REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 1, números 1, 2, 3; 1988.
http://dx.doi.org/10.5209/rev_REMA.1988.v1.n1.18176

Two geometric constants for operators acting on a separable Banach space

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ABSTRACT. The main result of this paper is the following: A separable Banach space X is reflexive if and only if the infimum of the Gelfand numbers of any bounded linear operator defined on X can be computed by means of just one sequence of nested, closed, finite codimensional subspaces with null intersection.

INTRODUCTION

Let A be an operator from a separable Banach space X into another Banach space Y. For every Markushevich basis of X, (a_n), we define two numbers $h_{A,(a_n)}$, $H_{A,(a_n)}$, which give some geometrical insight about the space X and also the operator A. In fact reflexivity of X is characterized by stability of $H_{A,(a_n)}$ through changes of the M-basis, for every operator A, as theorem 1 states.

In the framework of reflexive Banach spaces these constants will be denoted simply H_A and h_A and they coincide respectively with the infimum of the Gelfand numbers of A, and with a precise lower bound of the Bernstein numbers of A defined by Zemanek in [11]. So we come to the conclusion stated in theorem 2, that the infimum of the Gelfand numbers can be computed by means of a nested sequence of closed finite codimensional subspaces of null intersection.

In the third part we relate these numbers also with the spectral properties of A. Finally we see that for the particular case of a Hilbert space X, they are exactly the maximum and the minimum of the limit points of the spectrum of $(A^*A)^{1/2}$.

1980 Mathematics Subject Classification (1985 revision): 47A10, 47A53, 46B10, 40A05.
Editorial de la Universidad Complutense. Madrid, 1988.

NOTATIONS, DEFINITIONS AND REMARKS

We denote a Banach space by letters such as X, Y. The symbol [ ] will stand for closed linear span, and the symbol $\xrightarrow{w}$ for weak convergence. An M-basis will be the short name of a Markushevich basis, i.e. a fundamental sequence (a_n) in a separable Banach space X, for which there exists another sequence (a_n^*) in the dual X^*, called the conjugate sequence, with $a_m^*(a_n)=\delta_{mn}$ and such that $a_n^*(x)=0$ $\forall n\in N$, implies $x=0$. It is a well known result that *M-bases always exist* in a separable Banach space.

An M-basis (a_n) of X is called shrinking if for every f of X^*

$$\|f|_{[a_n,\ldots]}\| \underset{n\to\infty}{\to} 0$$

James has proven that this condition is equivalent to completeness of (a_n^*) in X^*, that is $[(a_n^*)]=X^*$.

The set of all bounded linear operators from X into Y will be called $B(X, Y)$, and $\Phi_+(X, Y)$ will stand for the set of semi-Fredholm + operators, i.e. those bounded linear operators with closed range and finite dimensional kernel.

1. Let X be a separable Banach space, $A\in B(X, Y)$ and let (a_n) be an M-basis of X. We define the two numbers:

$$H_{A,(a_n)}=\inf_n\|A|_{[a_n,\ldots]}\|$$

and

$$h_{A,(a_n)}=\sup_n m(A|_{[a_n,\ldots]})$$

where m stands for minimum modulus, i.e.

$$m(A)=\inf_{\|x\|=1}\|Ax\|$$

The first of these numbers leads us to a characterization of reflexivity of X, as can be seen in

Theorem 1. *The following statements are equivalent:*
1. *X is reflexive.*
2. *For every bounded linear operator from X into an arbitrary Banach space Y, $H_{A,(a_n)}$ is independent of the M-basis (a_n).*
3. *For every $f\in X^*$, $H_{f,(a_n)}$ is independent of the M-basis (a_n).*

Before giving the proof of the theorem we state three lemmas, which are easy to prove.

