

Two-Dimensional Regular Shock Reflection for the Pressure Gradient System of Conservation Laws

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Abstract We establish the existence of a *global* solution to a regular reflection of a shock hitting a ramp for the pressure gradient system of equations. The set-up of the reflection is the same as that of Mach's experiment for the compressible Euler system, i.e., a straight shock hitting a ramp. We assume that the angle of the ramp is close to 90 degrees. The solution has a reflected bow shock wave, called the diffraction of the planar shock at the compressive corner, which is mathematically regarded as a free boundary in the self-similar variable plane. The pressure gradient system of three equations is a subsystem, and an approximation, of the full Euler system, and we offer a couple of derivations.

Keywords Free boundary, oblique derivative, tangential oblique derivative, 2-D Riemann problem, regular reflection, Mach reflection, gas dynamics

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1 Introduction

We are interested in solving multi-dimensional systems of conservation laws. The primary system is the well-known Euler system for compressible gases ([22]). Much effort has been devoted to the system, see the survey papers [3,4,34,36,54], the conference proceedings [30] by Glimm and Majda, the monograph [29], and the recent progress Shuxing Chen^[16–18], Chen, Xin, and Yin^[19,20], Zhang^[69], and Guiqiang Chen and Feldman^[14,15]. However, the Euler system remains formidable because of its complexity. Simplified models that capture various isolated features of the Euler system may be proposed and studied to pave the road. Immediate models are the isentropic case, the irrotational (potential) flow equations, and the steady flows [53,64–66], some of these assumptions are already made in some of the aforementioned papers. A remarkable and distinct model is the unsteady transonic small disturbance system (UTSD) which was proposed and studied ([9–11,13,21,26,38,40,53,59,62]) for transonic solutions and the transition from Mach reflection to regular reflection. Despite the conveniences these models bring, all these models have their difficulties. In recent years, another model called the pressure gradient system is proposed ([68,70]), which seems to exhibit new and complementary conveniences. See [12] for a similar system. In this paper we present further motivational work on the pressure gradient system, and in particular we present the global existence of a self-similar solution that is similar to the regular reflection seen in Mach's experiment for the full Euler system, see Section 4.

We spend Section 2 deriving the pressure gradient model and providing supportive materials. Section 3 covers the set-up of the shock reflection problem and basic classical treatment. The precise statement of the entire result is given in Section 4. Sections 5–9 are devoted to the proof. Section 10 is on fine properties of the velocity. And in Section 11 we offer some discussions. In the Appendices we provide either details that are lengthy for the main body of the paper, or theory that are somewhat implied by theorems in various papers but not yet clearly stated in writing.

The highlights of the paper include the asymptotic derivation of the model, the use of Hörmander’s emergent type of degeneracy condition on the free shock boundary, the full three-component set-up, and the successful complete road-map for a global existence of the regular reflection.

2 The Pressure Gradient System

The *pressure gradient system* takes the form

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ E_t + (up)_x + (vp)_y = 0, \end{cases} \quad (1)$$

where $E = \frac{1}{2}(u^2 + v^2) + p$.

The pressure gradient system is a reduction of a subsystem appeared in the flux splitting scheme of Li and Cao^[45] and Agarwal and Halt^[1]. Its mathematical structure and numerical simulations were studied in Zhang, Li, and Zhang^[68], where one sees striking similarity to the Euler system. The pressure gradient system was intuitively justified for its own physical validity when the velocity is small and the gas constant γ is large in Zheng^[70], where the existence of a subsonic solution was also established, which adds on further attractiveness of the model. Its regime of physical validity is clarified further in the formal presentation of an asymptotic derivation to be presented later in this section. See the books by Li et. al.^[44] or Zheng^[71] for more background information.

We recall that the full Euler system for an ideal fluid takes the form

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u + pI) = 0, \\ (\rho E)_t + \nabla \cdot (\rho E u + p u) = 0, \end{cases} \quad (2)$$

where

$$E := \frac{1}{2}|u|^2 + e,$$

where e is the internal energy. For a polytropic gas, there holds

$$e = \frac{1}{\gamma - 1} \frac{p}{\rho},$$

where $\gamma > 1$ is a dimensionless gas constant (*the adiabatic exponent*). More precisely $\gamma = 1 + R c_v^{-1}$ where R is the constant in the equation of state for ideal gases and c_v is the specific heat at constant volume^[22].

2.1 The Flux-splitting Derivation

Separating the pressure from the inertia in the flux of the Euler equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uE + up \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ \rho vE + vp \end{pmatrix}_y = 0, \quad (3)$$

where

$$E = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1} \frac{p}{\rho},$$

we obtain two systems of equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho uv \\ \rho uE \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho v^2 \\ \rho vE \end{pmatrix}_y = 0 \quad (4)$$

and

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} 0 \\ p \\ 0 \\ up \end{pmatrix}_x + \begin{pmatrix} 0 \\ 0 \\ p \\ vp \end{pmatrix}_y = 0. \quad (5)$$

Agarwal and Halt^[1] have used this splitting (4)(5) to form a scheme in numerical computations for airfoil flows and observed a consistent improvement over other schemes (Roe, AUSM, CUSP, and Van Leer). System (4) is called the *zero pressure* system, or the *transport* or *convective* or *pressureless* system. System (5) is called the variable-density *pressure gradient system*. This splitting corresponds to the separation of the two mechanisms— pressure difference and inertia—that are causes for fluid motion.

We focus on system (5). Let us do some simple reduction. From the first equation of system (5) we obtain

$$\rho_t = 0.$$

Thus ρ is independent of time. For simplicity, let us assume $\rho = 1$. Then system (5) can be written as (1) with $E = (u^2 + v^2)/2 + p/(\gamma - 1)$. It can be seen easily that the transformation

$$\begin{cases} p = (\gamma - 1)P, \\ t = \frac{1}{\gamma - 1}T, \end{cases} \quad (6)$$

will effectively rescale the gas constant γ to 2. Thus system (1) with $E = (u^2 + v^2)/2 + p$ will be the primary system for us to study.

For smooth solutions or in regions where a solution is smooth, system (1) can be simplified to be

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ p_t + pu_x + pv_y = 0. \end{cases} \quad (7)$$

From system (7) we can find a decoupled equation

$$\left(\frac{p_t}{p} \right)_t = p_{xx} + p_{yy}. \quad (8)$$

2.2 The Asymptotic Derivation

Besides the previous somewhat brutal derivation, there is an asymptotic derivation. Let us write the energy in the form

$$E = \frac{1}{2}|u|^2 + \varepsilon p/\rho,$$

where

$$\varepsilon := \frac{1}{\gamma - 1}.$$

We propose to look for an asymptotic solution of the form

$$\begin{aligned} \rho &= \rho_0 + \varepsilon \rho_1 + O(\varepsilon^2), \\ u &= \varepsilon u_1 + O(\varepsilon^2), \\ p &= \varepsilon p_1 + O(\varepsilon^2). \end{aligned} \tag{9}$$

This scaling corresponds to sound speeds of the order $O(1)$:

$$c = \sqrt{\gamma p/\rho} = O(\varepsilon^0).$$

So we scale space and time variables by the same factor (order $O(1)$) to study acoustic phenomena.

The leading order perturbation equation from conservation of mass is

$$(\rho_0)_t = 0,$$

so

$$\rho_0 = \rho_0(x).$$

The leading order equation from conservation of momentum $O(\varepsilon)$ and conservation of energy $O(\varepsilon^2)$ constitute the variable-density pressure gradient system

$$\begin{cases} (\rho_0 u_1)_t + \nabla p_1 = 0, \\ \left(\frac{1}{2} \rho_0 |u_1|^2 + p_1 \right)_t + \nabla \cdot (p_1 u_1) = 0. \end{cases} \tag{10}$$

This is the same as in the flux-splitting derivation of the previous subsection.

John Hunter became interested in the system during the author's talk on its Riemann problem at a workshop of an FRG¹ in Pittsburgh in 2003, and wanted to fix the handwaving-type argument for the physical validity of the system loosely presented in the talk (based on Zheng^[70]). This formal derivation presented here is what he sent the author a few days after the meeting ([37]). We note that this asymptotic regime lacks strong physical sense because there is no such physical material for very large γ . There is no other nonphysical concerns though. For example, it is physical to have $p = \varepsilon p_1 + O(\varepsilon^2) \rightarrow 0$ since p is an independent variable and one can adjust the temperature to achieve it.

2.3 Progress of Research

Both Cauchy and Riemann problems for systems (5), (1), or (8) are open. The self-similar coordinates

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t}$$

can reduce the Riemann problem by one dimension. However, all three systems (5), (1), and (8), and even their linearized versions are of mixed type in the self-similar coordinates. Peng Zhang, Jiequan Li, and Tong Zhang^[68] have given a set of conjectures for solutions to the

¹Focused Research Group, which is a program of the National Science Foundation.

four-wave Riemann problem for these systems, see the book of Li, et.al.^[44], or Section 9.3 of [71]. Zheng has established the existence of solutions in the elliptic region [70,71].

More precisely, equation (8) in the self-similar coordinates (ξ, η) takes the form

$$(p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{1}{p}(\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0. \quad (11)$$

The eigenvalues of the coefficient matrix of the second order terms of (11) can be found to be p and $p - \xi^2 - \eta^2$. Zheng proved in [70] the existence of a weak solution for equation (11) in any open, bounded and convex region $\Omega \subset \mathbb{R}^2$ with smooth boundary and the degenerate boundary datum

$$p|_{\partial\Omega} = \xi^2 + \eta^2$$

provided that the boundary of Ω does not contain the origin $(0, 0)$.

Kyungwoo Song^[61] has removed the restriction on the origin and the smoothness of the boundary. Kim and Song^[43] have obtained regularity of the solution in the interior of the domain and continuity up to and including the boundary. Dai and Zhang^[23] have obtained the interaction of two rarefaction waves adjacent to the vacuum. Zheng has also obtained a global solution involving shocks as free boundaries, see [72]. For numerical simulations, see [44].

2.4 Closeness to the Euler

How good does the pressure gradient system approximate the full Euler system? We will show solid evidence in a future paper [74]. For now we notice that the full Euler system can be rewritten in a form in which the pressure gradient system plays a dominating role.

First let us show the *maximum principle* for p of the full Euler system, although it is not unexpected (see [60]). In the self-similar plane and for smooth solutions, the system takes the form:

$$\begin{cases} \frac{1}{\rho}\partial_s\rho + u_\xi + v_\eta = 0, \\ \partial_s u + \frac{1}{\rho}p_\xi = 0, \\ \partial_s v + \frac{1}{\rho}p_\eta = 0, \\ \frac{1}{\gamma p}\partial_s p + u_\xi + v_\eta = 0, \end{cases} \quad (13)$$

where

$$\partial_s := (u - \xi)\partial_\xi + (v - \eta)\partial_\eta.$$

By differentiating the fourth equation in (13) in the flow stream direction ∂_s and using the second and third equations, we obtain a second-order equation for p :

$$-\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) + \partial_s \frac{\partial_s p}{\gamma p} - \left(\frac{\partial_s p}{\gamma p} \right)^2 - \frac{\partial_s p}{\gamma p} + 2(u_\xi v_\eta - u_\eta v_\xi) = 0. \quad (14)$$

The combination $u_\xi v_\eta - u_\eta v_\xi$ can be manipulated to depend on (v_η, p_ξ, p_η) homogeneously with degree 2. In fact, using the last three equations in (13) we find that

$$u_\xi v_\eta - u_\eta v_\xi = \frac{v_\eta p_\eta}{(v - \eta)\rho} - \frac{v_\eta p_\xi}{(u - \xi)\rho} - \frac{p_\xi p_\eta}{(u - \xi)(v - \eta)\rho^2} + \frac{p_\eta p_s}{\gamma p \rho (v - \eta)}. \quad (15)$$

When ∇p vanishes, so does $u_\xi v_\eta - u_\eta v_\xi$. Therefore there holds the maximum principle for p in the subsonic domains.

Furthermore, we show that there holds the *ellipticity principle*. Let

$$\varphi := c^2 - (u - \xi)^2 - (v - \eta)^2. \quad (16)$$

Assuming the variables are in C^2 , we derive an equation for φ :

$$-c^2 \Delta \varphi + \partial_s^2 \varphi = O(\varepsilon) + \left(\frac{\partial_s \varphi}{c} \right)^2 - \frac{4\varphi \partial_s \varphi}{c^2} - 3\partial_s \varphi + \frac{3\varphi^2 + (\varphi - c^2)^2}{c^2} + c^2. \quad (17)$$

Thus so long as $|O(\varepsilon)| < c^2$, there holds the *ellipticity principle*:

$$\min_{\Omega} \varphi \geq \min_{\partial\Omega} \varphi. \quad (18)$$

This means that our solution will be subsonic in a region if we use a barrier to force the domain subsonic.

Since the right-hand side of the φ equation is so much positive, the ellipticity holds uniformly independent of $\varepsilon > 0$. That is, consider

$$F := \varphi - \beta\omega, \quad (19)$$

where $\omega > 0$ in Ω and

$$\omega = 0 \quad \text{on} \quad \partial\Omega. \quad (20)$$

Then there exists a small $\beta > 0$, independent of $\varepsilon > 0$, such that

$$F > 0 \quad \text{in} \quad \Omega, \quad (21)$$

thus

$$\varphi > \beta\omega > 0 \quad \text{in} \quad \Omega. \quad (22)$$

For ellipticity principle for the potential equation (see [25]).

