Two-Dimensional Stochastic Model for Interconnections in Master Slice Integrated Circuits

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***Abstract*—Two-dimensional stochastic models for interconnections in master slice LSI are described. Several limit theorems are derived for estimating the wiring area on large chips in terms of average wire length $\bar{R}$, average number of wires emanating from each logic block λ, and wire trajectory parameters. The expected value of the maximum number of tracks per channel on an $N \times N$ chip is shown to be less than $O(\ln N)$ as long as $\bar{R}$ does not grow faster than $O(\ln N)$. If $\bar{R} > O(\ln N)$, then the expected maximum number of tracks is $O(\bar{R})$. Simple bounds on the expected wiring area are given and numerical results compared to the earlier work by Heller et al.**

I. INTRODUCTION

THE RATE OF progress in integrated circuit technology indicates that it will soon be possible to implement on the order of several million devices on a single chip [1]. In contrast to implementing small or medium scale (MSI) integrated circuits, where a designer can quickly try different layout possibilities and choose the "best," the designer of a large system (LSI–VLSI) needs a way of predicting chip area and layout complexity without actually having to do complete layouts. These predictions are not only necessary for determining initial placements and layout, but, more importantly, for deciding whether a system can be implemented economically. In [2], M. Shima, the designer of the Z8000 microprocessor, emphasized this point by stating that, "If the designer (of the VLSI system) waits until the last minute to calculate chip size, his company may wind up with a chip too big to sell profitably," But how does one go about predicting the chip area? And what information must be available to the designer for such prediction? As far as we know no general answers to these questions exist. The problem of predicting the space requirement for wiring (PC boards, LSI) has, however, received some attention. Sutherland and Oestreicher [3] considered the case of wiring when the placement is done randomly. Their result, although useful for small to medium size integration, is too pessimistic for truly large systems.

In a recent paper, Heller *et al.* [4] gave a one-dimensional probabilistic model for wiring in master slice (gate array) LSI with the objective of predicting the wiring area. They heuristically extended their model to two dimensions, and applied their results to estimating the maximum number of tracks per channel needed for successful wiring. The results in [4] are currently being used by several design automation groups at IBM and elsewhere.

Briefly, the model in [4] considers a doubly infinite linear array of *equal size* logic blocks. The number of wires emanating from every block is taken to be randomly distributed according to a Poisson distribution with parameter λ, and is assumed to be independent among the different blocks. Each wire length is assumed to be independently chosen according to a geometric distribution with mean $\bar{R}$. The problem of estimating λ and $\bar{R}$ was studied in earlier work by Donath [5], [6] and in Feuer [7].

In this paper, we shall extend the work in [4] by formulating a two-dimensional probabilistic model for wiring. In addition to assuming knowledge of the distribution of the number of wires (which will also be taken Poisson) and the average wire length, we shall assume certain strategies for wire trajectories in two dimensions. It will be shown, however, that certain general results do not depend on the assumed wire trajectory distribution.

The organization of this paper is as follows: in Section II, we informally describe our model and state the main results. In Section III, a formal description of the model and several elementary results are derived. Limit theorems concerning the space requirements for wires are then given in Sections IV and V. In Section VI, similar results are stated for the case of "biased chips" (i.e., unequal average wire lengths in horizontal and vertical directions), and in Section VII, we consider the case when wires have at most one corner. In Section VIII, we assume that wire lengths have a geometric distribution and show that under this assumption a joint wiring distribution could be derived. Finally, several generalizations and other potential applications of our model are discussed in Section XI. Many of the important proofs in this paper are contained in [14].

II. INFORMAL DESCRIPTION OF MODEL AND RESULTS

The model of the chip is depicted in Fig. 1. It consists of a two-dimensional lattice; every lattice point represents the corner point of a logic block. Wires are assumed to start at lattice points, travel horizontally and vertically

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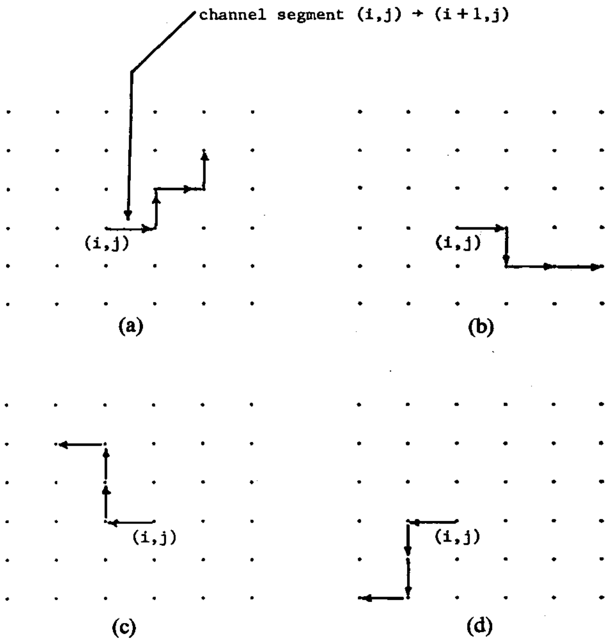

Fig. 1. The four minimum distance wire trajectories.

only (no diagonal wires), then stop at other lattice points (no loops). Each wire is assumed to take a minimum distance path in connecting two lattice points. This implies that any wire has to choose among the following four possible trajectory directions:

(a) a direction consisting of only (Right, Up) steps (in any order);

(b) a direction consisting of only (Right, Down) steps;

(c) a direction consisting of only (Left, Up) steps; or

(d) a direction consisting of only (Left, Down) steps.

These four directions are illustrated in Fig. 1.

We now describe the random generation of wires, wire lengths, and wire trajectories:

i) At every lattice point (i, j) the number of wires X_{ij} is independently drawn from a Poisson distribution with parameter λ.

ii) Given that the number of wires at (i, j) is $x_{ij}>0$, the length of every wire $L_{ij}(x)$, $1 \leqslant x \leqslant x_{ij}$, is drawn independently (of every other wire and of the X_{ij}'s), and according to a distribution P_L with mean $\bar{R}$. Observe that up to this point the generation of wire numbers X_{ij} and wire lengths is identical to that in [4].

iii) Now, to choose the trajectory of each wire, the wire first chooses one of the four directions (a)–(d) with equal probability and independently of i), ii). The specific trajectory is then decided by flipping a fair coin $L_{ij}(x)$ times and moving accordingly. For example, if a wire x of length $L_{ij}(x)=3$ starts at (i, j) and chooses direction (a), the coin is flipped three times. If the outcome of the coin flips is (HHT), the wire first moves to the right two steps then up one step, and stops at $(i+2, j+1)$.

It can be easily seen from the model assumptions that one can decompose the wiring process into four independent processes, each with X_{ij} Poisson distributed with parameter $\lambda/4$ (for every (i, j)), and such that the first process consists of wires in direction (a), the second consists of the wires in direction (b), and so on. Analysis of process (a) can then be carried out, and the results superposed (see Section III for details). It is also shown (Section VII) that the specific trajectory description given here is not necessary for deriving the limit theorems for wiring space. Rather, the coin flipping scheme is an intuitive way of describing wire trajectories (however, it is not the most realistic).

Before stating our main results, we introduce the following random variables:

1) For every (i, j) let T_{ij}^H be the number of wires in channel segment $(i, j)\to(i+1, j)$, and T_{ij}^V be the number of wires in channel segment $(i, j)\to(i, j+1)$.

2) Consider an $N \times N$ segment of the lattice, and define the random variable

$$S_N=\sum_{i=1}^{N}\sum_{j=1}^{N}\left(T_{ij}^H+T_{ij}^V\right)$$

to be the total number of wires segments on the $N \times N$ chip.

3) For prediction of wiring space consider the following random variables:

(i)
$$\hat{D}_N^V=\max_{1\leqslant i\leqslant N}\left\{\sum_{j=1}^{N}T_{ij}^H\right\}$$

$$\hat{D}_N^H=\max_{1\leqslant j\leqslant N}\left\{\sum_{i=1}^{N}T_{ij}^V\right\}$$

(ii)
$$\hat{T}_N=\max_{1\leqslant i,j\leqslant N}\max\left\{T_{ij}^H, T_{ij}^V\right\}$$

(iii)
$$\hat{D}_{iN}^V=\max_{1\leqslant j\leqslant N}\left\{T_{ij}^V\right\},\qquad \text{for } 1\leqslant i\leqslant N$$

$$\hat{D}_{jN}^H=\max_{1\leqslant i\leqslant N}\left\{T_{ij}^H\right\},\qquad \text{for } 1\leqslant j\leqslant N.$$

The random variables $\hat{D}_N^H$ and $\hat{D}_N^V$ represent, respectively, the maximum chip width and height needed (in terms of the total number of tracks). The random variable $\hat{T}_N$ represents the maximum number of tracks needed in any channel segment on the chip.

4) For every integer $0 \leqslant t \leqslant \infty$ define the random variable $K_N(t)$ to be the number of channel segments with t wires in an $N \times N$ chip.

We now summarize several key results in this paper.