Lemma 1. *If X is reflexive and separable, (a_n) an M-basis of X and $x_n \in [a_n, \ldots]$ with $\|x_n\|=1$, then $x_n \xrightarrow{w} 0$.*

Lemma 2. *Let (x_n) be a normalized sequence in a Banach space X. Then $x_n \xrightarrow{w} 0$ if and only if for every finite codimensional, closed subspace E, there exists (x'_n) in E, with $x'_n \xrightarrow{w} 0$, $\|x'_n\|=1$ and $\|x_n - x'_n\| \to 0$.*

It is a well known result, due to Zippin, that a Banach space X with a basis is reflexive, if and only if every basis of X is shrinking. The following lemma is a slight modification of this, not requiring the space to have a basis.

Lemma 3. *A separable Banach space X is reflexive if and only if every M-basis of X in shrinking.*

Proof. If X is reflexive a closed subspace F of X^* which is total on X must be the whole of X^*. Thus, for every M-basis (a_n), $[(a_n^*)] = X^*$ and equivalently (a_n) is shrinking.

Conversely, if X is non reflexive two cases arise. First, if X^* is non separable, no M-basis can be shrinking. Second, if X^* is separable, by the result of Gaposkin and Kadets, (see [4] p. 120), for every total subspace F of X^* there exists an M-basis (a_n) of X such that $[(a_n^*)] = F$. Thus (a_n) is a non shrinking M-basis.

Proof of theorem 1

1) $\to$2) Let (a_n), (b_n) be two different normalized, M-bases of X. Denote by $\mu_n = \|A|_{[a_n, a_{n+1}, \ldots]}\|$. *Clearly* $\mu_n \geqslant \mu_{n+1} \geqslant \cdots \geqslant 0$ and analogously $\mu'_n = \|A|_{[b_n, \ldots]}\|$.

Let (ε_n) be a decreasing sequence of real numbers with $\varepsilon_n \to 0$. Choose for every n, $x_n \in [a_n, \ldots]$ such that $\|x_n\|=1$ and $|\mu_n - \|Ax_n\|| < \varepsilon_n$. By lemma 1, $x_n \xrightarrow{w} 0$. For a fixed $p \in N$, take into account the descomposition $X = [b_1, \ldots, b_{p-1}] \oplus [b_p, \ldots]$ and choose (x'_n) in $[b_p, \ldots]$, such that $\|x'_n\|=1$ and $\|x_n - x'_n\| \to 0$.

Clearly $|\|Ax_n\| - \|Ax'_n\|| \leqslant \|A(x_n - x'_n)\| \to 0$ and $\|Ax'_n\| \to H_{A,(a_n)}$. For every $n \in N$, $\|Ax'_n\| \leqslant \|A|_{[b_p, \ldots]}\| = \mu'_p$ and thus $H_{A,(a_n)} \leqslant \mu'_p$ $(\forall p \in N)$. Therefore

$$H_{A,(a_n)} \leqslant \inf_p \mu'_p = H_{A,(b_n)}$$

Changing the roles of (a_n) and (b_n), the converse inequality is obtained, so $H_{A,(a_n)} = H_{A,(b_n)}$, which we will call H_A.

2$\to$3 is obvious.

$3 \rightarrow 1$ Assume X non reflexive. By lemma 3 there is a non shrinking M-basis (a_n) in X. Take $g \notin [(a_n^*)]$ and take a biorthogonal system (c_n, c_n^*) with

$[(c_n)_{n=1}^{\infty}] = \text{Ker } g$ and (c_n^*) total on Ker g. Choose then $b_1 \in X$ such that

$g(b_1) = 1$ and $b_n = c_{n-1}$ for $n \geqslant 2$. Now (b_n) is an M basis of X with conjugate system (b_n^*) given by $b_1^* = g$ and $b_{n+1}^* = c_n^* - c_n^*(b_1)g$, $n \geqslant 2$. Since $g \in [(b_n^*)]$, $H_{g,(b_n)} = 0$. On the other hand $g \notin [(a_n^*)]$, which implies $H_{g,(a_n)} \neq 0$. Thus $H_{g,(b_n)} \neq H_{g,(a_n)}$.