2.5 Comparison of Models

The Burgers' equation $u_t + (u^2/2)_x = \varepsilon u_{xx}$ has played an essential role in the theory of one-dimensional systems of conservation laws. No such a model has emerged for multi-dimensional systems. Current various models seem to have different physics captured. As the names suggest, the isentropic, the irrotational, and the steady flows are different. The UTSD model is regarded as describing the transition between Mach and regular reflections; i.e., it is best used locally at the triple point. The pressure gradient system can have global solutions that are similar to those of Euler system seen in numerical simulations and physical experiments, established in this paper. Thus this model possesses essential physics.

What separates this model from others is the series of features: It is a neat set of three evolutionary conservation laws, the pressure variable can decouple from the other two to form a single quasilinear equation whose quasilinearity is lower than that for the potential flow equation (or the potential formulation of the pressure gradient system). By quasilinearity we mean the order of the highest order of derivatives involved in the coefficients of the principal part of the second-order equation. In the pressure gradient system, the coefficients of the principal part of the second-order equation for the pressure do not depend on ∇p , thus its quasilinearity is zero. In the potential flow equation (p.248 of [22]), however, the coefficients of the principal part of the second-order equation for the potential depend on ∇p , thus its quasilinearity is one. This small difference makes the pressure gradient system more accessible. The success here, as intended as a model, may induce better utilization of the elliptic theory to handle the potential flow in the near future.

The model's simplicity has allowed us to establish the existence of a global solution in this paper that resembles the regular reflection of a straight shock hitting a ramp for the adiabatic Euler system. This is the first of such a result.

So this model is simple enough for mathematical treatment and yet captures essential physics. In addition, its potential to fully approximate a major part of the full Euler system is promising, see a conference proceeding [73] or a forthcoming paper [74].

3 Set-up of Regular Reflection

So we consider pressure gradient system (1). We consider self-similar solutions. In the self-similar plane $(\xi, \eta) = (x/t, y/t)$, the system of conservation laws becomes

$$\begin{cases} -\xi u_\xi - \eta u_\eta + p_\xi = 0, \\ -\xi v_\xi - \eta v_\eta + p_\eta = 0, \\ -\xi E_\xi - \eta E_\eta + (pu)_\xi + (pv)_\eta = 0. \end{cases} \quad (23)$$

For smooth solutions, we can simplify the system to

$$\begin{cases} -\xi u_\xi - \eta u_\eta + p_\xi = 0, \\ -\xi v_\xi - \eta v_\eta + p_\eta = 0, \\ -\xi p_\xi - \eta p_\eta + pu_\xi + pv_\eta = 0. \end{cases}$$

Consider a flat shock hitting a wedge with half angle $\theta_w \in (0, \pi/2)$. The state ahead of the shock is $(p, u, v) = (p_0, 0, 0)$ for some $p_0 > 0$. The state behind the shock is $(p_1, u_1, 0)$ with $p_1 > p_0$. To connect the two states with a single forward shock, we need the relation

$$u_1 = \frac{p_1 - p_0}{\sqrt{\bar{p}_{10}}}, \quad p_1 > p_0. \quad (25)$$

The overhead bar denotes the average: $\bar{p}_{10} = (p_1 + p_0)/2$. Assuming the shock reflection is a regular one, then it hits the ramp at the location

$$(\xi, \eta) = (\xi_{10}, \eta_{10}) := (\sqrt{\bar{p}_{10}}, \tan \theta_w \sqrt{\bar{p}_{10}}). \quad (26)$$

See Figure 1 for the set-up.

Two free parameters: We see that system (1) is invariant under the translation $(u, v) \rightarrow (u - a, v - b)$, which we have utilized in assuming that the velocity is zero ahead of the incident shock. The system enjoys another invariance: $(p, u, v, x, y) \rightarrow (\alpha^2 p, \alpha u, \alpha v, \alpha x, \alpha y)$ which we can use to scale $p_0 = 1$. Thus there is only one free variable p_1/p_0 in describing the data. The entire experiment can thus be characterized by the two parameters $(p_1/p_0, \theta_w)$.

3.1 Algebraic Portion:

To find the reflected shock and the state between it and the ramp, denoted by state 2, we need the Rankine-Hugoniot relation in 2-D. Let $\eta = \eta(\xi)$ with slope $\sigma = \eta'(\xi)$ be a shock curve. Then,

$$\begin{aligned} (\eta - \xi\sigma)[u] + \sigma[p] &= 0, \\ (\eta - \xi\sigma)[v] - [p] &= 0, \\ (\eta - \xi\sigma)[E] + \sigma[pv] - [pv] &= 0. \end{aligned} \quad (27)$$

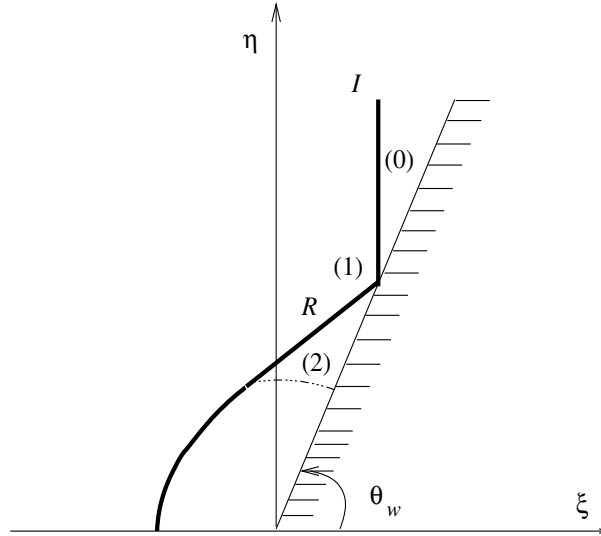


Figure 1. Regular reflection

We can solve them to obtain the contact discontinuity $\sigma = \eta/\xi = [v]/[u]$, $[p] = 0$ and the shocks

$$\begin{aligned} \frac{d\eta}{d\xi} &= \sigma_{\pm} := \frac{\xi\eta \pm \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\xi^2 - \bar{p}}, \\ [p] &= \xi[u] + \eta[v], \\ [p]^2 &= \bar{p}([u]^2 + [v]^2). \end{aligned} \tag{28}$$

We use the minus branch for the reflected shock. A useful and equivalent form for the Rankine-Hugoniot relation is

$$\frac{d\eta}{d\xi} = \sigma_{\pm} = \frac{\xi\eta \pm \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\xi^2 - \bar{p}}, \tag{29}$$

$$[u] = \frac{\xi\bar{p} \pm \eta\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\bar{p}(\xi^2 + \eta^2)} [p], \tag{30}$$

$$[v] = \frac{\eta\bar{p} \mp \xi\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\bar{p}(\xi^2 + \eta^2)} [p]. \tag{31}$$

We require that the state (p_2, u_2, v_2) be such that the velocity (u_2, v_2) be parallel to the wall; that is,

$$v_2 = \tan \theta_w u_2. \tag{32}$$

This requirement and the Rankine-Hugoniot relation determine the state 2.

Proposition 3.1. *(Regular reflection of the algebraic portion) There exists a critical $\theta_w = \theta_0 \in (0, \pi/2)$, depending only on p_1/p_0 , given by the formula*

$$\tan^2 \theta_0 = \frac{8p_1(p_1 - p_0)}{(p_1 + p_0)^2}, \tag{33}$$

such that there exist two states (p_2, u_2, v_2) for each $\theta_w > \theta_0$, given by

$$p_2 = p_1 + \bar{p}_{10} \tan^2 \theta_w \pm \tan \theta_w \sqrt{\bar{p}_{10}^2 \tan^2 \theta_w - 2p_1(p_1 - p_0)}, \quad (34)$$

$$u_2 = \frac{p_2 - p_0}{\sqrt{\bar{p}_{10}}} \frac{1}{1 + \tan^2 \theta_w}, \quad v_2 = u_2 \tan \theta_w. \quad (35)$$

Both values of the pressure of the state 2 are greater than p_1 ; the larger one goes to infinity while the smaller one approaches

$$p_2^* := p_1 + \frac{2p_1(p_1 - p_0)}{p_1 + p_0} \quad (36)$$

as $\theta_w \rightarrow \pi/2-$.

We comment that the p_2^* in (36) is the pressure that corresponds to the planar shock hitting a vertical wall with zero velocity $(u_2, v_2) = (0, 0)$ between the reflected backward shock at $\xi = -\sqrt{(p_1 + p_2^*)/2}$ and the vertical wall.

Proof. We manipulate the second equation in (28) to yield

$$p_2 = \xi_{10}(1 + \tan^2 \theta_w)u_2 + p_0. \quad (37)$$

Introducing the notation

$$\tilde{u}_2 := (1 + \tan^2 \theta_w)u_2, \quad (38)$$

we have

$$\tilde{u}_2 = \frac{p_2 - p_0}{\sqrt{\bar{p}_{10}}}, \quad (39)$$

or equivalently

$$\tilde{u}_2 - u_1 = \frac{p_2 - p_1}{\sqrt{\bar{p}_{10}}}. \quad (40)$$

Manipulating the third equation of (28), we obtain

$$(p_2 - p_1)^2/\bar{p}_{12} = (u_2 - u_1)^2 + \tan^2 \theta_w u_2^2. \quad (41)$$

This is a quadratic equation for u_2 , from which we find one branch

$$\tilde{u}_2 - u_1 = \sqrt{1 + \tan^2 \theta_w} \sqrt{(p_2 - p_1)^2/\bar{p}_{12} - \sin^2 \theta_w u_1^2}. \quad (42)$$

We have discarded the minus sign branch since it is irrelevant here. We equate the two equations (40) and (42) to eliminate \tilde{u}_2 and end up with the equation

$$(p_2 + p_1)(p_2 - p_1)^2 = \bar{p}_{10}(1 + \tan^2 \theta_w)[2(p_2 - p_1)^2 - (p_2 + p_1) \sin^2 \theta_w u_1^2]. \quad (43)$$

Let $x := p_2 - p_1$, we rewrite the above equation as

$$x^3 + (p_1 - p_0 - (p_1 + p_0) \tan^2 \theta_w)x^2 + (x + 2p_1) \tan^2 \theta_w(p_1 - p_0)^2 = 0. \quad (44)$$

We observe by inspection that the equation has a solution $x = p_0 - p_1$, which helps to reduce the equation to

$$x^2 - (p_0 + p_1) \tan^2 \theta_w x + 2p_1 \tan^2 \theta_w(p_1 - p_0) = 0.$$

We thus find the two roots easily when the discriminant is nonnegative. The asymptotic behavior for p_2 is obvious from the explicit formula. This completes the proof of the proposition.

We will use the weak reflection (with the smaller pressure value), and ignore the strong one (see p. 317 in [22] for a reason). We check easily to see that

$$\xi_{10}^2 + \eta_{10}^2 - \bar{p}_{12} > 0 \quad (46)$$

for the weak reflection as $\tan \theta_w \geq \tan \theta_0$, so the square root is well defined for the slope of the shock wave. In fact, we use the explicit formula (34) to first obtain $p_2 < p_1 + \bar{p}_{10} \tan^2 \theta_w$, then use $\tan \theta_w > \tan \theta_0$ to derive

$$\xi_{10}^2 + \eta_{10}^2 - \bar{p}_{12} > \frac{(p_1 - p_0)(3p_1 - p_0)}{2(p_1 + p_0)} > 0. \quad (47)$$

Further, the slope satisfies

$$\frac{d\eta}{d\xi} = \sigma < \tan \theta_w \quad (48)$$

through direct comparison for as long as $p_2 > p_1$. So locally at the reflection point, the regular reflection is possible when $\theta_w \in [\theta_0, \pi/2)$.

