

$$P\boldsymbol{h}(0,j) = 0; \quad [I - P_k]\boldsymbol{h}(s,j) = 0, \qquad j = 0,\cdots,r. \quad (3)$$

Given a PUP matrix, we define two subspaces of $\mathbb{R}^M$:

- $$\Omega_N \triangleq \bigcap_{j=0}^{r} \mathcal{N}(\boldsymbol{h}^T(0,j)) \quad (4a)$$

where $\mathcal{N}(\boldsymbol{h})$ is the null space of the matrix $\boldsymbol{h}$.

- $$\Omega_S \triangleq \bigcup_{j=0}^{r} \mathcal{S}(\boldsymbol{h}(s,j)) \quad (4b)$$

where $\mathcal{S}(\boldsymbol{h})$ is the space spanned by the columns of $\boldsymbol{h}$.

Equation (3) implies that there exists a symmetric projection matrix P onto a subspace Ω_p such that

$$\Omega_S \subseteq \Omega_p \subseteq \Omega_N.$$

We next discuss the relation between order-factorable, degree-factorable, and general (nonfactorable) orthogonal filter banks.

- Every $M \times M$ symmetric projection matrix P of rank R can be decomposed (nonuniquely) as

$$P = V_1V_1^T + V_2V_2^T + \cdots + V_RV_R^T$$

where V_i are mutually orthonormal vectors of length M. Thus

$$\det([I + (z^{-1} - 1)P]) = \det\left(\prod_{i=1}^{R} [I + (z^{-1} - 1)V_iV_i^T]\right) = z^{-R}.$$

- From this, it can be deduced that an order-factorable filter bank is likewise degree-factorable [6], [9].
- The converse is not true (although we wrongly asserted that this is the case in [11]). To prove this, we give the following counter example:

$$\boldsymbol{H}(z_1,z_2) = [I + (z_1^{-1} - 1)V_1V_1^T][I + (z_2^{-1} - 1)V_2V_2^T] \cdots [I + (z_2^{-1} - 1)V_3V_3^T][I + (z_1^{-1} - 1)V_4V_4^T]$$

where

$$V_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, V_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$
$$V_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, V_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}.$$

The above polyphase matrix has McMillan degrees (2, 2) and polynomial orders (1, 1). It is obviously degree-factorable, yet by explicitly obtaining the subspaces Ω_N and Ω_S as defined above (4a), (4b) for both possibilities

$$\boldsymbol{H}(z_1,z_2) = [I + (z_1^{-1} - 1)P_1][I + (z_2^{-1} - 1)P_2]$$

and

$$\boldsymbol{H}(z_1,z_2) = [I + (z_2^{-1} - 1)P_1][I + (z_1^{-1} - 1)P_2]$$

we find that $\Omega_S \nsubseteq \Omega_N$; hence, (3) cannot be satisfied.

Thus. it is concluded that order-factorable filter banks are included in the set of degree-factorable filter banks that are included in general 2-D filter banks.

III. Linear–Phase Filter Banks and Wavelets

The impulse response of a 2-D FIR filter that has linear phase has a center of symmetry (or antisymmetry) $\boldsymbol{c}$, that is

$$h(\boldsymbol{n}) = \pm h(2\boldsymbol{c} - \boldsymbol{n}), \forall \boldsymbol{n} \quad \text{or} \quad \boldsymbol{z}^{-2\boldsymbol{c}}H(\boldsymbol{z}^{-1}) = \pm H(\boldsymbol{z}).$$

To see how this symmetry reflects on the polyphase components of the filter, we must make a few assumptions that will not limit the generality but simplify the notation. Assuming the polyphase components of the M-band filters have zero coefficients for indexes $\boldsymbol{n} = (n_1,n_2) \notin \mathcal{A}$, where $\mathcal{A} =$

$\{(n_1,n_2)|\ 0 \le n_1 \le s, 0 \le n_2 \le r\}$, the coefficients of the filters themselves will vanish if

$$(n_1,n_2) \notin \mathcal{B} \triangleq \bigcup_{i=0}^{M-1} (D\mathcal{A} + \boldsymbol{k}_i)$$

where D is the sampling matrix, and $\boldsymbol{k}_i$ are the coset shift vectors. In order for a filter defined on the integer points included in $\mathcal{B}$ to have linear phase, the polygon defined by these points must have a center of symmetry itself. For this to be achieved, we choose the coset shift vectors so that they define a polygon that has a center of symmetry. Next, we rearrange the order of the vectors so that $\boldsymbol{k}_{M-1-i}$ is the reflection of $\boldsymbol{k}_i$ about their center of symmetry. With these conventions, we can translate most of the 1-D results [4] into our 2-D setting.