(1) (Section IV). If the average wire length $\bar{R}<\infty$ (independent of N), then

$$\frac{S_N}{N^2}\xrightarrow{\mathcal{P}}\lambda\bar{R}. \tag{1}$$

(2) (Section IV). If $EL^{2+\epsilon}<\infty$, for some $\epsilon>0$, then S_N satisfies the following central limit theorem:

$$\frac{S_N-N^2\lambda\bar{R}}{\sqrt{\operatorname{Var}(S_N)}}\xrightarrow{\mathcal{D}}\mathcal{N}(0,1) \tag{2}$$

where $\mathcal{N}(0,1)$ is the normal distribution with mean 0 and standard deviation 1.

(3) (Section V). If $\bar{R}<\infty$ (independent of N), the for every t

$$\frac{K_N(t)}{2N^2}\xrightarrow{\mathcal{P}}\frac{\left(\frac{\lambda\bar{R}}{2}\right)^t}{t!}e^{-\lambda\bar{R}/2}\triangleq P\left(\frac{\lambda\bar{R}}{2};t\right). \tag{3}$$

(4) (Section V). If $\bar{R} \leqslant 0(\ln N)$, then

i) $P\{\hat{T}_N \leqslant 0(\ln N)\} \underset{N\to\infty}{\longrightarrow} 1$, and

ii) $E\hat{T}_N \leqslant 0(\ln N)$. (4)

If $\bar{R} > 0(\ln N)$, then

i) $P\{\hat{T}_N \leqslant 0(\bar{R})\} \underset{N\to\infty}{\longrightarrow} 1$, and

ii) $E\hat{T}_N = 0(\bar{R})$. (5)

(5) (Section V).

i) $P\{\max(\hat{D}_N^H, \hat{D}_N^V) \leqslant 0(N\bar{R})\} \underset{N\to\infty}{\longrightarrow} 1$, and

ii) $E\{\max(\hat{D}_N^H, \hat{D}_N^V)\} = 0(N\bar{R})$. (6)

(6) (Section V). For $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant N$ and for $\bar{R} \leqslant 0(\ln N)$,

$$E\hat{D}_{iN}^V = E\hat{D}_{jN}^H \leqslant 0(\ln N)$$

and for $\bar{R} > 0(\ln N)$

$$E\hat{D}_{iN}^V = E\hat{D}_{jN}^H = 0(\bar{R}). \tag{7}$$

Statements (1) and (2) give with arbitrarily high probability the estimate of the area required for wiring large chips (large N). Statement (3) indicates that the distribution of the wires on the chip ($K_N(t)$) will converge to a Poisson distribution with parameter $\lambda\bar{R}/2$. Statement (4) implies that if $\bar{R} \leqslant 0(\ln N)$ then the widest channel segment (in terms of number of wires) on the $N \times N$ chip is upper bounded by $0(\ln N)$. Thus to ensure successful routing completion the designer of the master slice should leave $0(\ln N)$ tracks in every channel segment on the $N \times N$ chip. The total wiring area on the chip is, therefore, $0(N^2 \ln^2 N)$ (see Fig. 2(a)).

If $\bar{R}$ grows faster than $0(\ln N)$, then the designer should leave $0(\bar{R})$ tracks in every channels segment, and the resulting wiring area becomes $0(N^2\bar{R}^2)$. If the designer is allowed to change the dimensions of the master slice chip after wiring (this is not the common practice at the present time), he could in principal reduce the wiring area at least by a constant factor. One way to reduce the wiring area, while keeping the square shape of the chip, would be to first squeeze the horizontal wiring dimension of the chip from $N\hat{T}_N$ to $\hat{D}_N^H$, then the vertical dimension from $N\hat{T}_N$ to $\hat{D}_N^V$ (see Fig. 2(b)). In practice, this type of compaction may not be easy to do, however. An easier compaction could be done by keeping the row and column blocks lined up, as shown in Fig. 2(c). The wiring dimensions in this case are given by $\Sigma_{i=1}^N \hat{D}_{iN}^H$ and $\Sigma_{j=1}^N \hat{D}_{jN}^V$ in the vertical and horizontal directions, respectively.

Before proceeding with the formal presentation of the above results, we plot in Fig. 3 the upper bounds to $E\hat{T}_N$, $\frac{1}{N}E\hat{D}_N^H$, and $E\hat{D}_{1N}^H$ obtained in Section V versus $(N^2)^{1/6}$. The values of N, $\bar{R}$, and λ used are from [4] (where it was assumed that $\bar{R} = 0(N^{1/3})$). Consider the case $N = 30$. From Fig. 3, the expected width of the chip (in terms of the number of wire segments) when $E\hat{T}_N$ tracks are left in each channel segment $= 357$. If the compaction in Fig. 2(c) is performed, after wiring, the expected width of the chip $= E\hat{D}_N^H = 251$. If, on the other hand, the compaction in Fig. 2(b) is performed, the expected width becomes $= 139$.

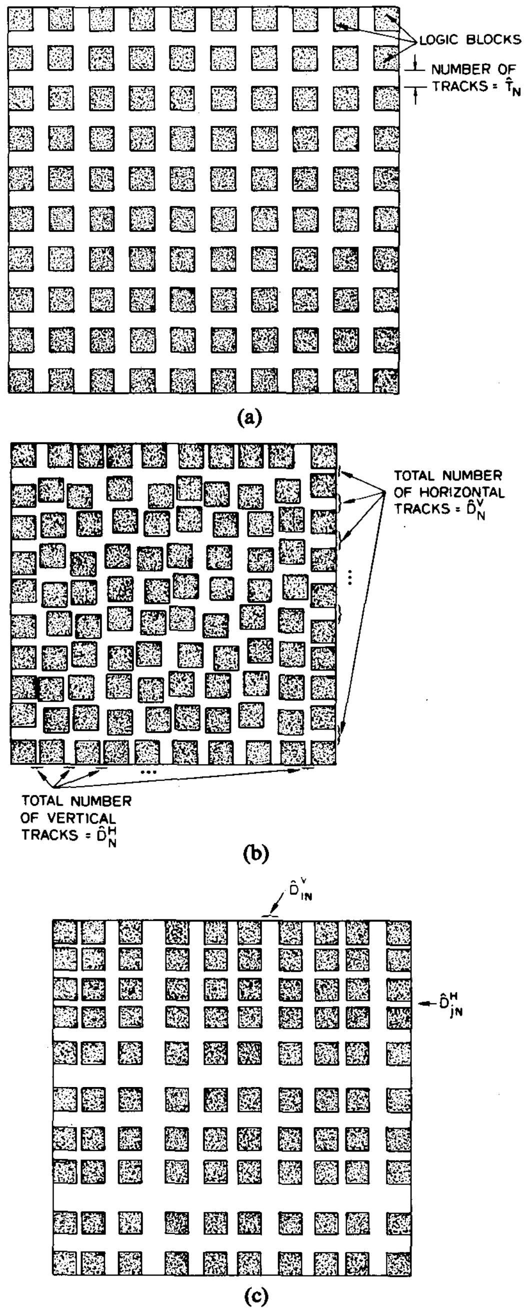

Fig. 2 (a). Master slice before wiring. (T_N tracks per channel segment). (b) Master slice after wiring and compaction. ($\hat{D}_N^H$ tracks in the horizontal direction and $\hat{D}_N^V$ tracks in the vertical direction). (c) Master slice after wiring and compaction leaving horizontal and vertical blocks lined up.

III. Formal Description of Process (a) and Elementary Results

Let $\mathbb{Z}^2 = \{(i,j)\}$ be the set of lattice points in $\mathbb{R}^2$, and define the set $\{X_{ij}^a, (i,j) \in \mathbb{Z}\}$ of independent identically distributed random variables each drawn according to a Poisson distribution with parameter $\lambda/4$, i.e.,

$$P\{X_{ij}^a = n\} = \frac{(\lambda/4)^n}{n!} e^{-\lambda/4}. \tag{8}$$

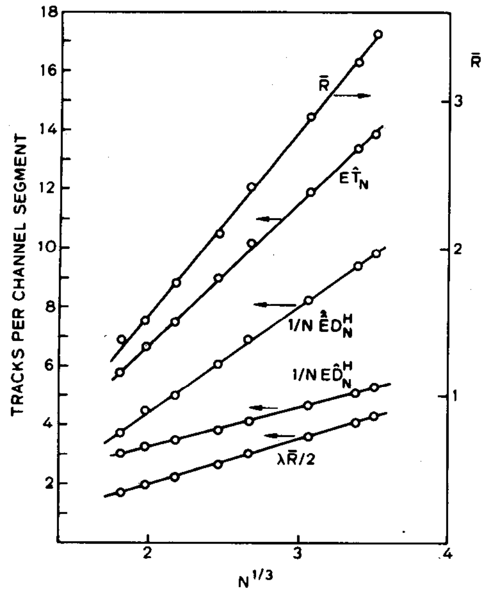

Fig. 3. Expected number of tracks with and without compaction.

