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TWO-GRID FINITE-ELEMENT SCHEMES FOR THE TRANSIENT NAVIER-STOKES PROBLEM

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Abstract. We semi-discretize in space a time-dependent Navier-Stokes system on a three-dimensional polyhedron by finite-elements schemes defined on two grids. In the first step, the fully non-linear problem is semi-discretized on a coarse grid, with mesh-size H. In the second step, the problem is linearized by substituting into the non-linear term, the velocity $\mathbf{u}_H$ computed at step one, and the linearized problem is semi-discretized on a fine grid with mesh-size h. This approach is motivated by the fact that, on a convex polyhedron and under adequate assumptions on the data, the contribution of $\mathbf{u}_H$ to the error analysis is measured in the L^2 norm in space and time, and thus, for the lowest-degree elements, is of the order of H^2. Hence, an error of the order of h can be recovered at the second step, provided $h = H^2$.

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0. Introduction

Let us consider a non-linear Partial Differential Equation (called PDE). We want to find an approximation of the solution, say u (or of *a* solution, properly defined). A very general strategy can be based on a *two-grid* approach. In a *first step*, our approximation, say u_H, is computed on a *coarse grid* of mesh-size H, using the fully non-linear PDE. In a *second step*, one linearizes the PDE "around" u_H (this can be done in infinitely many ways), and one computes an approximation of the linear problem on a *fine grid* of mesh-size h. Let us denote by u_h^{lin} this solution. Then under quite general circumstances, one can show that if h and H are chosen in an adequate fashion, then the error $\|u - u_h^{\text{lin}}\|$ is of the same order as $\|u - u_h\|$, where u_h denotes the approximation of the fully non-linear PDE computed on the fine grid. Of course, the computation of u_H and u_h^{lin} involves much less "work" than the direct computation of u_h!

The above strategy is valid for *stationary* PDE's and for *evolution* (transient) equations as well.

This "two-grid strategy" or "two-step strategy" has been widely studied for *steady semi-linear elliptic equations*, *cf.* for example the work of Xu [45], [46] and Niemistö [40], and for the steady Navier-Stokes problem, *cf.* the work of Layton [24], Layton and Lenferink [25], [26] and Girault and Lions [16]. *We want to apply here this strategy to non-linear PDE's of evolution.* More precisely, we have chosen to develop this strategy for the Navier-Stokes equations.

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Let Ω be a Lipschitz-continuous domain (*cf.* Grisvard [20]) of $\mathbb{R}^3$ with a polyhedral boundary $\partial\Omega$ and unit exterior normal $\mathbf{n}$, and let $[0,T]$ be a given time-interval. Consider the time-dependent Navier-Stokes equations:

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) - \nu\,\Delta\,\mathbf{u}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\,\mathbf{u}(\mathbf{x},t) + \nabla\,p(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) \quad \text{in } \Omega\times]0,T]\,, \tag{0.1}$$

with the incompressibility condition:

$$\operatorname{div}\mathbf{u}(\mathbf{x},t) = 0 \quad \text{in } \Omega\times]0,T]\,, \tag{0.2}$$

the homogeneous Dirichlet boundary condition:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{0} \quad \text{on } \partial\Omega\times]0,T]\,, \tag{0.3}$$

and the initial condition

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0} \quad \text{in } \Omega\,, \tag{0.4}$$

where the notation $\mathbf{u}\cdot\nabla\,\mathbf{u}$ means

$$\mathbf{u}\cdot\nabla\,\mathbf{u} = \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i}\mathbf{u}\,.$$

The existence of solutions to (0.1–0.4) is a fundamental question, and the existence-uniqueness of solutions is still an open problem. *Weak solutions* were introduced by Leray [27–29] in a series of classical papers. Further properties of the "Leray" solution (that he calls "turbulent solution") have been given by Ladyzenskaya [23] and Lions [30] and [31], Chapter 1. Under various hypotheses on the data, more or less "strong " solutions can be obtained, but whether or not singularities can develop in time and be accompanied by loss of uniqueness remains an open problem. Further results were obtained by Lions [33], who also dealt in [34] with many situations for compressible fluids (in particular, the reader can also refer to [35] for a list of interesting open questions).

Of course, due to the crucial importance of Navier-Stokes equations in very many applications, a huge amount of papers has been devoted to numerical schemes for approximating these equations. Approximation algorithms were already introduced in [23], [30] and [31], and much more developed in Temam [43], Girault and Raviart [17], and Pironneau [41] where in particular, the "pressure formulation" was studied in depth. The first "splitting" procedure (called the "projection method") for dissociating the incompressibility constraint from the non-linearity, was introduced by Chorin [8] and Temam [44]. Further on, Foias *et al.* [14] developed what became known later as the Nonlinear Galerkin Method (NLG), that is mildly related to the methods we shall study here. A good description of the NLG method can be found in Marion and Temam [38]; we also refer the reader to their previous works [36] and [37]. We mention here that a very complete reference on the numerical approximation of the steady or unsteady Navier-Stokes equations, by Glowinski, will appear in [19].

We recall now the classical formulation of the equations. Let V be the space of vector-valued functions $\mathbf{v} = (v_1, v_2, v_3)$ such that:

$$v_i \in H_0^1(\Omega)\,,\ \ i.e.\ \frac{\partial v_i}{\partial x_j} \in L^2(\Omega), 1 \le j \le 3\,,\, v_i = 0 \text{ on } \partial\Omega\,,$$

and such that

$$\operatorname{div}\mathbf{v} = 0\,,\ \text{ in } \Omega\,.$$

The "classical" variational formulation of (0.1–0.4) is as follows: Find $\mathbf{u} = \mathbf{u}(t)$ with values in V such that

$$\forall \mathbf{v} \in V,\ (\mathbf{u}'(t),\mathbf{v}) + \nu(\nabla\,\mathbf{u}(t),\nabla\,\mathbf{v}) + (\mathbf{u}(t)\cdot\nabla\,\mathbf{u}(t),\mathbf{v}) = \langle \mathbf{f}(t),\mathbf{v}\rangle \quad \text{in }]0,T]\,, \tag{0.5}$$

and

$$\mathbf{u}(0) = \mathbf{0}\,,$$

where $\mathbf{u}' = \frac{\partial \mathbf{u}}{\partial t}$. Of course, setting $X = H_0^1(\Omega)^3$ and

$$M = \{q \in L^2(\Omega)\,;\ \int_\Omega q\,\mathrm{d}\mathbf{x} = 0\}\,,$$

this formulation is equivalent to looking for $\mathbf{u}$ satisfying

$$\forall \mathbf{v} \in X\ ,\ (\mathbf{u}'(t), \mathbf{v}) + \nu(\nabla\,\mathbf{u}(t), \nabla\,\mathbf{v}) + (\mathbf{u}(t)\cdot\nabla\,\mathbf{u}(t), \mathbf{v}) - (p(t), \operatorname{div}\mathbf{v}) = \langle \mathbf{f}(t), \mathbf{v}\rangle \quad \text{in }]0, T]\,, \tag{0.6}$$

$$\forall q \in M\ ,\ (q, \operatorname{div}\mathbf{u}(t)) = 0\,,\ \text{ in }]0, T]\,, \tag{0.7}$$

and

$$\mathbf{u}(0) = \mathbf{0}\,.$$

It is well-known that, in general, the most straightforward semi-discrete analogues of the above formulations are *not equivalent*. Indeed, when discretized, condition (0.7) does not imply that the divergence is zero, unless polynomials of high degree are used. To achieve equivalence, we must work with an adequate approximation of space V. Without going into details, let us present quickly a semi-discretization. Let η be a discretization parameter, $\mathcal{T}_\eta$ a triangulation of $\overline{\Omega}$, and let V_η, X_η and M_η be finite-element spaces approximating respectively V, X and M (all this will be made precise below). The semi-discrete analogue of (0.5) is then: find $\mathbf{u}_\eta(t) \in V_\eta$ satisfying

$$\forall \mathbf{v}_\eta \in V_\eta\ ,\ (\mathbf{u}_\eta'(t), \mathbf{v}_\eta) + \nu(\nabla\,\mathbf{u}_\eta(t), \nabla\,\mathbf{v}_\eta) + (\mathbf{u}_\eta(t)\cdot\nabla\,\mathbf{u}_\eta(t), \mathbf{v}_\eta) = \langle \mathbf{f}(t), \mathbf{v}_\eta\rangle \quad \text{in }]0, T]\,, \tag{0.8}$$

and the semi-discrete analogue of (0.6),(0.7) is: find $\mathbf{u}_\eta(t) \in X_\eta$ satisfying

$$\forall \mathbf{v}_\eta \in X_\eta\ ,\ (\mathbf{u}_\eta'(t), \mathbf{v}_\eta) + \nu(\nabla\,\mathbf{u}_\eta(t), \nabla\,\mathbf{v}_\eta) + (\mathbf{u}_\eta(t)\cdot\nabla\,\mathbf{u}_\eta(t), \mathbf{v}_\eta) - (p_\eta(t), \operatorname{div}\mathbf{v}_\eta) = \langle \mathbf{f}(t), \mathbf{v}_\eta\rangle \quad \text{in }]0, T]\,, \tag{0.9}$$

and

$$\forall q_\eta \in M_\eta\ ,\ (q_\eta, \operatorname{div}\mathbf{u}_\eta(t)) = 0 \quad \text{in }]0, T]\,. \tag{0.10}$$

Of course, in both formulations, we add

$$\mathbf{u}_\eta(0) = \mathbf{0}\,. \tag{0.11}$$

Remark 0.1. The zero initial velocity is only a matter of simplification. The contents of this article can be readily adapted to non-zero initial data. □