- The polyphase components of a 2-D, FIR, linear-phase filter of polynomial orders (s,r) satisfy

$$H_i(z_1,z_2) = \pm z_1^{-s} z_2^{-r} H_{(M-1-i)}(z_1^{-1}, z_2^{-1}).$$

- The polyphase matrix of a bank of linear-phase filters with polynomial orders (s,r) satisfies

$$\boldsymbol{H}(z_1,z_2) = z_1^{-s} z_2^{-r} J^c \boldsymbol{H}(z_1^{-1}, z_2^{-1}) S \tag{5}$$

where J^c is a $M \times M$ symmetric permutation matrix, with $J^c_{i,j} = \delta_{i,(M-1-j)}$, and S is a diagonal matrix of $\pm 1, +1$ corresponding to symmetric filters and -1 to antisymmetric ones.

For M-band paraunitary filter banks, with M even, there are exactly $M/2$ symmetric filters and $M/2$ antisymmetric [3]. Since $J^c J^c = I$, (5) can be rewritten for PUP matrices as

$$\boldsymbol{H}_I(z_1,z_2)\boldsymbol{H}_0 = z_1^{-s} z_2^{-r} J^c \boldsymbol{H}_I(z_1^{-1}, z_2^{-1}) J^c J^c \boldsymbol{H}_0 S.$$

This condition, evaluated at the point $(z_1,z_2) = (1,1)$, is reduced to the constraint imposed on the characteristic matrix of linear phase PUP matrices

$$\boldsymbol{H}_0 = J^c \boldsymbol{H}_0 S, \tag{6}$$

and thus also yields

$$\boldsymbol{H}_I(z_1,z_2) = z_1^{-s} z_2^{-r} J^c \boldsymbol{H}_I(z_1^{-1}, z_2^{-1}) J^c. \tag{7}$$

Paraunitary polynomial matrices that satisfy the last equation are called *centrosymmetric* [4]. It can be easily seen that the projection matrix of a polynomial degree-1 centrosymmetric matrix satisfies $J^c P J^c = I - P$. In this paper, we limit the discussion to the case of even M, which is the simpler case. The 1-D results, for the case of M being odd, can be translated into our 2-D setting in a similar fashion.

The linear-phase imposes a two-fold symmetry, but for some applications, four-fold symmetry is required, that is, symmetry about both the horizontal and vertical axes; this, of course, is a special case of general linear phase. In order to achieve four-fold symmetry, we must choose the coset shift vectors defining the polyphase components in such a way that the polygon defined by the vectors has a four-fold symmetry. The polyphase components of a 2-D, FIR filter with polynomial orders (s,r) must satisfy the following.

- $H_i(z_1,z_2) = \pm z_1^{-s} H_j(z_1^{-1}, z_2)$ for all pairs (i,j) where the shift vector $\boldsymbol{k}_i$ is the symmetric reflection of $\boldsymbol{k}_j$ about the vertical axis of symmetry.
- $H_i(z_1,z_2) = \pm z_2^{-r} H_j(z_1, z_2^{-1})$ for all pairs (i,j) where the shift vector $\boldsymbol{k}_i$ is the symmetric reflection of $\boldsymbol{k}_j$ about the horizontal axis of symmetry.

Next, we define two rank-M symmetric permutation matrices.