The state 2 at the reflection point is also supersonic in the sense that

$$\xi^2 + \eta^2 - p > 0 \quad (49)$$

when θ_w is slightly more closer to $\pi/2$; i.e., $\tan \theta_w > \tan \theta_1$, where

$$\tan^2 \theta_1 := \frac{p_1 - p_0}{(p_1 + p_0)^2} \left(4p_1 + \sqrt{16p_1^2 + (p_1 + p_0)^2} \right). \quad (50)$$

The proof is straightforward so we omit it. The state (p_2, u_2, v_2) will remain supersonic in a neighborhood of the reflection point. So the constant state (p_2, u_2, v_2) will be used as a solution in the sector formed by the reflected straight shock and the wall until it hits the sonic circle

$$\xi^2 + \eta^2 = p_2. \quad (51)$$

3.2 Elliptic Portion:

Beyond the sonic arc, the state becomes nonconstant, and the reflected shock curls downward to meet the ground, vertically at the ground. We propose the problem as a boundary value problem with a free boundary on which the Rankine-Hugoniot relation holds. We convert the Rankine-Hugoniot relation into a (degenerate) oblique derivative boundary value problem, as in [13], see the next paragraph. On the wall, the boundary condition $v_2 = \tan \theta_w u_2$ becomes

$$\partial_{\mathbf{n}} p = 0 \quad (52)$$

where \mathbf{n} denotes the exterior unit normal to the wall. On the ground, we can use the Neumann condition (52) or equivalently we convert the ramp problem into a wedge problem, so there is no boundary. On the sonic arc, we use the Dirichlet problem. The decoupled p -equation in the subsonic domain is derived from (23) and mentioned in (11):

$$(p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{(\xi p_{\xi} + \eta p_{\eta})^2}{p} - 2(\xi p_{\xi} + \eta p_{\eta}) = 0. \quad (53)$$

Here is the derivation of the (degenerate) oblique derivative boundary value on the reflected shock, which we call Σ from now on. We require that all three equations (23) (taken the limit from the inside) and all three Rankine-Hugoniot relations (29–31) hold. We differentiate the

last two equations in the Rankine-Hugoniot relations along the shock wave so we have five differential equations for six derivatives $(u_\xi, v_\xi, p_\xi, u_\eta, v_\eta, p_\eta)$. Holding p_η fixed, we solve for the other five $(u_\xi, v_\xi, u_\eta, v_\eta, p_\xi)$. The relation between p_ξ and p_η decouples from the other derivatives and is used as the oblique derivative condition on the shock wave:

$$\begin{aligned} & \dot{p} \left(\frac{[p]}{4\bar{p}} (\xi^2 + \eta^2) - (\xi^2 + \eta^2 - \bar{p}) \right) \\ & + (\xi\eta + \eta'(\bar{p} - \xi^2)) \left\{ (-\sigma_\pm p_\xi + p_\eta) + (\xi p_\xi + \eta p_\eta) \frac{\xi\sigma_\pm - \eta}{p} \right\} = 0, \end{aligned} \quad (54)$$

where $[p] = p - p_1$, and the term \dot{p} denotes the tangential differentiation along the shock: $\dot{p} := p_\xi + \sigma_\pm p_\eta$. The other relations are used for determining (u, v) once p is obtained. The first equation in the R-H relation (29) is symbolically “reserved” for use to determine the location of Σ . The oblique derivative condition (54) is in the form $\mathbf{I} \cdot \nabla p = 0$ for a smooth vector field $\mathbf{I}(\xi, \eta, \eta')$, which is tangent to the shock boundary at its tip. The apparent p -dependence of \mathbf{I} is removed by the formula

$$\bar{p} = \frac{(\eta - \xi\sigma_\pm)^2}{1 + \sigma_\pm^2} \quad (55)$$

obtained by inverting (29).

The sufficiency of the oblique derivative boundary condition is seen as follows. The second-order equation for p plus the two first-order equations for (u, v) implies

$$(\xi\partial_\xi + \eta\partial_\eta)L + L = 0 \quad (56)$$

for

$$L := (\xi\partial_\xi p + \eta\partial_\eta p)/p - u_\xi - v_\eta. \quad (57)$$

This ODE (56) in the form $r\partial_r L + L = 0$ has the only solution $L = 0$ if $L(r_0) = 0$ at any point $r = r_0 > 0$. Thus the third equation of (23) will hold if it holds on the boundary Σ . From our previous derivation of the oblique derivative boundary condition, we see that $L = 0$ on the free boundary, provided that the oblique derivative boundary condition holds along with the other four relations between $(u_\xi, v_\xi, u_\eta, v_\eta)$ and p_η in obtaining (u, v) . We will use the four relations to find (u, v) . Thus the oblique derivative boundary condition is a condition that guarantees that solutions of the second-order equation for p are solutions for the first-order systems for (p, u, v) in the sectoral domain spanned by Σ . We point out that $L = 0$ holds automatically on the sectoral domain spanned by the sonic arc if p is Lipschitz.

The free boundary problem: Find p satisfying the degenerate elliptic equation (53) in the domain Ω bounded by the curved reflected shock Σ , the wall and the ground, and the sonic arc (51). On the wall and the ground, it is the Neumann condition (52). On the circular arc, it is the Dirichlet problem. On the free boundary Σ , it is the degenerate oblique derivative boundary condition (54). The free boundary satisfies (29).

A trivial solution to the above free boundary problem is $p = p_2$ in the entire domain Ω , whose boundary Σ consists of two parts: The first part is the extension of the reflected shock R to the tangent point with the circle $\xi^2 + \eta^2 = \bar{p}_{12}$ while the second part is the circle $\xi^2 + \eta^2 = \bar{p}_{12}$ between the tangent point and the ground. This constant solution does not result in a solution (p, u, v) for the original problem. The reason is that the tangential oblique derivative boundary value condition is completely degenerate on the circular free boundary which fails to impart information from the second-order equation (53) for p to the first-order equation for p (i.e., the third equation in (23)). In short, the velocity field (u, v) caused by a circular shock wave connecting two constant pressure is necessarily such that $u_\xi + v_\eta \neq 0$. We avoid this trivial solution by finding only nearly flat free boundary shocks.

Corner condition. Further restriction is possible. We can require our solution to have zero speed at the corner of the wedge; i.e., $u(0, 0) = 0$. We can use the equation $-\xi u_\xi - \eta u_\eta + p_\xi = 0$ along the ground to find

$$u(0, 0) = u_m - \int_{\xi_m}^0 \frac{p_\xi(x, 0)}{-x} dx, \tag{58}$$

where (u_m, ξ_m) are the velocity and position at the shock foot on the ground, and are both determined by the pressure there, denoted by p_m , via the Rankine-Hugoniot relation

$$u_m = u_1 - \frac{p_m - p_1}{\sqrt{\bar{p}_{m1}}}; \quad \xi_m = -\sqrt{\bar{p}_{m1}}. \tag{59}$$

In short, the value p_m of the pressure at the foot of the shock satisfies

$$u_1 - \frac{p_m - p_1}{\sqrt{\bar{p}_{m1}}} = \int_{-\sqrt{\bar{p}_{m1}}}^0 \frac{p_\xi(x, 0)}{-x} dx \tag{60}$$

in order for $u(0, 0) = 0$.

4 Result

Main Theorem. *There exists an (entropy) solution (p, u, v) defined for all $(\xi, \eta) \in \mathbb{R}^2$, provided that the ramp angle θ_w is close to $\pi/2$. The structure is a regular reflection consisting of a reflection point on the wall, a reflected bow shock, and a subsonic region. The shock curve is C^1 and piecewise C^∞ smooth. The pressure p is C^∞ smooth in the subsonic domain. The velocity is bounded and Hölder continuous on the closure of the subsonic domain. Pseudo-velocity stagnates only at the origin and is vorticity free everywhere.*

See Figure 1 for an illustration, where only half of the wedge is presented. Our result is complementary to (and consistent with) the asymptotic results of Keller and Blank^[41] and Hunter and Keller^[39].

The proof is given in the next sections.

5 Linear Theory with Fixed Boundary

We consider a linear problem with a fixed boundary that forms the basis for our nonlinear problem. The domain Ω is the region between the wedge and a pre-selected shock curve and its corresponding sonic arcs. See Figure 2, where the shock curve is denoted by the curve AGB , and also by Σ . The sonic arcs are AC and BD . We are interested in the case where AC and BD do not degenerate to single points.

More precisely, for a given $\theta_w \in (0, \pi/2)$, p_1, p_m ($p_1 < p_m$), and a $\beta \in (0, \beta_0)$, where

$$\beta_0 := \frac{1}{\sqrt{\bar{p}_{m1}}}, \quad \bar{p}_{m1} := (p_1 + p_m)/2 \tag{61}$$

let

$$\mathbb{K} = \{ \eta = \eta(\xi) \in C^{2,\gamma}(\mathbb{R}) \mid \eta(0) = \eta_{m1}, \eta'(0) = 0, \eta''(0) = \beta, 0 \leq \eta'' \leq \beta_0, \eta \text{ even} \}. \tag{62}$$

Here $\gamma \in (0, 1)$ and

$$\eta_{m1} := -\sqrt{\bar{p}_{m1}}. \tag{63}$$

For an $\eta \in \mathbb{K}$, we introduce

$$\bar{p} = \frac{(\eta - \xi\eta'(\xi))^2}{1 + (\eta'(\xi))^2}, \tag{64}$$

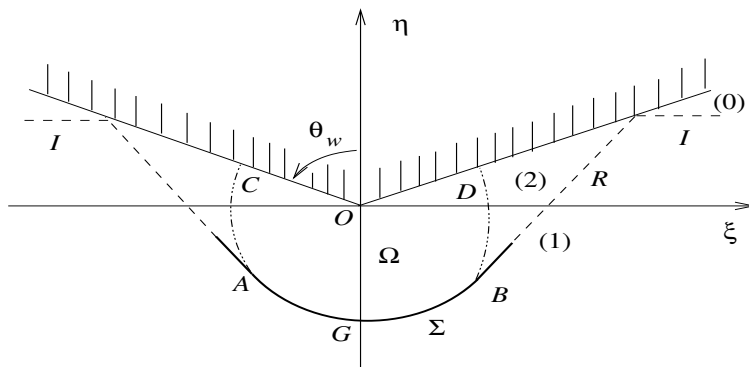


Figure 2. Domain for the linear theory

where \bar{p} stands for the average of p_1 and the p from the inside of the shock. This \bar{p} is always well-defined and belongs to $C^{1,\gamma}(\mathbb{R})$ and $C^{2,\gamma}$ at the point $\xi = 0$ magically. In fact, it has the expansion

$$\bar{p} = \bar{p}_{m1} + \beta\sqrt{\bar{p}_{m1}}(1 - \beta\sqrt{\bar{p}_{m1}})\xi^2 + O(\xi^{2+\gamma}) \tag{65}$$

at $\xi = 0$. So \bar{p} has a local minimum at $\xi = 0$ for the choices of β and β_0 . Furthermore, we have

$$\xi^2 + \eta^2 - \bar{p} = \xi^2((1 - \beta\sqrt{\bar{p}_{m1}})^2 + O(\xi^\gamma)) \tag{66}$$

at $\xi = 0$. Thus the square root function

$$F(\xi) = \begin{cases} \sqrt{\xi^2 + \eta^2 - \bar{p}}, & \xi > 0, \\ -\sqrt{\xi^2 + \eta^2 - \bar{p}}, & \xi < 0 \end{cases}$$

is a smooth function at $\xi = 0$ along the curve $\eta(\xi)$. Finally, through pure algebraic manipulations, we find

$$\bar{p}(\xi^2 + \eta^2 - \bar{p}) = (\eta'(\xi^2 - \bar{p}) - \xi\eta)^2. \tag{67}$$

We would like to invert (67) to find

$$\frac{d\eta}{d\xi} = \sigma_{\pm} = \frac{\xi\eta + \xi\sqrt{\bar{p}(1 + (\eta^2 - \bar{p})/\xi^2)}}{\xi^2 - \bar{p}} =: \sigma, \tag{68}$$

which is the standard Rankine-Hugoniot locus, but we need to prove that

$$\eta'(\xi^2 - \bar{p}) - \xi\eta > 0, \quad \text{for } \xi > 0. \tag{69}$$

In fact, we use algebraic manipulations to find

$$\eta'(\xi^2 - \bar{p}) - \xi\eta = \frac{1}{1 + (\eta')^2}(\xi\eta' - \eta)(\xi + \eta\eta'). \tag{70}$$

The term $\xi\eta' - \eta > 0$ because it is positive at $\xi = 0$ and its derivative along the curve is $\xi\eta'' > 0$. The other term $\xi + \eta\eta'$ is positive:

$$\xi + \eta\eta' > 0, \tag{71}$$

because it is zero at $\xi = 0$ and its derivative is $1 + (\eta')^2 + \eta\eta''$ which is positive since $\eta'' \geq 0$ if $\eta > 0$, or $|\eta\eta''| \leq (\max |\eta|)\beta_0 = 1$. So indeed (69) holds and we have (68).

The pre-selected shock will be a curve from the set \mathbb{K} . It is possible that only a portion of an $\eta(\xi)$ in \mathbb{K} is actually used: We use the elliptic portion, corresponding to AGB . Let us explain. For each $\eta(\xi) \in \mathbb{K}$, we use the equation $\eta' = \sigma_-$ for $\xi < 0$ and $\eta' = \sigma_+$ for $\xi \geq 0$ to locate p on the boundary; i.e., formula (64). Along this curve as ξ increases, both p and $\xi^2 + \eta^2$ increase (see (71)) until $p - \xi^2 - \eta^2 = 0$ from which point we stop using this boundary, i.e., the point B . Use p_2 to denote this value of p (at B) along the boundary where the ellipticity first vanishes. Use this p from (64) in the vector field \mathbf{I} of the oblique derivative condition. We then have a linear tangential oblique derivative boundary value problem for $p^{(1)}$:

$$\begin{aligned} & (p_\xi^{(1)} + \eta' p_\eta^{(1)}) \left(\frac{[p]}{4\bar{p}} (\xi^2 + \eta^2) - (\xi^2 + \eta^2 - \bar{p}) \right) \\ & + (\xi\eta + \eta'(\bar{p} - \xi^2)) \left\{ (-\eta' p_\xi^{(1)} + p_\eta^{(1)}) + (\xi p_\xi^{(1)} + \eta p_\eta^{(1)}) \frac{\xi\eta' - \eta}{p} \right\} = 0. \end{aligned} \tag{72}$$

The coefficients of (72) all enjoy $C^{1,\gamma}$ regularity or higher. Note carefully that we do *not* use the value p in (64) as Dirichlet boundary value. We use Dirichlet value p_2 on the sonic line and p_m at G :

$$p = p_2 \quad \text{on } \xi^2 + \eta^2 = p_2; \quad p = p_m \quad \text{at } G. \tag{73}$$

The extra condition p_m at G will be justified later by identifying the problem as an *emergent type* (see [6,35]) since the obliqueness fails in an emergent type.

