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Two Discrete Forms of the Jordan Curve Theorem

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The Jordan curve theorem is one of those frustrating results in topology: it is intuitively clear but quite hard to prove. In this note we will look at two discrete analogs of the Jordan curve theorem that are easy to prove by an induction argument coupled with some geometric intuition. One of the surprises is that when we discretize the plane we get two Jordan curve theorems rather than one, a consequence of the interplay between two natural products in the category of graphs. Topology in this context has been studied by Farmer in [2].

To state the discrete versions, we need to know what the discrete analog of the plane is and what plays the role of a simple closed curve. Since the plane is the topological product of two lines, we take as our discrete analog the product of two discrete lines. We will use undirected graphs for our analogs of spaces, with vertices for points and edges connecting points which are to be thought of as touching.

DEFINITION 1. A discrete n point line $[1, n]$ *is* a graph with vertices $\{1, 2, \dots, n\}$ and edges connecting each vertex to itself and to its successor. The discrete line L is a similar graph based on all of the integers.

DEFINITION 2. A discrete n point circle is a discrete n point line with n and 1 connected by an edge.

There are two important products in the category of graphs: the categorical product and the tight product. The tight product is used in building graphs using a sort of prime factorization in Behzad and Chartrand [1].

DEFINITION 3. The *product* of two graphs (V_1, E_1)fl(V_2, E_2) has the set $V_1 \times V_2$ as vertices and has (v_1, v_2) connected to $(vi, v2)$ by the edge (e_1, e_2) if e_1 connects v_i and vi and e_2 connects v_2 and $v2$.

DEFINITION 4. The *tight product* of two graphs has $V_1 \times V_z$ as its set of vertices and has an edge connecting (v_1, v_2) and $(vi, v2)$ if and only if $U_1 = vi$ and there is an edge connecting v_2 and $v2$, or $U_z = v2$ and there is an edge connecting U_1 and v. We denote this as $(V_1, E_1) 0 (V_Z, E2)$.

If we take the product of two lines we get a patch of the plane with points connected which are nearest neighbors vertically, horizontally, or diagonally. If we take the tight product we leave out the diagonal connections.

The analog of continuous functions will be mappings of graphs: vertices are taken to vertices and edges to edges. A closed curve is the image of a circle under a graph map. It is simple if the map also reflects adjacency; that is, if $c(v)$ has an edge connecting it with c(v') then U and v' had an edge connecting them too.

Simple curves then are forbidden to touch themselves, not just forbidden to cross themselves. This puts us in a position to state the two forms of the Jordan curve theorem.

THEOREM (Jordan curve theorem for tight closed curves). *If s is a simple closed curve with domain having at least 8 points in $L \square L$, then $L \times L \setminus \mathrm{im}(s)$ has exactly two product path components.*

THEOREM (Jordan curve theorem for product closed curves). *If s is a simple closed curve with domain having at least 4 points in $L \sqcap L$, then $L \times L \setminus \mathrm{im}(s)$ has exactly two tight path components.*

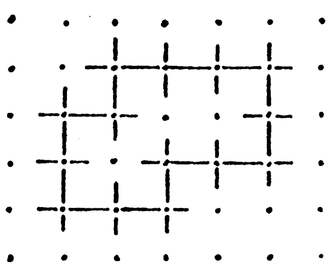

FIG. 1.

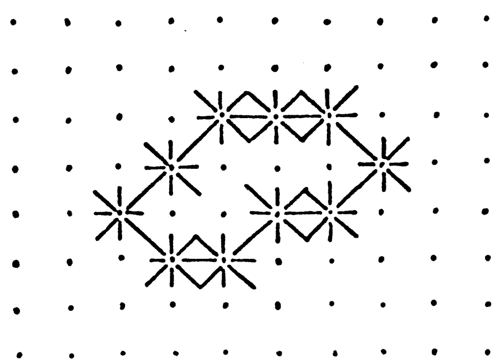

FIG. 2.