We present now our *two-grid scheme* for the pressure approximation (0.9–0.11). Let $\mathcal{T}_H$ be a coarse triangulation of $\overline{\Omega}$ and let X_H and M_H be suitable finite-element spaces for discretizing the velocity $\mathbf{u}$ and pressure p. Similarly, let $\mathcal{T}_h$ be a fine triangulation, with corresponding finite-element spaces X_h and M_h. The two-grid algorithm for semi-discretizing (0.1–0.4) is:
Step One (non-linear problem on coarse grid): Find $(\mathbf{u}_H, p_H)$ with values in $X_H \times M_H$ for each $t \in [0, T]$, solution of

$$\forall \mathbf{v}_H \in X_H\ ,\ (\mathbf{u}_H'(t), \mathbf{v}_H) + \nu(\nabla\,\mathbf{u}_H(t), \nabla\,\mathbf{v}_H) + (\mathbf{u}_H(t)\cdot\nabla\,\mathbf{u}_H(t), \mathbf{v}_H) - (p_H(t), \operatorname{div}\mathbf{v}_H) = \langle \mathbf{f}(t), \mathbf{v}_H\rangle \quad \text{in }]0, T]\,, \tag{0.12}$$

$$\forall q_H \in M_H \ , \ (q_H, \operatorname{div} \mathbf{u}_H(t)) = 0 \quad \text{in }]0,T] \,, \tag{0.13}$$

$$\mathbf{u}_H(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega \,. \tag{0.14}$$

Step Two (linearized problem on fine grid): Find $(\mathbf{u}_h, p_h)$ with values in $X_h \times M_h$ for each $t \in [0,T]$, solution of

$$\forall \mathbf{v}_h \in X_h \ , \ (\mathbf{u}_h'(t), \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h(t), \nabla \mathbf{v}_h) + (\mathbf{u}_H(t) \cdot \nabla \mathbf{u}_h(t), \mathbf{v}_h) - (p_h(t), \operatorname{div} \mathbf{v}_h) = \langle \mathbf{f}(t), \mathbf{v}_h \rangle \quad \text{in }]0,T] \,, \tag{0.15}$$

$$\forall q_h \in M_h \ , \ (q_h, \operatorname{div} \mathbf{u}_h(t)) = 0 \quad \text{in }]0,T] \,, \tag{0.16}$$

$$\mathbf{u}_h(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega \,. \tag{0.17}$$

Remark 0.2. *One has to be careful with the notation.* The function $\mathbf{u}_H$ in (0.12–0.14) coincides with the function $\mathbf{u}_\eta$ in (0.9–0.11) if $\eta = H$. But the function $\mathbf{u}_h$ defined in Step Two does *not* coincide with $\mathbf{u}_\eta$ for $\eta = h$, *i.e.* $\mathbf{u}_h \neq \mathbf{u}_{\eta,\eta=h}$. □

Remark 0.3. The convection terms in (0.12) and (0.15) are not antisymmetric. Indeed, as mentioned above, (0.13) does not necessarily imply that $\operatorname{div} \mathbf{u}_H = 0$; therefore in general

$$(\mathbf{u}_H \cdot \nabla \mathbf{u}_H, \mathbf{u}_H) \neq 0 \,.$$

Then deriving *a priori* estimates for the solution of Step One is not a matter of routine. We shall see below how this can be settled.

Of course, one can make the whole approximation antisymmetric, thus simplifying the analysis. But *since it can be justified,* formulation (0.12) is simpler and it is actually widely used in practice. □

The main result in the present article is that if one chooses:

$$h = H^2 \,, \tag{0.18}$$

and the mesh is regular, then the error (estimated in suitable norms) is the same for $\mathbf{u} - \mathbf{u}_h$ *as for* $\mathbf{u} - \mathbf{u}_{\eta,\eta=h}$*. Interestingly, this result for the velocity is obtained without requiring a uniformly regular (or quasi-uniform) triangulation.* The proof uses in the first place precise error estimates for $\mathbf{u} - \mathbf{u}_H$, where $\mathbf{u}_H$ is the solution (whose existence we establish) of Step One. They are somewhat similar to those derived by Heywood and Rannacher in [22], but they are slightly more complex because the convection term in (0.12) is not antisymmetric. Then the proof relies on estimates which are more of the Functional Analysis type, using among others, Sobolev inequalities.

Of course, we have at our disposal the choice of the finite elements. In this article, we have chosen the "mini-element" (*cf.* Arnold *et al.* [3], Brezzi and Fortin [7] or Girault and Raviart [18]) for the spaces X_H and M_H, and X_h and M_h, but the subsequent analysis can be adapted to other stable pairs of finite-element spaces. In addition, for convenient computation, we assume that the fine grid is a refinement of the coarse grid, and hence $X_H \subset X_h$ and $M_H \subset M_h$.

A few remarks are now in order.

Remark 0.4. As we have already mentioned, the "two-grid strategy" has been widely studied before. In [16], we have obtained a result analogous to the above with the choice:

$$h = H^{3/2} \,, \tag{0.19}$$

a result which appears *less* favourable at first sight than (0.18). For instance, if we take $H = 2^{-4}$ then (0.18) gives $h = 2^{-8}$, a much better precision than (0.19) which only gives $h = 2^{-6}$. But there is a subtle difference. In the stationary case, the result (0.19) of [16] is obtained under hypotheses on the data which *imply* the desired regularity properties for the velocity and for the pressure, whatever the angles of the domain. In this respect, this result is optimal. *This is not the case in what follows.* Here, the main result for the velocity is obtained *on a convex domain and a regular triangulation, assuming* that some minimal regularity properties for the velocity and pressure *are satisfied.* Of course, these can be guaranteed by imposing stronger regularity assumptions on the force $\mathbf{f}$, such as in Lemma 4.6 and Remark 5.3, but we do not know whether or not they are really necessary.

Depending on the error estimates available for $\mathbf{u} - \mathbf{u}_H$ (where $\mathbf{u}_H$ is given by Step One), one can obtain the main result for $h = H^{3/2}$ or for $h = H^2$. This is presented in Section 1 below for the variational formulation *without pressure*, in order to simplify the presentation, and assuming that V_H is contained in V. □

Remark 0.5. For the transient Navier-Stokes equations studied in the present paper, there does not appear to be much literature on the two-grid algorithm, but we can quote two references in a related direction, both inspired by the Nonlinear Galerkin Method (NLG); as mentioned above, we refer to [38] for a good description of NLG.

The first one is the work by Ait Ou Amni and Marion [2] on the two-dimensional Navier-Stokes equations, that is also based on a coarse-grid space X_H and a fine-grid space X_h. More precisely, they introduce X_h^H, the L^2 orthogonal complement of X_H in X_h:

$$X_h = X_H \oplus X_h^H \,,$$

they define the bilinear form:

$$a(\mathbf{u}, \mathbf{v}) = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) \,,$$

and the antisymmetric trilinear form:

$$c(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \frac{1}{2}\big((\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})\big) \,,$$

and their scheme is:

NLG - Preliminary Step Solve (0.1–0.4) starting with $\mathbf{u}(0) = \mathbf{u}_0$ given at time $t = 0$ (instead of $\mathbf{u}(0) = \mathbf{0}$), up a given time t_0, by a classical Galerkin semi-discrete method with the pair of spaces (X_h, M_h); let $(\mathbf{u}_h, p_h)$ be the solution.

NLG - Steps One and Two For $t \geq t_0$, find $\mathbf{v}_H \in X_H$, $\mathbf{w}_h \in X_h^H$ and $p_h \in M_h$, solution of the coupled system:

$$\forall \varphi \in X_H \,,\; (\mathbf{v}'_H, \varphi) + a(\mathbf{v}_H + \mathbf{w}_h, \varphi) + c(\mathbf{v}_H + \mathbf{w}_h; \mathbf{v}_H, \varphi) + c(\mathbf{v}_H; \mathbf{w}_h, \varphi) - (p_h, \operatorname{div} \varphi) = (\mathbf{f}, \varphi) \,, \tag{0.20}$$

$$\forall \chi \in X_h^H \,,\; a(\mathbf{v}_H + \mathbf{w}_h, \chi) + c(\mathbf{v}_H; \mathbf{v}_H, \chi) - (p_h, \operatorname{div} \chi) = (\mathbf{f}, \chi) \,, \tag{0.21}$$

$$\forall q \in M_h \,,\; (\operatorname{div}(\mathbf{v}_H + \mathbf{w}_h), q) = 0 \,, \tag{0.22}$$

$$\mathbf{v}_H(t_0) = P_H(\mathbf{u}_h(t_0)) \,, \tag{0.23}$$

where P_H is the orthogonal projection for the L^2 norm onto X_H. Note that (0.20) and (0.21) are coupled and for this reason, we do not dissociate Step One from Step Two.

For $\mathbf{u}_0$ given in $H^2(\Omega)^2 \cap V$, $\mathbf{f}$ given in $L^2(\Omega)^2$, independent of t, and (X_h, M_h) a stable pair of finite element spaces with either constant or $\mathbb{P}_1$ pressure in each element, on a *uniformly-regular* triangulation, Ait Ou Amni and Marion prove that

$$\forall t \geq t_0 \,,\; \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{H^1(\Omega)} \leq \kappa(t)(H^2 + h) \,,$$

$$\forall t > t_0\,,\ \|p(t) - p_h(t)\|_{L^2(\Omega)} \le \tau(t)^{-1/2}\kappa(t)(H^2 + h)\,,$$

where $\tau(t) = t - t_0$ and $\kappa(t)$ is a continuous function of t for $t > 0$ (that depends on $\mathbf{u}_0$). Thus, in two dimensions, the error of this NLG scheme is of the order of h, provided $h = H^2$, a result that is similar to Theorem 24.1, p. 648 of [38]. Note that we achieve the same order of accuracy for the velocity in three dimensions, without requiring a uniformly regular triangulation and with the advantage that our two steps are decoupled.

