Two Extremal Problems in Graph Theory

Richard A. Brualdi* and Stephen Mellendorf†
Department of Mathematics
University of Wisconsin
Madison, WI 53706

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Abstract

We consider the following two problems. (1) Let t and n be positive integers with $n \geq t \geq 2$. Determine the maximum number of edges of a graph of order n that contains neither K_t nor $K_{t,t}$ as a subgraph. (2) Let r, t and n be positive integers with $n \geq rt$ and $t \geq 2$. Determine the maximum number of edges of a graph of order n that does not contain r disjoint copies of K_t. Problem 1 for $n < 2t$ is solved by Turán's theorem and we solve it for $n = 2t$. We also solve Problem 2 for $n = rt$.

1 Introduction

One of the best known results in extremal graph theory is the following theorem of Turán.

Theorem 1 *Let t and n be positive integers with $n \geq t \geq 2$. Then the maximum number of edges of a graph of order n that does not contain a complete subgraph K_t of order t equals*

$$\binom{n}{2} - \sum_{i=1}^{t-1} \binom{n_i}{2} \tag{1}$$

where $n = n_1 + \cdots + n_{t-1}$ is a partition of n into $t-1$ parts which are as equal as possible. Furthermore, the only graph of order n whose number of edges equals (1) that does not contain a complete subgraph K_t is the complete $(t-1)$-partite graph $K_{n_1,\ldots,n_{t-1}}$ with parts of sizes $n_1, \ldots, n_{t-1}$, respectively.

In general, the extremal graph $K_{n_1,\ldots,n_{t-1}}$ in Theorem 1 contains a complete bipartite subgraph $K_{t,t}$. This suggests the following problem.

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Problem 1 *Let t and n be positive integers with $n \geq t \geq 2$. Determine the maximum number of edges of a graph of order n that contains neither K_t nor $K_{t,t}$ as a subgraph.*

If $n < 2t$, then Problem 1 is equivalent to Turán's theorem. The case $n = 2t$ is settled in the next theorem.

If G and H are graphs, then their *sum* is the graph $G+H$ obtained by taking disjoint copies of G and H and putting an edge between each vertex of G and each vertex of H. A path of order n is denoted by P_n. The complement of a graph G is denoted by $\overline{G}$. A connected graph is *unicyclic* provided it has a unique cycle. It follows easily that a connected graph of order n is unicyclic if and only if it has exactly n edges. Recall that a set of vertices of a graph is *independent* provided no two of its vertices are joined by an edge. If n is an odd integer, then $\mathcal{H}_n$ denotes the collection of all unicyclic graphs of order n for which the maximum cardinality of an independent set equals $(n-1)/2$. Note that $\mathcal{H}_1$ is empty. The graphs in $\mathcal{H}_n$ are characterized in the final section.

Theorem 2 *Let t be a positive integer with $t \geq 3$. Then the maximum number of edges of a graph of order $2t$ that contains neither K_t nor $K_{t,t}$ as a subgraph equals*

$$\binom{2t}{2} - \frac{3t}{2} - 1 \text{ if } t \text{ is even,} \tag{2}$$

and equals

$$\binom{2t}{2} - t - 4 \text{ if } t \text{ is odd.} \tag{3}$$

If t is even, then the only graphs of order $2t$ that contain neither K_t nor $K_{t,t}$ as a subgraph and whose number of edges equals (2) are the graphs of the form $K_{2,\ldots,2} + \overline{H_a} + \overline{H_b}$ where a and b are odd integers with $a+b = t+2$, H_a is in $\mathcal{H}_a$ and H_b is in $\mathcal{H}_b$. If t is odd, then the only graphs of order $2t$ that contain neither K_t nor $K_{t,t}$ as a subgraph and whose number of edges equals (3) are the graphs of the form $K_{2,\ldots,2,4}$ and $K_{2,\ldots,2} + U$ where U is the graph obtained from $K_{3,3}$ by removing an edge, and the graphs $K_{1,3,3,3}$ and $K_{3,3} + \overline{P_4}$ for $t = 5$.

We prove Theorem 2 in the equivalent complementary form stated in the next theorem.

If G and H are graphs, then their *union* is the graph $G \cup H$ consisting of disjoint copies of G and H. If m is a positive integer, then mG is the graph consisting of m disjoint copies of G. We call a graph *bisectable* provided its vertices can be partitioned into two parts of equal size such that there are no edges between the two parts.

