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TWO-GRAPHS, SWITCHING CLASSES AND EULER GRAPHS ARE EQUAL IN NUMBER*

C. L. MALLOWS AND N. J. A. SLOANE†

Abstract. Seidel has shown that the number t_n of two-graphs on n nodes is equal to the number of switching classes of graphs on n nodes. Robinson, and independently Liskovec, have given an explicit formula for the number e_n of Euler graphs on n nodes. It is shown here that $t_n = e_n$ for all n.

1. Introduction.

DEFINITION 1. A *two-graph* [9] consists of a set Ω of nodes together with a collection of triples of elements of Ω such that each 4-element subset of Ω contains an even number of these triples.

DEFINITION 2. Let G be an (ordinary) graph with node set Ω (containing no loops or multiple edges). Let ω be a node in Ω and suppose ω is adjacent in G to the (possibly empty) subset $X \subseteq \Omega$. The operation of *switching* at node ω ([5], [9]) transforms G by deleting the edges from ω to X, and adding edges from ω to $\Omega - X$. In other words, take the complement of the edges incident with node ω.

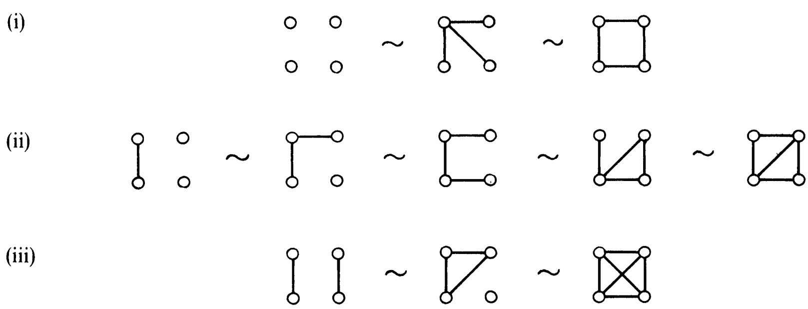

FIG. 1. *The three switching classes for* $n = 4$

We define an equivalence relation on graphs by saying that G' is equivalent to G if there exists a sequence of nodes $\omega_1, \omega_2, \cdots$ such that G' is obtained from G by switching at $\omega_1, \omega_2, \cdots$. Figure 1 shows the three equivalence classes (or *switching classes*) of graphs with $n = 4$ nodes.

Seidel [9] showed that the number t_n of switching classes of graphs with n nodes is equal to the number of two-graphs with n nodes. The values of $t_1, \cdots, t_7$ were given in [5], and of t_8, t_9 (found by F. C. Bussemaker) in [9].

DEFINITION 3. An *Euler* graph ([7], [11, p. 20]) is a graph in which every node has even degree.

Robinson [8] and independently Liskovec [6] have given formulas for the number e_n of Euler graphs on n nodes. Robinson's formula is

$$e_n = \sum_{(\sigma)} \frac{2^{\nu(\sigma) - \lambda(\sigma)}}{\prod_i i^{\sigma_i} \sigma_i!}, \tag{1}$$

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† Bell Laboratories, Murray Hill, New Jersey 07974.

the sum being over all ordered n-tuples $\sigma = (\sigma_1, \cdots, \sigma_n)$ such that $n = \sum_i i\sigma_i$, where

(2) $$v(\sigma) = \sum_{i<j} \sigma_i\sigma_j(i,j) + \sum_i i\left(\sigma_{2i} + \sigma_{2i+1} + \binom{\sigma_i}{2}\right),$$

(3) $$\lambda(\sigma) = \sum_i \sigma_i - \operatorname{sgn}\left(\sum_i \sigma_{2i+1}\right),$$

$$\operatorname{sgn}(x) = 0 \quad \text{if } x = 0, \quad = 1 \text{ if } x > 0,$$

and (i, j) is the greatest common divisor of i and j. Both [8] and [6] give the values of $e_1, \cdots, e_8$. (The value of e_8 is given incorrectly in [4, p. 117].) Values of e_n for $1 \leqq n \leqq 21$, obtained from (1), are given in Table 1.

TABLE 1
The number t_n of two-graphs, switching classes, or Euler graphs on n nodes

n	t_n
1	1
2	1
3	2
4	3
5	7
6	16
7	54
8	243
9	2,038
10	33,120
11	1,182,004
12	87,723,296
13	12,886,193,064
14	3,633,057,074,584
15	1,944,000,150,734,320
16	1,967,881,448,329,407,496
17	3,768,516,017,219,786,199,856
18	13,670,271,807,937,483,065,795,200
19	94,109,042,015,724,412,679,233,018,144
20	1,232,069,666,043,220,685,614,640,133,362,240
21	30,739,974,599,837,035,594,494,908,346,443,726,272

In collecting data for a supplement to [12] it was observed that the numbers e_n and t_n agree for $1 \leqq n \leqq 8$. It is the object of this note to prove the following.