The second one is the Post-Processing Method (PP) of Garcia-Archilla and Titi [15] for semi-linear scalar elliptic equations in any dimension. In terms of operators, let A be the linear elliptic operator in the equation and F the non-linear operator:

$$\frac{\mathrm{d}}{\mathrm{d}t}u + \nu A\,u + F(u) = 0\ ,\ u(0) = u_0\,.$$

The operator F is either of the form $F(u) = g(u)$ for a smooth real-valued function g or of the form

$$F(u) = g(u) + \mathbf{b}(u)\cdot\nabla\,u\,, \tag{0.24}$$

for a smooth vector-valued function $\mathbf{b}$. Let S_H be a space of continuous finite element functions on a coarse grid; then the scheme is:

PP - Step One Find u_H with values in S_H for all $t \in [0, T]$ solution of the non-linear elliptic equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}u_H + \nu A_H u_H + P_H(F(u_H)) = 0\ ,\ u_H(0) = R_H(u_0)\,, \tag{0.25}$$

where P_H and R_H are respectively the L^2 and H_0^1 orthogonal projection operators onto S_H. Although we use the same notation for P_H as in NLG, the operator P_H is not the same here since the functions of S_H are scalar-valued.

PP - Step Two Find $\tilde{u}$ solution of the linear elliptic equation:

$$\nu A\,\tilde{u} = -F(u_H(T)) - \frac{\mathrm{d}}{\mathrm{d}t}u_H(T)\,, \tag{0.26}$$

where $\tilde{u}$ is approximated in a suitable fine-grid space. There are similarities between (0.20), (0.21) and (0.25), (0.26), but the great advantage of this post-processing approach is that (0.25) and (0.26) are decoupled.

Among other results, Garcia-Archilla and Titi show that, when F has the form (0.24), for all H sufficiently small and for smooth enough u, if the functions of S_H are polynomials of degree r with $r \ge 2$ on a uniformly regular triangulation, this post-processed Galerkin approximation $\tilde{u}$ satisfies the error bound:

$$\|u(T) - \tilde{u}\|_{H^1(\Omega)} \le CH^{r+1}|\log(H)|\,.$$

When $\tilde{u}$ is approximated by polynomials of degree r in a fine-grid space, say $\tilde{u}_h \in S_h$, the error $\|u(T) - \tilde{u}_h\|_{H^1(\Omega)}$ is of the order of h^r provided $h^r = H^{r+1}|\log(H)|$. Thus, if $r = 2$, h and H must be related by

$$h = H^{3/2}|\log(H)|^{1/2}\,.$$

This result is not as sharp as ours, considering that the degree of the polynomials must be at least two and the triangulation must be uniformly regular. If this scheme were to be applied to the Navier-Stokes equations, the first step would correspond to a formulation without pressure, whereas the second step would read: Find $\tilde{\mathbf{u}}_h(T) \in V_h$ such that

$$\forall \mathbf{v}_h \in V_h\,,\ \nu(\nabla\,\tilde{\mathbf{u}}_h(T), \nabla\,\mathbf{v}_h) = -(\mathbf{u}_H(T)\cdot\nabla\,\mathbf{u}_H(T), \mathbf{v}_h) - (\mathbf{u}'_H(T), \mathbf{v}_h) + (\mathbf{f}(T), \mathbf{v}_h)\,. \tag{0.27}$$

□

Remark 0.6. In the present paper, only semi-discretizations are studied and no aim at effective computation has been pursued. The emphasis for the time being, is on *error estimates*, presented under precise assumptions on the regularity of the velocity and pressure. Nevertheless, here is a simple fully discrete scheme. Let $\Delta t = T/(N+1)$ for some positive integer N, let $t_n = n\Delta t$, and suppose that $\mathbf{u}_h^n$ has been computed. Then, we set:

$$\mathbf{u}_H^n = \mathcal{R}(\mathbf{u}_h^n)\,,$$

where $\mathcal{R}$ is a suitable restriction from X_h into X_H, and we propose the following two-grid algorithm:
Step One (non-linear problem on coarse grid): Find $(\mathbf{u}_H^{n+1}, p_H^{n+1})$ with values in $X_H \times M_H$, solution of

$$\forall \mathbf{v}_H \in X_H\ ,\ \frac{1}{\Delta t}(\mathbf{u}_H^{n+1} - \mathbf{u}_H^n, \mathbf{v}_H) + \nu(\nabla\,\mathbf{u}_H^{n+1}, \nabla\,\mathbf{v}_H) + (\mathbf{u}_H^{n+1}\cdot\nabla\,\mathbf{u}_H^{n+1}, \mathbf{v}_H) - (p_H^{n+1}, \operatorname{div}\mathbf{v}_H) = \langle \mathbf{f}(t_{n+1}), \mathbf{v}_H\rangle\,, \tag{0.28}$$

$$\forall q_H \in M_H\ ,\ (q_H, \operatorname{div}\mathbf{u}_H^{n+1}) = 0\,. \tag{0.29}$$

Step Two (linearized problem on fine grid): Find $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ with values in $X_h \times M_h$, solution of

$$\forall \mathbf{v}_h \in X_h\ ,\ \frac{1}{\Delta t}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \nu(\nabla\,\mathbf{u}_h^{n+1}, \nabla\,\mathbf{v}_h) + (\mathbf{u}_H^{n+1}\cdot\nabla\,\mathbf{u}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \operatorname{div}\mathbf{v}_h) = \langle \mathbf{f}(t_{n+1}), \mathbf{v}_h\rangle\,, \tag{0.30}$$

$$\forall q_h \in M_h\ ,\ (q_h, \operatorname{div}\mathbf{u}_h^{n+1}) = 0\,. \tag{0.31}$$

This is a simple example in which both equations use the same time step and are both of order one with respect to time. A more elaborate idea for Step Two would be to use a scheme of second-order in time with the same time step, or some time-splitting scheme of order one. □

Remark 0.7. Among other applications, this two-grid scheme can be used for solving higher-order equations such as the Kuramoto-Shivashinsky's equation. It could also be used, for instance, in all the Global Circulation Models appearing in climatology, including the coupled ocean-atmosphere models. This is not developed here. □

The remainder of this article is organized as follows. In Section 1, we describe the main steps in the proof of the error estimates for the two-grid method in the simplified case where V_H and V_h are contained in V, *i.e.* the discrete functions have exactly zero divergence. The technically more difficult, though more important and more realistic, formulations with the pressure and discrete velocities with non-zero divergence are studied in the next sections. In Section 2, we derive by two different methods *a priori* estimates for the (unique) solution of Step One. Section 3 is devoted to proving error estimates and Sections 4 and 5 to proving the duality argument. The pressure is estimated in Section 6 and Step Two is studied in Section 7. Finally, an Appendix discusses briefly the approximation properties of a regularization operator acting on L^1 functions in three dimensions.

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a,b[$ with values in a functional space, say X (*cf.* Lions and Magenes [32]). More precisely, let $\|\cdot\|_X$ denote the norm of X; then for any number r, $1 \le r \le \infty$, we define

$$L^r(a,b;X) = \{f \text{ measurable in }]a,b[\,;\ \int_a^b \|f(t)\|_X^r \mathrm{d}t < \infty\}$$

equipped with the norm

$$\|f\|_{L^r(a,b;X)} = \left(\int_a^b \|f(t)\|_X^r \mathrm{d}t\right)^{1/r}\,,$$

with the usual modification if $r = \infty$. It is a Banach space if X is a Banach space. Here X is usually a Sobolev space, such as (*cf.* Adams [1] or Nečas [39]):

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega)\,;\, \partial^k v \in L^r(\Omega)\ \forall |k| \le m\}\,,$$

where (k_1, k_2, k_3) is a triple of non-negative integers, $|k| = k_1 + k_2 + k_3$ and

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}\,.$$

This space is equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^r \,\mathrm{d}\mathbf{x}\right]^{1/r}\,,$$

and is a Banach space for the norm

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0 \le k \le m} |v|^r_{W^{k,r}(\Omega)}\right]^{1/r}\,.$$

When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. Similarly, $L^2(a,b;H^m(\Omega))$ is a Hilbert space and in particular $L^2(a,b;L^2(\Omega))$ coincides with $L^2(\Omega\times]a,b[)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{u} = (u_1, u_2, u_3)$; then we set

$$\|\mathbf{u}\|_{L^r(\Omega)} = \left[\int_\Omega \|\mathbf{u}(\mathbf{x})\|^r \,\mathrm{d}\mathbf{x}\right]^{1/r}\,,$$

where $\|\cdot\|$ denotes the Euclidean vector norm.

For vanishing boundary values, we define

$$H_0^1(\Omega) = \{v \in H^1(\Omega)\,;\, v|_{\partial\Omega} = 0\}\,,$$

and its dual space, $H^{-1}(\Omega)$. Recall Sobolev's imbeddings: in three dimensions, for any real number $1 \le r \le 6$, there exists a constant S_r such that

$$\forall v \in H_0^1(\Omega)\ ,\ \|v\|_{L^r(\Omega)} \le S_r |v|_{H^1(\Omega)}\,, \tag{0.32}$$

where

$$|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}\,. \tag{0.33}$$

When $r = 2$, (0.32) reduces to Poincaré's inequality and S_2 is Poincaré's constant. Owing to Poincaré's inequality, the seminorm $|\cdot|_{H^1(\Omega)}$ is a norm on $H_0^1(\Omega)$ and we use it to define the dual norm:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v\rangle}{|v|_{H^1(\Omega)}}\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Also, recall the spaces we introduced at the beginning:

$$V = \{\mathbf{v} \in H_0^1(\Omega)^3 \, ; \, \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\} \, ,$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega) \, ; \, \int_\Omega q \, \mathrm{d}\mathbf{x} = 0\} \, ,$$

and the orthogonal complement of V in $H_0^1(\Omega)^3$:

$$V^\perp = \{\mathbf{v} \in H_0^1(\Omega)^3 \, ; \, \forall \mathbf{w} \in V \, , \, (\nabla \mathbf{v}, \nabla \mathbf{w}) = 0\} \, .$$

Finally, the next theorem and its corollary collect some regularity properties of the solution of the steady Stokes problem:

$$-\nu \, \Delta \mathbf{v} + \nabla q = \mathbf{g} \, , \; \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \, , \tag{0.34}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \, , \tag{0.35}$$

and of every solution of the steady Navier-Stokes problem:

$$-\nu \, \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla q = \mathbf{g} \, , \; \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \, , \tag{0.36}$$

where $\mathbf{v}$ is subject to boundary condition (0.35).