The random variable X_{ij}^a represents the number of wires that emanate from (i, j) and that move right (R) and up (U) only (as indicated in Fig. 1). For $X_{ij}^a \geq 1$, the length of each wire x, $1 \leq x \leq X_{ij}^a$, is an independent integer-valued random variable $L_{ij}(x) \geq 1$ drawn according to some distribution P_L, with *finite mean* $\bar{R}$ (in the proof of the central limit theorem (Theorem 3) we shall also require the finiteness of $EL^{2+\epsilon}$). In Section VIII we will choose P_L to be geometric and derive several simplified results.

To determine the trajectory of each wire x, first define the sequence of independent identically distributed random variables

$$Z_{ij1}(x), Z_{ij2}(x), \cdots, Z_{ijL_{ij}(x)}(x)$$

such that

$$Z_{ijk}(x) = \begin{cases} +1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2. \end{cases}$$

The wire x, starting at the point (i, j), moves according to the outcomes of the $Z_{ijk}(x)$'s, $1 \leq k \leq L_{ij}(x)$; to the right (R) if the outcome is $+1$ and up (U) if the outcome is -1.

Now, for every $(i, j) \in \mathbb{Z}^2$, the number of wires, their lengths and their trajectories is completely specified by the following set of random variables.

$$\Lambda_{ij}^a = \Big\{X_{ij}^a; L_{ij}(1), L_{ij}(2), \cdots, L_{ij}(X_{ij}^a); \\ Z_{ij1}(1), \cdots, Z_{ijL_{ij}(1)}(1), \cdots, \\ Z_{ij1}(X_{ij}^a), \cdots, Z_{ijL_{ij}(X_{ij}^a)}(X_{ij}^a)\Big\}.$$

The random process $\{\Lambda_{ij}^a, (i, j) \in \mathbb{Z}^2\}$ thus consists of independent identically distributed components, and completely specifies wiring process (a).

We shall be interested in estimating the number of wires crossing from say, point (i, j) to $(i+1, j)$, or (i, j) to $(i, j+1)$. Therefore, for $-\infty \leq l \leq i$, $-\infty \leq k \leq j$ we define random variables

$$T_{ij}^{aH}(l, k) = \sum_{x=1}^{X_{lk}^a} 1\Big\{L_{lj}(x) \geq (i+j+1-l-k), \\ \sum_{m=1}^{i+j-l-k} Z_{ijm}(x) = (i-l)-(j-k) \\ \text{and } Z_{ij(i+j+1-l-k)}(x) = +1\Big\}$$

and

$$T_{ij}^{aV}(l, k) = \sum_{x=1}^{X_{lk}^a} 1\Big\{L_{lj}(x) \geq (i+j+1-l-k), \\ \sum_{m=1}^{i+j-l-k} Z_{ijm}(x) = (i-l)-(j-k) \\ \text{and } Z_{ij(i+j+1-l-k)}(x) = -1\Big\} \quad (9)$$

where $1\{A\}$ is the indicator function of the event A. The random variables $T_{ij}^{aH}(l, k)$ and $T_{ij}^{aV}(l, k)$ represent the number of wires starting at (l, k) and going through (i, j) to $(i+1, j)$ and to $(i, j+1)$, respectively. Therefore, the total number of wires crossing from point (i, j) to $(i+1, j)$ is the sum of the random variables $T_{ij}^{aH}(l, k)$ over $-\infty \leq l \leq i$, $-\infty \leq k \leq j$. We denote that sum by

$$T_{ij}^{aH} = \sum_{l=-\infty}^{i} \sum_{k=-\infty}^{j} T_{ij}^{aH}(l, k). \quad (10)$$

Similarly, we denote the total number of wires from (i, j) to $(i, j+1)$ by

$$T_{ij}^{aV} = \sum_{l=-\infty}^{i} \sum_{k=-\infty}^{j} T_{ij}^{aV}(l, k). \quad (11)$$

Observe that the random vectors $(T_{ij}^{aH} T_{ij}^{aV})$ are *not independent*. However, it is easy to see that the two-dimensional random field $\{(T_{ij}^{aH} T_{ij}^{aV}); (i, j) \in \mathbb{Z}^2\}$ is translation-invariant in the sense that if $A \subset \mathbb{Z}^2$, then the joint distribution of $\{(T_{ij}^{aH} T_{ij}^{aV}), (i, j) \in A + (l, k)\}$ is identical to the distribution of $\{(T_{ij}^{aH} T_{ij}^{aV}), (i, j) \in A\}$ for any integer pair (l, k). It is also easy to see that two random variables, say T_{lk}^{aH} and T_{mn}^{aV}, from the collection $\{T_{ij}^{aH}, T_{ij}^{aV}\}$ are independent if no wire can simultaneously go through channel segments $(l, k) \rightarrow (l+1, k)$ and $(m, n) \rightarrow (m+1, n)$.

We shall now state several elementary lemmas concerning the random variables $T_{ij}^{aH}(l, k), T_{ij}^{aV}(l, k)$ and their sums T_{ij}^{aH}, T_{ij}^{aV}.

Lemma 1: For $(i, j) \in \mathbb{Z}^2$, the random variables $\{T_{ij}^{aH}(l, k), T_{ij}^{aV}(l, k) \ -\infty \leq l \leq i, \ -\infty \leq k \leq j\}$ are mutually independent and the distribution of $T_{ij}^{aH}(l, k)$ is Poisson with the parameter

$$\lambda_{lk} = \frac{\lambda}{8} P(L \geq i+j+1-l-k) \\ \cdot \binom{i+j-l-k}{i-l} 2^{-(i+j-l-k)} \quad (12)$$

Lemma 2: The random variables T_{ij}^{aH} and T_{ij}^{aV} are independent and identically Poisson distributed with parameter $\lambda\bar{R}/8$. Moreover, $(T_{ij}^{aH}T_{ij}^{aV})$ are independent of the random variables $\{(T_{lk}^{aH}T_{lk}^{aV});\ i+1\leq l\leq\infty,\ -\infty\leq k\leq j-1$ or $-\infty\leq l\leq i-1,\ j+1\leq k\leq\infty\}$.

Lemma 3: If (i,j) and (l,k) are such that $i>l$ and $j>k$, then

$$\text{(i)}\quad P\{T_{lk}^{aH}=t_1,\ T_{ij}^{aH}=t_2\} = P\{T_{lk}^{aH}=t_1,\ T_{ij}^{aV}=t_2\} = P\{T_{lk}^{aH}=t_1,\ T_{ij}^{aH}=t_2\} = P\{T_{lk}^{aV}=t_1,\ T_{ij}^{aV}=t_2\}$$
$$= \sum_{d=0}^{\min(t_1,t_2)} P\left(\frac{\lambda}{8}\left(\bar{R}-\bar{R}_{(l,k)}^{(i,j)}\right);\ t_1-d\right)\cdot P\left(\frac{\lambda}{8}\left(\bar{R}-\bar{R}_{(l,k)}^{(i,j)}\right);\ t_2-d\right)P\left(\frac{\lambda}{8}\bar{R}_{(l,k)}^{(i,j)};\ d\right) \tag{13}$$

where

$$P(A;x)=\frac{A^x}{x!}e^{-A}$$

and

$$\bar{R}_{(l,k)}^{(i,j)}=\binom{i+j-l-k-1}{i-l}2^{-(i+j-l-k)}\sum_{m=(i+j-l-k)}^{\infty}P\{L\geq m+1\}. \tag{14}$$

Remarks:

1) The independence in Lemmas 1 and 2 is a result of the infinite divisibility of the Poisson distribution and the fact that no wire contributes to more than one of the random variables considered. In particular, observe that for any (i,j) and (l,k) such that either $i+1\leq l\leq\infty$, $-\infty\leq k\leq j-1$ or $-\infty\leq l\leq i-1$, $j+1\leq k\leq\infty$ no wire can pass through both (i,j) and (l,k); also, no wire can go both horizontally and vertically at (i,j).

2) If (i,j) and (l,k) in Lemmas 3 are far apart, it is easy to see that $\bar{R}_{(l,k)}^{(i,j)}$ approaches zero. This is true since the larger the distance between (i,j) and (l,k), the fewer the number of wires that pass through both points. To illustrate this important fact, we derive the following approximations of $\bar{R}_{(l,k)}^{(i,j)}$.

Lemma 4: (See [8]). For large $K=i+j-l-k-1$,

(a) if $\frac{K}{2}-K^{1/2+\epsilon}\leq i-l\leq\frac{K}{2}+K^{1/2+\epsilon}$

then

$$\bar{R}_{(l,k)}^{(i,j)}\sim\frac{e^{-(i-l-(K/2))^2/(K/2)}}{\sqrt{\pi\frac{K}{2}}}\sum_{m=K}^{\infty}P\{L\geq m+1\}. \tag{15}$$

$$\text{(b)}\quad \sum_{\substack{i-l>(K/2)+K^{1/2+\epsilon}\\ \text{or}\ \ i-l<(K/2)-K^{1/2+\epsilon}}}\bar{R}_{(l,k)}^{(i,j)}\lesssim\frac{2e^{-2K^{2\epsilon}}}{\sqrt{\pi\frac{K}{2}}}\cdot\sum_{m=K}^{\infty}P\{L\geq m+1\}. \tag{16}$$

The symbol $\sim$ means asymptotic equality, $\lesssim$ means asymptotic inequality.