- $J^H_{i,j} \triangleq \begin{cases} 1 & \text{if } \boldsymbol{k}_i \text{ is the horizontal symmetric reflection of } \boldsymbol{k}_j \\ 0 & \text{otherwise.} \end{cases}$
- $J^H_{i,j} \triangleq \begin{cases} 1 & \text{if } \boldsymbol{k}_i \text{ is the horizontal symmetric reflection of } \boldsymbol{k}_j \\ 0 & \text{otherwise.} \end{cases}$

With these two matrices, we can formulate two matrix constraints on the polyphase matrix just as in (5)

$$\boldsymbol{H}(z_1,z_2) = z_1^{-s} J^V \boldsymbol{H}(z_1^{-1}, z_2) S^V \tag{8a}$$
$$\boldsymbol{H}(z_1,z_2) = z_2^{-r} J^H \boldsymbol{H}(z_1, z_2^{-1}) S^H \tag{8b}$$

where S^V and S^H are diagonal matrices representing the sign of symmetry about the vertical (z_2) and horizontal (z_1) axes, respectively, in the same manner as S^s. Note that if the shift vectors are ordered as in the linear-phase case, (8a) and (8b) must together yield (5), that is

$$J^H J^V = J^V J^H = J^c; \quad S^H S^V = S^V S^H = S.$$

Moreover, as with the anti-diagonal matrix S, S^H and S^V must have $M/2$ terms equal to $+1$ and $M/2$ equal to -1, and the terms must be arranged in such a way that this will hold for S as well. Note that since M is even, if $\boldsymbol{k}_i$ and $\boldsymbol{k}_j$ are a symmetric pair, then $i \neq j$; hence, all the diagonal terms of J are zero. This is, of course, true for J^c as well, and in the rest of the paper, it will be assumed that $\text{diag}(J) = 0$ when referring to a symmetric permutation matrix J.

IV. Factorable Linear-Phase Filter Banks and Wavelets

We now give a generalization of [4, Lemma 3.1], which will be used in the following theorems.

Lemma 1: Given a $M \times r$ matrix U with M even and $r \le M/2$ and a symmetric $M \times M$ full rank permutation matrix J such that all of the $2r$ columns of U and JU are orthogonal to each other, a $M \times (M/2 - r)$ matrix V exists such that the M columns of the four matrices $\{U, JU, V, JV\}$ form an orthogonal matrix.

Proof: The permutation matrix defines $M/2$ pairs of indices $(k_i, q_i); i = 1, \cdots M/2$, for which $J_{k_i,q_i} = 1$. Let us define a transformation T_J that maps a $M \times p$ matrix into two $M/2 \times p$ matrices in the following way:

$$T_J(A) = (B,C) \Rightarrow B_{i,j} = A_{k_i,j} \quad \text{and} \quad C_{i,j} = A_{q_i,j}.$$

We begin the proof by using T_J to partition the matrix—$U, T_J(U) = (U_1, U_2)$. It can be easily seen that in both matrices $(U_1 + U_2)$ and $(U_1 - U_2)$, all the columns are orthogonal to each other. Next, we define two $M/2 \times (M/2 - r)$ matrices V_1, V_2 to be matrices that complete $(U_1 + U_2)$ and $(U_1 - U_2)$, respectively, to orthogonal matrices of rank $M/2$. We claim that the matrix

$$V \triangleq T_J^{-1}((V_1 + V_2)/2, (V_1 - V_2)/2)$$

satisfies the requirements of Lemma 1. This can be easily checked using the definitions of U and V, for example

$$\begin{aligned} V^T V &= \tfrac{1}{4}\,(V_1 + V_2)^T (V_1 + V_2) + \tfrac{1}{4}\,(V_1 - V_2)^T (V_1 - V_2) \\ &= \tfrac{1}{2}\,(V_1^T V_1 + V_2^T V_2) = I \end{aligned}$$

and

$$\begin{aligned} V^T (JU) &= (JV)^T U \\ &= \tfrac{1}{2}\,(V_1 - V_2)^T U_1 + \tfrac{1}{2}\,(V_1 + V_2)^T U_2 \\ &= \tfrac{1}{2}\,V_1^T (U_1 + U_2) - \tfrac{1}{2}\,V_2^T (U_1 - U_2) = 0. \end{aligned}$$

■

From the next theorem, we see that imposing the constraint of linear phase on the filter bank imposes a simple constraint on each of the order-1 polynomial factors.