Theorem 0.8. *If $\mathbf{g} \in L^{3/2}(\Omega)^3$, the solution $(\mathbf{v}, q)$ of (0.34), (0.35) belongs to $H^{3/2}(\Omega)^3 \times H^{1/2}(\Omega)$, without restrictions on the angles of $\partial\Omega$. If $\mathbf{g} \in L^2(\Omega)^3$ and Ω is convex then $(\mathbf{v}, q) \in H^2(\Omega)^3 \times H^1(\Omega)$.*

Here $H^{3/2}(\Omega)$ and $H^{1/2}(\Omega)$ denote the interpolation spaces just "in the middle" between $H^2(\Omega)$ and $H^1(\Omega)$, and respectively $H^1(\Omega)$ and $L^2(\Omega)$ (*cf.* [32]). The proof of the first part is due to Dauge and Costabel; it is presented in [16]. The proof of the second part is due to Dauge [12]. Then the corollary below is established by a bootstrap argument.

Corollary 0.9. *The statement of Theorem 0.8 is valid for any solution $(\mathbf{v}, q)$ of (0.36), (0.35).*

Proof. Let $\mathbf{v} \in H^1(\Omega)^3$ and $q \in L^2(\Omega)$ be any solution of (0.36), (0.35). Then $\mathbf{v} \cdot \nabla \mathbf{v}$ belongs to $L^{3/2}(\Omega)^3$ and

$$\|\mathbf{v} \cdot \nabla \mathbf{v}\|_{L^{3/2}(\Omega)} \le \|\mathbf{v}\|_{L^6(\Omega)} \|\nabla \mathbf{v}\|_{L^2(\Omega)} \, .$$

Hence the pair $(\mathbf{v}, q)$ is the solution of the Stokes problem:

$$-\nu \, \Delta \mathbf{v} + \nabla q = \mathbf{g} - \mathbf{v} \cdot \nabla \mathbf{v} \, , \; \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \, ,$$

with data in $L^{3/2}(\Omega)^3$. Thus Theorem 0.8 implies that $(\mathbf{v}, q)$ belongs to $H^{3/2}(\Omega)^3 \times H^{1/2}(\Omega)$. Furthermore by Sobolev's imbedding theorem for interpolated spaces in three dimensions, we have

$$H^{3/2}(\Omega) \subset W^{1,3}\Omega) \, .$$

As a consequence, $\mathbf{v} \cdot \nabla \mathbf{v}$ belongs to $L^2(\Omega)^3$ and

$$\|\mathbf{v} \cdot \nabla \mathbf{v}\|_{L^2(\Omega)} \le \|\mathbf{v}\|_{L^6(\Omega)} \|\nabla \mathbf{v}\|_{L^3(\Omega)} \, .$$

Then the fact that $(\mathbf{v}, q) \in H^2(\Omega)^3 \times H^1(\Omega)$ follows from another application of Theorem 0.8. $\square$

1. Error estimates for the variational formulation without pressure

In this section, we shall exceptionally work with divergence-free discrete velocities, in other words, we shall work with subspaces V_H and V_h of V. This will not be the case in the subsequent sections, but we propose this here in order to highlight some important steps in the derivation of the error estimates without too many technical details. Let us denote by $\mathbf{w} \in V$ any given approximation of $\mathbf{u}$ over the interval $]0,T[$; in the application we have in mind, $\mathbf{w}$ will be given by

$$\mathbf{w} = \mathbf{u}_H\,,$$

where $\mathbf{u}_H(t) \in V_H$ is a semi-discrete approximation of $\mathbf{u}(t)$ on the coarse grid with mesh-size H. We shall specify below the error estimates satisfied by $\mathbf{w}$. As $\mathbf{w}$ is known, we define $\mathbf{u}_h$ as being the solution of: $\mathbf{u}_h(t) \in V_h$, such that

$$\forall \mathbf{v}_h \in V_h\,,\ (\mathbf{u}_h', \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{w}; \mathbf{u}_h, \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle\,,\ \text{ in }]0,T[\,, \tag{1.1}$$

$$\mathbf{u}_h(t)|_{t=0} = \mathbf{0}\,,$$

where

$$a(\mathbf{u}_h, \mathbf{v}_h) = \nu \int_\Omega \nabla\, \mathbf{u}_h \cdot \nabla\, \mathbf{v}_h \,\mathrm{d}\mathbf{x}\,,$$

$$b(\mathbf{w}; \mathbf{u}_h, \mathbf{v}_h) = \int_\Omega (\mathbf{w} \cdot \nabla\, \mathbf{u}_h) \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}\,.$$

Since $\mathbf{w}$ is given, (1.1) is a square system of linear ordinary differential equations in the finite-dimensional space V_h. It is easy to check that it has a unique solution. We want to estimate $\mathbf{u} - \mathbf{u}_h$ in suitable norms, and under appropriate regularity assumptions on $\mathbf{u}$ (and $\mathbf{w}$).

Let us choose $\mathbf{v} = \mathbf{v}_h$ in (0.5), a choice which is possible because $V_h \subset V$, then subtract (1.1) from (0.5):

$$\forall \mathbf{v}_h \in V_h\,,\ (\mathbf{u}' - \mathbf{u}_h', \mathbf{v}_h) + a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}; \mathbf{u}, \mathbf{v}_h) - b(\mathbf{w}; \mathbf{u}_h, \mathbf{v}_h) = 0\,. \tag{1.2}$$

Let $\mathbf{z}_h$ be given arbitrarily in V_h. We can rewrite (1.2) in the equivalent form:

$$\begin{aligned}\forall \mathbf{v}_h \in V_h\,,\ ((\mathbf{u} - \mathbf{z}_h + \mathbf{z}_h - \mathbf{u}_h)', \mathbf{v}_h) + a(\mathbf{u} - \mathbf{z}_h + \mathbf{z}_h - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u} - \mathbf{w}; \mathbf{u}, \mathbf{v}_h)\\ + b(\mathbf{w}; \mathbf{u} - \mathbf{z}_h + \mathbf{z}_h - \mathbf{u}_h, \mathbf{v}_h) = 0\,.\end{aligned} \tag{1.3}$$

We choose $\mathbf{v}_h = \mathbf{z}_h - \mathbf{u}_h$ in (1.3). As $\operatorname{div} \mathbf{w} = 0$, we have

$$\forall \mathbf{v}_h \in V_h\,,\ b(\mathbf{w}; \mathbf{v}_h, \mathbf{v}_h) = 0\,. \tag{1.4}$$

Therefore (1.3) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{z}_h - \mathbf{u}_h\|^2_{L^2(\Omega)} + \nu|\mathbf{z}_h - \mathbf{u}_h|^2_{H^1(\Omega)} + Z_1 + Z_2 = 0\,, \tag{1.5}$$

where

$$Z_1 = ((\mathbf{u} - \mathbf{z}_h)', \mathbf{z}_h - \mathbf{u}_h) + a(\mathbf{u} - \mathbf{z}_h, \mathbf{z}_h - \mathbf{u}_h)\,,$$

and in view of (1.4),

$$Z_2 = b(\mathbf{u} - \mathbf{w}; \mathbf{u}, \mathbf{z}_h - \mathbf{u}_h) + b(\mathbf{w}; \mathbf{u} - \mathbf{z}_h, \mathbf{z}_h - \mathbf{u}_h)\,.$$

For estimating the linear term Z_1, we use the fact that

$$|((\mathbf{u} - \mathbf{z}_h)', \mathbf{z}_h - \mathbf{u}_h)| \le \|(\mathbf{u} - \mathbf{z}_h)'\|_{H^{-1}(\Omega)}|\mathbf{z}_h - \mathbf{u}_h|_{H^1(\Omega)}\,.$$

Hence

$$|((\mathbf{u}-\mathbf{z}_h)',\mathbf{z}_h-\mathbf{u}_h)| \le \frac{\nu}{8}|\mathbf{z}_h-\mathbf{u}_h|^2_{H^1(\Omega)} + \frac{2}{\nu}\|(\mathbf{u}-\mathbf{z}_h)'\|^2_{H^{-1}(\Omega)}\,.$$

Similarly,

$$|a(\mathbf{u}-\mathbf{z}_h,\mathbf{z}_h-\mathbf{u}_h)| \le \frac{\nu}{8}|\mathbf{z}_h-\mathbf{u}_h|^2_{H^1(\Omega)} + 2\nu|\mathbf{u}-\mathbf{z}_h|^2_{H^1(\Omega)}\,.$$

Thus

$$|Z_1| \le \frac{\nu}{4}|\mathbf{z}_h-\mathbf{u}_h|^2_{H^1(\Omega)} + \frac{2}{\nu}\|(\mathbf{u}-\mathbf{z}_h)'\|^2_{H^{-1}(\Omega)} + 2\nu|\mathbf{u}-\mathbf{z}_h|^2_{H^1(\Omega)}\,. \tag{1.6}$$

For estimating the non-linear term Z_2, we split it into two parts: $Z_2 = Z_{21} + Z_{22}$, where

$$Z_{21} = b(\mathbf{u}-\mathbf{w};\mathbf{u},\mathbf{z}_h-\mathbf{u}_h)\ ,\ Z_{22} = b(\mathbf{w};\mathbf{u}-\mathbf{z}_h,\mathbf{z}_h-\mathbf{u}_h)\,. \tag{1.7}$$