Proofs (Lemmas 1, 2, 3, and 4):

The independence claimed in Lemma 1 is evident since the random variables in Lemma 1 are functions of independent random variables (see definitions (8), (9)). Next, (12) can be proved by a simple combinatorial argument.

Lemma 2 is clearly true since no wire can go through both channel segments $(i,j)\to(i+1,j)$ and $(i,j)\to(i,j+1)$, simultaneously.

To prove Lemma 3, observe that the number of wires T_{lk}^{aH} going through channel segment $(l,k)\to(l+1,k)$ is the sum of two independent Poisson distributed random variables; the first T_{lk1}^{aH} has parameter $(\lambda/8)(\bar{R}-\bar{R}_{(l,k)}^{(i,j)})$ and represents the number of wires that go through $(l,k)\to(l+1,k)$, but not through $(i,j)\to(i+1,j)$; the second $T_0^{aH}(i,j;l,k)$ has parameter $(\lambda/8)\bar{R}_{(l,k)}^{(i,j)}$ and represents the number of wires that go through both $(l,k)\to(l+1,k)$ and $(i,j)\to(i+1,j)$. Similarly, $T_{ij}^{aH}=T_{ij1}^{aH}+T_0^{aH}(i,j;l,k)$, where T_{ij1}^{aH} is independent of $T_0^{aH}(i,j;l,k)$ and is Poisson with parameter $(\lambda/8)(\bar{R}-\bar{R}_{(l,k)}^{(i,j)})$. Computing $\bar{R}_{(i,k)}^{(i,j)}$ can be done by a simple combinatorial argument.

Lemma 4(a) follows from the DeMoivre–Laplace limit theorem [8, p. 168]. Lemma 4(b) follows from Lemma 2 VII.1 in [8] and the DeMoivre–Laplace theorem. This completes the outline of the proofs of Lemmas 1–4.

Following the lines of the above discussion, we can define the wiring stochastic processes (b), (c), and (d) and their associated random variables $T_{ij}^{bH}, T_{ij}^{bV},\cdots$, etc., and derive identical results to Lemmas 1–4. The total number of wires in channel segments $(i,j)\to(i+1,j)$ and $(i,j)\to(i,j+1)$ is now defined by

$$T_{ij}^{H}=T_{ij}^{aH}+T_{ij}^{bH}+T_{ij}^{cH}+T_{ij}^{dH} \tag{17}$$
$$T_{ij}^{V}=T_{ij}^{aV}+T_{ij}^{bV}+T_{ij}^{cV}+T_{ij}^{dV} \tag{18}$$

respectively.

IV. Estimation of Wiring Space

The number of wire segments (horizontal and vertical) generated by process (a) on an $N\times N$ segment of the doubly infinite chip is given by the random variable

$$S_N(a)=\sum_{i=1}^{N}\sum_{j=1}^{N}\left(T_{ij}^{aH}+T_{ij}^{aV}\right). \tag{19}$$

In this section, we give a weak law of large numbers and a central limit theorem for $S_N(a)$.

First, it is easy to see that

$$ES_N(a)=\frac{N^2\lambda\bar{R}}{4}. \tag{20}$$

Next we compute the variance of $S_N(a)$.

Lemma 5:

(a) $\mathrm{Var}(S_N(a))\geq 0(N^2)$.

(b) If $\bar{R}<\infty$, $\dfrac{\mathrm{Var}(S_N(a))}{N^4}\to 0$.

(c) If $EL^{2+\epsilon}<\infty$, $\mathrm{Var}(S_N(a))=0(N^2)$.

It is now easy to prove the following weak law of large numbers for $S_N(a)$.

Theorem 1: If $\bar{R}<\infty$, then

$$\frac{S_N(a)}{N^2} \xrightarrow{\mathcal{P}} \frac{\lambda \bar{R}}{4}. \tag{21}$$

Proof of Lemma 5: (Appendix A in [14]).

Proof of Theorem 1:

The proof follows directly from Lemma 5 and Chebychev's inequality

$$P\left\{\left|\frac{S_N}{N^2}-\frac{\lambda \bar{R}}{4}\right| \geqslant \epsilon\right\} \leqslant \frac{\operatorname{Var}(S_N(a))}{N^4 \epsilon^2}. \tag{22}$$

The right side of (22) goes to zero as $N \to \infty$, thus proving the weak law of large numbers.

To prove that $(S_N(a)-(\lambda \bar{R}/4)N^2)/\sqrt{\operatorname{Var}(S_N(a))}$ converges to a normal distribution with zero mean and variance 1 (i.e., central limit theorem) requires more work. First the random variables $(T_{ij}^{aH}+T_{ij}^{aV})$ are clearly not independent, and, therefore, the standard central limit theorem for sums of independent random variables cannot be directly applied. Secondly, little is known about general limit theorems for random fields (see, for example [9], [10]). However, a fair amount of work has been done in proving limit theorems for one-dimensional stationary stochastic processes (see, for example, [11], [14]). In particular, if the stochastic process is uniformly or weakly mixing (elements far apart are weakly dependent), central limit theorems can be proved under a variety of conditions [12].

For the random field $(T_{ij}^{aH} T_{ij}^{aV})$ under consideration, we could show that if $EL^{2+\epsilon}<\infty$ it is possible to prove a central limit theorem. This is done by first considering the underlying independent process Λ_{ij}^a, and then observing that the $(T_{ij}^{aH} T_{ij}^{aV})$'s are functions of that independent process. The proof follows theorem 7.5 in [11] and theorem 18.6.1 in [12] with the necessary generalizations to two-dimensional processes and several simplifications resulting from the special structure of $(T_{ij}^{aH} T_{ij}^{aV})$.

We first introduce a two-dimensional generalization of the m-dependent stochastic process [12].

Definition: The translation-invariant random field $\{Y_{ij},(i,j)\in \mathbb{Z}^2\}$ is said to be m-dependent if for any (i,j) and (l,k) such that $l>i+m$ or $l<i-m$ or $k>j+m$ or $k<j-m$, the random variables Y_{ij} and Y_{lk} are independent.

Theorem 2: For the m-dependent translation-invariant random field $Y_{ij}\geqslant 0$, if $\operatorname{Var}(\sum_{i=1}^N \sum_{j=1}^N Y_{ij})=0(N^2)$, and $EY_{11}^3<\infty$, then

$$\frac{\left(\sum_{i=1}^{N}\sum_{j=1}^{N} Y_{ij}-N^2 EY_{11}\right)}{N\sigma_N} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \tag{23}$$

where

$$\sigma_N = 1/N\sqrt{\operatorname{Var}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} Y_{ij}\right)}.$$

Proof: (Appendix B in [14]).

We now utilize Theorem 2 to prove a central limit theorem for $S_N(a)$.

Theorem 3: If $EL^{2+\epsilon}<\infty$, for some $\epsilon>0$, then

$$\frac{S_N(a)-N^2\lambda \bar{R}/4}{N\sigma_N} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \tag{24}$$

where

$$\sigma_N = 1/N\sqrt{\operatorname{Var}(S_N(a))}.$$

Proof: (Appendix B in [14]).

The number of wire segments $S_N(a)$ on a large $N\times N$ portion of the doubly infinite chip is therefore less than (or equal to)

$$\left(N^2 \frac{\lambda \bar{R}}{4}+5\sigma_N N\right)$$

with probability over 0.999.

Finally, by superimposing the four processes that constitute the total wiring of the chip it is easy to prove that

1) If $\bar{R}<\infty$, then

$$\frac{S_N}{N^2} \xrightarrow{\mathcal{P}} \lambda \bar{R} \tag{25}$$

where $S_N=S_N(a)+S_N(b)+S_N(c)+S_N(d)$ is the total number of wire segments on the $N\times N$ chip, and

2) if $EL^{2+\epsilon}<\infty$, then

$$\frac{S_N-N^2\lambda \bar{R}}{2\sigma_N N} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \tag{26}$$

i.e.,

$$S_N \leqslant N^2 \lambda \bar{R}+10\sigma_N N \tag{27}$$

with probability higher than 0.999.

Remark: The above theorems could be generalized for $\bar{R}<\infty$ growing with N. For example, if $\bar{R}=cN^{1/3}$, then it can be proved that

$$\frac{S_N}{N^2 \cdot N^{1/3}} \xrightarrow{\mathcal{P}} \lambda c.$$

V. Distribution of Wires

In the previous section, we found that with very high probability there will be $\lesssim (N^2\lambda \bar{R}+10\sigma_N N)$ wire segments on a chip of size $N\times N$. The question we shall address in this section is how these segments are distributed among the different channels. In particular, we shall determine the number of channels on the chip with t wire segments. Thus define the random variable $K_N(t)$ that represents the number of channels in an $N\times N$ portion of the finite chip, with t wires

$$K_N(t)=\sum_{i=1}^{N}\sum_{j=1}^{N}\left(1\{T_{ij}^H=t\}+1\{T_{ij}^V=t\}\right). \tag{28}$$

It is easy to see that

$$EK_N(t)=2N^2 P\left(\frac{\lambda \bar{R}}{2};t\right). \tag{29}$$

Thus the average number of channels with t wire segments is proportional to $P(\lambda \bar{R}/2;t)$.