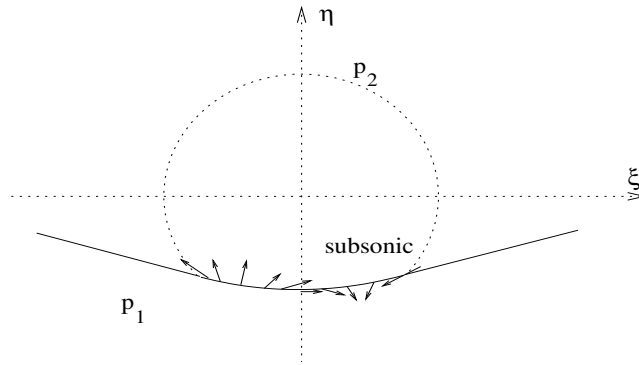


Figure 3. Directions of the tangential oblique derivatives.

The direction field $\mathbf{I}(\xi, \eta, \eta'(\xi))$ of the oblique derivative condition is drawn for a flat shock in Figure 3. This vector field has a natural extension into the domain by simply letting η free everywhere including in the formula for \bar{p} , while holding $\eta'(\xi)$ fixed as the slope of the fixed boundary.

We linearize equation (53). Fix an $\alpha \in (0, 1)$. And fix a positive constant $\varepsilon_e > 0$. For any

$$Q \in C^{1,\alpha}(\bar{\Omega}), \quad p_m < Q < p_2 \quad \text{in } \Omega, \tag{74}$$

we consider the equation

$$\begin{aligned} & \varepsilon_e \Delta p + ((Q - \xi^2 - \eta^2)^+ + \eta^2) p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + ((Q - \xi^2 - \eta^2)^+ + \xi^2) p_{\eta\eta} \\ & + \left\{ \frac{\xi Q_\xi + \eta Q_\eta}{Q} - 2 \right\} (\xi p_\xi + \eta p_\eta) = 0, \end{aligned} \tag{75}$$

where the regularization term $\varepsilon_e \Delta p$ is added to ensure uniform ellipticity. Or in short, we consider an equation like this:

$$Lu := \sum_{ij} a^{ij} u_{x_i x_j} + \sum_i b^i u_{x_i} = 0 \quad \text{in } \Omega. \tag{76}$$

We point out that we have to linearize the equation to utilize the established theory on (tangential) oblique derivative boundary value problems of Lieberman and Guan and Sawyer, although some quasilinear and even fully nonlinear theory, which unfortunately do not apply to our case, is available (see [32,58]).

The boundary condition on the wall is the Neumann condition (52)

$$\partial_{\mathbf{n}} p^{(1)} = 0. \tag{77}$$

We introduce a couple of notations. Let

$$d_x = \text{distance } \{x, \{O, A, B, C, D\}\},$$

and for $\delta > 0$,

$$\Omega_\delta := \{x \in \Omega \mid d_x > \delta\}.$$

Let $|u|_a$ be the C^a norms on the usual C^a functions, $a \geq 0$.

Theorem 5.1. (Existence for the linearized and fixed boundary problem). *There exists a classical smooth solution $p^{(1)} \in C^{1,\alpha'}(\overline{\Omega}) \cap C^2(\Omega \cup \{\Sigma \setminus \{A, B\}\})$ for some $\alpha' > 0$ to the linearized equation (75) with the Dirichlet (73), Neumann (77), and the tangential oblique derivative boundary conditions (72). The solution satisfies the following basic estimates:*

$$\begin{aligned} p_m < p^{(1)} < p_2 & \quad \text{in } \Omega, \\ |p^{(1)} - p_m|_{1+\alpha'}(\overline{\Omega}) & \leq C|p_2 - p_m| \quad \text{for } \alpha' \leq \alpha_1 \left(\frac{\theta_w}{\pi - \theta_w} \right), \\ |p^{(1)} - p_m|_{2+\gamma'}(\overline{\Omega}_\delta) & \leq C_\delta |p_2 - p_m| \quad \text{for } \gamma' \leq \min\{\gamma, \alpha\}, \quad 0 < \delta \ll 1, \end{aligned} \tag{78}$$

where α_1 is a positive number depending on $\frac{\theta_w}{\pi - \theta_w}$.

Remark: We remark that it is possible to express these estimates more neatly by using the intermediate spaces $H_a^{(b)}(\Omega)$, introduced in Gilbarg-Hörmander^[27]. The $H_a^{(b)}(\Omega)$ consists of functions u such that

$$|u|_a^{(b)}(\Omega) := \sup_{\delta > 0} \{ \delta^{a+b} |u|_a(\overline{\Omega}_\delta) \} < \infty. \tag{79}$$

The numbers a and b satisfy $a \geq 0$ and $a + b \geq 0$.

Proof. The proof is based on a series of four papers [46–49] from Lieberman, one paper from Azzam^[2], and one paper from Guan and Sawyer^[32]. The paper of Guan and Sawyer helps us to solve the tangential oblique derivative problem at the point G locally. The paper of Azzam provides higher regularity of solutions at corners C and D, where higher compatibility conditions than Lieberman’s^[49] are satisfied. All other difficulties are handled by the papers of Lieberman. Paper [46] provides the frame-work of Perron’s method and handles the oblique derivative part. Paper [47] handles the mixed case and in particular points A, B, C , and D . Paper [49] gives optimal regularity at those points. And paper [48] handles the point O where two oblique derivative boundary conditions are satisfied simultaneously. As for the interior and the Dirichlet boundary condition on the sonic arcs, they are classical, see Gilbarg and Trudinger^[28] or Zheng^[70].

We mention here the key points of the proof, details are given in Appendix A. First we verify the obliqueness of (72) on the boundary. An interior normal of Σ is $(-\sigma_{\pm}, 1)$. The obliqueness is defined by the inner product of a unit (interior) normal with the oblique direction $(-\sigma_{\pm}, 1) + \frac{\xi\sigma_{\pm} - \eta}{p}(\xi, \eta)$. We ignore the tangential direction. We ignore the normalization of the interior normal for now as well. Thus

$$\text{Obliqueness} = (-\sigma_{\pm}, 1) \cdot \left\{ (-\sigma_{\pm}, 1) + \frac{\xi\sigma_{\pm} - \eta}{p}(\xi, \eta) \right\}. \quad (80)$$

We do some simple algebra to yield,

$$\text{Obliqueness} = \frac{p - \xi^2}{p} \left\{ \left(\sigma_{\pm} + \frac{\xi\eta}{p - \xi^2} \right)^2 + \frac{p(p - \xi^2 - \eta^2)}{(p - \xi^2)^2} \right\}. \quad (81)$$

The obliqueness is obvious provided that $p - \xi^2 - \eta^2 > 0$ along the curve. It is true in a neighborhood of G , and recall we stop when it ceases to be true. The other factor $\xi\eta + \eta'(\bar{p} - \xi^2)$, which we do not place in (80), has been shown in (69) to be less than zero for ξ between G and B .

Regarding point O , we realize that the interior angle COD is not less than π , thus Lieberman's theory does not apply directly to yield $C^{1,\alpha}$ estimate. But, we realize that the solution is symmetric with respect to the η axis. Hence, the left-half of the domain enjoys an interior angle COG which is $\pi - \theta_w \leq \pi$. Therefore Lieberman's theory can apply in this case to yield C^{1,α_1} solutions where α_1 is a positive number depending on $\frac{\theta_w}{\pi - \theta_w}$.

Regarding point B , where $p_2 = \xi^2 + \eta^2$, the oblique derivative boundary condition becomes

$$K^2(p_{\xi} + \eta'p_{\eta}) - \eta'p_{\xi} + p_{\eta} + (\xi p_{\xi} + \eta p_{\eta})(\xi\eta' - \eta)/(\xi^2 + \eta^2) = 0, \quad (82)$$

where

$$K^2 = \frac{p_1}{p_1 + p_2} \sqrt{\frac{p_2 - p_1}{p_2 + p_1}}.$$

We rewrite (82)

$$p_{\xi} + Hp_{\eta} = 0, \quad \text{where } H := \frac{K^2\eta' + 1 + \eta(\xi\eta' - \eta)/(\xi^2 + \eta^2)}{K^2 - \eta' + \xi(\xi\eta' - \eta)/(\xi^2 + \eta^2)}. \quad (83)$$

We claim that the direction of the oblique derivative points in between the tangential directions of the shock curve and the sonic circle; i.e.,

$$\xi/(-\eta) > H > \eta', \quad (84)$$

assuming $\eta_B < 0$. To prove the claim, we consider the two separate cases $\eta_B \geq 0$ and $\eta_B < 0$. First let $\eta_B < 0$. Then the denominator of H is positive since

$$-\eta' + \xi(\xi\eta' - \eta)/(\xi^2 + \eta^2) = -\eta(\xi + \eta\eta')/p_2 > 0.$$

(See (71) for $\xi + \eta\eta' > 0$). Using cross multiplication, the inequality $H > \eta'$ becomes $(\xi + \eta\eta')^2 > 0$, while the inequality $\xi/(-\eta) > H$ becomes $K^2(\xi + \eta\eta') > 0$. So they are both valid. For the case $\eta_B \geq 0$, we note that the numerator of H is always positive: $1 + \eta(\xi\eta' - \eta)/p_2 = \xi(\xi + \eta\eta')/p_2 > 0$. When the denominator is positive or zero, the proof for inequality $H > \eta'$ is the same as before. When the denominator is negative, there holds $H < 0 < \eta'$. When the denominator is positive or zero, the inequality $\xi/(-\eta) < 0 < H$ is trivial. When the denominator is negative, then both denominators of $\xi/(-\eta)$ and H are negative, thus the proof

for the case $\eta < 0$ is still applicable. Thus, in all cases, we have proved our claim: The vector field \mathbf{l} points from the exterior to the exterior of the domain, see Figure 3. By Lieberman's^[47], the solution $p^{(1)}$ enjoys C^{1,α_2} regularity at the point for some $\alpha_2 > 0$. This α_2 depends on p_1, p_m , and the upper bound β_0 of β , as well as ε_e .

Regarding point C , Lieberman's yields only Hölder continuity C^{α_3} for any $\alpha_3 < 1$. But the constant value of p on the sonic arc has zero tangential derivative while the normal of the wall is tangent to the arc, which allows for higher C^{1,α_4} regularity for some $\alpha_4 > 0$ by Azzam^[2].

Regarding point G , we realize that Theorem 1.1 of Guan and Sawyer does not apply directly since its requirement include that the boundary Σ be in $C^{3+\lambda}$ ($\lambda > 0$). Fortunately, its Remark 1.3 indicates that the boundary regularity can be reduced to $C^{2+\gamma}$ when the structure of the tangential manifold is simple, which is our case.

We use the Perron method for existence, as framed in Lieberman^[46]. The basic local existence at point G is the only new case we need to provide a proof for. It is formulated as a Dirichlet and tangential oblique derivative boundary value problem, which is slightly different from a pure tangential oblique derivative problem. More precisely, let B_2 be a neighborhood of G with smooth boundary. Let h be any continuous function on $\partial B_2 \cap \Omega$. Consider the problem

$$p = h \quad \text{on } \partial B_2 \cap \Omega \tag{85}$$

for equation (75) restricted to the domain $B_2 \cap \Omega$ with the oblique derivative boundary conditions (72) restricted to $\Sigma \cap B_2$. The local existence for some such B_2 implies global existence of solution in Ω by Lieberman^[46]. We know that the elliptic estimates are local, so Guan and Sawyer's estimate applies in this case. More details on the local existence are provided in Appendix A.

We comment that there is a lot of interesting work on the topic of tangential oblique derivative boundary value problems, (see [5–7,24,32,35,42,50,52,55–58,63,67]).

6 Quasilinear Theory with Fixed Boundary

Now we take away the linearizing function Q . This step is simple by the estimates of the previous theorem. So we consider

$$Q \in C^{1,\alpha}(\overline{\Omega}), \quad p_m < Q < p_2 \quad \text{in } \Omega. \tag{86}$$

Let α_1 be the least of the three exponents given at the points A, O , and C . Recall that α_1 is determined by the local geometry and is independent of $\alpha \in [0, 1)$, i.e., we can obtain α_1 by setting $\alpha = 0$. We choose α so that $0 < \alpha < \alpha_1$. So our previous section yields a solution $p^{(1)} \in C^{1,\alpha_1}(\overline{\Omega})$ and satisfies the same lower and upper bounds in Ω .

For each fixed positive ε_e , the estimates make the mapping $Q \in C^{1,\alpha} \rightarrow p^{(1)} \in C^{1,\alpha_1}$ compact.