Theorem 3 *Let t be a positive integer with $t \geq 3$. Then the minimum number of edges of a graph of order $2t$ that does not contain an independent set of t vertices and is not bisectable equals*

$$\frac{3t}{2} + 1 \text{ if } t \text{ is even,} \tag{4}$$

and equals

$$t + 4 \text{ if } t \text{ is odd.} \tag{5}$$

If t is even, then the only graphs of order $2t$ that do not contain an independent set of t vertices and are not bisectable and whose number of edges equals (4) are the graphs of the form $(t/2-1)K_2 \cup H_a \cup H_b$ where a and b are odd integers with $a+b=t+2$, H_a is in $\mathcal{H}_a$ and H_b is in $\mathcal{H}_b$. If t is odd, then the only graphs of order $2t$ that do not contain an independent set of t vertices and are not bisectable and whose number of edges equals (5) are the graphs $(t-2)K_2 \cup K_4$ and $(t-3)K_2 \cup W$ where W is the graph obtained from $K_3 \cup K_3$ by inserting an additional edge, and the graphs $K_1 \cup 3K_3$ and $2K_3 \cup P_4$ when $t=5$.

Problem 2 *Let r, t and n be positive integers with $n \geq rt$ and $t \geq 2$. Determine the maximum number of edges of a graph of order n that does not contain r disjoint copies of K_t.*

If n is sufficiently large, then the solution to Problem 2 is contained in the following theorem of Simonovits [5] (see also page 346 of [1]).

Theorem 4 *Let r, t and n be positive integers with $t \geq 2$ and n sufficiently large. Then the unique graph of order n with the maximum number of edges that does not contain r disjoint copies of K_t is the graph $G = K_{r-1} + H$ where $H = K_{n_1,\ldots,n_{t-1}}$ and $n-r+1 = n_1 + \cdots + n_{t-1}$ is a partition of $n-r+1$ into $t-1$ parts as equal as possible.*

The smallest instance of Problem 2 occurs when $n = rt$ and this is settled in the next theorem. By considering complements, we obtain the following equivalent formulation of Problem 2 in this case: *Determine the minimum number of edges of a graph of order rt that is not r-partite with parts of size t.*

Theorem 5 *Let r and t be positive integers with $t \geq 2$. Then the minimum number of edges of a graph of order rt that is not r-partite with parts of size t equals*

$$\min\{\binom{r+1}{2}, rt-t+1\} = \begin{cases} \binom{r+1}{2} & \text{if } r \leq 2t-2 \\ rt-t+1 & \text{if } r \geq 2t-2. \end{cases} \tag{6}$$

The only graphs of order rt that are not r-partite with parts of size t and whose number of edges equals (6) are the graphs of the form

$$K_{r+1} \cup (rt-r-1)K_1 \text{ for } r \leq 2t-2, \tag{7}$$

and the graphs of the form

$$K_{1,rt-t+1-p} \cup pK_2 \cup (t-2-p)K_1, \; (0 \leq p \leq t-2) \text{ for } r \geq 2t-2. \tag{8}$$

In the proof of Theorem 5 we shall make use of the following difficult result of Hajnal and Szemerédi [3] (see also page 351 of [1]).

Theorem 6 *Let r and t be positive integers, and let G be a graph of order rt each of whose vertices has degree at most $r-1$. Then G is r-partite with parts of size t.*

To conclude this introduction we note that by use of the adjacency matrix, each of Problems 1 and 2 can be formulated in terms of matrices. If A is a matrix of order n and α and β are subsets of $\{1, 2, \ldots, n\}$, then $A[\alpha, \beta]$ is the submatrix of A determined by the rows indexed by α and columns indexed by β.

Problem 1 is equivalent to the following.

Problem 3 *Let t and n be positive integers with $n \geq t \geq 2$. Determine the minimum number $s(n, t)$ such that every symmetric $(0, 1)$-matrix of order n with 0's on the main diagonal and with at least $s(n, t)$ 0's above the main diagonal contains a zero submatrix $A[\alpha, \beta]$ of order t where either $\alpha = \beta$ or $\alpha \cap \beta = \emptyset$.*

From Theorem 2 we obtain that $s(2t, t) = \binom{2t}{2} - \frac{3t}{2}$ if t is even and $s(2t, t) = \binom{2t}{2} - t - 3$ if t is odd.