Theorem 1: A order-factorable PUP $M \times M$ matrix of polynomial orders (s, r) has linear phase if and only if

1) its characteristic matrix satisfies $\boldsymbol{H}_0 = J^c \boldsymbol{H}_0 S$; and
2) there exists a polynomial factorization, where all the polynomial order-1 factors are centrosymmetric, that is, all the corresponding projection matrices satisfy$J^c P_k J^c = I - P_k$.

Proof: In view of the conditions in (6) and (7), the *if* part is obvious since a product of centrosymmetric factors is also centrosymmetric; we now prove the *only if* part. As in the above discussion, we assume that the first order-1 factor is of the variable z_1. Let us define another subspace

$$\overline{\Omega}_S \triangleq \bigcup_{j=0}^{r} \mathcal{S}(\boldsymbol{h}(0, j))$$

and define U to be a $M \times q$ matrix whose columns form an orthogonal basis of $\overline{\Omega}_S$. The centrosymmetric condition (7) of linear-phase wavelet matrices implies that $\boldsymbol{h}(s, r - i) = J^c \boldsymbol{h}(0, i) S$, that is, the columns of $\boldsymbol{h}(s, r - i)$ are the same as in $\boldsymbol{h}(0, i)$ modulo the sign but turned from top to bottom. Thus, the columns of $J^c U$ form an orthogonal basis of Ω_S. From the definition of the subspaces, one has

$$\overline{\Omega}_S \bigcap \Omega_N = \emptyset$$

and since we also have $\Omega_S \subseteq \Omega_N$, together, we get

$$\overline{\Omega}_S \bigcap \Omega_S = \emptyset.$$

Hence, all the columns of U are orthogonal to all columns of $J^c U$, that is $U^\dagger J^c U = 0$. If we define the complementary space Ω_C to be $\Omega_C \triangleq \mathbb{R}^M \ominus \overline{\Omega}_S \ominus \Omega_S$, then Lemma 1 assures us that we can find (for M even) a $M \times (M/2 - q)$ matrix V such that the $(M - 2q)$ columns of V and JV form an orthogonal basis of Ω_C; hence, the M columns of the four matrices $U, J^c U, V, J^c V$ constitute an orthogonal rank M matrix. We next define the matrix W to be a $M \times M/2$ matrix that includes the columns of V and U. It can be easily seen that the projection matrix P, which is defined as $P \triangleq (J^c W)(J^c W)^\dagger$, can be chosen as the first matrix in the polynomial factorization, since

$$[I - P]\boldsymbol{h}(s, j) = 0, P\boldsymbol{h}(0, j) = 0; j = 0, \cdots r$$

and

$$I - P = WW^\dagger = J^c[(J^c W)(J^c W)^\dagger] J^c = J^c P J^c. \tag{9}$$

By repeating this process on $\boldsymbol{H}' = [I + (z_1 - 1)P]\boldsymbol{H}$, we can obtain the next order-1 centrosymmetric factor. This, of course, can be repeated until we obtain all the order-1 centrosymmetric factors, thus proving the *only if* part. ■

One should note that the projection matrix of a M-band, PUP order-1, centrosymmetric matrix must be of rank $r = M/2$ since

$$\text{Rank}(P) = \text{Rank}(JPJ) = \text{Rank}(I - P) = M - \text{Ramk}(P).$$

Hence, the McMillan degrees of a order-factorable, linear-phase PUP matrix of polynomial degrees (s, r) is $((M/2)s, (M/2)r)$. Every unit vector V_i used to construct the projection matrix P must be orthogonal to all other $(M/2 - 1)$ vectors and to their reflections about their center. Thus, the number of degrees of freedom of a PUP, linear-phase matrix of order-1 is

$$((M - 2) + (M - 4) + \cdots + 2) = 2\binom{M/2}{2}.$$

From this, we should subtract $\binom{M/2}{2}$ redundant degrees of freedom, corresponding to the freedom in choosing the orthogonal vectors that span the subspace projected on by P. Finally, we see that the true number of free parameters of an order factorable, linear-phase PUP matrix $\boldsymbol{H}_I$ with McMillan degrees $((M/2)s, (M/2)r)$ is $N_{fp} = (s + r)\binom{M/2}{2}$ as compared with $N_{fp} = (s + r)(M - 1)$, including some redundant parameters for general PUP matrices with the same McMillan degrees. To this, we must add the $\binom{M}{2}$ free parameters defining the characteristic matrix for general filter banks, or $\binom{M-1}{2}$ for the Haar wavelet matrix in the case of wavelets. We, therefore, have a simple and complete method of designing 2-D orthogonal, linear-phase, order-factorable wavelets with given polynomial degrees (for M even).