There is not much choice for estimating Z_{22}, since (1.6) already includes the terms $|\mathbf{u}-\mathbf{z}_h|^2_{H^1(\Omega)}$ and $|\mathbf{z}_h - \mathbf{u}_h|^2_{H^1(\Omega)}$. Therefore

$$|Z_{22}| \le \|\mathbf{w}\|_{L^3(\Omega)}\|\mathbf{z}_h-\mathbf{u}_h\|_{L^6(\Omega)}|\mathbf{u}-\mathbf{z}_h|_{H^1(\Omega)}\,.$$

Since by the Sobolev imbedding (0.32),

$$\|\mathbf{z}_h-\mathbf{u}_h\|_{L^6(\Omega)} \le S_6|\mathbf{z}_h-\mathbf{u}_h|_{H^1(\Omega)}\,,$$

and if we assume that

$$\mathbf{w} \in L^\infty(0,T;L^3(\Omega)^3)\,, \tag{1.8}$$

we have finally

$$|Z_{22}| \le \frac{\nu}{8}|\mathbf{z}_h-\mathbf{u}_h|^2_{H^1(\Omega)} + \frac{2}{\nu}S_6^2\|\mathbf{w}\|^2_{L^\infty(0,T;L^3(\Omega)^3)}|\mathbf{u}-\mathbf{z}_h|^2_{H^1(\Omega)}\,. \tag{1.9}$$

Estimates for Z_{21} can be obtained in several different ways, depending on the information we have on $\mathbf{u}$. For instance, we can write:

$$|Z_{21}| \le |\mathbf{u}|_{H^1(\Omega)}\|\mathbf{u}-\mathbf{w}\|_{L^3(\Omega)}\|\mathbf{z}_h-\mathbf{u}_h\|_{L^6(\Omega)}\,, \tag{1.10}$$

or we can write:

$$|Z_{21}| \le |\mathbf{u}|_{W^{1,3}(\Omega)}\|\mathbf{u}-\mathbf{w}\|_{L^2(\Omega)}\|\mathbf{z}_h-\mathbf{u}_h\|_{L^6(\Omega)}\,. \tag{1.11}$$

Therefore, if we assume that

$$\mathbf{u} \in L^\infty(0,T;H^1(\Omega)^3)\,, \tag{1.12}$$

we derive from (1.10)

$$|Z_{21}| \le \frac{\nu}{8}|\mathbf{z}_h-\mathbf{u}_h|^2_{H^1(\Omega)} + \frac{2}{\nu}S_6^2\|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\|\mathbf{u}-\mathbf{w}\|^2_{L^3(\Omega)}\,, \tag{1.13}$$

or if $\mathbf{u}$ satisfies the *stronger* assumption:

$$\mathbf{u} \in L^\infty(0,T;W^{1,3}(\Omega)^3)\,, \tag{1.14}$$

we derive from (1.11)

$$|Z_{21}| \leq \frac{\nu}{8}|\mathbf{z}_h - \mathbf{u}_h|^2_{H^1(\Omega)} + \frac{2}{\nu}S_6^2\|\mathbf{u}\|^2_{L^\infty(0,T;W^{1,3}(\Omega)^3)}\|\mathbf{u}-\mathbf{w}\|^2_{L^2(\Omega)}\,. \tag{1.15}$$

Now, substituting (1.6), (1.9) and (1.13) or (1.15) into (1.5), we obtain after integration over $]0,t[$:

$$\begin{aligned}\|(\mathbf{z}_h - \mathbf{u}_h)(t)\|^2_{L^2(\Omega)} + \nu\int_0^t |(\mathbf{z}_h - \mathbf{u}_h)(s)|^2_{H^1(\Omega)}\mathrm{d}s \leq \frac{4}{\nu}\int_0^t \|(\mathbf{u}-\mathbf{z}_h)'(s)\|^2_{H^{-1}(\Omega)}\mathrm{d}s \\ + 4\Big(\nu + \frac{S_6^2}{\nu}\|\mathbf{w}\|^2_{L^\infty(0,T;L^3(\Omega)^3)}\Big)\int_0^t |(\mathbf{u}-\mathbf{z}_h)(s)|^2_{H^1(\Omega)}\mathrm{d}s \\ + 4\frac{S_6^2}{\nu}C_\rho\int_0^t \|(\mathbf{u}-\mathbf{w})(s)\|^2_{L^\rho(\Omega)}\mathrm{d}s\,,\end{aligned} \tag{1.16}$$

where $\rho = 3$ and $C_\rho = \|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega)^3)}$ if we assume (1.12), or $\rho = 2$ and $C_\rho = \|\mathbf{u}\|^2_{L^\infty(0,T;W^{1,3}(\Omega)^3)}$ if we assume (1.14). We shall see in the next sections that if the finite-element space V_h is well-chosen and if $\mathbf{u}$ has the regularity:

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)^3)\;,\;\; \mathbf{u}' \in L^2(\Omega\times]0,T[)^3\,,$$

then by applying the triangular inequality and taking the infimum with respect to $\mathbf{z}_h \in V_h$ in (1.16), we obtain

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|^2_{L^2(\Omega)} + \nu\int_0^t |(\mathbf{u}-\mathbf{u}_h)(s)|^2_{H^1(\Omega)}\mathrm{d}s \leq C_1h^2 + C_2\int_0^t \|(\mathbf{u}-\mathbf{w})(s)\|^2_{L^\rho(\Omega)}\mathrm{d}s\,. \tag{1.17}$$

We now use estimates on $\|\mathbf{u}-\mathbf{w}\|_{L^2(0,T;L^\rho(\Omega)^3)}$ that can "reasonably" be expected, when $\mathbf{w}$ is a "reasonable" approximation of $\mathbf{u}$ computed on the coarse grid with mesh-size H on a convex domain. We may expect that (and this is the tricky part of the proof):

$$\|\mathbf{u}-\mathbf{w}\|_{L^2(0,T;L^2(\Omega)^3)} \leq C_3H^2\,, \tag{1.18}$$

and

$$\|\mathbf{u}-\mathbf{w}\|_{L^2(0,T;H^1(\Omega)^3)} \leq C_4H\,. \tag{1.19}$$

Now if we use the space $H^{1/2}(\Omega)$, we have:

$$\|\varphi\|_{H^{1/2}(\Omega)} \leq C_5\|\varphi\|^{1/2}_{H^1(\Omega)}\|\varphi\|^{1/2}_{L^2(\Omega)}\,,$$

so that (1.18) and (1.19) imply that

$$\|\mathbf{u}-\mathbf{w}\|_{L^2(0,T;H^{1/2}(\Omega)^3)} \leq C_6H^{3/2}\,. \tag{1.20}$$

But using Sobolev's imbedding theorem for interpolated spaces, we have in three dimensions

$$H^{1/2}(\Omega) \subset L^3(\Omega)\,,$$

so that we obtain finally

$$\|\mathbf{u}-\mathbf{w}\|_{L^2(0,T;L^3(\Omega)^3)} \leq C_7H^{3/2}\,.$$

When substituted into (1.17), these estimates imply:

$$\|\mathbf{u}-\mathbf{u}_h\|^2_{L^\infty(0,T;L^2(\Omega)^3)} + \nu\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(0,T;H^1(\Omega)^3)} \leq \begin{array}{ll} C(h^2+H^3) & \text{under } (1.12) \\ C(h^2+H^4) & \text{under } (1.14)\,, \end{array} \tag{1.21}$$

hence the choice

$$h = H^{3/2} \text{ under (1.12)} \ , \tag{1.22}$$

and

$$h = H^{2} \text{ under (1.14)} \ . \tag{1.23}$$

Remark 1.1. The choice (1.22) is the "transient analogue" of what we have obtained in [16] for the steady (stationary) case, where the regularity properties for $\mathbf{u}$ *follow* exclusively from standard hypotheses on the data, without restriction on the angles of $\partial\Omega$. Similarly, the choice (1.23) was also obtained in [16] for the steady problem on a convex domain. □

Remark 1.2. In what follows, we shall obtain (1.23) for the formulation with pressure and with discrete velocities with non-zero divergence. □

2. *A priori* estimates for the solution of Step One

First, we describe the finite-element spaces. Since the notation H is somewhat cumbersome for a discretization parameter, we denote it by η as in (0.9–0.11). Thus, let $\eta > 0$ be a discretization parameter, that will tend to zero, and for each η, let $\mathcal{T}_\eta$ be a regular triangulation of $\overline{\Omega}$, consisting of tetrahedra with diameters bounded by η. As usual, any pair of tetrahedra of $\mathcal{T}_\eta$ are either disjoint or share a whole face, a whole edge or a vertex. For any tetrahedron κ, we denote by η_κ the diameter of κ and by ρ_κ the diameter of its inscribed sphere. By regular we mean (*cf.* Ciarlet [9]): there exists a constant $\sigma > 0$, independent of η such that

$$\forall \kappa \in \mathcal{T}_\eta \ , \ \frac{\eta_\kappa}{\rho_\kappa} = \sigma_\kappa \leq \sigma \, . \tag{2.1}$$

Let $\mathbb{P}_k$ denote the space of polynomials in three variables with total degree less than or equal to k. In each tetrahedron κ, the pressure p is a polynomial of $\mathbb{P}_1$ and each component of the velocity is the sum of a polynomial of $\mathbb{P}_1$ and a "bubble" function. Denoting the vertices of κ by $\mathbf{a}_i$, $1 \leq i \leq 4$, and its corresponding barycentric coordinate by λ_i, the basic bubble function b_κ is the polynomial of degree four

$$b_\kappa(\mathbf{x}) = \lambda_1(\mathbf{x})\lambda_2(\mathbf{x})\lambda_3(\mathbf{x})\lambda_4(\mathbf{x}) \, ,$$

that vanishes on the boundary of κ. Thus, we take

$$X_\eta = \{\mathbf{v}_\eta \in H_0^1(\Omega)^3 \, ; \, \forall \kappa \in \mathcal{T}_\eta \, , \, \mathbf{v}_\eta|_\kappa \in (\mathbb{P}_1 \oplus \mathrm{Vect}(b_\kappa))^3\} \, , \tag{2.2}$$

$$M_\eta = \{q_\eta \in H^1(\Omega) \cap L_0^2(\Omega) \, ; \, \forall \kappa \in \mathcal{T}_\eta \, , \, q_\eta|_\kappa \in \mathbb{P}_1\} \, . \tag{2.3}$$