It is now easy to show that in fact the proportion of channel segments with t wires is asymptotically equal to $P(\lambda \bar{R}/2; t)$.

Theorem 4: If $\bar{R}<\infty$, then

$$\frac{K_N(t)}{2N^2} \xrightarrow{\mathcal{P}} P\left(\frac{\lambda \bar{R}}{2}; t\right). \tag{30}$$

Proof: Follows by computing $\mathrm{Var}(K_N(t))$ and applying Chebychev's inequality.

Remark: A central limit theorem can also be derived for $K_N(t)$.

We now turn to the problem of bounding $E\hat{T}_N$. We first bound the probability that $\hat{T}_N$ exceeds some large number t_N.

Lemma 6: For $t_N > \lambda \bar{R}/2$

$$P\left\{T_N = \max_{1 \le i,j \le N} \max\left(T_{ij}^H, T_{ij}^V\right) \le t_N\right\} \ge 1 - 2N^2 e^{-(t_N+1)\ln((t_N+1)/\lambda \bar{R}/2)+t_N+1-(\lambda \bar{R}/2)}. \tag{32}$$

Proof: Define the random variable

$$M_N(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(1\{T_{ij}^H > t\} + 1\{T_{ij}^V > t\}\right).$$

It is easy to see that

$$\left\{\max_{1 \le i,j \le N} \max\left(T_{ij}^H, T_{ij}^V\right) \le t_N\right\} = \{M_N(t_N) < 1\}. \tag{33}$$

Therefore,

$$P\{\hat{T}_N \le t_N\} = P\{M_N(t_N) < 1\} = 1 - P\{M_N(t_N) \ge 1\}. \tag{34}$$

But

$$EM_N(t_N) = 2N^2 \sum_{t=t_N+1}^{\infty} P\left(\frac{\lambda \bar{R}}{2}; t\right).$$

Using Markov inequality, it follows that

$$P\{M_N(t_N) \ge 1\} \le 2N^2 P\{T_{11}^H > t_N\}. \tag{35}$$

Now using the Chernoff bound on $P\{X \ge A\}$ (see [13]), we obtain

$$P\{T_{11}^H > t_N\} \le e^{-(t_N+1)\ln(t_N+1)/\lambda \bar{R}/2 + t_N + 1 - \lambda \bar{R}/2}.$$

Combining this bound with (34) and (35) completes the proof of the Lemma.

Corollary:

i) If $\bar{R} \le 0(\ln N)$, then

$$E\hat{T}_N \le 0(\ln N). \tag{36}$$

ii) If $\bar{R} > 0(\ln N)$, then

$$E\hat{T}_N \le 0(\bar{R}). \tag{37}$$

Proof: First, it is easy to see that

$$E\hat{T}_N = \sum_{t=1}^{\infty} P\{\hat{T}_N \ge t\}.$$

Now, let t^* be the smallest t such that

$$2N^2 e^{-(t+1)\ln(t+1)/\lambda \bar{R}/2 + t + 1 - (\lambda \bar{R}/2)} < 1.$$

TABLE I
ONE-DIMENSIONAL CHANNEL WIDTH PREDICTIONS

N	$\bar{R}$	$\hat{T}_N$ from [4]	$\hat{T}_N$ from (41)	$\hat{T}_N$ from (42)
6	1.387	9.0	10.7	10
8	1.59	10.5	12.1	11
10	1.771	11.7	13.4	12
15	2.117	13.9	15.7	14
20	2.41	15.6	17.6	16
30	2.889	18.4	20.5	19
40	3.276	20.6	22.8	21
44	3.469	21.5	23.8	22

$\lambda = 2.5$

It then follows that

$$E\hat{T}_N \le t^* + 2N^2 \sum_{t=t^*+1}^{\infty} e^{-t\ln(t/\lambda \bar{R}/2)+t+1-(\lambda \bar{R}/2)}. \tag{38}$$

Statements i) and ii) now follow by investigating (38).

Remarks:

1) Alternative bound to (32) can be obtained by using Chebychev inequality instead of Markov inequality. Thus

$$P\{M_N(t_N) \ge 1\} \le P\{|M_N(t_N) - EM_N(t_N)| > 1 - EM_N(t_N)\} \le \frac{\mathrm{Var}(M_N(t_N))}{(1 - EM_N(t_N))^2}. \tag{39}$$

Evaluation of (39) requires knowledge of $\mathrm{Var}(M_N(t_N))$.

2) Lemma 6 gives a bound on $P\{T_N \le t\}$ which is completely independent of the routing strategy (coin flipping, one corner wires, $\cdots$, etc.). It is possible to improve that bound when the routing strategy is specified.

Using the union of events bound it can be easily shown that

$$P\{\hat{T}_N \le t\} \ge 1 - N^2\left(1 - P\{T_{11}^H \le t_N, T_{11}^V \le t_N\}\right). \tag{40}$$

Now, given a routing strategy, $P\{T_{11}^H \le t_N,\ T_{11}^V \le t_N\}$ can be evaluated (Remarks 3) and 4) below).

3) Lemma 6 could be easily used to give a lower bound to the probability of success P_S, as defined in [4], for the one-dimensional model. Thus consider an N^2 logic blocks segment of a doubly infinite linear array. Define T_i to be the number of tracks in channel $1 \le i \le N^2$. The random variables T_i, $1 \le i \le N^2$ are Poisson distributed with parameter $\lambda \bar{R}$. Therefore,

$$P_S \triangleq P\left\{\hat{T}_N \triangleq \max_{1 \le i \le N^2} T_i \le t_N\right\} \ge 1 - N^2 \exp\left(-(t_N+1)\ln\left(\frac{t_N+1}{\lambda \bar{R}}\right) + t_N + 1 - \lambda \bar{R}\right). \tag{41}$$

We can also use (40) to obtain the following tighter bound to P_S:

$$P_S \ge 1 - \frac{N^2}{2}\left(1 - P(T_1 \le t_N, T_2 \le t_N)\right). \tag{42}$$

A comparison of $\hat{T}_N$ obtained from (41), (42), and from [4], for $P_S = 0.9$, is given in Table I.

TABLE II
TWO-DIMENSIONAL CHANNEL WIDTH PREDICTIONS

N	$\bar{R}$	$E\,\hat{T}_N$	$E\max(T^H_{ij}+T^V_{ij})$	$E\max(T^H_{ij}+T^V_{ij})$ from [4], for $P_S = .9$
6	1.387	5.8	8.65	11.1
8	1.59	6.7	10.09	12
10	1.771	7.5	11.42	13
15	2.117	9	13.86	14.6
20	2.41	10.2	15.83	15.9
30	2.889	11.9	18.88	17.7
40	3.276	13.4	21.23	19.1
44	3.469	13.9	22.30	19.7

$\lambda = 2.5$

TABLE III
CHIP DIMENSIONS AFTER COMPACTION

N	$\bar{R}$	$1/N\,E\,\hat{\hat{D}}^H_N$	$1/N\,E\,\hat{D}^H_N$	Average Number of wire segments $\frac{\lambda\bar{R}}{2}$
6	1.387	3.79	3.05	1.73
8	1.59	4.54	3.27	1.99
10	1.771	5.02	3.47	2.21
15	2.117	6.11	3.82	2.65
20	2.41	6.97	4.14	3.01
30	2.889	8.37	4.66	3.61
40	3.276	9.45	5.09	4.10
44	3.469	. 9.89	5.32	4.34

$\lambda = 2.5$

The values of $\hat{T}_N$ given by (42) and by the algorithm in [4] are almost identical. The advantage of using (42), in addition to being computationally simpler than the method in [4], is that it provides a lower bound to P_S which is independent of the wire length distribution P_L (the algorithm in [4] gives an approximation to P_S when P_L is geometric).

4) We now find an upper bound to $E\hat{T}_N$, for the two-dimensional case, using (40) and the method in the Corollary to Lemma 6. Let t^* be the smallest integer t such that

$$N^2\left(1-P\left(T^H_{11}<t+1, T^V_{11}<t+1\right)\right)<1$$

then we obtain

$$E\hat{T}_N \leqslant t^* + N^2 \sum_{t=t^*+1}^{\infty} \left(1-P\left(T^H_{11}<t, T^V_{11}<t\right)\right). \quad (43)$$

In Table II, we list the upper bounds to $E\hat{T}_N$ obtained from (43) for values of λ, $\bar{R}$, and N from [4]. No corresponding values to $E\hat{T}_N$ were given in [4]. Rather an estimate of the maximum of the sum of a horizontal plus a vertical channel widths were given. For reasons of comparison, we give in Table II upper bounds to $E(\max_{1\leqslant i,j\leqslant N}(T^H_{ij}+T^V_{ij}))$ and list the corresponding values from [4].