Problem 3 can be viewed as a symmetric version of the famous problem of Zarankiewicz: Let $1 \leq c \leq a$ and $1 \leq d \leq b$. Determine the minimum number $Z(a, b; c, d)$, such that each $a \times b$ matrix with $Z(a, b; c, d)$ zeros contains an $c \times d$ zero submatrix.

2 Proofs

In this section we give the proofs of Theorems 3 and 5.

Lemma 7 *Let G be a graph of order n. If G is a tree, then G has an independent set of size $\lceil n/2 \rceil$. If G is a unicyclic graph, then G has an independent set of size $\lfloor n/2 \rfloor$.*

Proof. A tree of order n is a bipartite graph and has either disjoint independent sets of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, or an independent set of size $\lceil n/2 \rceil + 1$. If G is unicyclic, then G can be obtained from a tree of order n by adding an edge and hence G contains an independent set of size $\lfloor n/2 \rfloor$. □

Lemma 8 *Let G be a graph of order $2t$ such that G is not bisectable. Assume that G has a component T which is a tree of odd order k and a component B of order l which is not a tree. Let G' be the graph obtained from G as follows:*

(i) *If B is a unicyclic graph of odd order, then replace $T \cup B$ with P_{k+l};*

(ii) *Otherwise, remove any edge of B which does not disconnect B and replace T by a cycle of order k.*

Then G' is not bisectable and G' does not contain a larger independent set than G.

Proof. The nonbisectability of G clearly implies the nonbisectability of G'. First assume that B is unicyclic of odd order. By Lemma 7, T has an independent set of size $(k+1)/2$ and B has an independent set of size $(l-1)/2$. Hence $T \cup B$ has an independent set of size $(k+l)/2$. Since the maximum size of an independent set of P_{k+l} equals $(k+l)/2$, G' does not contain a larger independent set than G. Now assume that B is not unicyclic of odd order. Removing an edge of a graph increases the maximum size of an independent set by at most 1. Since T has an independent set of size $(k+1)/2$ and the maximum size of an independent set of a cycle of odd order k equals $(k-1)/2$, G' does not contain a larger independent set than G. □

In the proof of Theorem 3 we shall make use of the following result [2].

Lemma 9 *Let $a_1, a_2, \ldots, a_m$ be positive integers with $\sum_{i=1}^{m} a_i = b$. If $m > \lfloor b/2 \rfloor$, then for each positive integer k with $k \leq b$ there exists a subset I of $\{1, 2, \ldots, m\}$ such that $k = \sum_{i \in I} a_i$.*

Proof of Theorem 3 for t odd. Let G be a graph of order $2t$ with at most $t+3$ edges. By applying Theorem 1 to $\overline{G}$, if G does not contain an independent set of size t then $G = 2K_3 \cup (t-3)K_2$ and hence G is bisectable. The graphs for t odd given in the statement of the theorem have $t+4$ edges, do not contain an independent set of size t and are not bisectable. This proves the first assertion of the theorem for t odd.

We now assume that G has exactly $t+4$ edges, and that G does not contain an independent set of size t and is not bisectable.

Case 1: Each component of G which is a tree equals K_2. Then at least $t-4$ of the components of G are trees and thus are K_2's. Since G has $t+4$ edges, at least one component of G is not a tree and hence G has at least $t-3$ components. Since G does not have an independent set of size t, G has at most $t-1$ components.

Case 1a: G has exactly $t-3$ components. Thus $G = (t-4)K_2 \cup F$ where F is a unicyclic graph of order 8. By Lemma 7, F has an independent set of size 4, and thus G has an independent set of size t, a contradiction.

Case 1b: G has exactly $t-2$ components. Then either $t-4$ or $t-3$ of the components of G are trees. Suppose that G has $t-4$ trees. Then G has exactly two components G_1 and G_2 which are not trees (and so are unicyclic). If G_1 and G_2 have even order (and so order equal to 4), then using Lemma 7, we see that G has an independent set of size t, a contradiction. If G_1 and G_2 have odd order (and so of orders 3 and 5), then G is bisectable, another contradiction.

Now suppose that G has $t-3$ trees. Then G has exactly one component E which is not a tree, and this component has order 6 and has 7 edges. Since G does not have an independent set of size t, E does not have an independent set of size 3. It is now easy to check that E must be the graph H in the statement of the theorem. Thus $G = (t-3)K_2 \cup H$.