We quote from Gilbarg and Trudinger^[28] the following fixed point theorem: Corollary 11.2, p. 280:

Let \mathcal{C} be a closed convex set in a Banach space \mathcal{B} and let T be a continuous mapping of \mathcal{C} into itself such that the image $T\mathcal{C}$ is precompact. Then T has a fixed point.

We let \mathcal{B} be the Banach space $C^{1,\alpha}(\overline{\Omega})$. We let \mathcal{C} be all $Q \in \mathcal{B}$ such that $p_m \leq Q \leq p_2$ in $\overline{\Omega}$, $Q = p_m$ at the point G , $Q = p_2$ on the sonic arcs AC and BD , and the \mathcal{B} -distance from p_2 be less or equal to 1. This \mathcal{C} is a bounded, closed, and convex set.

Let T be the mapping from $Q \in \mathcal{C}$ to \mathcal{B} constructed above. Each TQ satisfies $p_m < TQ < p_2$ in Ω and the corresponding boundary values at G, AC and BD . The image $T\mathcal{C}$ is in $C^{1,\alpha_1}(\overline{\Omega})$ for $\alpha_1 > \alpha$ and the $C^{1,\alpha_1}(\overline{\Omega})$ -norm is bounded by a constant times the factor $p_2 - p_m$. Letting $p_2 - p_m$ be sufficiently small, we obtain $T\mathcal{C} \subset\subset \mathcal{C}$. The continuity of T follows from the precompactness and the linearity. So there exists a fixed point $Q = p^{(1)} \in C^{1,\alpha_1}(\overline{\Omega})$.

The maximum principle still holds for the quasilinear problem:

$$p_m < p^{(1)} < p_2 \quad \text{in } \Omega. \tag{87}$$

We still have that $p^{(1)}$ is $C^{2,\gamma'}$ in $\Omega \cup \{\Sigma \setminus \{A, B\}\}$. The ellipticity holds

$$p^{(1)} - \xi^2 - \eta^2 > 0 \quad \text{in } \Omega. \tag{88}$$

To prove this, we write an equation for the variable $w =: p^{(1)} - \xi^2 - \eta^2$ and show that there is no interior minimum for w . Thus the sign restriction is redundant and our solution satisfies

$$\varepsilon_\varepsilon \Delta p + (p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \tag{89}$$

The solution $p^{(1)}$ is monotone on either arms of the edge Σ , $\xi < 0$, or $\xi > 0$. The proof is similar to that for Proposition 2.4 of [12]. The idea is to use contradiction method. Suppose $p^{(1)}$ is not monotone on the right arm. Then a typical case would be that there will be a local max at ξ_N and a local min at a $\xi_n > \xi_N$ in the interior of the arm, with the local max greater than the local min. Since these values are not permitted to be global extremes in the closure of the domain, we can find a curve leading from the max into the domain along which the value of $p^{(1)}$ will be increasing; Similarly we find a curve starting from the min and leading into the domain along which the value of $p^{(1)}$ will be decreasing. These two curves will have no where to end due to the geometry and data we have for our problem. A more elaborate proof is given in Appendix B.

The solution satisfies

$$|\nabla p^{(1)}| \leq C_1(p_2 - p_m), \quad \text{in } \bar{\Omega} \tag{90}$$

for some constant $C_1 = C_1(\varepsilon_\varepsilon)$.

7 Free Boundary

We handle the free boundary problem. Instead of considering the set \mathbb{K} and a mapping from \mathbb{K} to itself, we will use a different set, set \mathbb{B} .

We first deal with an idealized situation. We take two values of pressure $p_2 > p_1 > 0$ and a small number $\delta \in (0, \delta_0]$ where

$$\delta_0 := 1 - \frac{\sqrt{\bar{p}_{12}}}{\sqrt{p_2}}. \tag{91}$$

We introduce the point B with coordinates

$$(\xi, \eta)|_B = (\sqrt{(p_2 - p_1)/2}, -\sqrt{(p_2 + p_1)/2}) =: (\xi_{12}, \eta_{12}), \tag{92}$$

and A is its symmetric counterpart. Let B_δ be the intersection point of the circle

$$\xi^2 + \eta^2 = p_2 \tag{93}$$

with a straight line of slope δ that is tangent to the circle

$$\xi^2 + \eta^2 = \bar{p}_{12}. \tag{94}$$

Denote the coordinates of B_δ by $(\xi_{12\delta}, \eta_{12\delta})$ that is the closest to the point B . Let A_δ be its symmetric part. We consider the pressure equation in the domain Ω bounded by the upper boundary (93) and the free boundary connecting A_δ and B_δ . The δ will be the slope of the free boundary at the point B_δ . See Figure 4, where a is inside or on the circle (93) for $\delta \leq \delta_0$.

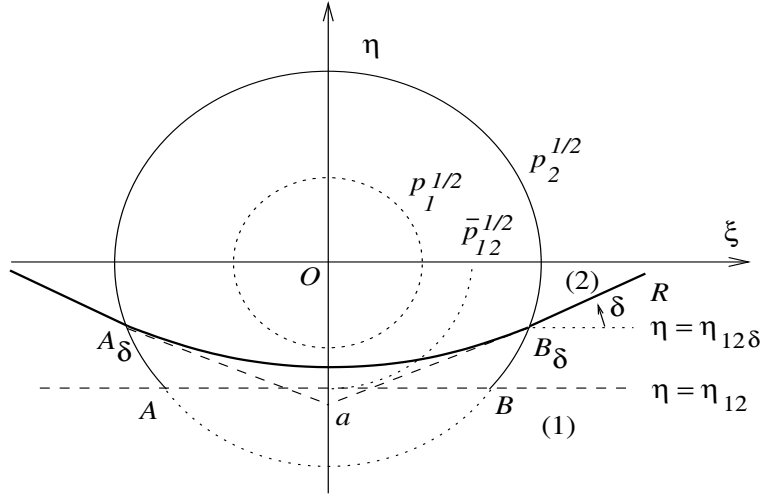


Figure 4. Domain for the free boundary.

On the upper boundary we impose $p = p_2$; Below the shock wave we take $p = p_1$. The free boundary is required to start at point B_δ , have slope δ at point B_δ , be even and convex in ξ . A typical approximate free boundary is this

$$\eta = \eta^X(\xi) := \eta_{12\delta} + \frac{\delta}{2\xi_{12\delta}}(\xi^2 - \xi_{12\delta}^2), \quad \xi \in [-\xi_{12\delta}, \xi_{12\delta}]. \quad (95)$$

We now introduce a set

$$\mathbb{B} = \left\{ \eta \in C^{2,\gamma}[-\xi_{12\delta}, \xi_{12\delta}] \mid \eta(\xi_{12\delta}) = \eta_{12\delta}, \right. \\ \left. \eta'(\xi_{12\delta}) = \delta, 0 \leq \eta'' \leq \frac{1}{4}p_2^{-1/2}, [\eta'']_\gamma \leq 1, \eta(\xi) = \eta(-\xi) \right\}, \quad (96)$$

where $\gamma \in (0, 1)$, $\delta \in (0, \delta_0]$, and $[\eta'']_\gamma$ denotes the γ -Hölder modulus of continuity. The set is closed and convex in $C^{2,\gamma}[-\xi_{12\delta}, \xi_{12\delta}]$.

For any $\eta \in \mathbb{B}$, we find \bar{p} on it by (64). We estimate the minimum of p , called p_m^δ . We differentiate (64) along the boundary to find

$$\frac{d\bar{p}}{d_\Sigma \xi} = \frac{2\sigma'}{(1 + \sigma^2)^2}(\xi\sigma - \eta)(\xi + \sigma\eta). \quad (97)$$

We show that this derivative is nonnegative for $\xi \in [0, \xi_{12\delta}]$. First $\sigma' \geq 0$ is given. Second $\xi\sigma - \eta \geq 0$ trivially, assuming $\eta < 0$. Third we estimate that $\sigma \leq p_2^{-1/2}\xi$ and

$$|\eta| \leq \delta_0 \xi_{12\delta} + |\eta_{12\delta}| \leq \delta_0 \sqrt{p_2} + |\eta_{12}| \leq p_2^{1/2}$$

by the choice of δ_0 , thus $\xi + \sigma\eta \geq 0$. So the derivative is nonnegative. And the derivative is bounded from above by $C\sigma'$ from which we integrate (97) in ξ to find that p_m^δ deviates from p_2 at most by $C\delta$. In terms of the previous sections, we have $p_m = p_m^\delta = p_2 - O(\delta)$. This is a crucial estimate as it is the starting point for a mapping to be defined to take \mathbb{B} to itself. Thus, there exists a $\delta^* \in (0, \delta_0]$, such that $p_m - p_1 > C\delta_0$ for all $\delta \in (0, \delta^*)$.

We show that $p - \xi^2 - \eta^2 > 0$ between A_δ and B_δ . At B_δ it is zero. We show that its derivative is nonpositive for $\xi \in (0, \xi_{12\delta})$. The derivative along the shock is

$$\frac{d}{d_\Sigma \xi}(p - \xi^2 - \eta^2) = \frac{4\eta''}{(1 + (\eta')^2)^2}(\xi\eta' - \eta)(\xi + \eta\eta') - 2(\xi + \eta\eta'). \quad (98)$$

Its nonpositivity is equivalent to

$$2\eta''(\xi\eta' - \eta) < (1 + (\eta')^2)^2.$$

We use the bound $\eta'' \leq \frac{1}{4}p_2^{-1/2}$, $\delta \leq \delta_0$ to derive

$$2\eta''(\xi\eta' - \eta) < 2 \cdot \frac{1}{4} \frac{1}{\sqrt{p_2}}(\sqrt{p_2}\delta + \sqrt{p_2}) < \frac{1}{2}(\delta + 1) < \frac{1}{2}(\delta_0 + 1) < 1.$$

Thus the nonpositivity holds and $p - \xi^2 - \eta^2 > 0$ between A_δ and B_δ .

And the choice of the upper bound of the second-order derivative of η is such that $\xi\eta + \eta'(\bar{p} - \xi^2) < 0$ for $\xi > 0$, see (69)(70), thus the obliqueness holds.

For completeness we mention that $\xi^2 + \eta^2 - \bar{p} > 0$ for $\xi \in (0, \xi_{12\delta})$ because the second-order derivative of η is less than β_0 defined in (61), see (66), (67), (69).

We impose the condition that the solution p takes on the old value at the point G : $p_m = p_m^\delta$, which is part of the condition for the emergent type of tangential oblique derivative boundary condition. By the previous sections, there exists a $\delta^{**} > 0$ such that a solution p exists for all $\delta \in (0, \delta^{**})$. The solution is smooth every where in the closure of the domain including A and B since the higher order compatibility condition is satisfied. The oscillation $\max_\Omega |p - p_2| \leq p_2 - p_m^\delta$ is bounded by $K_1\delta$ where K_1 is determined by the upper bound of the $C^{2,\gamma}$ norm of $\eta \in \mathbb{B}$.

Now we define a mapping on \mathbb{B} : The mapping J . Given a boundary $\Sigma^{(0)}$ from \mathbb{B} , we let $p^{(1)}$ denote the unique solution. Restricted to the old $\Sigma^{(0)}$, we can regard $p^{(1)}$ as a function of the single variable ξ . We use $p^{(1)}(\xi)$ to define $\Sigma^{(1)}$ in the standard formula:

$$\frac{d\eta^{(1)}}{d\xi} = \frac{\xi\eta^{(1)} \pm \sqrt{\bar{p}^{(1)}(\xi^2 + (\eta^{(1)})^2 - \bar{p}^{(1)})}}{\xi^2 - \bar{p}^{(1)}}; \quad \eta^{(1)}(\xi_{12\delta}) = \eta_{12\delta}, \quad (99)$$

up to a point $\xi_9 \in (0, \xi_{12\delta})$ where either

$$\eta^{(1)} = -\sqrt{\bar{p}^{(1)} - 4K_3\delta\xi^2}, \quad (100)$$

or

$$\eta^{(1)} = -\sqrt{\bar{p}^{(1)}(0)}. \quad (101)$$

The constant K_3 is explained below.