1) Choose the matrix S, i.e., decide which $M/2$ filters will be symmetric (obviously, the scaling filter cannot be anti-symmetric) and which will be antisymmetric.
2) Choose a characteristic (Haar wavelet) matrix that satisfies condition (6).
3) Choose $(s + r)$ sets of $M/2$ unit vectors of length M, V_i, orthogonal to each other and to their reflections about their center JV_j.

In Section IV we demonstrate, for the four-band case, how steps 2) and 3) can be done simply by fixing the value of a minimal set of free parameters.

Repeating the reasoning in the proof of Theorem 1, we can prove the following theorem.

Theorem 2: A order-factorable PUP $M \times M$ matrix of orders (s, r) has four-fold symmetry if and only if

1) its characteristic matrix satisfies
$$\boldsymbol{H}_0 = J^V \boldsymbol{H}_0 S^V = J^H \boldsymbol{H}_0 S^H. \tag{10}$$
and
2) there exists a polynomial factorization, where all the polynomial order-1 factors in the variable z_1 satisfy
$$\boldsymbol{H}_k(z_1) = z_1^{-1} J^V \boldsymbol{H}_k(z_1^{-1}) J^V = J^H \boldsymbol{H}_k(z_1) J^H$$
that is, for the corresponding projection matrix, we have
$$P_k = I - J^V P_k J^V = J^H P_k J^H \tag{11}$$
and all factors in the variable z_2 satisfy
$$\boldsymbol{H}_q(z_2) = z_2^{-1} J^H \boldsymbol{H}_q(z_2^{-1}) J^H = J^V \boldsymbol{H}_q(z_2) J^V$$
that is
$$P_q = I - J^H P_q J^H = J^V P_q J^V. \tag{12}$$

Thus, the design procedure for 2-D orthogonal, factorable filter banks with four-fold symmetry follows.

1) Choose the matrices S^V, S^H subjected to the constraints mentioned in this section.
2) Choose a characteristic (Haar wavelet) matrix that satisfies (10).
3) Choose s rank $M/2$ symmetric projection matrices P_k that satisfy (11) and r matrices P_q that satisfy (12).

In the following section, we show that in the four-band case, these constraints leave us with very little freedom; for one, there are no continuous wavelet solutions.

V. Design Examples

In this section, we give a few design examples for the four-band separable sampling scheme, which is the sampling scheme used in virtually all 2-D applications of the DWT. The examples are limited to wavelets with a certain degree of regularity achieved by requiring as many vanishing moments as possible for the given polynomial degree. In accordance with Theorem 1 and the discussion thereafter, every four-band, order-factorable, linear-phase wavelet matrix of orders (s, r) can be decomposed as
$$\boldsymbol{H}(z_1, z_2) = \prod_{l=1}^{s+r} \left(I + (z_{i_l}^{-1} - 1) P_l\right) \mathcal{H}^{(4)}.$$

Each symmetric projection matrix is of rank two and satisfies $JPJ = I - P$. Solving this matrix equation, we find that there are two types of possible projection matrices, shown at the bottom of the page, each defined by one free parameter.

In the following design examples, we used only 1-D factors of type P^I, and the symmetry matrix is chosen to be $S = \text{diag}(1, -1, -1, 1)$, that is, the scaling and last wavelet functions are symmetric, and the other two wavelet functions are anti-symmetric. The HWM must satisfy $\mathcal{H}^{(4)} = J\mathcal{H}^{(4)}S$ using the parameterization of HWM's [6], [10]; we find that the HWM has one continuous free parameter and has the form
$$\mathcal{H}^{(4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \pm 1 & \pm 1 \\ 1 & -1 & \pm 1 & \mp 1 \\ 1 & 1 & \mp 1 & \mp 1 \\ 1 & -1 & \mp 1 & \pm 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) & 0 \\ 0 & \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