THEOREM 1. *$e_n = t_n$ for all n.*

This result had already been established by Seidel [10] for n odd. However, the result when n is even seems to lie much deeper. Indeed, when n is odd, it is not difficult to show that there is a unique Euler graph in each switching class, thus establishing a 1–1 correspondence between the two families. But for n even, we have been unable to find such a correspondence.

2. The labeled case. Before proving Theorem 1, we consider the corresponding problem when the nodes of the graph are labeled, say with the labels $\{1, 2, \cdots, n\}$. Define

$\mathscr{G}_n$ = set of all labeled graphs on n nodes,
$\mathscr{E}_n$ = set of all Euler graphs in $\mathscr{G}_n$,
$\mathscr{T}_n$ = set of switching classes in $\mathscr{G}_n$,
$\mathscr{C}_n$ = set of all cocycles in $\mathscr{G}_n$ ([3, p. 38], [1, p. 13]. In the notation of [11, p. 28], $\mathscr{C}_n$ = cutsets of the complete graph on $\{1, \cdots, n\}$).

For $A, B \in \mathscr{G}_n$, define $A + B \in \mathscr{G}_n$ to be the graph containing the edges in either A or B but not both. With this definition of addition, $\mathscr{G}_n$, $\mathscr{E}_n$, $\mathscr{C}_n$ are elementary abelian groups of type $(1, 1, \cdots, 1)$ of orders $G_n = 2^{\binom{n}{2}}$, $E_n = 2^{\binom{n-1}{2}}$, $C_n = 2^{n-1}$ respectively [1], [11].

Observe that two graphs are in the same switching class if and only if their sum is a cocycle. Hence $\mathscr{T}_n \cong \mathscr{G}_n/\mathscr{C}_n$, and has order $T_n = 2^{\binom{n}{2}}/2^{n-1} = 2^{\binom{n-1}{2}} = E_n$.

It is easy to construct a 1–1 correspondence between $\mathscr{E}_n$ and $\mathscr{T}_n$. Let τ denote the tree with edges $12, 13, \cdots, 1n$, and let the edges ij $(1 < i < j)$ not in τ be called (in this section only) *chords*. As a basis for $\mathscr{E}_n$ take the *fundamental circuits* [11, p. 27] $\Phi_1, \cdots, \Phi_d$, $d = \binom{n-1}{2}$, each consisting of a chord and the two coterminal edges of τ. Any element of $\mathscr{E}_n$ can be written uniquely as $\sum_{i=1}^{d} a_i\Phi_i$, $a_i = 0$ or 1. Each switching class contains exactly one *chordal* graph that has no edges of τ; as a basis for these graphs we take the *unichordal* graphs $\psi_1, \cdots, \psi_d$ that each consist of a single chord. Then each chordal graph can be written uniquely as $\sum_{i=1}^{d} a_i\psi_i$, $a_i = 0$ or 1. Thus we have proved the following.

THEOREM 2. *There is a 1–1 correspondence between the $2^{\binom{n-1}{2}}$ labeled Euler graphs and labeled switching classes on n nodes.*

Figure 2 shows the correspondence when $n = 4$.

3. The unlabeled case: Proof of Theorem 1. For a subset $C \subseteq \{1, \cdots, n\}$, let $X(C)$ denote the operation of switching at all the nodes of C (in any order). $X(C)$ induces a permutation (which we also denote $X(C)$) on the set $\mathscr{G}_n$ of all labeled graphs. Note that $X(C)$ and $X(\bar{C})$ are the same permutation (where $\bar{C}$ is the complement of C).

The number t_n of switching classes is the number of equivalence classes in $\mathscr{G}_n$ under the combined action of all $X(C)$ and all permutations π of the n nodes. The set of all $X(C)$ and all π generate a group of order $2^{n-1}n!$, consisting of pairs (π, C). By Burnside's lemma [2],

$$t_n = \frac{1}{2^n n!} \sum_{\pi \in \mathscr{S}_n} \sum_{C \subseteq \{1, \cdots, n\}} f(\pi, C), \tag{4}$$

where $f(\pi, C)$ is the number of graphs that are fixed under the operation of first permuting the nodes according to π and then switching at the nodes in C. We proceed to calculate $f(\pi, C)$.

Suppose π permutes the nodes in σ_i cycles (called node-cycles) of length i, for $1 \leqq i \leqq n$. Let $\beta_1, \beta_2, \cdots, \beta_c$ be the (node-) cycles of π, with lengths

$b_1, b_2, \cdots, b_c$ respectively ($c = \sum \sigma_i, \sum b_j = n$), and let $\gamma(\beta_i, C)$ be the number of nodes of β_i that are in C.