As X_η contains all polynomials of degree one in each κ, there exists an operator $\Pi_\eta \in \mathcal{L}(H_0^1(\Omega)^3; X_\eta)$ such that for any real number $s \in [1,2]$:

$$\forall \mathbf{v} \in \left[H^s(\Omega) \cap H_0^1(\Omega)\right]^3 \ , \ |\Pi_\eta(\mathbf{v}) - \mathbf{v}|_{H^m(\Omega)} \leq C\, \eta^{s-m} |\mathbf{v}|_{H^s(\Omega)} \ , \ m = 0, 1 \, . \tag{2.4}$$

Similarly, as M_η contains all polynomials of degree zero in each κ, there exists an operator $r_\eta \in \mathcal{L}(L_0^2(\Omega); M_\eta)$ such that for any real number $s \in [0,1]$:

$$\forall q \in H^s(\Omega) \cap L_0^2(\Omega) \ , \ \|r_\eta(q) - q\|_{L^2(\Omega)} \leq C\, \eta^s |q|_{H^s(\Omega)} \, . \tag{2.5}$$

Furthermore, the pair (X_η, M_η) satisfies a uniform inf-sup condition: there exists a constant $\beta^* > 0$, independent of η, such that:

$$\forall q_\eta \in M_\eta\ ,\ \sup_{\mathbf{v}_\eta \in X_\eta} \frac{1}{|\mathbf{v}_\eta|_{H^1(\Omega)}} \int_\Omega q_\eta \mathrm{div}\, \mathbf{v}_\eta \,\mathrm{d}\mathbf{x} \geq \beta^* \|q_\eta\|_{L^2(\Omega)}\,. \tag{2.6}$$

In addition, we can construct an operator $P_\eta \in \mathcal{L}(H_0^1(\Omega)^3; X_\eta)$ such that (*cf.* [18]):

$$\forall \mathbf{v} \in H_0^1(\Omega)^3\,,\ \forall q_\eta \in M_\eta\,,\ \int_\Omega q_\eta \mathrm{div}(P_\eta(\mathbf{v}) - \mathbf{v})\,\mathrm{d}\mathbf{x} = 0\,, \tag{2.7}$$

and for $k = 0$ or 1 (*cf.* for instance [16]):

$$\forall \mathbf{v} \in \left[H^{1+k}(\Omega) \cap H_0^1(\Omega)\right]^3\ ,\ \|P_\eta(\mathbf{v}) - \mathbf{v}\|_{L^2(\Omega)} \leq C\,\eta^{1+k} |\mathbf{v}|_{H^{1+k}(\Omega)}\,, \tag{2.8}$$

and for any number $r \geq 2$, $k = 0$ or 1,

$$\forall \mathbf{v} \in \left[W^{1+k,r}(\Omega) \cap H_0^1(\Omega)\right]^3\ ,\ |P_\eta(\mathbf{v}) - \mathbf{v}|_{W^{1,r}(\Omega)} \leq C_r\,\eta^k |\mathbf{v}|_{W^{1+k,r}(\Omega)}\,, \tag{2.9}$$

with constants independent of η. In particular, if we approximate V by:

$$V_\eta = \{\mathbf{v}_\eta \in X_\eta\,;\, \forall q_\eta \in M_\eta\,,\ \int_\Omega q_\eta \mathrm{div}\, \mathbf{v}_\eta \,\mathrm{d}\mathbf{x} = 0\}\,, \tag{2.10}$$

then $P_\eta \in \mathcal{L}(V; V_\eta)$.

Remark 2.1. Note that V_η is not a subspace of V, because M_η does not contain enough functions to enforce a zero divergence. Thus V_η with $\eta = H$ does not refer to the space V_H of Section 1. □

Remark 2.2. The operator P_η is not unique. It is constructed by suitably correcting a regularization operator and it depends upon the regularization operator chosen. In particular if we choose an extension of the regularization operator of Scott and Zhang (*cf.* [42]) as in Section 3, then P_η can be defined on $L^1(\Omega)^3$, instead of $H^1(\Omega)^3$. □

Now, let us reformulate Step One more conveniently. First, we assume that $\mathbf{f} \in L^2(0, T; H^{-1}(\Omega)^3)$. Let N_η be the dimension of V_η and let $\{\mathbf{w}_j\}_{j=1}^{N_\eta}$ be a basis of V_η. Then $\mathbf{u}_\eta$ has the form:

$$\mathbf{u}_\eta(t) = \sum_{j=1}^{N_\eta} g_j(t)\mathbf{w}_j\,, \tag{2.11}$$

and the initial condition (0.14) reads

$$g_j(0) = 0\ ,\ 1 \leq j \leq N_\eta\,. \tag{2.12}$$

Thus choosing the test functions $\mathbf{v}_\eta$ in V_η, the equations of Step One become: Find $\mathbf{u}_\eta$ of the form (2.11) with the unknown coefficients $g_j \in \mathcal{C}^0([0, T])$ satisfying (2.12) and

$$\forall \mathbf{v}_\eta \in V_\eta\ ,\ (\mathbf{u}_\eta'(t), \mathbf{v}_\eta) + \nu(\nabla\, \mathbf{u}_\eta(t), \nabla\, \mathbf{v}_\eta) + (\mathbf{u}_\eta(t) \cdot \nabla\, \mathbf{u}_\eta(t), \mathbf{v}_\eta) = \langle \mathbf{f}(t), \mathbf{v}_\eta\rangle \quad \text{in }]0, T]\,. \tag{2.13}$$

The inf-sup condition (2.6) implies that (2.11–2.13) is equivalent to (0.12–0.14) (*cf.* Babuška [4], Brezzi [6], [18] or Brenner and Scott [5]). Now, (2.11–2.13) is a square system of N_η ordinary non-linear differential equations of order one, with a constant non-singular matrix multiplying the derivative and the remaining coefficients at least in $L^2(0,T)$. Therefore, Carathéodory's Theorem (*cf.* Coddington and Levinson [10]) implies that it has a local maximal solution $\mathbf{u}_\eta(t)$ in an interval $[0,T_\eta[$, where $0 < T_\eta \le T$. It remains to prove that, at best $T_\eta = T$, or at least T_η can be bounded below by a constant $T^\star > 0$, independent of η. This is achieved by means of *a priori* estimates.

But the trouble is that V_η is not a subspace of V, and the choice $\mathbf{v}_\eta = \mathbf{u}_\eta$ in (2.13) does not eliminate the non-linear term as in (1.4). More precisely, for $\mathbf{u}_\eta \in V_\eta$,

$$\forall \mathbf{w}_\eta \in X_\eta\,,\ \int_\Omega \mathbf{u}_\eta \cdot \nabla\, \mathbf{w}_\eta \cdot \mathbf{w}_\eta \mathrm{d}\mathbf{x} = -\frac{1}{2}\int_\Omega \operatorname{div} \mathbf{u}_\eta \|\mathbf{w}_\eta\|^2 \mathrm{d}\mathbf{x}\,,$$

and it is shown in [16] that there exists a constant $\hat{C}$, independent of η such that, for all $\mathbf{u}_\eta \in V_\eta$,

$$\forall \mathbf{w}_\eta \in X_\eta\,,\ |\int_\Omega \mathbf{u}_\eta \cdot \nabla\, \mathbf{w}_\eta \cdot \mathbf{w}_\eta \mathrm{d}\mathbf{x}| \le \frac{1}{2}\hat{C}\eta^{1/2}\|\operatorname{div} \mathbf{u}_\eta\|_{L^2(\Omega)}|\mathbf{w}_\eta|^2_{H^1(\Omega)}\,. \tag{2.14}$$

(We also refer to Crouzeix [11], Lemme 5.1, p. 214, for a related inequality). Thus for deriving an *a priori* estimate from (2.13), we must establish the uniform bound

$$\eta^{1/2}\|\operatorname{div} \mathbf{u}_\eta\|_{L^\infty(0,T_\eta;L^2(\Omega))} \le C\,. \tag{2.15}$$

Actually, we are going to prove that

$$\|\mathbf{u}_\eta\|_{L^\infty(0,T_\eta;H^1(\Omega)^3)} \le C\ ,\ C \text{ independent of } \eta\,, \tag{2.16}$$

which is stronger than (2.15). It is likely that (2.15) is true under *weaker* hypotheses than those used in proving (2.16), but this is an open problem. Here we propose two proofs of (2.16), each one with a different set of assumptions.

Theorem 2.3. *Let* $\mathbf{f}$ *belong to* $H^1(0,T;H^{-1}(\Omega)^3)$ *and assume that* $\mathbf{f}(0)$ *belongs to* $L^2(\Omega)^3$. *If* $\eta \le \eta_0$, *where*

$$\eta_0 = \frac{1}{3}\Big(\frac{S_3S_6}{\hat{C}}\Big)^2\,, \tag{2.17}$$

$\hat{C}$ *is the constant of (2.14) and if*

$$\sqrt{\frac{2}{3\nu}}\|\mathbf{f}\|_{L^2(0,T;H^{-1}(\Omega)^3)}\big(\|\mathbf{f}(0)\|^2_{L^2(\Omega)} + \frac{2}{\nu}\|\mathbf{f}'\|^2_{L^2(0,T;H^{-1}(\Omega)^3)}\big)^{1/2} + \frac{1}{\nu}\|\mathbf{f}\|^2_{L^\infty(0,T;H^{-1}(\Omega)^3)} < \Big(\frac{\nu}{2}\Big)^3\Big(\frac{1}{S_3S_6}\Big)^2\,, \tag{2.18}$$

then $T_\eta = T$, *the discrete solution* $\mathbf{u}_\eta$ *is unique and is bounded uniformly with respect to* η *in* $L^\infty(0,T;H^1(\Omega)^3)$. *In addition,* $\mathbf{u}'_\eta$ *is bounded uniformly with respect to* η *in* $L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;H^1(\Omega)^3)$.