Case 1c: G has exactly $t-1$ components. Since G does not have an independent set of size t, each of its components is a complete graph. It follows easily that $G = (t-2)K_2 \cup K_4$.

Case 2: There is a component T of G which is a tree of order $2m$ with $m \geq 2$. We replace T in G by mK_2 and obtain a graph G' of order $2t$ with at most $t+3$ edges. It follows from Lemma 7 that the maximum size of an independent set of G' is at most $t-1$. By Theorem 1, $G' = 2K_3 \cup (t-3)K_2$ and hence $G = 2K_3 \cup P_4 \cup (t-5)K_2$. Since G is not bisectable, $t = 5$ and $G = 2K_3 \cup P_4$.

Case 3: There is a component of G which is a tree of odd order. We repeatedly apply the transformation in Lemma 8 to obtain a graph $G^\dagger$ none of whose components is a tree of odd order. By Lemma 8, $G^\dagger$ is a graph of order $2t$ with $t+4$ edges which does not contain an independent set of size t and is not bisectable. Applying what we have proved in Cases 1 and 2 to $G^\dagger$, we conclude that $G^\dagger$ equals $(t-2)K_2 \cup K_4$, $(t-3)K_2 \cup W$, or $2K_3 \cup P_4$. First suppose that $G^\dagger$ was obtained from a graph G^* by applying the transformation (i) in Lemma 8. Then one of the components of $G^\dagger$ is a path of even length at least 4. Hence $G^\dagger = 2K_3 \cup P_4$. This implies that $G^* = K_1 \cup 3K_3$. Since G^* cannot be obtained by applying a transformation in Lemma 8, $G = K_1 \cup 3K_3$. Now suppose that $G^\dagger$ was obtained from a graph G^* by applying the transformation (ii) in Lemma 8. Then one of the components of $G^\dagger$ must be a cycle of odd length and again $G^\dagger = 2K_3 \cup P_4$. This implies that G^*, and hence G, has an independent set of size 5. Since G has 10 vertices, this is a contradiction. □

Proof of Theorem 3 for t even. Let G be a graph of order $2t$ with at most $3t/2$ edges. Suppose that G does not contain an independent set of size t and G is not bisectable. By arbitrarily adding new edges to G we obtain a graph G^* with $3t/2+1$ edges with the same properties. Suppose G^* is one of the graphs $(t/2-1)K_2 \cup H_a \cup H_b$ given in the theorem. If we remove an edge of one of the K_2's of G^*, then we obtain an independent set of size t. Suppose that we remove an edge from, say, H_b. If the removal disconnects H_b, we obtain a bisectable graph. Otherwise we obtain an independent set of size t by Lemma 7. Therefore to complete the proof of the theorem it suffices to show that the only graphs of order $2t$ with $3t/2+1$ edges which do not contain an independent set of size t and are not bisectable are the graphs $(t/2-1)K_2 \cup H_a \cup H_b$ given in the theorem.

We now assume that G has exactly $3t/2+1$ edges, and that G does not contain an independent set of size t and is not bisectable. Then G has at least $t/2-1$ components which are trees. Since G does not have an independent set of size t, Lemma 7 implies that G has at least $t/2+1$ components.

First suppose that G has at least $t/2$ components of even order. Let $2m_1, \ldots, 2m_{t/2}$ be the orders of $t/2$ components of G with even order. By the pigeonhole principle there is a subset I of $\{1, \ldots, t/2\}$ such that $\sum_{i \in I} m_i$ is a multiple of $t/2$. Since $\sum_{i=1}^{t/2} 2m_i < 2t$, it follows that $\sum_{i \in I} 2m_i = t$ and hence G is bisectable, a contradiction. Thus G has at most $t/2-1$ components of even order.

Case 1: No component of G is a tree of odd order. Thus exactly $t/2-1$ of the components of G are trees and each has even order, and all other components are unicyclic of odd order. If there are at least four components of odd order, then replacing the orders of two of these components by their sum and arguing as above, we again contradict the nonbisectability of G. Hence G has exactly two components of odd order. Let the order of the trees be $2m_1, \ldots, 2m_{t/2-1}$. Suppose that at least one tree has order greater than 2 and hence $\sum_{i=1}^{t/2-1} 2m_i \geq t$. Since also $\sum_{i=1}^{t/2-1} 2m_i \leq 2t-6$, we have

$$\frac{t}{2} \leq \sum_{i=1}^{t/2-1} m_i \leq t-3.$$

It follows from Lemma 9 that there exists a subset I of $\{1, \ldots, t/2-1\}$ such that $\sum_{i \in I} m_i = t/2$. Once again we contradict the nonbisectability of G. We now conclude that $G = (t/2-1)K_2 \cup H_a \cup H_b$ where a and b are odd integers with $a+b = t+2$, H_a is in $\mathcal{H}_a$ and H_b is in $\mathcal{H}_b$.