Using the technique of Lemma 6, we now provide bounds on the dimensions of wiring space when the square shape of the chip is to be preserved (Fig. 2(b)). Thus consider the random variables

$$\hat{D}^V_N = \max_{1\leqslant i\leqslant N}\left\{\sum_{j=1}^{N} T^H_{ij}\right\}$$

and

$$\hat{D}^H_N = \max_{1\leqslant j\leqslant N}\left\{\sum_{i=1}^{N} T^V_{ij}\right\}.$$

The width of the square chip (occupied by wires) is, therefore,

$$\hat{D}_N = \max\left\{\hat{D}^H_N, \hat{D}^V_N\right\}.$$

For every $1\leqslant i\leqslant N$, $\sum_{j=1}^{N} T^H_{ij}$ is Poisson distributed with parameter $N\lambda\bar{R}/2$. Similarly, for every $1\leqslant j\leqslant N$, $\sum_{i=1}^{N} T^V_{ij}$ is Poisson distributed with parameter $N\lambda\bar{R}/2$. Thus

$$P\{\hat{D}_N \leqslant Nd_N\} \geqslant 1-2N \sum_{t=Nd_N+1}^{\infty} P\left(\frac{\lambda\bar{R}N}{2};t\right)$$
$$\geqslant 1-\exp\left[-(d_N N+1)\ln\left[\frac{d_N+\frac{1}{N}}{\lambda\bar{R}/2}\right]+\ln 2N + Nd_N + 1 - N\lambda\bar{R}/2\right]. \quad (44)$$

By an argument similar to that in the Corollary of Lemma 6, it follows that

$$E\hat{D}_N \leqslant 0(N\bar{R}). \quad (45)$$

We can also estimate the width of chip when the compaction in Fig. 2(c) is done (rows and columns of logic blocks remain lined up). We define for $1\leqslant i\leqslant N$, $1\leqslant j\leqslant N$ the random variables

$$\hat{D}^V_{iN} = \max_{1\leqslant j\leqslant N}\left\{T^V_{ij}\right\}$$
$$\hat{D}^H_{jN} = \max_{1\leqslant i\leqslant N}\left\{T^H_{ij}\right\}.$$

Now, the dimensions of the chip (in terms of the number of wires) after compaction are given by $\hat{\hat{D}}^H_N = \sum_{i=1}^{N} D^V_{iN}$ in the horizontal direction, and by $\hat{D}^V_N = \sum_{j=1}^{N} \hat{D}^H_{jN}$ in the vertical direction. It can now be shown that

$$E\hat{\hat{D}}^V_N = NE\hat{D}^V_{1N} \leqslant Nt^* + \frac{N}{2} \cdot \sum_{t=t^*+1}^{\infty} \left(1-P\left(T^V_{11}<t, T^V_{21}<t\right)\right). \quad (46)$$

The estimates of the chip dimensions after the compaction of Fig. 2(b); $E\hat{D}^H_N$, and after the compaction of Fig. 2(c); $E\hat{D}^H_N$, are given in Table III.

VI. BIASED CHIPS

Suppose now that the average length of each wire is larger in, for example, the vertical direction than it is in the horizontal. Our model can still be applied with the following simple modification in the definition of the random variables $Z_{ijk}(x)$:

Given that average length in the vertical direction is $p\bar{R}$ and that the average length in the horizontal direction is

$q\bar{R}$, where $0 \leqslant q \leqslant p \leqslant 1$ and $p+q=1$, define

$$Z_{ijk}(x)=\begin{cases} +1, & \text{with probability } q \\ -1, & \text{with probability } p. \end{cases}$$

Other definitions introduced in Section III are unchanged. Lemmas 1–3 can be generalized in an obvious way to yield the following.

Lemma 1': For $(i, j) \in \mathbb{Z}^2$, the random variables $\{T_{ij}^{aH}(l,k), T_{ij}^{aV}(l,k),\ -\infty \leqslant l \leqslant i,\ -\infty \leqslant k \leqslant j\}$ are mutually independent, and

(a) $T_{ij}^{aH}(l,k)$ is Poisson distributed with parameter

$$q\frac{\lambda}{4}P(L \geqslant i+j+1-l-k)\binom{i+j-l-k}{i-l}p^{j-k}q^{i-l} \quad (47)$$

(b) $T_{ij}^{V}(l,k)$ is Poisson distributed with parameter

$$p\frac{\lambda}{4}P(L \geqslant i+j+1-l-k)\binom{i+j-l-k}{i-l}p^{j-k}q^{i-l}. \quad (48)$$

Lemma 2': The random variables T_{ij}^{aH} and T_{ij}^{aV} are independent and Poisson distributed with parameters $q\lambda\bar{R}/4$ and $p\lambda\bar{R}/4$, respectively.

Lemma 3': If (i, j) and (l, k) are such that $i>l$ and $j>k$, then

(i) $$P\{T_{lk}^{aH}=t_1, T_{ij}^{aH}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} P\left(q\frac{\lambda}{4}\left(\bar{R}-\bar{R}'^{(i,j)}_{(l,k)}\right); t_1-d\right) \cdot P\left(q\frac{\lambda}{4}\left(\bar{R}-\bar{R}'^{(i,j)}_{(l,k)}\right); t_2-d\right)P\left(q\frac{\lambda}{4}\bar{R}'^{(i,j)}_{(l,k)}; d\right)$$

(ii) $$P\{T_{lk}^{aH}=t_1, T_{ij}^{aV}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} P\left(\frac{\lambda}{4}\left(q\bar{R}-p\bar{R}'^{(i,j)}_{(l,k)}\right); t_1-d\right) \cdot P\left(p\frac{\lambda}{4}\left(\bar{R}-\bar{R}'^{(i,j)}_{(l,k)}\right); t_2-d\right)P\left(p\frac{\lambda}{4}\bar{R}'^{(i,j)}_{(l,k)}; d\right)$$

(iii) $$P\{T_{lk}^{aV}=t_1, T_{ij}^{aH}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} P\left(\frac{\lambda}{4}\left(p\bar{R}-q\bar{R}'^{(i,j)}_{(l,k)}\right); t_1-d\right) \cdot P\left(q\frac{\lambda}{4}\left(\bar{R}-\bar{R}'^{(i,j)}_{(l,k)}\right); t_2-d\right)P\left(q\frac{\lambda}{4}\bar{R}'^{(i,j)}_{(l,k)}; d\right)$$

(iv) $$P\{T_{lk}^{aV}=t_1, T_{ij}^{aV}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} P\left(p\frac{\lambda}{4}\left(\bar{R}-\bar{R}'^{(i,j)}_{(l,k)}\right); t_1-d\right) \cdot P\left(p\frac{\lambda}{4}\left(\bar{R}-\bar{R}'^{(i,j)}_{(l,k)}\right); t_2-d\right)P\left(p\frac{\lambda}{4}\bar{R}'^{(i,j)}_{(l,k)}; d\right)$$

where

$$\bar{R}'^{(i,j)}_{(l,k)}=\binom{i+j-l-k-1}{i-l}p^{j-k}q^{i-l} \cdot \sum_{m=(i+j-l-k)}^{\infty} P\{L \geqslant m+1\}. \quad (49)$$

Results similar to those in Section V can be readily derived.

VII. Wires with at Most One Corner

The model described in Sections II and III allows each wire to have an average of $(\bar{R}-1)/2$ corners. In practice, the number of corners is usually constrained such that each wire may have at most one corner. We now modify our model to conform to this constraint.

The random variables $\{X_{ij}^a\}$ remain unchanged. However, the trajectory and length of each wire is chosen in the following way:

For every $1 \leqslant x \leqslant X_{ij}^a$, define the random variables

$$Z_{ij}(x)=\begin{cases} +1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2, \end{cases} \quad (50)$$

$L_{ij}^V(x) \sim P_L^V$, and $L_{ij}^H(x) \sim P_L^H$ where P_L^V and P_L^H are two distributions on the set of nonnegative integers $\{0,1,2,\cdots\}$ with means $p(\bar{R}-1)$ and $q(\bar{R}-1)$, respectively.