Remark 1. A parallel to the theorem, setting $h_{A,(a_n)}$ instead of $H_{A,(a_n)}$, does not hold. Although 1)→2) could be proved with the same argument, in 3)→1) we would obtain $h_{f,(a_n)} = 0$ for every (a_n), not depending on reflexivity of X.

Remark 2. Recall the following result of Milman and Tumarkin: «For a separable Banach space X and a sequence (E_n) of closed subspaces of X such that codim $E_n = n$, $E_n \supset E_{n+1}$ $\forall n \in N$, and $\bigcap_n E_n = \{0\}$, there exists a biorthogonal system (a_n, a_n^*) such that $[a_{n+1}, a_{n+2}, \ldots] = E_n$». (see [4], thm 1.6). According to it, theorem 1 could be stated without reference to an M-basis, only with a sequence of nested subspaces of finite codimension and null intersection.

From now on we abreviate h_A and H_A for $h_{A,(a_n)}$ and $H_{A,(a_n)}$ respectively, taking for granted that we are dealing with a separable, reflexive, Banach space X.

2. We relate now h_A and H_A with s-numbers and some characteristics given by Zemanek in [11]. First recall that the Gelfand numbers of an operator A are $c_n(A) = \inf\{\|A|_W\|, \text{ codim } W < n\}$, and that the Bernstein numbers are $u_n(A) = \sup\{m(A|_W), \dim W \geqslant n\}$.

The sequences $c_n(A)$ and $u_n(A)$ are decreasing, bounded from below by

$$c(A) = \inf\{\|A|_W\|, \text{ codim } W < \infty\}$$

and

$$u(A) = \sup\{m(A|_W), \dim W = \infty\}$$

respectively. Also note that

$$G(A) = \inf\{\|A|_W\|, \dim W = \infty\} \leqslant c(A)$$

and

$$B(A) = \sup\{m(A|_W), \text{ codim } W < \infty\} \leqslant u(A)$$

In the quoted paper Zemanek obtains nice results concerning these numbers, and Kolmogorof numbers, Mityagin numbers, and their infima; he compares them with each other and with geometrical concepts such as Semi-Fredholm radius, reduced minimum modulus, etc. In the frame of separable reflexive Banach spaces we relate them with h_A and H_A as follows:

Proposition 1. *Let X be a separable reflexive Banach space and A an operator from X into any Banach space Y. Then $H_A = c(A)$ and $h_A = B(A)$.*

Proof. Fix an M-basis (a_n) and define as before $\mu_n = \|A|_{[a_n,\dots]}\|$. Since $c_n(A) = \inf\{\|A|_W\|, \text{ codim } W<n\}$, it follows that $c_n(A) \leqslant \mu_n$, $\forall n \in N$. Therefore

$$c(A) = \inf_n c_n(A) \leqslant \inf_n \mu_n = H_A$$

For the converse inequality, fix a real number $\varepsilon>0$. Take $K\in N$ such that $c_K(A)<c(A)+\varepsilon$ and take also a subspace W with codim $W<k$ and $\|A|W\|<c(A)+\varepsilon$. Any M-basis of W can be extended in order to obtain an M-basis of X (see [4] thm. 1.8). Thus, assume $(a_n)_{n=k}^{\infty}$ is an M-basis of W and extend it to an M-basis of X, $(a_n)_{n=1}^{\infty}$. We have $H_A \leqslant \|A|_{[a_k,\dots]}\| < c(A)+\varepsilon$. This implies $H_A < c(A)$, and so $H_A = c(A)$.

A similar argument proves $h_A = B(A)$.

Now we can rewrite theorem 1 as follows:

Theorem 2. *A separable Banach space X is reflexive if and only if for every decreasing sequence of closed subspaces (E^n) such that*

a) $n = \text{codim } E^n$.

b) $\bigcap_n E^n = \{0\}$.

and for every operator A from X into another Banach space Y, $\inf_n \|A|_{E^n}\| = c(A)$, where $c(A)$ is the infimum of the Gelfand numbers of A.