In order to obtain wavelets with a certain degree of regularity, we require that the scaling filter/function should have the maximum possible number of vanishing moments for given degrees. In [6], the relation between the number of vanishing moments and the regularity of the wavelet function, as well as methods to measure the regularity, were discussed. A simple technique, which is based on the rate of decrease of the $L\infty$ norm of the finite difference as a function of the approximation level, was used to get an estimate of the Hölder exponent, and another, more exact, method due to Villemoes [12] was used to bound the Hölder exponent. The higher the degrees of the wavelet matrix, the larger the number of vanishing moments that can be obtained, and hence, the better the regularity of the resulting wavelet functions. In order to obtain wavelets with L vanishing moments, we require the scaling filter to have a zero of order L at the three aliasing frequencies. Since each 1-D factor of type P^I is block diagonal, the total wavelet matrix $\boldsymbol{H}_I$ is block diagonal as well. The upper and lower blocks are related to each other; if we denote the upper block by $W_{\{\theta\}_i}(z_1, z_2)$, then the lower block will be $W_{\{-\theta\}_i}(z_1, z_2)$. The 2×2 block of a matrix of orders (s, r) defined by $(s + r)$ parameters $\{\theta\}_i$ is exactly the matrix obtained in [6] for a quincunx wavelet matrix of McMillan degrees (s, r) defined by the same set of parameters. Thus, the form of the vanishing moments constraints on the free parameters $\{\theta\}_i$

$$P^I = \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) & 0 & 0 \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) & 0 & 0 \\ 0 & 0 & \cos^2(\theta) & -\sin(\theta)\cos(\theta) \\ 0 & 0 & -\sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}$$

$$P^{II} = \begin{pmatrix} \cos^2(\theta) & 0 & \sin(\theta)\cos(\theta) & 0 \\ 0 & \cos^2(\theta) & 0 & -\sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & 0 & \sin^2(\theta) & 0 \\ 0 & -\sin(\theta)\cos(\theta) & 0 & \sin^2(\theta) \end{pmatrix}.$$

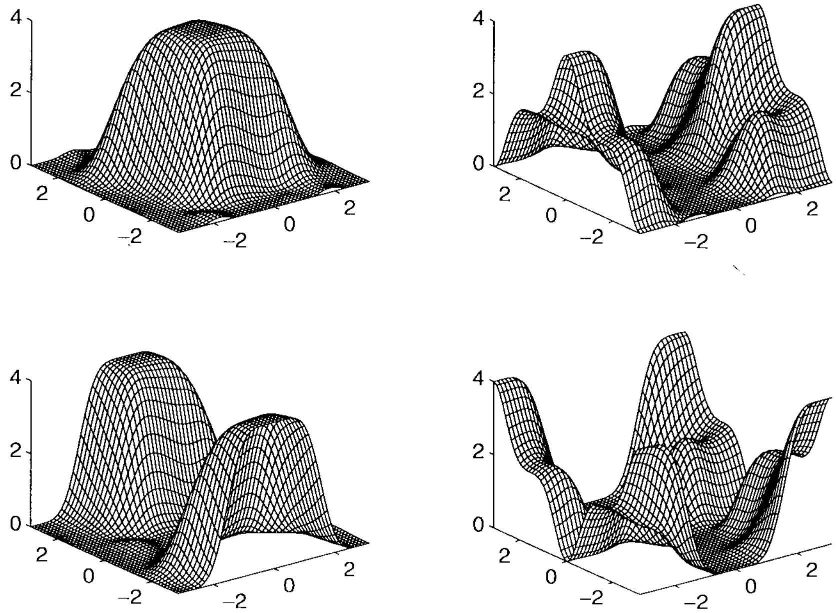

Fig. 1. Frequency response of the four wavelet filters, where the top left filter is the scaling filter of Example 2. The filters are of size 10×10 and have three vanishing moments.

is very closely related to the constraints developed in the appendix of [6].