Case 2: There is a component of G which is a tree of odd order. We repeatedly apply the transformation in Lemma 8 to obtain a graph $G^\dagger$ none of whose components is a tree of odd order. By Lemma 8, $G^\dagger$ is a graph of order $2t$ with $3t/2+1$ edges which does not contain an independent set of size t and is not bisectable. Applying what we have proved in Case 1 to $G^\dagger$, we conclude that $G^\dagger$ is of the form $(t/2-1)K_2 \cup H_a \cup H_b$ given in the theorem. Since $G^\dagger$ does not contain a component which is a path of even order at least 4, it was not obtained by applying the transformation (i) in Lemma 8. Thus $G^\dagger$ was obtained from a graph G^* by applying the transformation (ii) in Lemma 8. Then one of H_a and H_b, say H_a, is a cycle of odd length. Hence $G^* = T \cup H_b^* \cup (t/2-1)K_2$ where T is a tree of order a and H_b^* is obtained by adding a new edge to H_b. By Lemma 7, T has an independent set of size $(a+1)/2$, and by an extension of the proof of Lemma 7, H_b^* has an independent set of size $(b-1)/2$. Therefore G^*, and hence G, has an independent set of size t, a contradiction. □

Proof of Theorem 5. We first prove by induction on r that a graph G of order rt whose number of edges is at most

$$\min\{\binom{r+1}{2} - 1, rt - t\}$$

is r-partite with parts of size t. If $r = 1$, then G has no edges and the conclusion holds. Now let $r > 1$. By Theorem 6 we can assume that G has a vertex v whose degree is at least r. Since G has at most $rt - t$ edges, the number of connected components of G is at least t. Thus there is an independent set A of vertices such that $v \in A$ and $|A| = t$. Let G' be the graph obtained from G by removing the vertices in A. Since the degree of v is at least r, G' has at least r fewer edges than G. Hence the number of edges of G' is at most

$$\binom{r+1}{2} - 1 - r = \binom{r}{2} - 1.$$

If $r - 1 \leq 2t - 2$, then

$$\min\{\binom{r}{2} - 1, (r-1)t - t\} = \binom{r}{2} - 1$$

and hence by induction G' is $(r-1)$-partite with parts of size t. Assume that $r-1 > 2t-2$. Since $t \geq 2$, this implies that $r \geq t$ and thus G' has at least t fewer edges than G. Hence the number of edges of G' is at most

$$(rt - t) - t = (r-1)t - t = \min\{\binom{r}{2} - 1, (r-1)t - t\}.$$

Again by induction G' is $(r-1)$-partite with parts of size t. Since A is an independent set of t vertices of G, G is r-partite with parts of size t.

The graphs in (7) and (8) have order rt, are not r-partite with parts of size t, and their number of edges is given by (6), and hence the first assertion of the theorem follows. We now prove that the graphs in (7) and (8) are the only graphs with these properties.

Let G be a graph of order rt with number of edges given by (6) which is not r-partite with parts of size t. Since G has at most $rt-t+1$ edges, G has at least $t-1$ connected components. By Theorem 6, G has a vertex v of degree at least r. First suppose that $r < 2t-2$. Then

$$\binom{r+1}{2} < rt-t+1$$

implying that G has at most $rt-t$ edges and hence at least t connected components. Since G has at least t components, for each vertex $u \neq v$ which is not adjacent to v, there is an independent set of size t containing both u and v. G cannot have an independent set of size t which is incident with at least $r+1$ edges, since otherwise by the first assertion of the theorem, G is r-partite with parts of size t. We now conclude that the degree of v equals r, the component containing v has order $r+1$, and every other component has order one. It now follows that G is of the form (7).

Now suppose that $r > 2t-2$. Then G has exactly $rt-t+1$ edges and at least $t-1$ of its components are trees. Also G cannot have an independent set of size t containing v, since otherwise by the first assertion of the theorem, G is r-partite with parts of size t. Thus G has exactly $t-1$ components, v is adjacent to each vertex in its component, and every other component is a complete graph. Hence G has the form (8).