Proof. The proof is based on differentiating (2.13) with respect to t. Taking the t-derivative of the variational approximation is a standard procedure, *cf.* for instance [30], Chapter 10, Theorem 6.2, and the bibliography therein. A slightly different approach to t-differentiation can be found in [23], Chapter 6, Section 4, pp.162,163, and in its references. We give here a detailed proof in order to arrive at the precise estimate (2.18).

Note that $\mathbf{f}$ belongs to $\mathcal{C}^0([0,T];H^{-1}(\Omega)^3)$. Therefore $\mathbf{u}'_\eta$ is in $\mathcal{C}^0([0,T_\eta[;V_\eta)$; hence we can differentiate (2.13) with respect to t and choose $\mathbf{v}_\eta = \mathbf{u}'_\eta(t)$. Applying (2.14) and Hölder's inequality, we obtain at any time $t \in [0,T_\eta[$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}'_\eta(t)\|^2_{L^2(\Omega)} + 2\nu\,|\mathbf{u}'_\eta(t)|^2_{H^1(\Omega)} - 2\,|\mathbf{u}_\eta(t)|_{H^1(\Omega)}\|\mathbf{u}'_\eta(t)\|_{L^6(\Omega)}\|\mathbf{u}'_\eta(t)\|_{L^3(\Omega)}\\ - \hat{C}\eta^{1/2}\|\operatorname{div} \mathbf{u}_\eta(t)\|_{L^2(\Omega)}|\mathbf{u}'_\eta(t)|^2_{H^1(\Omega)} \le 2\|\mathbf{f}'(t)\|_{H^{-1}(\Omega)}|\mathbf{u}'_\eta(t)|_{H^1(\Omega)}\,.\end{aligned} \tag{2.19}$$

But

$$\|\mathbf{u}'_\eta(t)\|_{L^6(\Omega)}\|\mathbf{u}'_\eta(t)\|_{L^3(\Omega)} \le S_6 S_3 |\mathbf{u}'_\eta(t)|^2_{H^1(\Omega)}\,,$$

and since $\mathbf{u}_\eta(0) = \mathbf{0}$, by continuity there exists a time $\tilde{T}_\eta$, $0 < \tilde{T}_\eta \le T_\eta$ such that

$$\forall t \in [0, \tilde{T}_\eta]\,,\ |\mathbf{u}_\eta(t)|_{H^1(\Omega)} < \frac{\nu}{2S_3S_6}\,. \tag{2.20}$$

Moreover, we derive from (2.20) and (2.14) that, if $\eta \le \eta_0$, with η_0 defined by (2.17), then

$$\forall \mathbf{w}_\eta \in X_\eta\,,\ |\int_\Omega \mathbf{u}_\eta(t)\cdot\nabla\,\mathbf{w}_\eta\cdot\mathbf{w}_\eta \mathrm{d}\mathbf{x}| \le \frac{\nu}{4}|\mathbf{w}_\eta|^2_{H^1(\Omega)}\,. \tag{2.21}$$

Let us prove that, if $\eta \le \eta_0$ and the data satisfies (2.18), then (2.20) holds on $[0, T_\eta]$, which in turn implies that $T_\eta = T$. The proof proceeds by contradiction. Suppose there exists $T^\star$, $0 < T^\star \le T_\eta$ such that

$$\forall t \in [0, T^\star[\,,\ |\mathbf{u}_\eta(t)|_{H^1(\Omega)} < \frac{\nu}{2S_3S_6} \ \text{ and } |\mathbf{u}_\eta(T^\star)|_{H^1(\Omega)} = \frac{\nu}{2S_3S_6}\,. \tag{2.22}$$

Then, (2.19) and (2.22) imply for all $t \in [0, T^\star]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}'_\eta(t)\|^2_{L^2(\Omega)} + \frac{\nu}{2}|\mathbf{u}'_\eta(t)|^2_{H^1(\Omega)} \le \varepsilon\,|\mathbf{u}'_\eta(t)|^2_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\mathbf{f}'(t)\|^2_{H^{-1}(\Omega)}\,. \tag{2.23}$$

In addition, substituting $\mathbf{v}_\eta = \mathbf{u}'_\eta(0)$ into (2.13) at time $t = 0$, we obtain

$$\|\mathbf{u}'_\eta(0)\|_{L^2(\Omega)} \le \|\mathbf{f}(0)\|_{L^2(\Omega)}\,.$$

Then the choice $\varepsilon = \nu/2$ in (2.23) yields

$$\forall t \in [0, T^\star]\,,\ \|\mathbf{u}'_\eta(t)\|^2_{L^2(\Omega)} \le \|\mathbf{f}(0)\|^2_{L^2(\Omega)} + \frac{2}{\nu}\|\mathbf{f}'\|^2_{L^2(0,T;H^{-1}(\Omega)^3)}\,. \tag{2.24}$$

Similarly, the choice $\varepsilon = \nu/4$ in (2.23) yields:

$$\forall t \in [0, T^\star]\,,\ \int_0^t |\mathbf{u}'_\eta(s)|^2_{H^1(\Omega)}\mathrm{d}s \le \frac{4}{\nu}\big(\|\mathbf{f}(0)\|^2_{L^2(\Omega)} + \frac{4}{\nu}\|\mathbf{f}'\|^2_{L^2(0,T;H^{-1}(\Omega)^3)}\big)\,. \tag{2.25}$$

Next, let us choose $\mathbf{v}_\eta = \mathbf{u}_\eta(t)$ in (2.13); in view of (2.21), we obtain for $\eta \le \eta_0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_\eta(t)\|^2_{L^2(\Omega)} + \frac{3}{2}\nu\,|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)} \le \varepsilon\,|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\mathbf{f}(t)\|^2_{H^{-1}(\Omega)}\,.$$

Therefore

$$\forall t \in [0, T^\star]\,,\ \|\mathbf{u}_\eta(t)\|^2_{L^2(\Omega)} \le \frac{2}{3\nu}\|\mathbf{f}\|^2_{L^2(0,T;H^{-1}(\Omega)^3)}\,. \tag{2.26}$$

Finally, the same choice in (2.13) also gives

$$(\mathbf{u}'_\eta(t), \mathbf{u}_\eta(t)) + \frac{3}{4}\nu\,|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)} \le \frac{1}{2}(\varepsilon\,|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)} + \frac{1}{\varepsilon}\|\mathbf{f}(t)\|^2_{H^{-1}(\Omega)})\,.$$

Thus the choice $\varepsilon = \nu/2$ implies

$$\frac{\nu}{2}|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)} \le \frac{1}{\nu}\|\mathbf{f}(t)\|^2_{H^{-1}(\Omega)} + \|\mathbf{u}'_\eta(t)\|_{L^2(\Omega)}\|\mathbf{u}_\eta(t)\|_{L^2(\Omega)}\,.$$

This inequality together with (2.26) and (2.24) give for all $t \in [0, T^\star]$

$$|\mathbf{u}_\eta(t)|_{H^1(\Omega)} \leq \sqrt{\frac{2}{\nu}}\Big(\frac{1}{\nu}\|\mathbf{f}\|^2_{L^\infty(0,T;H^{-1}(\Omega)^3)} + \sqrt{\frac{2}{3\nu}}\|\mathbf{f}\|_{L^2(0,T;H^{-1}(\Omega)^3)}\big(\|\mathbf{f}(0)\|^2_{L^2(\Omega)} + \frac{2}{\nu}\|\mathbf{f}'\|^2_{L^2(0,T;H^{-1}(\Omega)^3)}\big)^{1/2}\Big)^{1/2}.$$

Since this inequality is valid at $t = T^\star$, (2.18) contradicts the equality in (2.22).

This establishes global existence of a discrete solution. Uniqueness follows easily from the above estimates and Gronwall's Lemma. □

Remark 2.4. The solution $(\mathbf{u}, p)$ of (0.1–0.4) is unique if the data $\mathbf{f}$ and ν satisfy the conditions of Theorem 2.3; in fact, the smallness condition on the data is a little less restrictive than (2.18) because the non-linear term is exactly antisymmetric. Furthermore $\mathbf{u} \in L^\infty(0,T;H^1(\Omega)^3)$ and $\mathbf{u}' \in L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;H^1(\Omega)^3)$ (*cf.* [43]). Therefore, it follows from Corollary 0.9 that if in addition $\mathbf{f} \in L^r(0,T;L^{3/2}(\Omega)^3)$ for any number $r \geq 2$, then $(\mathbf{u}, p) \in L^r(0,T;H^{3/2}(\Omega)^3) \times L^r(0,T;H^{1/2}(\Omega))$, without restrictions on the angles of $\partial\Omega$. Indeed, passing $\mathbf{u}'(t)$ to the right-hand side, we can write for almost every $t \in]0, T[$:

$$-\nu\Delta\,\mathbf{u}(t) + \mathbf{u}(t)\cdot\nabla\,\mathbf{u}(t) + \nabla\,p(t) = \mathbf{f}(t) - \mathbf{u}'(t) \text{ in } \Omega\,,$$

$$\operatorname{div}\mathbf{u}(t) = 0 \text{ in } \Omega\,,$$
$$\mathbf{u}(t) = \mathbf{0} \text{ on } \partial\Omega\,,$$

i.e. the pair $(\mathbf{u}(t), p(t))$ is the solution of a Navier-Stokes equation parametrized by t. Similarly, if in addition, $\mathbf{f} \in L^r(0,T;L^2(\Omega)^3)$ and Ω is convex then $(\mathbf{u}, p) \in L^r(0,T;H^2(\Omega)^3) \times L^r(0,T;H^1(\Omega))$. □