Notice that in Figure 1 the interior of the tight product closed curve is not connected in the tight product space. The interior is, however, connected in the product space, which allows diagonal connections. In the second illustration we have a simple closed curve in the product sense which fails to disconnect the categorical product space. If we use the tight product instead, then the interior is not connected to the exterior and each forms a connected set. The minimum size restriction eliminates the trivial cases in the next illustration.

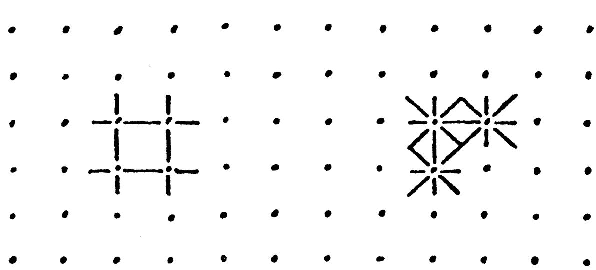

FIG. 3.

Proof (for product closed curves). Since a simple closed curve involves only a finite number of points we can move it into the first quadrant and guarantee that the coordinates of points are bigger than 0 and less than m for sufficiently large m. We define the *rank* of s as the triple (N, X, Y) where N is the number of distinct points in the closed curve and (X, Y) is the point in the closed curve with largest first coordinate X and largest second coordinate Y of the points of $\mathrm{im}(s)$ with that first coordinate. Ranks are ordered lexicographically. This is a well-ordering, so strong induction on rank is a valid proof technique.

The smallest simple closed curve for this theorem has $N = 4$. It forms a diamond surrounding a single point which forms the inside component. All other points are connected to the point $(0, 0)$ by a tight path. The requirement that a simple curve reflect adjacency eliminates other possible curves of length four. Thus the theorem is true for closed curves with length 4.

Now suppose that the theorem has been proved for all closed curves with rank less than (N, X, Y) and that s is a simple closed curve with rank (N, X, Y). We will reduce the rank by moving the point (X, Y) to $(X - 1, Y)$. The points in the closed curve s which were adjacent to (X, Y) could only be among $(X, Y - 1)$, $(X - 1, Y - 1)$, and $(X - 1, Y + 1)$. (Two points are adjacent to (X, Y) and they must be nonadjacent, hence, $(X - 1, Y)$ is not one of the possible points.) All of these are adjacent to $(X - 1, Y)$ so the result is still a closed curve, though it may not be a simple closed curve. Observe that moving this point reduces the rank. If the new closed curve is a simple closed curve then we are done since the interior of the original curve is the interior of the curve of lower rank with the point $(X - 1, Y)$, which is tight adjacent to it, added. The exterior of the original curve is the exterior of the new curve with the point (X, Y) removed. This is still tight connected since any tight path passing through (X, Y) in the exterior of the lower rank curve can take a detour through $(X, Y + 1)$, $(X + 1, Y + 1)$ and $(X + 1, Y)$.

There are two ways for the resulting closed curve to fail to be simple: either the point $(X - 1, Y)$ is adjacent to one of the points two steps away from (X, Y) in s, or it is adjacent to a point more than two steps away. If (X, Y) was $s(h)$ and $(X - 1, Y)$ is adjacent to $s(h - 2)$ then we can remove $s(h - 1)$. If (X, Y) was $s(h)$ and $(X - 1, Y)$ is adjacent to $s(h + 2)$ then we can remove $s(h + 1)$. Removing these points, if necessary, will further reduce the rank. The interior of the resulting curve is tight connected to (X, Y), so the interior of the original curve is tight connected. Any tight path passing through one of the points removed has a detour which avoids them and stays in the exterior. Figure 4 shows how this works for a typical case.

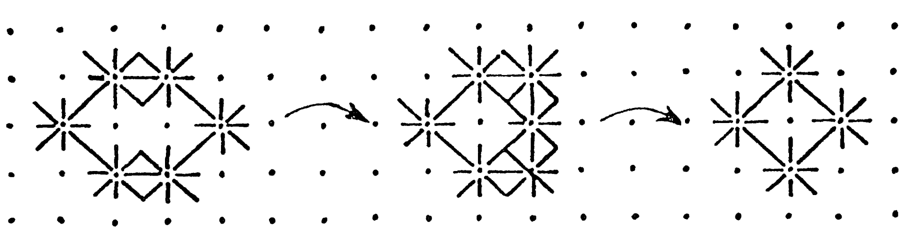

FIG. 4.