Example 1: In order to obtain two vanishing moments, we must satisfy six constraints: two first-order derivatives at three aliasing frequencies. Two of the constraints are satisfied for all linear-phase matrices, and we are left with four. The minimal orders are $(2, 2)$, and in this case, the constraints can be solved analytically. The two z_1 factors are defined by the parameters $\cos(2\theta_1) = 1/4$ and $\theta_2 = \pi - \theta_1$, and the z_2 parameters are given by $\sin(2\theta_3) = 1/4$ and $\theta_4 = \pi/2 - \theta_3$. By using Villemoes' method, we were able to show that these wavelets are at least continuous.

Example 2: Figs. 1 and 2 depict the frequency response of the four wavelet filters and the four wavelet functions of an example with three vanishing moments. The orders of the wavelet matrix are (4, 4); hence, the wavelet filters are of size 10×10. This solution was obtained by numerically solving the vanishing moment constraints for the specific order $\{z_1, z_1, z_2, z_2, z_1, z_1, z_2, z_2\}$ of order-1 factors, which gave

$$\{\theta\}_{i=1}^{8} = \{1.755200, 0.000177, 0.989755, 2.845917, 2.772\,468, 1.063\,203, 1.763\,801, 0.387\,635\}$$

and using the separable Haar wavelet matrix, i.e., $\beta = 0$. Using the method introduced by Villemoes, we found that the Hölder exponent can be bounded by $\alpha \geq 0.38$, yet our estimate indicated that $\alpha \approx 0.7$.

A. Wavelets with Four-Fold Symmetry

Since four-fold symmetry wavelets are a special case of linear phase wavelets, we can fined the appropriate order-1 factors by adding the extra constraints to the results of the last section. Using the coset shift vectors $\{(0,0), (1,0), (0,1), (1,1)\}$, we see that the matrices J^V and J^H are

$$J^V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J^H = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The symmetry matrices were chosen to be $S^V = \text{diag}(1, -1, 1, -1)$ about the vertical axis and $S^H = \text{diag}(1, 1, -1, -1)$ about the horizontal axis. Putting the constraints (11) on P^{I} and P^{II}, we see that the z_1 factors can only be of type II with $\theta = \pi/4$ or $3\pi/4$. In a similar manner, (12) forces the z_2 factors to be of type I, again with $\theta = \pi/4$ or $3\pi/4$; hence

$$P_1^{\pm} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & \mp 1 \\ \pm 1 & 0 & 1 & 0 \\ 0 & \mp 1 & 0 & 1 \end{pmatrix}$$
$$P_2^{\pm} = \tfrac{1}{2}\begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ \pm 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mp 1 \\ 0 & 0 & \mp 1 & 1 \end{pmatrix}.$$

Equation (10) forces β in (13) to be 0 or π; hence, the HWM is the four-band separable HWM or a variation of it with the sign of one or more of the columns changed. Thus, four-band wavelets with four-fold symmetry have no free parameters except for the polynomial degree and the order of multiplication of the order-1 factors. It can be easily verified that

- $P_1^+ P_1^- = P_2^+ P_2^- = 0$;
- $P_1 P_2 = P_2 P_1$; and
- $P_1^+ + P_1^- = P_2^+ + P_2^- = I$.