First, the tangential oblique derivative boundary condition (72) implies

$$|p_\xi^{(1)} + \eta^{(0)'} p_\eta^{(1)}| \left| \frac{[p]}{4\bar{p}}(\xi^2 + \eta^2) - (\xi^2 + \eta^2 - \bar{p}) \right| \leq |\xi\eta + \eta'(\bar{p} - \xi^2)| K_2 |\nabla p^{(1)}|. \quad (102)$$

Recalling (67), we can move the term

$$\xi^2 + \eta^2 - \bar{p} = O(|\xi\eta + \eta'(\bar{p} - \xi^2)|)$$

to the other side to obtain

$$|p_\xi^{(1)} + \eta^{(0)'} p_\eta^{(1)}| \frac{[p]}{4\bar{p}}(\xi^2 + \eta^2) \leq |\xi\eta + \eta'(\bar{p} - \xi^2)| K_2' |\nabla p^{(1)}|. \quad (103)$$

We have

$$|\xi\eta + \eta'(\bar{p} - \xi^2)| \leq |\xi| \left(\sqrt{p_2} + \frac{1}{4\sqrt{p_2}}(\bar{p} + p_2) \right) \leq K_2|\xi|.$$

Thus

$$|p_\xi^{(1)} + \eta^{(0)'} p_\eta^{(1)}| \leq K_2' |\xi| |\nabla p^{(1)}| \tag{104}$$

for all $\xi \in [0, \xi_{12\delta}]$ and all $\delta \in (0, \delta^{**})$. Now, our solution $p^{(1)}$ is a smooth solution satisfying

$$|\nabla p^{(1)}| \leq K_2(p_2 - p_m^\delta), \tag{105}$$

where K_2 is independent of δ , but it may depend on the parameter ε_e , the size of the domain Ω , and the C^2 norm of $\Sigma^{(0)}$ which is bounded. So,

$$|p_\xi^{(1)} + \eta^{(0)'} p_\eta^{(1)}| \leq K_3 \xi \delta \tag{106}$$

for all $\xi \in [0, \xi_{12\delta}]$ and all $\delta \in (0, \delta^{**})$.

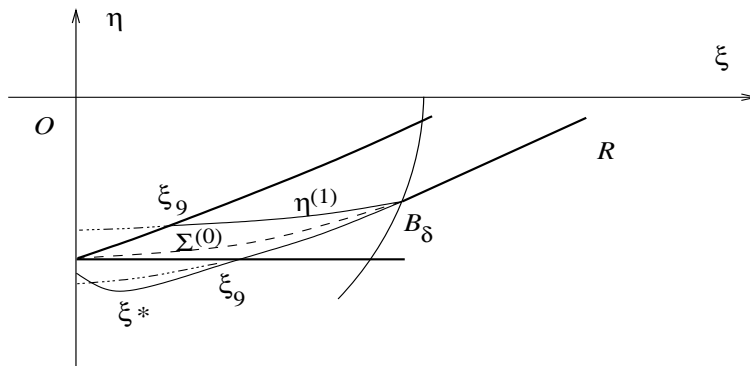


Figure 5. Definition of the mapping J.

Thus in the definition of the mapping, if the curve $\eta^{(1)}$ hits the upper boundary (100), then it is still a distance away from the singularity boundary $\eta^{(1)} = -\sqrt{\bar{p}^{(1)} - \xi^2}$, assuming $4K_3\delta < 1$. So we can and will continue the curve from the upper boundary (100) to $\xi = 0$, $C^{3,\gamma}$ -smoothly, keeping convexity, $(\eta^{(1)})'(0) = 0$, and with second-order derivative bounded by β_0 . This can be done, we omit the details. See the upper thin curve in Figure 5.

Second, if the curve goes below the line $\eta = \bar{p}^{(1)}(0)$ at a point ξ_9 , see the lower thin curve in Figure 5, then from the definition formula expressed as

$$\frac{d\eta^{(1)}}{d\xi} = \frac{\bar{p}^{(1)} - (\eta^{(1)})^2}{\sqrt{\bar{p}^{(1)}(\xi^2 + (\eta^{(1)})^2 - \bar{p}^{(1)}) + \xi(-\eta^{(1)})}} > 0, \quad \text{for } \xi > \xi_9, \tag{107}$$

we find that there will be a point $\xi^* \in (0, \xi_9)$ such that the curve $\eta^{(1)}$ has zero derivative at that point. We find that the second-order derivative formula

$$\frac{d^2\eta^{(1)}}{d\xi^2} = \frac{1}{\bar{p}^{(1)} - \xi^2} \left(\frac{2\bar{p}^{(1)} - (\xi^2 + (\eta^{(1)})^2)}{\sqrt{\bar{p}^{(1)}(\xi^2 + (\eta^{(1)})^2 - \bar{p}^{(1)})}} - \frac{d\eta^{(1)}}{d\xi} \right) \frac{d\bar{p}^{(1)}}{d_{\Sigma^{(0)}}\xi} \tag{108}$$

and the estimate (106) yield a uniform estimate

$$0 \leq \frac{d^2\eta^{(1)}}{d\xi^2} \leq K_5\delta, \quad \text{for all } \xi \geq \xi^* \tag{109}$$

for all $\xi^* \geq 0$. We can then start at the point ξ_9 (not ξ^*) and modify the curve so that it remains convex with convexity less than or equal to β_0 , gets to $\xi = 0$ with ending slope zero.

From the construction it is clear that the mapping is defined on \mathbb{B} and takes \mathbb{B} to itself when δ is small enough. In addition, the solution $p^{(1)}$ is in $C^{2,\gamma}$, thus $\eta^{(1)}$ is $C^{3,\gamma}$. Hence the mapping J is pre-compact. We obviously have continuity of J , assuming that our modification at the upper and lower boundaries are smooth. So J is a continuous mapping from a closed and convex set \mathbb{B} of the Banach space $C^{2,\gamma}$ to itself whose image $J\mathbb{B}$ is pre-compact, thus, by Schauder Fixed Point Theorem (see [28], Corollary 11.2, p.280) quoted already earlier, we have a fixed point for J , which is the existence of the free boundary.

7.1 Free Boundary for the Wedge Problem

We now consider the wedge problem. Since the corners C, D, O are a fixed distance away from the free boundary, we can similarly obtain the existence of a free boundary for the wedge problem. The mild dependence of p_2 on $\tan\theta_w$ is trivially allowed. The main change is the handling of the value p_m . We build an extra mapping to handle it.

The value p_m is from an equation derived from the requirement that $v(0, 0) = 0$: i.e.,

$$v_1 - \frac{p_m - p_1}{\sqrt{p_{m1}}} = \int_{-\sqrt{p_{m1}}}^0 \frac{p_\eta(0, y)}{-y} dy. \tag{110}$$

We observe that the v_m formula

$$v_m = v_1 - \frac{p_m - p_1}{\sqrt{p_{m1}}} \tag{111}$$

is monotone with respect to p_m and vanishes at $p_m = p_2^*$. Recall that p_2^* is the limit of p_2 as $\theta_w \rightarrow \pi/2$. Now for a given free boundary $\eta \in \mathbb{B}$, we can find its foot location on the η -axis with coordinate η_m , velocity v_m , and a pressure p_m^δ , which are related by the Rankine-Hugoniot relation. Suppose we take p_m to be a value less than p_2^* , we build a solution p to the oblique derivative boundary value problem with $p(G) = p_m$, we calculate the integral

$$\int_{-\sqrt{(p_m^\delta + p_1)/2}}^0 \frac{p_\eta(0, y)}{-y} dy =: vv, \tag{112}$$

then use this value to invert

$$v_1 - \frac{p_m - p_1}{\sqrt{p_{m1}}} = vv \tag{113}$$

for a new p_m , thus define a mapping from p_m to p_m , and it preserves the upper bound $p_m < p_2^*$. To find a lower bound, we observe that p_m^δ approaches p_2^* as $\delta \rightarrow 0$, and thus v_m approaches zero. Thus we can manage to have p_m in such a way that vv goes to zero slower than v_m as $\delta \rightarrow 0$, thus the mapping will map a p_* to a number greater than p_* . Thus we have a mapping that maps $[p_*, p_2^*]$ to itself. The continuity of the mapping follows from the fact that the solution p is C^{1,α_1} , and hence we have a fixed point, which is a solution to the equation for p_m .

Once we have a fixed point for the free boundary Σ , the new value of p on the shock is, on the one hand, p_m , but also the old value p_m^δ on the other hand. Thus they are equal, and the fixed point is a solution to the free boundary problem and the velocity v is zero at the origin.

8 De-regularization $-\epsilon_e$ and Lipschitz Continuity

We obtain uniform estimates with respect to $\epsilon_e > 0$. We use barriers to find the bound of the gradient of p on the sonic boundary. Then we use maximum principle to establish global bound of the gradient of p on the entire closure of the subsonic domain. Thus the solution p is Lipschitz at the sonic arcs.

First, the maximum principle implies $p \leq p_2$. Second, the uniform ellipticity implies $p \geq \xi^2 + \eta^2$. Combining the two, we obtain that $0 \leq \partial_{\mathbf{n}} p \leq 2\sqrt{p_2}$ on the sonic arcs where \mathbf{n} is the unit exterior normal to the arcs, independent of $\epsilon_e \in (0, 1]$.

We let (r, θ) denote the polar coordinates of the (ξ, η) plane. In polar coordinates, the p equation can be written as

$$(p - r^2)p_{rr} + \frac{p}{r^2}p_{\theta\theta} + \frac{p}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0. \quad (114)$$

We omit the term $\epsilon_e \Delta p$ since it does not present any help or trouble. Multiplying with $\frac{r^2}{p}$, we obtain

$$r^2 \left(1 - \frac{r^2}{p}\right) p_{rr} + p_{\theta\theta} + rp_r + \frac{r^2}{p^2} (rp_r)^2 - \frac{2r^3}{p} p_r = 0. \quad (115)$$

Taking the derivative ∂_r , letting $Z := p_r$, we obtain

$$r^2 \left(1 - \frac{r^2}{p}\right) Z_{rr} + Z_{\theta\theta} + rZ_r + \frac{2r^4}{p^2} p_r Z_r - \frac{2r^3}{p} Z_r + \partial_r \left(r^2 - \frac{r^4}{p}\right) Z_r - ZF(Z, r, p) = 0, \quad (116)$$

where

$$F(Z, r, p) := \frac{6r^2}{p} - 1 - \frac{6r^3}{p^2} Z + \frac{2r^4}{p^3} Z^2. \quad (117)$$

Near $p = r^2 = p_2$, the function F is bounded from below by $1/2$ for all $Z \in \mathbb{R}$. We select a thin shell domain immediately inside the sonic arc. We choose δ small, so all the derivatives are small depending on δ and ϵ_e . We then apply the maximum/minimum principle on Z . Thus $|Z| < 2\sqrt{p_2}$ in a small neighborhood of the sonic arc.

In the interior of the domain Ω , the limit $\epsilon_e \rightarrow 0+$ does not cause trouble since there holds uniform ellipticity.

9 Recovering the Velocity

Now that we have obtained p in the entire subsonic domain, we can integrate the first two linear equations in (23) for u and v to obtain (u, v) . More precisely, we integrate

$$ru_r = \xi u_\xi + \eta u_\eta = p_\xi, \quad u = u_2 \text{ at } r = \sqrt{p_2} \quad (118)$$

in the subsonic sector spanned by radial rays of the sonic arc to find

$$u(r \cos \alpha, r \sin \alpha) = \int_{\sqrt{p_2}}^r p_\xi(\tau \cos \alpha, \tau \sin \alpha) / \tau \, d\tau + u_2 \quad (119)$$

for $0 < r < \sqrt{p_2}$ and α stretching between θ_w and θ_A (the polar angle of the point A , see Figure 2). For α between θ_A and π , we use in (118) as data the velocity obtained by (30)(31) on the free boundary shock to obtain u in the subsonic sector spanned by radial rays of the free boundary Σ . We evaluate v similarly. Thus (u, v) are defined, and the first two equations in (23) are satisfied in the subsonic domain.

We verify the third equation in (23). As previously discussed, the second-order equation written in the form $r\partial_r L + L = 0$ implies the third equation if the third equation holds at a single point on each ray. Thus it suffices to show that $L = 0$ (defined in (57)) on the inside edge of the sonic arc and the free boundary Σ . To achieve that for the sonic arc we first note that our p is Lipschitz up to the boundary from the previous section. Our (u, v) are also Lipschitz at the boundary. Note further that (u, v) are constant on the arc, thus their derivatives along the sonic arc are zero:

$$-\eta u_\xi + \xi u_\eta = 0, \quad -\eta v_\xi + \xi v_\eta = 0. \quad (120)$$

Using these relations in the equation

$$\xi u_\xi + \eta u_\eta = p_\xi, \quad \xi v_\xi + \eta v_\eta = p_\eta \quad (121)$$

we obtain

$$(\xi^2 + \eta^2)(u_\xi + v_\eta) = \xi p_\xi + \eta p_\eta. \quad (122)$$

Recall that $\xi^2 + \eta^2 = p$ on the sonic arc, we obtain

$$L := (\xi p_\xi + \eta p_\eta)/p - (u_\xi + v_\eta) = 0 \quad (123)$$

on the sonic arc! Thus the third equation of (23) holds in the entire sector spanned by the sonic arc.

The oblique derivative boundary condition ensures, as previously mentioned, that $L = 0$ on Σ . Recall that we derived the oblique derivative boundary condition by assuming that $L = 0$ on Σ . Now we want to reverse the process; i.e., we want to derive that $L = 0$ on Σ from p and the oblique derivative boundary condition. From the three facts that (i) p is known and the free boundary satisfies equation (29), (ii) the values of (u, v) on the inside edge are obtained from (30), (31), (iii) the first two equations of (23) hold on the edge, we can differentiate (30), (31) along (29) to solve five variables $(u_\xi, u_\eta, v_\xi, v_\eta, p_\xi)$ from the five equations in terms of the sixth p_η , using the condition $\xi^2 + \eta^2 - \bar{p} \neq 0$ for $\eta > 0$. It can be verified that the five variables depend on the sixth linearly, and the relation $L = 0$ holds identically on the inside edge of Σ , where we use the fact that (p_1, u_1, v_1) is a constant state. Then the second-order equation $r\partial_r L + L = 0$ implies $L = 0$ in the interior of the domain. Thus the solution p of the second-order equation is a solution to the system of first order equations in the sectoral domain spanned by Σ .