The value of $Z_{ij}(x)$ determines the *first step* in the trajectory; if $Z_{ij}(x)=+1$ the first step is right, if $Z_{ij}(x)=-1$ the first step is up. If the first step is to the right $L_{ij}^H(x) \sim P_L^H$ determines the remaining length of the horizontal part of the wire, then $L_{ij}^V(x) \sim P_L^V$ determines the length of the vertical part. Similarly, if $Z_{ij}(x)=-1$, $L_{ij}^V(x) \sim P_L^V$ determines the remaining length of the vertical part of the wire, and $L_{ij}^H(x) \sim P_L^H$ determines the length of the horizontal part. Observe that the average length of every wire is $\bar{R}$. However, if $Z_{ij}(x)$ is given say to be $+1$, the average length of the wire in the horizontal direction is $q(\bar{R}-1)+1$, and in the vertical direction is $p(\bar{R}-1)$. Similarly, if $Z_{ij}(x)=-1$, the average length in the vertical direction is $p(\bar{R}-1)+1$, and in the horizontal direction is $q(\bar{R}-1)$. Thus the average length of the wire in the horizontal direction is $\bar{R}^H \triangleq (q(\bar{R}-1)+1/2)$ and in the vertical direction is $\bar{R}^V \triangleq (p(\bar{R}-1)+1/2)$. Lemmas 1–3 can be generalized as follows,

Lemma 1'': For $(i, j) \in \mathbb{Z}^2$, the random variables $\{T_{ij}^{aH}(l,k), T_{ij}^{aV}(l,k);\ -\infty \leqslant l \leqslant i,\ -\infty \leqslant k \leqslant j\}$ are mutually independent and

(a) For $-\infty \leqslant l < i,\ -\infty \leqslant k < j$, $T_{ij}^{aH}(l,k)$ is Poisson with parameter

$$\lambda/8 \cdot P^V\{L=j-k-1\}P^H\{L \geqslant i-l+1\} \quad (51)$$

and $T_{ij}^{aV}(l,k)$ is Poisson with parameter

$$\lambda/8 \cdot p^H\{L=i-l-1\}P^V\{L \geqslant j-k+1\}. \quad (52)$$

(b) For $l=i$ and $-\infty \leqslant k \leqslant j$, $T_{ij}^{aH}(l,k)$ is Poisson with the same parameter in (51), and $T_{ij}^{aV}(l,k)$ is Poisson with parameter

$$\lambda/8P^V\{L \geqslant j-k\}. \quad (53)$$

Similarly, for $k=j$ and $-\infty \leqslant l \leqslant i$, $T_{ij}^{aH}(l,k)$ is Poisson with parameter

$$\lambda/8P^H\{L \geqslant i-l\} \quad (54)$$

and $T_{ij}^{aV}(l,k)$ is Poisson with parameter in (52).

Lemma 2'': The random variables T_{ij}^{aH} and T_{ij}^{aV} are independent and Poisson distributed with parameters $\lambda/4\bar{R}^H$ and $\lambda/4\bar{R}^V$, respectively.

Notice that on the average half the wires contributing to T_{ij}^{aH} come from the horizontal points (l,k), $-\infty < l < i$,

$j=k$. Similarly, for T_{ij}^{aH}, on the average half the wires come from the vertical points (l, k), $-\infty \leq k \leq j$, $i=l$.

Lemma 3″:

(a) If (i, j) and (l, k) are such that $i>l$ and $j>k$, then $T_{lk}^{aH}, T_{lk}^{aV}, T_{ij}^{aH}, T_{ij}^{aV}$ are *independent*.

(b) For $i=l$ and $j>k$,

$$P\{T_{ij}^{aV}=t_1, T_{ik}^{aV}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} p\left(\lambda/4\left(\bar{R}^V-\bar{R}_{1_{(k)}}^{V_{(j)}}\right); t_1-d\right) \cdot p\left(\lambda/4\left(\bar{R}^V-\bar{R}_{1_{(k)}}^{V_{(j)}}\right); t_2-d\right) p\left(\lambda/4\bar{R}_{1_{(k)}}^{V_{(j)}}; d\right)$$

$$P\{T_{ij}^{aH}=t_1, T_{ik}^{aV}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} P\left(\lambda/4\left(\bar{R}^H-\bar{R}_{1_{(k)}}^{H_{(j)}}\right); t_1-d\right) \cdot P\left(\lambda/4\left(\bar{R}^V-\bar{R}_{1_{(k)}}^{H_{(j)}}\right); t_2-d\right) P\left(\lambda/4\bar{R}_{1_{(k)}}^{H_{(j)}}; d\right)$$

and T_{ij}^{aH}, T_{ij}^{aV}, and T_{ik}^{aH} are independent, where

$$\bar{R}_{1_{(k)}}^{V_{(j)}} = 1/2\left[\sum_{i=j-k}^{\infty} P^V\{L\geq i\}\right] + 1/2\left[\sum_{i=j-k}^{\infty} P^V\{L\geq i+1\}\right]$$

and

$$\bar{R}_{1_{(k)}}^{H_{(j)}} = 1/2\left[P^V\{L\geq j-k-1\}P^H\{L>0\}\right].$$

Similarly, for $j=k$ and $i>l$,

$$P\{T_{ij}^{aH}=t_1, T_{lj}^{aH}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} P\left(\lambda/4\left(\bar{R}^H-\bar{R}_{2_{(l)}}^{H_{(i)}}\right); t_1-d\right) \cdot P\left(\lambda/4\left(\bar{R}^H-\bar{R}_{2_{(l)}}^{H_{(i)}}\right); t_2-d\right) \cdot P\left(\lambda/4\bar{R}_{2_{(l)}}^{H_{(i)}}; d\right)$$

$$P\{T_{ij}^{aV}=t_1, T_{lj}^{aH}=t_2\} = \sum_{d=0}^{\min(t_1,t_2)} P\left(\lambda/4\left(\bar{R}^V-\bar{R}_{2_{(l)}}^{V_{(i)}}\right); t_1-d\right) \cdot P\left(\lambda/4\left(\bar{R}^H-\bar{R}_{2_{(l)}}^{V_{(i)}}\right); t_2-d\right) P\left(\lambda/4\bar{R}_{2_{(l)}}^{V_{(i)}}; d\right)$$

and T_{ij}^{aV}, T_{ij}^{aH}, and T_{lj}^{aV} are independent where

$$\bar{R}_{2_{(l)}}^{H_{(i)}} = 1/2\left[\sum_{j=i-l}^{\infty} P^H\{L\geq j\}\right] + 1/2\left[\sum_{j=i-l}^{\infty} P^H\{L\geq j+1\}\right]$$

and

$$\bar{R}_{2_{(l)}}^{V_{(i)}} = 1/2\left[P^H\{L\geq i-l-1\}P^V\{L>0\}\right].$$

The theorems and lemmas of Sections IV and V can also be generalized for the present model.

VIII. $P_L \sim$ GEOMETRIC

In our treatment up to this point, we have not assumed any specific probability distribution on the wire length (except for conditions on moments). This is both an advantage and a disadvantage. It is an advantage since our results on the size of the chip and wiring distribution are quite general, and it is a disadvantage because no joint probability distribution on $\{(T_{ij}^{aH}T_{ij}^{aV}), (i, j)\in A\subset \mathbb{Z}^2\}$ can be easily derived and hence only bounds on the probability $P\{\hat{T}_N \leq t\}$ can be given (see Sections IV and V). In this section, we assume that the wire length is distributed according to a *geometric distribution* with parameter ϵ. The distribution of $L_{ij}(x)$ in Section III is, therefore, given by

$$P\{L_{ij}(x)=l\}=\epsilon^{l-1}(1-\epsilon), \qquad 1\leq l\leq\infty$$

$$\bar{R}=\frac{\epsilon}{(1-\epsilon)}. \tag{55}$$

The simplication resulting from the above assumption can perhaps be best understood by an example.

Consider the one-dimensional model, in [4]. Let i, $i+1$, $i+2$, be three consecutive points on a doubly infinite one-dimensional array and let T_i be the number of wires in channel segment $i\to i+1$. Assuming that $L=3$ with probability one, then from A (5) in [4]:

$$P(T_{i+1}=0/T_i=2)=(1/3)^2P(\lambda;0)\neq 0. \tag{56}$$

On the other hand, it is clear that

$$P(T_{i+1}=0/T_i=2, T_{i-1}=0)=0$$

thus $T_{i+1}\to T_i\to T_{i-1}$ *do not* form a Markov chain.[1]

On the other hand, if P_L is geometric, it can be easily shown that these random variables form a Markov chain thus simplifying the joint probability distribution of the channel widths $\{T_i\}$.

Remark: An equivalent choice to the geometric distribution is to assume that at each point (i, j) a wire will decide to continue with probability ϵ and terminate with probability $(1-\epsilon)$.

Now, returning to our two-dimensional model, it can be shown that conditional on the number of wires incident to any point (i, j), $(T_{i-1j}^{aH}, T_{ij-1}^{aV})$, the number of wires leaving (i, j) are independent of $\mathfrak{L}_{ij} \triangleq \{(T_{lk}^{H}, T_{lk}^{V});\ -\infty\leq l\leq i-1, -\infty\leq k\leq j-1$ or $-\infty\leq l\leq i-2$, $k=j$ or $l=i$, $-\infty\leq k\leq j-2\}$. The conditional distribution of $(T_{ij}^{aH}, T_{ij}^{aV})$ given $(T_{i-1j}^{aH}, T_{ij-1}^{aV})$ is now stated.