Suppose that (X, Y) is $s(h)$ and $(X - 1, Y)$ is adjacent to $s(k)$ where k is more than two away from h. Then by moving to $(X - 1, Y)$ we pinch the closed curve into two closed curves which have a tight path connecting their interiors which passes through the point $(X-1, Y)$ and each of which is strictly shorter than our original loop. (See Figure 5.) Since they have smaller ranks they each divide the product into exactly two tight pieces. The interior of s is then the union of the interiors of these two new closed curves plus the point $(X-1, Y)$. It remains to show that the exterior is tight path connected.

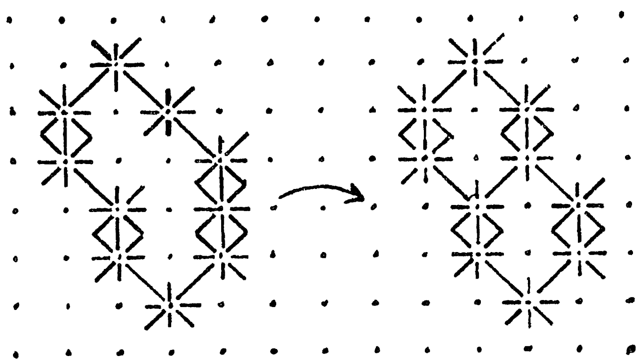

FIG. 5.

The exterior is the intersection of the exteriors of the two new closed curves. Call the new closed curves s_1 and s_2 and renumber so that the intersection points are at $t=0$ and $t=1$, with $(X-1, Y)=s_2(1)$. Let p and q be in $\operatorname{ext}(s_1) \cap \operatorname{ext}(s_2)$. Since $\operatorname{ext}(s_1)$ is tight path connected there is a tight path in $\operatorname{ext}(s_1)$ from p to q. If that path is also in $\operatorname{ext}(s_2)$ then nothing more needs to be done. If not then there are points p' and q' such that p' is the last point in the path for which the segment from p to p' is in $\operatorname{ext}(s_2)$ and q' is the first so that the segment from q' to q is in $\operatorname{ext}(s_2)$. It follows that both p' and q' are adjacent to points in s_2. Thus to prove the theorem it will suffice to show that the set E of all points adjacent to s_2 and in the exterior of both curves is tight connected.

Since the original curve was simple we know that $s_2(0)$ and $s_2(1)$ are the only points in s_2 that are adjacent to points in s_1. We will show that E is tight path connected by walking around s_2 starting at (X, Y) and observing what happens in each nine point patch with an element of s_2 at the center. It is not difficult to list all of the ways that a product path can pass through a nine-point patch (see Figure 6) and in all cases the points on either side of the path form tight path connected sets. Since s_2 is of finite length we can piece together such patches to see that the set E is tight path connected.

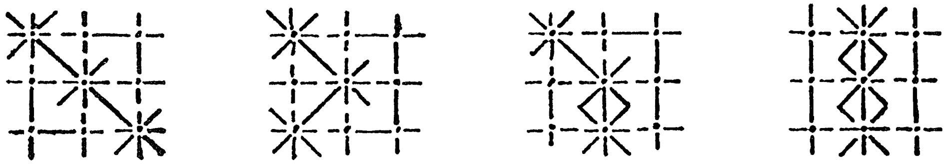

FIG. 6.

The proof of the theorem for simple closed curves in the tight product is a similar, though slightly less difficult, induction argument. The rank is defined the same way as in the product case. The reduction is done by moving the point (X, Y) to $(X - 1, Y - 1)$ which either gives a simple closed curve of lower rank or pinches the curve in two or gives a curve which can be shortened. Analysis of the possible nine-point patches again allows us to show that the exterior is connected.

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2. F. Farmer, Homology of Reflexive Relations, *Math. Japonica*, V. 20 #1 (1975) 21–28.