Finally, suppose that $r = 2t-2$. If G has at least t components, then as in the case $r < 2t-2$, G is of the form (7). Thus we may assume that G has exactly $t-1$ components. Since G has $rt-t+1$ edges, each of its components are trees. G cannot have an independent set of size t which is incident with at least $r+1$ edges, since otherwise by the first assertion of the theorem, G is r-partite with parts of size t. This implies that v is adjacent to every vertex in its component, and every other component is a complete graph. Therefore G has the form (8). □

3 Characterization of $\mathcal{H}_n$

In our characterization of the graphs in $\mathcal{H}_n$ we use the following lemma which is a simple consequence of the well known theorem of König. Recall that a *perfect matching* of a graph is a set of pairwise vertex disjoint edges which touch every vertex of the graph.

Lemma 10 *Let G be a bipartite graph of order $2t$. Then the maximum cardinality of an independent set of G equals t if and only if G has a perfect matching.*

Let H be a unicyclic graph of order n. Then H contains a unique cycle $C = (x_1, x_2, \ldots, x_p, x_1)$. The connected components of the subgraph of H induced by the vertices not belonging to C are trees. These trees are called the *trees of the unicyclic graph H*. Each such tree is joined by an edge to exactly one vertex x_i of C.

Theorem 11 *Let $n \geq 3$ be an odd integer. Then a unicyclic graph of order n is in $\mathcal{H}_n$ if and only if its unique cycle has odd length and each of its trees is of even order and has a perfect matching.*

Proof. Let G be a unicyclic graph of order n and let the unique cycle C of G have length p. First assume that p is odd and each of the trees of G is of even order and has a perfect matching. Each independent set of G contains at most half of the vertices of each of its trees by Lemma 10 and at most $(p-1)/2$ vertices of C. Thus the maximum cardinality of an independent set of G is at most $(n-1)/2$ and by Lemma 7, equals $(n-1)/2$. Therefore G is in $\mathcal{H}_n$.

Now assume that G is in $\mathcal{H}_n$. Suppose to the contrary that p is even. Then C has exactly two independent sets A and B of size $p/2$. Since n is odd, the number of trees of odd order of G is also odd. Without loss of generality assume that A is joined to fewer trees of odd order than B. Each tree of order b joined to B has by Lemma 7 an independent set of size $\lceil b/2 \rceil$. Each tree of order a joined to A has by (the proof of) Lemma 7 an independent set of size $\lfloor a/2 \rfloor$ none of whose vertices is joined to A. These independent sets along with A give an independent set of G of size greater than $(n-1)/2$, contradicting the assumption that G is in $\mathcal{H}_n$. Hence p is odd.

Now suppose to the contrary that at least one of the trees of G has odd order.Let q_i be the number of trees of odd order joined to vertex x_i of C $(i = 1, \ldots, p)$, and let $q = q_1 + \cdots + q_p$ be the number of odd trees of G. Let $\mathcal{I}$ be the set consisting of the p independent sets of C of size $(p-1)/2$. Each vertex of C is contained in exactly $(p-1)/2$ sets of $\mathcal{I}$. The average number of trees of odd order joined to the sets in $\mathcal{I}$ equals

$$\begin{aligned}
\frac{1}{p}\sum_{I\in\mathcal{I}}\sum_{x_i\in I} q_i &= \frac{1}{p}\sum_{i=1}^{p}\sum_{\{I\in\mathcal{I}:x_i\in I\}} q_i \\
&= \frac{1}{p}\sum_{i=1}^{p}\frac{p-1}{2}\,q_i \\
&= \frac{p-1}{p}\cdot\frac{q}{2} \\
&< \frac{q}{2}.
\end{aligned}$$

Hence there exists a set A in $\mathcal{I}$ which is joined to fewer than $q/2$ trees of odd order. As in the preceding paragraph we obtain an independent set of G of size greater than $(n-1)/2$, contradicting the assumption that G is in $\mathcal{H}_n$. Thus all the trees of G have even order.

If one of the trees of G does not have a perfect matching, then using Lemma 10 we again construct an independent set of size greater than $(n-1)/2$. Hence each tree of G has a perfect matching. □

Various characterizations of trees of even order are listed in [4].

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