Proof. Just take into account Remark 2, and Proposition 2.

We can say further that when X is separable and reflexive, $B(A)$ can also be computed by means of a nested sequence of closed subspaces satisfying the same conditions as those of theorem 2.

Open questions. 1) What can be said when separability of X is dropped? 2) Why does reflexivity establish such a difference?

3. We relate these constants now with spectral theory. The real numbers of the interval $[h_A, H_A]$ can be characterized as follows:

Theorem 3. *Let X be a separable reflexive Banach space, A an operator defined from X into an arbitrary Banach space Y. Then $\lambda \in [h_A, H_A]$ if and only if there exists (x_n) in X, with $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$, such that $\|Ax_n\| \to \lambda$.*

Proof. Let $\lambda \in (h_A, H_A)$ and let (a_n) be an M-basis of X. For every $n \in N$, let $v_n = \inf\{\|Ax\|;\ x \in [a_n, ...],\ \|x\| = 1\}$. Then $v_n \leqslant h_A < \lambda < H_A \leqslant \mu_n = \|A|_{[a_n, ...]}\|$. Take now $z_n,\ y_n \in [a_n, ...]$, with $\|z_n\| = \|y_n\| = 1$ such that $v_n \leqslant \|Az_n\| < \lambda$ and $\mu_n \geqslant \|Ay_n\| > \lambda$. Because the unit sphere of $[a_n, ...]$ is connected, one can find $x_n \in [a_n, ...]$, $\|x_n\| = 1$ such that $\|Ax_n\| = \lambda$. Iterating for $n \in N$ we obtain a sequence (x_n), which by lemma 1 converges weakly to zero, and $\|Ax_n\| = \lambda$.

Slight modifications in this argument give the proof for $\lambda = h_A$ and $\lambda = H_A$.

Conversely, let $x_n \xrightarrow{w} 0$, $\|x_n\| = 1$ and $\|Ax_n\| \to \lambda$.

By lemma 2, for every fixed $p \in N$ we can choose $(z_n^{(p)})_{n=1}^{\infty} \subset [a_p, a_{p+1}, ...]$ with $\|z_n^{(p)}\| = 1$ $\forall n \in N$, $z_n^{(p)} \underset{n\to\infty}{\xrightarrow{w}} 0$ and $\|x_n - z_n^{(p)}\| \underset{n\to\infty}{\to} 0$ wich implies $|\ \|Az_n^{(p)}\| - \|Ax_n\|\ | \to 0$. Thus $\|Az_n^{(p)} \to \lambda$.

Select now a diagonal sequence $(z_{n(p)}^{(p)})_{n=1}^{\infty}$, with $\|Az_{n(p)}^{(p)}\| \underset{p\to\infty}{\to} \lambda$.
For every $p \in N$, $v_p \leqslant \|Az_{n(p)}^{(p)}\| \leqslant \mu_p$. Therefore, $\lambda \in [h_A, H_A]$.

Equality of h_A and H_A, for an operator satisfying the former conditions, is determined by:

Corollary 1. *The following are equivalent:*

1) $h_A = H_A = r$.
2) *Every normalized sequence* (x_n) *of* X, *with* $x_n \xrightarrow{w} 0$ *verifies* $\|Ax_n\| \to r$.
3) *For every normalized* $(x_n) \subset X$ *with* $x_n \xrightarrow{w} 0$, $(\|Ax_n\|)$ *is convergent.*

Corollary 2. *The operator A is compact if and only if* $h_A = H_A = 0$.

Corollary 3. *The operator A is semi-Fredholm+ if and only if* $h_A > 0$.

We omit the proof of these corollaries which are straight-forward, and do the same with the following propositions.