Furthermore, the velocity (u, v) are smooth in the interior of the subsonic domain. On the wall and the ground we show that the velocity is parallel to the solid surface. Our p satisfies $\partial_{\mathbf{n}} p = 0$. Our (u, v) from (121) satisfies the equation

$$\xi(u_{\mathbf{n}})_\xi + \eta(u_{\mathbf{n}})_\eta = \partial_{\mathbf{n}} p$$

where $u_{\mathbf{n}} := (u, v) \cdot \mathbf{n}$. Thus, $u_{\mathbf{n}} = \text{constant}$. The constant is zero in the state (2) and at the tip of the bow shock. So $u_{\mathbf{n}} = 0$. Thus, the velocity is parallel to the solid surface. Finally, the velocity is bounded at the origin O since ∇p is Hölder continuous and vanishes at O .

10 Fine Properties of the Velocity

One might ask if there is a stagnation point of the flow in the subsonic domain, or at the corner in particular. One might also ask if the velocity remains uniformly small to fulfill the condition posteriorly, under which the model is derived. Furthermore, one might ask what exactly determines the position of the free boundary shock, if the pressure equation decouples from the velocity field.

We first observe that a key condition to locate the free boundary shock is the condition $u(0, 0) = 0$ (in the vertical set-up of Figure 1), although this condition can be expressed entirely

in terms of the pressure p_m . To answer the other two questions, we find it is necessary to study qualitatively the velocity field of the solution in the subsonic domain.

Proposition 10.1 (State 2). *As $\cos \theta_w \rightarrow 0+$, we have*

$$p_2 = p_2^* + \frac{p_1^2(p_1 - p_0)^2}{2\bar{p}_{10}^3 \tan^2 \theta_w} + O(\tan^{-4} \theta_w), \tag{124}$$

where p_2^* is defined in (36).

Proof. This is an easy consequence from the explicit formula in Proposition 3.1.

Proposition 10.2 (State 2 on Σ). *The velocity on the free boundary Σ is nonnegative*

$$u \geq 0, \quad v \geq 0 \quad \text{on } \Sigma \tag{125}$$

for large wedge angles.

Proof. In fact, we find from (31) that

$$\frac{[v]}{[p]} = \frac{\bar{p} - \xi^2}{\eta\bar{p} - \xi\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}} > 0$$

on the shock, thus $v > 0$ on the inside of the shock. We present the calculation for u , at the point G only for brevity. At the point G , we use the R-H relation (30) to find

$$u_{2G} = u_1 - \frac{p_{2G} - p_1}{\sqrt{(p_{2G} + p_1)/2}}. \tag{126}$$

In (126), we see quickly that u_{2G} is a decreasing function of p_{2G} , holding p_1 and u_1 fixed and unrelated to p_{2G} . In addition, if we use $p_{2G} = p_2^*$ from (36), we find $u_{2G} = 0$. So, to show $u_{2G} > 0$, it suffices to show that $p_{2G} < p_2^*$ for all large wedge angles.

From Proposition 10.1, we have $p_2 = p_2^* + O(\cos^2 \theta_w)$. On the other hand, we show that p_{2G} will be smaller than p_2 by an amount $C_2\delta$ at the least, which is $C_2 \cot \theta_w$, $C_2 \neq 0$. This follows from the oblique derivative boundary condition written in the form (in the swapped coordinate where the free boundary shock is basically horizontal)

$$\frac{\eta''(\bar{p} - \xi^2) \left\{ \frac{[p]}{\bar{p}} (\xi^2 + \eta^2) - 4(\xi^2 + \eta^2 - \bar{p}) \right\}}{2\bar{p} - \xi^2 - \eta^2 + 2\eta' \{ \xi\eta + \eta'(\bar{p} - \xi^2) \}} = -\eta' p_\xi^{(1)} + p_\eta^{(1)} + (\xi p_\xi^{(1)} + \eta p_\eta^{(1)}) \frac{\xi\eta' - \eta}{p}. \tag{127}$$

We have $0 \leq p_\eta \leq C_2\delta$ since $p_2 - p_{2G} \leq C_2\delta$. So $\eta''(\xi) \leq C_2\delta$. From formula (97) we have

$$\frac{d\bar{p}}{d_\Sigma \xi} \geq C_2 \xi \sigma' \tag{128}$$

which implies

$$\bar{p}(B) - \bar{p}(G) \geq C_2 \int_0^{\xi_B} \xi \sigma' d\xi = C_2 \left(\xi_B \delta - \int_0^{\xi_B} \sigma d\xi \right). \tag{129}$$

But we have

$$\begin{aligned} \int_0^{\xi_B} \sigma d\xi &\leq \int_0^{1/C_2} \xi \sigma'(\xi) d\xi + \int_{1/C_2}^{\xi_B} \delta d\xi \\ &\leq \frac{1}{2} \left(\frac{1}{C_2} \right)^2 (C_2\delta) + \left(\xi_B - \frac{1}{C_2} \right) \delta = \xi_B \delta - \frac{1}{2C_2} \delta. \end{aligned} \tag{130}$$

Thus $p_{2G} = p_2 - C_2 \cos \theta_w < p_2^*$. Thus $u_{2G} > 0$. This completes the proof.

As fluid particles flow toward the wedge wall, the build-up of fluid raises the pressure in the subsonic region. In the extreme case of a vertical wall, there is no escape for the fluid, thus the velocity $(u, v) = (0, 0)$ and the pressure is $p = p_2^*$ in the subsonic region. For a wall with $\theta_w \in (0, \pi/2)$, fluid particles can slide along the wedge surface: The fluid particles get deflected at the corner. Both the zero pressure gradient on the wedge surface and the zero head-on velocity at the corner are used to send a signal to the decoupled second-order pressure gradient equation to take care of the build-up of fluid at the corner.

To explain the changes of velocity in the subsonic domain in response to the pressure gradient, we find that our (u, v) from (121) satisfies the equation

$$\xi(u_\alpha)_\xi + \eta(u_\alpha)_\eta = \partial_\alpha p$$

where $u_\alpha := u \cos \alpha + v \sin \alpha$ and $\partial_\alpha p = p_\xi \cos \alpha + p_\eta \sin \alpha$. Thus, for example u_α for $\alpha = \pi/4$ is decreasing from the shock wave toward the origin, provided that the pressure is increasing in the direction $(1, 1)$, see Figure 6. Overall, the velocity remains small, which answers our second question proposed earlier.

We need to find more information to answer the question of stagnation points.

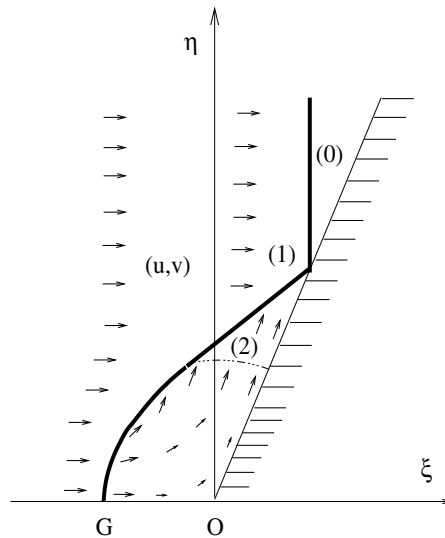


Figure 6. Flow pattern of the regular reflection

10.1 Vorticity

We study the vorticity in the subsonic region to reveal more refined properties of the velocity. As is usual, the vorticity is defined by $\omega = \partial_x v - \partial_y u$. We can show that the vorticity is zero at time $t = 0$ for our configuration. Furthermore, it is easy to check that

$$\partial_t \omega = 0, \quad t > 0$$

for smooth solutions of the pressure gradient system. So we expect that our solution has zero vorticity.

We show that our solution indeed has zero vorticity. Although it might seem unnecessary to do the proof, we feel that our construction of the solution has been a guess work, and piecewise,

so it is a good idea to verify the end result. In addition, curved oblique shock waves in a gas are known to generate vorticity, see e.g. the appendix of [33], or p.431 of [51]. Thus, first, from (120), (121) we obtain

$$v_\xi = \frac{\xi}{p} p_\eta, \quad u_\eta = \frac{\eta}{p} p_\xi \quad (131)$$

on the sonic arc, and thus

$$v_\xi - u_\eta = \frac{\xi p_\eta - \eta p_\xi}{p} = 0 \quad (132)$$

on the sonic arc, since $p = p_2$ is a constant there. Thus, there is no vorticity along the sonic arc in the subsonic region.

In the self-similar plane we denote for convenience $\omega = v_\xi - u_\eta$ although the physical quantity $t(\partial_x v - \partial_y u)$ scales to be $v_\xi - u_\eta$. We use equations (23) to easily find that ω satisfies the equation

$$-\xi\omega_\xi - \eta\omega_\eta + \omega = 0 \quad (133)$$

in any region where the solution is smooth. Using equation (133) and boundary data (132), we obtain that vorticity is zero in the subsonic domain spanned by the sonic arc.

We show then that the vorticity is zero on the free boundary shock. From the original form of Rankine-Hugoniot relation we can obtain that

$$[u] = -\frac{\sigma}{\eta - \xi\sigma}[p], \quad [v] = \frac{1}{\eta - \xi\sigma}[p]. \quad (134)$$

We differentiate the $[u]$ equation along the shock to find

$$\frac{d[u]}{d_\Sigma \xi} = \frac{-\eta\sigma'}{(\eta - \xi\sigma)^2}[p] - \frac{\sigma}{\eta - \xi\sigma} \frac{dp}{d_\Sigma \xi}. \quad (135)$$

Using the fact that u_1 is a constant and the first equation in (23) we obtain

$$(\xi\sigma - \eta)u_\eta = -p_\xi - \frac{\xi\eta\sigma'}{(\eta - \xi\sigma)^2}[p] - \frac{\xi\sigma}{\eta - \xi\sigma} \frac{dp}{d_\Sigma \xi}. \quad (136)$$

Similarly we obtain

$$(\eta - \xi\sigma)v_\xi = \frac{\xi\eta\sigma'}{(\eta - \xi\sigma)^2}[p] + \frac{\eta}{\eta - \xi\sigma} \frac{dp}{d_\Sigma \xi} - \sigma p_\eta. \quad (137)$$

Combining the two terms we obtain $v_\xi - u_\eta = 0$ on the shock. Thus vorticity is zero in the subsonic domain spanned by the free boundary. So the flow is irrotational in the entire subsonic domain.

We can now use $v_\xi = u_\eta$ to manipulate the first two equations of (23) into

$$u = (\xi u + \eta v - p)_\xi, \quad v = (\xi u + \eta v - p)_\eta. \quad (138)$$

We call

$$\psi := \xi u + \eta v - p \quad (139)$$

the potential. Thus we have a nice formula

$$p + \psi = \xi u + \eta v. \quad (140)$$

So we have

$$\Delta\psi = u_\xi + v_\eta = \frac{\xi p_\xi + \eta p_\eta}{p} \quad (141)$$

by the third equation of (23). Replacing the derivatives p_ξ and p_η from the first two equations of (23), we obtain

$$p\Delta\psi - (\xi^2\psi_{\xi\xi} + 2\xi\eta\psi_{\xi\eta} + \eta^2\psi_{\eta\eta}) = 0. \tag{142}$$

We can replace p to obtain a decoupled equation

$$(\xi\psi_\xi + \eta\psi_\eta - \psi)\Delta\psi - (\xi^2\psi_{\xi\xi} + 2\xi\eta\psi_{\xi\eta} + \eta^2\psi_{\eta\eta}) = 0. \tag{143}$$

The boundary data for ψ are the Dirichlet on the free boundary and the sonic arc, and Neumann $\nabla\psi \cdot \mathbf{n} = 0$ on the ground and the surface of the wall. Since we have obtained p already, we choose to use equation (142). We are mainly interested in regularity of the solution. First we find that *the only stagnation point is at the origin*. Second, the solution ψ is in $C^\alpha(\bar{\Omega})$ where $\alpha = \pi/(\pi - \theta_w) \in (1, 2)$, and the solution ψ is unique, see Grisvard^[31]. Further, we have the asymptotic formula

$$\psi = r^\alpha \cos(\alpha(\theta - \theta_w)) + o(r^\alpha) \tag{144}$$

in polar coordinates (r, θ) , at the origin. This formula is useful for explaining the existence of a stationary point in the pseudo-velocity $(u - \xi, v - \eta)$ of the full Euler system on the wall (but not the corner), see [74].

Thus we conclude that the velocity is uniformly small in the region, the only stagnation point of the pseudo-velocity is at the origin, and the pseudo-flow field is vorticity free.

11 Further Discussions

It remains open to calculate the threshold at which regular reflection gives way to Mach reflection in the whole (not just locally at the Mach stem). Further in depth questions include solutions to the Riemann problems and the Euler system.