Lemma 7: Under the assumption that the distribution of the length of the wire is geometric, and for any $(i, j)\in \mathbb{Z}^2$,

$$\begin{aligned} P\left(T_{ij}^{aV}=t_1, T_{ij}^{aH}=t_2/T_{i-1j}^{aH}=t_3, T_{ij-1}^{aV}=t_4, \mathfrak{L}_{ij}\right) &= P\left(T_{ij}^{aV}=t_1, T_{ij}^{aH}=t_2/T_{i-1j}^{aH}+T_{ij-1}^{aV}=t_3+t_4\right) \\ &= 2^{-(t_1+t_2)}\binom{t_1+t_2}{t_1}\sum_{j=0}^{\min(t_3+t_4,t_1+t_2)}\binom{t_3+t_4}{j} \\ &\quad\cdot\left(\frac{1}{\bar{R}}\right)^{t_3+t_4-j}\left(1-\frac{1}{\bar{R}}\right)^{j} p(\lambda/4; t_1+t_2-j). \end{aligned} \tag{57}$$

Proof: (Appendix C in [14]).

Remark: The conditional probability $P(T_{ij}^{aV}, T_{ij}^{aH}/T_{i-1j}^{aH}, T_{ij-1}^{aV})$ *for any* P_L with mean $\bar{R}$ is given by (57).

The joint probability distribution of $\{(T_{ij}^{aH}T_{ij}^{aV}), 1\leq i\leq N, 1\leq j\leq N\}$, i.e., of an $N\times N$ chip can be easily derived (see [14]).

[1] This is contrary to the claim in [4] that for all P_L, T_i form a Markov chain.

IX. Generalizations and Remarks

In this paper, we have analyzed a two-dimensional stochastic model for wiring in master slice LSI. In this section, we conclude the analysis by briefly discussing (i) the validity of the different assumptions made in constructing the model, (ii) the effect of relaxing some of these assumptions on the derived results, (iii) possible immediate generalizations of the results, and (iv) alternative applications for the model.

We begin with (i) and (ii):

1) X_{ij} independent identically Poisson distributed: The Poisson assumption does not seem to be unreasonable in view of the empirical data collected by Heller *et al.* (personal discussions with Heller, Donath, and Mikhail). Relaxing this assumption will in general make the analysis very difficult. However, it is the infinite divisibility property of the Poisson distribution that is most needed. Thus assuming that X_{ij} is distributed as *mixtures of Poissons*, i.e.,

$$X_{ij} \sim \int P_{\lambda}(\cdot)\, dF \tag{58}$$

where F is a distribution function on $[0, \infty]$ with mean $\bar{\lambda}$, it appears that generalizations to most of the results in this paper can be derived.

One possible objection to the Poisson assumption is in deriving the maximum channel width. Under the Poisson assumption, the maximum number of wires emanating from a logic block $\max_{1 \leqslant i, j \leqslant N} X_{ij}$ grows with the number of blocks N (rate close to $\log N$). In practice, the I/O of any fixed size logic block is bounded by some number M that does not necessarily depend on N. Thus for large N, our estimate of the maximum channel width may be pessimistic. To incorporate this fact in the analysis one could assume that X_{ij} has binomial distribution with parameters $0 \leqslant p \leqslant 1$ and M. Following steps similar to the derivations in Sections III and V, a bound on $P\{\hat{T}_N > t\}$ could be obtained.

2) The independence of the X_{ij}'s is not completely realistic. In practice, strong dependence may exist among neighboring X_{ij}'s. It should be pointed out, however, that the independence assumption allows us to compute upper bounds to the desired estimates of area.

3) Independence of $L_{ij}(x)$'s of the X_{ij}'s: this assumption may not be completely realistic. However, we are not aware of any attempts to determine the nature of the above dependence, either empirically or theoretically.

If such dependence is assumed, the derivation of many of the results in this paper may become very difficult.

4) Independence of wires trajectories: most master slice routing algorithms attempt to accomplish the following objectives:

a) route all wires in minimum distance fashion;

b) minimize the number of necessary vias (or corners);

c) minimize the variations in the numbers of wires in the routing channels.

Our model takes into consideration objectives a) and b), but by assuming the independence of wires trajectories, it does not completely accomplish objective c). The results in this paper, however, suggest that at least for $\bar{R} > 0(\ln N)$, the saving in wiring area by relaxing the wires trajectories independence assumption is only a constant factor.

5) Wires travel minimum distance: This assumption is crucial for the "causality" of our model. Moreover, it is in general a desired feature to have the shortest possible wire lengths on the chip. We do not, therefore, feel that it is necessary to relax this assumption.

(iii) Generalizations of the results.

Now recall Lemma 6. It is interesting to notice that only knowledge of the individual (marginal) distribution of the random variables (T_{ij}^H, T_{ij}^V) is needed for the proof. In fact, one can easily derive the following general result:

Let $X_1, X_2, \cdots, X_n$ be nonnegative random variables (not necessarily independent) such that $X_i \sim P_x$, for all i. Assume that $P(X > x) \triangleq a(x) \underset{x \to \infty}{\longrightarrow} 0$ faster than $1/x$, then if $Na(t_N) \to 0$ as $N \to \infty$,

$$p\Big\{ \max_{1 \leqslant i \leqslant N} X_i < t_N \Big\} \to 1, \qquad \text{as } N \to \infty. \tag{59}$$

In the particular case that P_x is Poisson, $t_N = 0(\ln N)$.

A generalization of our two-dimensional model to three dimensions (or to K dimensions) is straightforward. In this case, one decomposes the wiring process into the sum of 6 processes (or in general $2K$ processes) in the diagonal directions.

(iv) Possible alternative applications.

Our model may apply to traffic flow in a large city of $N \times N$ blocks. In such a case X_{ij} represents the number of cars per unit time starting at any block (i, j). $\bar{R}$ is the average number of blocks traveled by cars. An interesting observation is that the widest street should be $0(\ln N)$ lanes for a city of N^2 blocks!

Communication networks are another potential application for the model. The lattice points would represent the different communications terminals, X_{ij} is the number of packets emanating from terminal (i, j) per unit time, and $\bar{R}$ is the average number of hops traveled by a packet. Again, the largest bandwidth of any channel should be $0(\ln N)$.

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Qualitative Analysis of Nonlinear Quasi-Monotone Dynamical Systems Described by Functional-Differential Equations

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***Abstract*—This paper discusses properties related to the stability of a nonlinear quasi-monotone dynamical system described by a functional-differential equation $\dot{x}=F(x_t,t)+u(t)$. Specially, mathematical conditions which guarantee the same qualitative behavior inherent in a nonlinear off-diagonally monotone dynamical system $\dot{x}=f(x(t),t)+u(t)$ are discussed. We first consider the basic properties of solutions: lower and upper bound preservation and ordering preservation of solutions. By using these properties, we estimate the trajectory behavior by means of a partial ordering relation, and derive the following results: If F is independent of t, and u is a constant input, then every bounded solution converges to a unique equilibrium point x^* under some natural conditions. In addition, if F is a nonlinear functional with separate variables, then every solution converges to x^* under the same conditions; If $F(x_t,\cdot)$ and $u(\cdot)$ are periodic and have the same period ω, then, under certain natural conditions, there is a ω-periodic solution $x^*(\cdot)$, and every solution converges to it if it is a unique ω-periodic solution.**

I. Introduction

In the study of social-process models [1], price adjustment process models [2]–[4], and compartmental systems [5]–[11], we often encounter a dynamical system $\dot{x}=f(x,t)$ in which f has the off-diagonally monotone property. Furthermore, in also the stability analysis of nonlinear large scale composite systems by means of vector Lyapunov's functions, we encounter a differential inequality $\dot{v}\leqslant f(v)$ (the inequality means that $\dot{v}_i\leqslant f_i(v)$ for each i) in which f has the off-diagonally monotone property [11]–[14], and the problem of estimating its behavior is reduced to that of a differential equation $\dot{x}=f(x)$ by applying the comparison principle.

On the other hand, there are many systems which are not well modeled by ordinary differential equation. An often observed feature in the real world is the existence of time delays.

In this paper, we deal with a nonlinear dynamical system described by a functional-differential equation $\dot{x}=F(x_t,t)+u(t)$ in which u is an input, and F has the quasi-monotone property (which coincides with the off-diagonally monotone property if $F(x_t,t)=f[x(t),t]$).

It is well known that a linear off-diagonally monotone time-invariant system $\dot{x}_i=\sum_{j=1}^n a_{ij}x_j+u_i$ $(i=1,\cdots,n)$ has an asymptotically stable equilibrium point $x^*=-A^{-1}u\geqslant\theta$ (θ is the zero vector) for all $u\geqslant\theta$ if and only if $-A=-(a_{ij})$ is an M-matrix (see, for example, [15] and/or [16] concerning M-matrices). At the same time, it is also shown that a linear quasi-monotone time-invariant system $\dot{x}_i(t)=a_{ii}x_i(t)+\sum_{j\neq i}a_{ij}x_j(t-\tau_{ij})+u_i$ $(i=1,\cdots,n)$ has an asymptotically stable equilibrium point $x^*\geqslant\theta$ for all $u\geqslant\theta$ if $-A$ is an M-matrix [17], [18]. The significance of this is that the existence of a stable equilibrium point is not affected by that of time delays. One suspects that this insensitivity property is an intrinsic one not only of linear systems but also of nonlinear systems. In fact, it is shown that a nonlinear compartmental system with constant time

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