Euler graph	Chordal representative of switching class	a_1	a_2	a_3
		0	0	0
		1	0	0
		0	1	0
		0	0	1
		1	1	0
		1	0	1
		0	1	1
		1	1	1

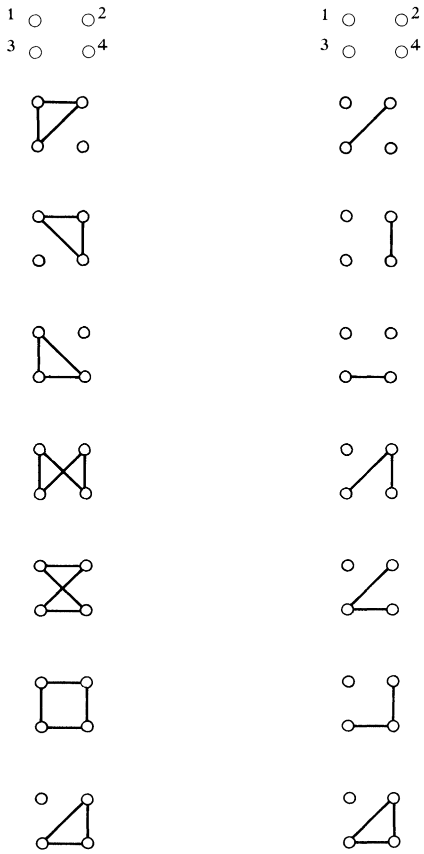

FIG. 2. 1–1 *correspondence between labeled Euler graphs and labeled switching classes on four nodes*

Under the action of π, the edges of the complete graph are permuted in cycles (called edge-cycles). It is shown in [8] that the total number of edge-cycles is $v(\sigma)$.

(i) If $\beta = (a_1, \cdots, a_{2i})$ is a node-cycle of length $b = 2i$, the edges $a_\mu a_\nu$ ($1 \leqq \mu < \nu \leqq 2i$) are permuted in $i - 1$ distinct edge-cycles of length $2i$, and one edge-cycle (the cycle of diameters) of length i. Then as a diameter traverses the edge-cycle of length i under repeated applications of (π, C), it is switched $\gamma(\beta, C)$ times. Thus in order for it to be possible for a graph to be fixed under (π, C), $\gamma(\beta, C)$ must be even. An edge which is not a diameter will be switched $2\gamma(\beta, C)$ times, which is always even. Thus if $\gamma(\beta, C)$ is even, there are 2^i ways of inserting the chords of β so that this subgraph is fixed under (π, C); if $\gamma(\beta, C)$ is odd, there are no such ways.

(ii) The edges joining the nodes of a node-cycle of length $2i + 1$ comprise i distinct edge-cycles each of length $2i + 1$. An edge in one of these cycles is switched an even number of times during $2i + 1$ successive applications of (π, C). Thus there are 2^i ways of inserting the chords of such a node-cycle, for any C.

(iii) Let $\beta = (a_1, \cdots, a_b)$, $\beta' = (a'_1, \cdots, a'_{b'})$ be distinct node-cycles. Let $t = (b, b')$, and write

$$b = rt, \quad b' = r't, \quad \text{where } (r, r') = 1.$$

The bb' edges $a_\mu a'_{\mu'}$ are permuted in t distinct edge-cycles of length $rr't$ each.

As an edge traverses its edge-cycle, it is switched $p = r'\gamma(\beta, C) + r\gamma(\beta', C)$ times. If both b and b' are even, and an invariant subgraph is possible, both $\gamma(\beta, C)$ and $\gamma(\beta', C)$ must be even by (i). If b is even and b' odd, r and $\gamma(\beta, C)$ must be even. But if both b and b' are odd, so r, r' are odd, then for p to be even, $\gamma(\beta, C)$ and $\gamma(\beta', C)$ must both be even or both odd. Therefore if an invariant subgraph is to be possible, all node-cycles of odd length must contain an even number of points of C, or all must contain an odd number of points of C. If this holds, the number of ways of drawing an invariant subgraph of edges from β to β' is 2^t.

(iv) Putting all this together, the number of ways of drawing a graph that is invariant under (π, C) is $f(\pi, C) = 2^{\nu(\sigma)}\xi(\pi, C)$, where $\xi(\pi, C) = 0$ or 1, and $=1$ if and only if $\gamma(\beta_i, C)$ is even whenever b_i is even, and $\{\gamma(\beta_j, C): b_j \text{ odd}\}$ are all even or all odd together. It follows that

$$\sum_C \xi(\pi, C) = 2^{n-\lambda(\sigma)}.$$

Hence from (4) and (1),

$$t_n = \sum_\pi 2^{\nu(\sigma)-\lambda(\sigma)}$$

which collapses to (1), and the theorem is proved.

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