Appendix A Linear Theory

A1 Local Existence

We provide details on the issue of local existence. Recall that B_2 is a neighborhood of G with smooth boundary and h is a continuous function on $\partial B_2 \cap \Omega$. Consider the problem

$$p = h \quad \text{on } \partial B_2 \cap \Omega \tag{145}$$

for equation (75) restricted to the domain $B_2 \cap \Omega$ with the tangential oblique derivative boundary condition (72) restricted to $B_2 \cap \Sigma$. This is a mixed-type boundary value problem on which we do not find any clear literature. So we give an existence proof here. We chop off the tip G ; i.e., we replace Ω by Ω_δ which is δ -distance shorter than Ω from the point G upward. See Figure A1. On the bottom straight boundary of Ω_δ , we impose the Dirichlet boundary condition

$$p = p_m \quad \text{on bottom of } \Omega_\delta.$$

Now by Lieberman^[47], there exists a solution p_δ in $C(\overline{\Omega_\delta} \cap \overline{B_2}) \cap C^{2,\alpha}(\Omega_\delta \cap B_2)$. The maximum principle holds for p_δ , thus there is a subsequence of p_δ , which converges locally in $C^2(\Omega \cap B_2)$ to a solution in $C^{2,\alpha}(\Omega \cap B_2)$ as $\delta \rightarrow 0+$.

We use a barrier function at G to obtain continuity of p at G . We note that our \mathbf{l} is not degenerate ($|\mathbf{l}| \neq 0$) and well-defined for $(\xi, \eta) \in B_2$ provided that B_2 is sufficiently small and

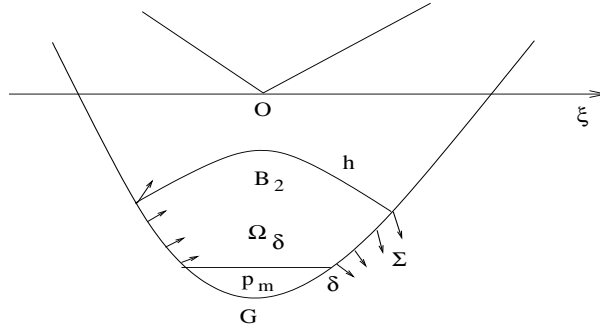


Figure A1. Domain with tip G removed.

the variable \bar{p} is regarded as given by formula (64) where $\eta'(\xi)$ is given, but η can be free (well, we choose $\eta = \eta(\xi)$ to be given). So we can introduce integral curves in B_2 by

$$\left(\frac{d\xi}{ds}, \frac{d\eta}{ds}\right) = \mathbf{l}(\xi, \eta(\xi), \eta'(\xi)).$$

The vector field enjoys $C^{1,\gamma}(B_2)$ regularity, so the integral curves enjoy $C^{2,\gamma}$ regularity. We now perform a coordinate transformation to straighten the vector field so that the boundary condition becomes

$$\frac{\partial p}{\partial \xi} = 0 \tag{147}$$

and the equation becomes

$$a^{ij}D_{ij}p + b^iD_i p = 0, \tag{148}$$

where the usual summation convention is used. We omit introducing new notation for p for the new coordinate system. We consider the auxiliary function

$$v = p_m + c(1 - e^{-N(\eta - \bar{\eta}_{m1})}), \tag{149}$$

where $c > 0$ and $N > 0$ are to be chosen large. This v satisfies the oblique boundary condition and is greater than p_m on each of the bottom boundary of Ω_δ , $\delta > 0$. For the equation, we have

$$a^{ij}D_{ij}v + b^iD_iv = -ce^{-N(\eta - \bar{\eta}_{m1})}(N^2a^{11} - Nb^1),$$

which can be made less than a negative constant by choosing $N > |b^1|_0/\lambda$ where $\lambda \leq a^{11}(\xi, \eta)$. Now we can choose c so that v is greater than $\sup |h|$. Thus all our solutions p_δ are bounded from above by the super solution v and below by the constant p_m (we consider the case $h \geq p_m$ only). So p is continuous at the point G . The continuity of p at other points follows from Lieberman's aforementioned work.

We establish the regularity $C^{2,\alpha'}(\bar{\Omega} \cap B_2)$ for p for $\alpha' = \min(\alpha, \gamma)$. Take a domain $B_3 \subset \subset B_2$ so that ∂B_3 intersects Σ with infinite order of contact. Extend the vector field \mathbf{l} from the (only two) contact points to $\partial B_3 \cap B_2$ C^∞ -smoothly so that there is only one point on the boundary $\partial B_3 \cap B_2$ that is tangentially degenerate, thus this portion is rather like the Σ . Propose the corresponding tangential oblique derivative boundary value problem on $B_3 \cap \Omega$: The equation is the same, the boundary condition on Σ is the same as before, but the (nonhomogeneous) tangential oblique derivative boundary condition on $\partial B_3 \cap B_2$ is the newly-invented one. We know that this problem has a solution p that is continuous on the closure of $B_3 \cap \Omega$ by design. But this problem has maximum principle, so any $C^{2,\alpha}$ solution (keep the nonhomogeneous data

unchanged) will coincide with the solution p . The homogeneous problem will have only the zero solution, thus the null space has dimension zero. Hence, there is no compatibility condition for this problem, since this problem is Fredholm (p. 158, [24]). Thus smooth data imply existence of smooth solutions by Theorem 5.3 of Egorov and Kondrat'ev^[24]; I.e., there exists a solution p in the Sobolev space $W^{1,2}(B_3 \cap \Omega)$. By the smoothness theorem, Theorem 5.2, of the same paper [24], the solution is as regular as the data, which we can make very smooth to start with the curve $\eta(\xi)$ and the linearization Q . For smooth solutions, we then use Guan and Sawyer^[32] to obtain uniform estimates in the space $C^{2,\gamma}(B_2 \cap \overline{\Omega})$ in terms of $C^{2,\gamma}$ regularity of $\eta(\xi)$ and Q .

A2 Corner O

The angle $2\theta_w$ will be close to π , but strictly less than π . Thus the interior angle is more than π , so Lieberman^[48] does not apply. We extract an example from Grisvard^[31].

Example Consider the Laplace equation

$$\Delta p = 0$$

in a sector of angle $\omega \in (0, 2\pi)$ in \mathbb{R}^2 , in the polar coordinate (r, ϕ) . See Figure A2. Consider the function

$$p_1 = r^{\pi/\omega} \cos(\phi\pi/\omega).$$

It is harmonic in the sector and satisfies $\partial_\phi p = 0$ on the edges of the sector (see [31]). If $\omega \in (\pi, 2\pi)$, this function is only Hölder continuous at the origin. This example illustrates the requirement $0 < \phi \leq \pi$ in Theorem 1 of [48] in order for p to be in $C^1(\overline{\Omega}) \cup C^2(\Omega)$.

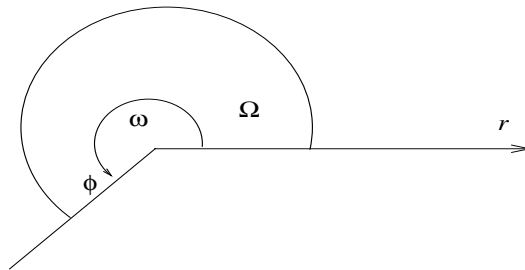


Figure A2. Laplace at a big corner.

So we use the symmetry of the problem with respect to ξ to cut the large angle in half. Then the solution at the corner has Hölder continuous gradient: $p \in C^{1,\theta_w/(\pi-\theta_w)}$.

Appendix B Monotonicity

We establish in details the fact that p (i.e., $p^{(1)}$) is monotone along the shock boundary Σ from the point G to B , See Figure 2. The procedure is similar to that of [12].

Along the boundary Σ , the pressure p is a function of a single variable, say ξ . Assuming that p is not monotone, we can then find two points c and d on Σ such that p has a maximum at c over the interval of Σ from G to d , p has a minimum at d over the interval of Σ from c to B , and $p(c) > p(d)$.

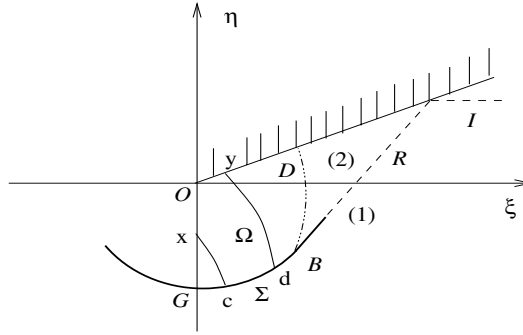


Figure B1. Hypothetical curves cx and dy .

Now we consider p in the domain Ω , formed by the boundary G, B, D, O, G , see Figure B1. We will partition Ω into three subdomains with two curves Γ_c and Γ_d from c and d respectively to points x and y respectively on the η -axis or the wall, in such a way that $p(x) > p(y)$ and so we can conclude that there is a point z on the η -axis or the wall at which the pressure obtains a maximum on either the subdomains, and there is a point m on the η -axis or the wall at which the pressure obtains a minimum on either the subdomains, thus at least one of the pair (z, m) will violate Hopf maximum principle, even if one of them happens to be the origin O .

One wishes to find smooth curves on which p is monotone, but we know of no way to achieve that. An approximation is, however, possible. We construct Lipschitz curves on which p is monotone on the joints. Let

$$\mu := \frac{1}{4} \min \{p(c) - p(d), p(B) - p(c), p(d) - p(G)\}. \quad (150)$$

We plan to construct curves on which

$$\begin{aligned} p(x) &\geq p \geq p(c) - \mu, \quad \text{on } \Gamma_c, \quad \text{and } p(x) > p(c), \\ p(y) &\leq p \leq p(d) + \mu, \quad \text{on } \Gamma_d, \quad \text{and } p(y) < p(d). \end{aligned} \quad (151)$$

We let M be a bound of $|\nabla p|$ in Ω , which may depend on ε_e . Let $R = \mu/(2M)$. Then the oscillation of p in any ball of radius R inside Ω is bounded by μ . We can construct Γ_c now. Consider a ball $B_R(c)$. In $B_R(c) \cap \Omega$, $p(c)$ cannot be the maximum value of p (because c is not a point of local maximum in Ω), therefore there are points of $\partial B_R(c) \cap \Omega$ where $p > p(c)$. Let X_1 be a point at which p attains its maximum value in $\overline{B_R(c) \cap \Omega}$. The first segment of Γ_c is a straight line from c to X_1 . There holds $p(X_1) > p(c)$, and $p(X_1) > p \geq p(c) - \mu$ on the segment.

We use induction to form a sequence of line segments with corners at $\{X_i\}$ (where $c = X_0$), along which holds $p \geq p(c) - \mu$ and such that $p(X_1) < p(X_2) < \dots$. Let

$$\Omega_j = \Omega \setminus \overline{\{\cup_0^{j-1} B_R(X_i)\}}; \quad (152)$$

we have $X_j \in \partial\Omega_j$ and we consider $B_R(X_j)$. We note that $p(X_j)$ is the largest value of p on the part of $B_R(X_j)$ inside the complement of Ω_j . However, $p(X_j)$ is less than the maximum value of p on $B_R(X_j)$, by the maximum principle. Therefore, there is a point $X_{j+1} \in \partial B_R(X_j) \cap \Omega_j$ at which p attains its maximum value in $\overline{B_R(X_j) \cap \Omega_j}$. Also, along the straight line from X_j to X_{j+1} there holds $p \geq p(X_j) - \mu > p(c) - \mu$.

So we have $\text{dist}(X_{i-1}, \Omega_i) = R$, and $\Omega_j \subset \Omega_{j-1} \subset \dots \subset \Omega_1$, thus $\text{dist}(X_i, \Omega_k) \geq R$ for $k \geq i + 1$, and since $X_k \in \partial\Omega_k$, there holds the estimate

$$\text{dist}(X_{j+1}, X_i) \geq R, \quad \text{for all } i < j. \quad (153)$$

Hence the distance between any pair of points X_i, X_j in the sequence is greater or equal to R . The domain Ω is finite, so the process will end at a finite step when we reach a point $X_L = x \in \partial\Omega$. By construction, Γ_c has the property (151). Similarly we construct Γ_d with termination point $y \in \partial\Omega$.

We now locate x and y . We note that the two curves cannot cross each other because at every point on Γ_c , $p \geq p(c) - \mu > p(d) + \mu$, while at every point on Γ_d there holds $p < p(d) + \mu$. Further, Γ_d cannot terminate on the sonic arc, because of its lower value, and it cannot come back to Σ between the points c and B because it is a minimum to start with. So it has to end on the wall or the η -axis. Similarly, Γ_c cannot terminate on the sonic arc and must terminate as Γ_d does.

We drive further to find a contradiction. Since $p(x)$ is greater than $p(G)$ and $p(y)$, there is a z along the boundary $G - O - y$ at which p attains a maximum. Assume first that z is not the origin, then z cannot be a local maximum for the domain Ω by the Hopf Lemma. But along the entire boundary of the domain formed by G, c, d, y, G , the value of p are all less or equal to $p(z)$, so it is a maximum. This is a contradiction. Now if z coincides with O , then a similar minimum point m resembling y cannot coincide with O , and we can find that there is no place for such an m , either. Thus there is always a contradiction. So we conclude that p is monotone along Σ from G to B .

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