Proposition 2. *Let* $A \in B(X, Y)$. *The following properties hold:*

1) h_A *and* H_A *are stable through compact perturbations, i.e.,* $h_{A+K} = h_A$ and $H_{A+K} = H_A$, *for every compact operator* K.
2) *If* $A, T \in B(X, X)$
   $H_{AT} < \|A\| H_T$
   $h_{AT} \geqslant h_A h_T$
   If $T \in \Phi_+$, $H_{AT} \leqslant H_A H_T$.
3) $H_{A+B} \leqslant H_A + H_B$ *and* $h_{A+B} \leqslant h_A + h_B$.

Proposition 3. *Let* $A \in B(X, Y)$. *The following properties are equivalent:*

1) $\|A|_E\| = \|A\|$ *for every closed, linear finite-codimensional subspace E.*
2) $\|A\| = H_A$.
3) *There exists a normalized weakly null sequence,* (x_n), *such that* $\|Ax_n\| \to \|A\|$.

4. GEOMETRICAL MEANING OF THE CONSTANTS h_A AND H_A FOR A HILBERT SPACE OPERATOR A

Now let X be a separable Hilbert space, A a selfadjoint operator on X, and E the spectral measure of A. Recall that a number λ is a limit point of the spectrum of A, $\sigma(A)$, iff $E(V)$ has infinite rank for every neighborhood V of λ.

Theorem 4. *If A is a selfadjoint operator on X, H_A and h_A are respectively, the maximum and the minimum of the limit points of the spectrum of A.*

Proof. Suppose λ is a limit point. According to Weyl's result (s. [3], p. 266) there exists (x_n) in X, $\|x_n\|=1$ $x_n \xrightarrow{w} 0$ and $\|(A-\lambda I)x_n\| \to 0$. In particular, $\|Ax_n\| \to \lambda$ and therefore $h_A \leqslant \lambda \leqslant H_A$, although obviously not all the points of $[h_A, H_A]$ are limit points. Taking into account that addition of a self adjoint compact operator does not change the limit points on the spectrum, and by the results of Weyl and Von Neumann (s. [3], th. 2.1 p. 253), without loss of generality, we assume A is a diagonal operator (obviously referred to an orthonormal basis of eigenvectors (e_n)), say

$$A = \begin{bmatrix} \lambda_1 \ldots 0 & \ldots \\ 0 \ldots \lambda_n & \ldots \\ \ldots\ldots\ldots \end{bmatrix} \quad \lambda_i \geqslant 0. \ \forall i \in N$$

Let $\mu_n = \sup\limits_{\substack{x \in [e_n, \ldots] \\ \|x\|=1}} \|Ax\|$.

Clearly $\mu_n = \sup\limits_{i>n} \lambda_i$

Choose now $(\lambda_{p_n}) \subset (\lambda_n)$ *such that* $\mu_n - \lambda_{p_n} \underset{n\to\infty}{\to} 0$. We have $\lambda_{p_n} \to H_A$.

The fact $Ae_{p_n} = \lambda_{p_n} e_{p_n}$, $\forall n \in N$, implies $\|(A - H_A)e_{p_n}\| \to 0$, and therefore H_A is a limit point, actually the largest of them. A similar argument proves that h_A is the minimum of the limit points of A.

Remark 3. If A is not selfadjoint then H_A and h_A are not necessarily limit points of $\sigma(A)$. Nevertheless it can be easily proved, by polar descomposition and theorem 2, that h_A and H_A are exactly the maximum and the minimum of the limit points of $(A^*A)^{1/2}$.

Remark 4. Other geometric properties of operators A with $h_A = H_A$, can be seen in [7] and [8], where such operators are called «asymptotic similarities», because of their geometric behaviour.

Acknowledgements. The first author is indebted to Professor J. Zemanek for valuable conversations at Georgenthal, during the 8th-GDR-Polish-Seminar on Banach Spaces, where part of this paper was presented as a talk.

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Recibido: 3 de marzo de 1988
Revisado: 24 de junio de 1988