Hence, without loss of generality, a wavelet matrix of orders

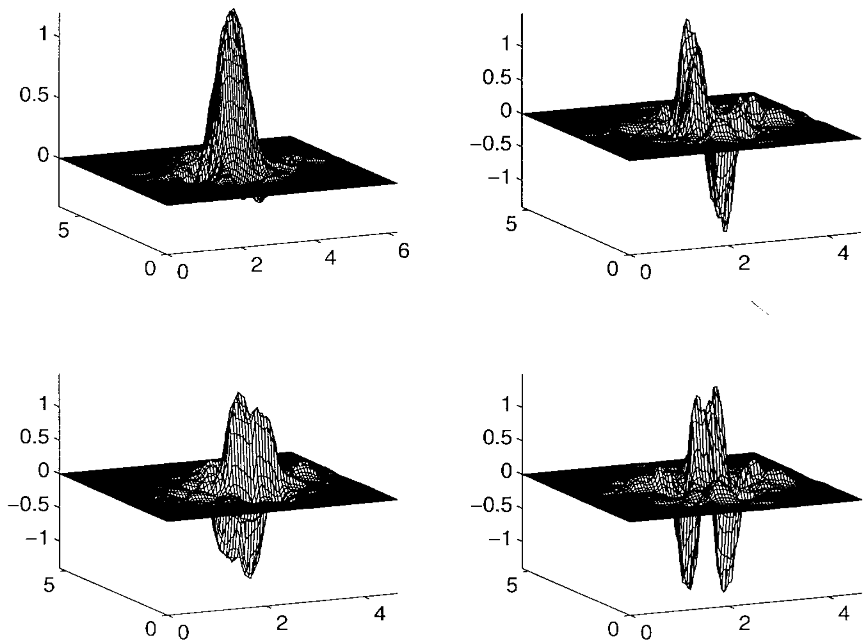

Fig. 2. Scaling function (top left) and the other three wavelet functions of example 2. It is quite evident from the graphs that the scaling function and third wavelet function are symmetric, whereas the other two wavelets are antisymmetric.

(s, r) will have the form

$$\boldsymbol{H}(z_1, z_2) = z_1^{t-s} z_2^{q-r} [I + (z_1^{-t} - 1)P_1^{\pm}] \cdot [I + (z_2^{-q} - 1)P_2^{\pm}]\mathcal{H}^{(4)} \quad (14)$$

where $t \leq s$, and $q \leq r$. From here, it can be readily seen that the impulse response of the scaling filter has one of the following two forms (after shifting it to the origin):

$$\frac{1}{2}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad \text{or} \quad \frac{1}{4}\begin{pmatrix} a & b & \cdots & b & a \\ c & d & \cdots & d & c \\ \vdots & \vdots & & \vdots & \vdots \\ c & d & \cdots & d & c \\ a & b & \cdots & b & a \end{pmatrix}$$

where one out of $\{a, b, c, d\}$ equals -1, and the rest are equal to 1, and the dots stand for an even number of zeros. Both classes of solutions, the first of which is the 2-D separable Haar scaling filter (or a version of it zero padded in the center), have only one vanishing moment (the zero-order moment) and are discontinuous. This result is reminiscent of the 1-D, well-known fact that the only the orthogonal, symmetric scaling filter is the Haar scaling filter, or the Haar scaling filter padded with zeros in the center, which, however, does not result in an orthogonal scaling function. Thus, we arrive at the conclusion.

Proposition 1: In the case of four-band separable sampling, no order-factorable orthogonal wavelets exist that are both continuous and have four-fold symmetry.

Our conjecture is that this statement is true not only for factorable wavelets, but this has not been proven yet.

VI. Summary

We have shown that there exists a simple parameterization of the space of 2-D, orthogonal, order-factorable filter banks and wavelets with linear phase. The space of orthogonal, order-factorable filter banks is a subspace of degree-factorable filter banks that in turn is a subspace of general (non factorable) filter banks. Every corresponding polyphase matrix can be decomposed as a product of order-1 linear-phase factors, which can easily be designed. Obviously, adding the constraints of linear phase to the parameterization of orthogonal filter banks and wavelets reduces the number of free parameters for a given filter size; in the four-band case, this number is reduced, roughly speaking, by a factor of three. In order to obtain more regular wavelet functions, we require the wavelet filters to have the maximum possible number of vanishing moments for given filter size. This is achieved by imposing a set of nonlinear constraints on otherwise free parameters. Thus, to obtain the same number of vanishing moments, linear-phase wavelets will generally be of larger size compared with ordinary orthogonal wavelets. In some applications, linear phase, that is, two-fold symmetry, is not enough, and four-fold symmetry is required in order to facilitate symmetrical boundary conditions. In the four-band sampling scheme dealt with here, we have shown that adding the appropriate constraints leaves no free continuous parameters in the order-factorable case. One unfortunate consequence is that order-factorable, orthogonal wavelets cannot have four-fold symmetry and be continuous at the same time. Only two types of wavelets with four-fold symmetry are possible, and both are discontinuous, with only one vanishing moment. A similar result is well known in the two-band 1-D case; the only symmetric/antisymmetric orthogonal wavelet is the Haar wavelet. One should bear in mind that this result has only been shown for order-factorable wavelets and not for general wavelets, yet to the best of our knowledge, no general wavelet counter example has been given.

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