The second proof of (2.16) follows an idea introduced by Heywood in [21] for the continuous Navier-Stokes problem, and used by [22] for a discrete Navier-Stokes problem on a single grid. Beforehand, recall the definition of a uniformly regular triangulation: in addition to (2.1), there exists a constant $\tau > 0$, independent of η, such that

$$\forall\kappa \in \mathcal{T}_\eta\,,\ \tau\,\eta \leq \eta_\kappa \leq \sigma_\kappa\rho_\kappa\,. \tag{2.27}$$

Theorem 2.5. *Let* $\mathbf{f}$ *belong to* $L^2(\Omega\times]0,T[)^3$ *and assume that* Ω *is convex and the triangulation* $\mathcal{T}_\eta$ *satisfies (2.27). Then there exists a time* $T^\star > 0$*, depending on the data, but independent of* η*, such that* $\mathbf{u}_\eta$ *is unique and is bounded uniformly with respect to* η *in* $L^\infty(0,T^\star;H^1(\Omega)^3)$*. In addition,* $\mathbf{u}_\eta$ *and* $\mathbf{u}'_\eta$ *are bounded uniformly with respect to* η *in* $L^2(0,T^\star;W^{1,6}(\Omega)^3)$ *and* $L^2(\Omega\times]0,T^\star[)^3$ *respectively.*

Proof. The idea of the proof in [21] consists in taking the scalar product of (0.1) by the Helmholtz decomposition of $\nu\Delta\,\mathbf{u}(t)$. In the discrete case, this is not possible because $\Delta\,\mathbf{u}_\eta(t)$ does not belong to $L^2(\Omega)^3$. We replace it by the unique solution $\mathbf{w}_\eta(t) \in V_\eta$ of

$$\forall\mathbf{v}_\eta \in V_\eta\,,\ (\mathbf{w}_\eta(t), \mathbf{v}_\eta) = \nu\,(\nabla\,\mathbf{u}_\eta(t), \nabla\,\mathbf{v}_\eta)\,. \tag{2.28}$$

As V_η is a finite-dimensional space, $\mathbf{w}_\eta(t)$ is uniquely defined by (2.28). Note that $\mathbf{w}_\eta$ depends on t because $\mathbf{u}_\eta$ depends on t and thus $\mathbf{w}_\eta \in \mathcal{C}^0([0,T_\eta[;V_\eta)$. For establishing the theorem, we need to extend to $\|\nabla\,\mathbf{u}_\eta\|_{L^3(\Omega)}$ the well-known consequence of Sobolev's inequality:

$$\forall g \in H^1(\Omega)\,,\ \|g\|_{L^3(\Omega)} \leq C\|g\|^{1/2}_{L^2(\Omega)}|g|^{1/2}_{H^1(\Omega)}\,. \tag{2.29}$$

Specifically, let us prove that there exists a constant K, independent of η and t, such that

$$\|\nabla\,\mathbf{u}_\eta(t)\|_{L^3(\Omega)} \leq K\|\nabla\,\mathbf{u}_\eta(t)\|^{1/2}_{L^2(\Omega)}\|\mathbf{w}_\eta(t)\|^{1/2}_{L^2(\Omega)}\,. \tag{2.30}$$

Since

$$\|\nabla \mathbf{u}_\eta(t)\|_{L^3(\Omega)} \le \|\nabla \mathbf{u}_\eta(t)\|_{L^2(\Omega)}^{1/2} \|\nabla \mathbf{u}_\eta(t)\|_{L^6(\Omega)}^{1/2}\,, \tag{2.31}$$

we must find a suitable estimate for $\|\nabla \mathbf{u}_\eta(t)\|_{L^6(\Omega)}$.

To simplify the notation, we drop t for the time being. We associate with $\mathbf{w}_\eta$ the solution $(\mathbf{u}(\eta), p(\eta)) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ of the Stokes problem:

$$-\nu\,\Delta\,\mathbf{u}(\eta) + \nabla\, p(\eta) = \mathbf{w}_\eta\ ,\ \operatorname{div}\mathbf{u}(\eta) = 0\ \text{ in } \Omega\,.$$

Then $\mathbf{u}(\eta)$ and $\mathbf{u}_\eta$ are related by:

$$\forall \mathbf{v}_\eta \in V_\eta, \forall q_\eta \in M_\eta\,,\ \nu\,(\nabla(\mathbf{u}_\eta - \mathbf{u}(\eta)), \nabla\,\mathbf{v}_\eta) + (p(\eta) - q_\eta, \operatorname{div}\mathbf{v}_\eta) = 0\,.$$

Hence,

$$|\mathbf{u}_\eta - P_\eta(\mathbf{u}(\eta))|_{H^1(\Omega)} \le |\mathbf{u}(\eta) - P_\eta(\mathbf{u}(\eta))|_{H^1(\Omega)} + \frac{\sqrt{3}}{\nu}\|p(\eta) - r_\eta(p(\eta))\|_{L^2(\Omega)}\,. \tag{2.32}$$

Now the convexity assumption on Ω implies that $(\mathbf{u}(\eta), p(\eta)) \in H^2(\Omega)^3 \times H^1(\Omega)$ (*cf.* [12]) and there exists a constant c_1 that depends only on Ω such that

$$|\mathbf{u}(\eta)|_{H^2(\Omega)} + |p(\eta)|_{H^1(\Omega)} \le c_1\|\mathbf{w}_\eta\|_{L^2(\Omega)}\,. \tag{2.33}$$

Therefore, applying (2.5) with $s = 1$, (2.9) with $r = 2$ and $k = 1$, and (2.33) and substituting into (2.32), we obtain, with a constant c_2 independent of η:

$$|\mathbf{u}_\eta - P_\eta(\mathbf{u}(\eta))|_{H^1(\Omega)} \le c_2\eta\|\mathbf{w}_\eta\|_{L^2(\Omega)}\,. \tag{2.34}$$

Let us write

$$\|\nabla\,\mathbf{u}_\eta\|_{L^6(\Omega)} \le \|\nabla(\mathbf{u}_\eta - P_\eta(\mathbf{u}(\eta)))\|_{L^6(\Omega)} + \|\nabla\,P_\eta(\mathbf{u}(\eta))\|_{L^6(\Omega)}\,.$$

The uniform regularity (2.27) of the triangulation implies the inverse inequality: there exists a constant c_3, independent of η, such that

$$\|\nabla(\mathbf{u}_\eta - P_\eta(\mathbf{u}(\eta)))\|_{L^6(\Omega)} \le \frac{1}{\eta}c_3\|\nabla(\mathbf{u}_\eta - P_\eta(\mathbf{u}(\eta)))\|_{L^2(\Omega)}\,. \tag{2.35}$$

Together with (2.34) and (2.9) with $r = 6$ and $k = 0$, this inequality implies

$$\|\nabla\,\mathbf{u}_\eta\|_{L^6(\Omega)} \le c_4\|\mathbf{w}_\eta\|_{L^2(\Omega)}\,, \tag{2.36}$$

with a constant c_4, independent of η and t. Then (2.30) with $K = \sqrt{c_4}$ follows by substituting (2.36) into (2.31).

Now, we choose $\mathbf{v}_\eta = \mathbf{w}_\eta(t)$ in (2.13). On one hand, (2.28) yields

$$(\mathbf{u}_\eta'(t), \mathbf{w}_\eta(t)) = \nu(\nabla\,\mathbf{u}_\eta'(t), \nabla\,\mathbf{u}_\eta(t)) = \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_\eta(t)|_{H^1(\Omega)}^2\,,$$

$$\nu(\nabla\,\mathbf{u}_\eta(t), \nabla\,\mathbf{w}_\eta(t)) = \|\mathbf{w}_\eta(t)\|_{L^2(\Omega)}^2\,,$$

and on the other hand, (2.30) implies

$$\left|\int_\Omega \mathbf{u}_\eta(t)\cdot\nabla\,\mathbf{u}_\eta(t)\cdot\mathbf{w}_\eta(t)\mathrm{d}\mathbf{x}\right| \le \frac{3}{4}\|\mathbf{w}_\eta(t)\|_{L^2(\Omega)}^2 + \frac{1}{4}(KS_6)^4|\mathbf{u}_\eta(t)|_{H^1(\Omega)}^6\,. \tag{2.37}$$

Hence

$$\nu\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)}+2\|\mathbf{w}_\eta(t)\|^2_{L^2(\Omega)}\le\frac{3}{2}\|\mathbf{w}_\eta(t)\|^2_{L^2(\Omega)}+\frac{1}{2}(KS_6)^4|\mathbf{u}_\eta(t)|^6_{H^1(\Omega)}+\varepsilon\|\mathbf{w}_\eta(t)\|^2_{L^2(\Omega)}+\frac{1}{\varepsilon}\|\mathbf{f}(t)\|^2_{L^2(\Omega)}\,,$$

and the choice $\varepsilon = 1/2$ gives

$$|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)}\le\psi(t)\,,$$

where ψ is the solution of the differential equation

$$\psi'(t)=\frac{2}{\nu}\|\mathbf{f}(t)\|^2_{L^2(\Omega)}+\frac{1}{2\nu}(KS_6)^4\psi(t)^3\quad\text{for } t>0\,,$$

$$\psi(0)=0\,.$$

Then, if $\mathbf{f}\in L^2(\Omega\times]0,T[)^3$, for any constant $C>0$, there exists a time $T^\star$ depending only on C and the constant coefficients of this equation (and hence independent of η) such that

$$\forall t\in[0,T^\star]\,,\ |\mathbf{u}_\eta(t)|^2_{H^1(\Omega)}\le C\,. \tag{2.38}$$

In turn, the choice $\varepsilon = 1/4$ and (2.38) give

$$\|\mathbf{w}_\eta\|_{L^2(\Omega\times]0,T^\star[)}\le c_5\,, \tag{2.39}$$

and then (2.36) implies that

$$\|\mathbf{u}_\eta\|_{L^2(0,T^\star;W^{1,6}(\Omega)^3)}\le c_4c_5\,. \tag{2.40}$$

As far as $\mathbf{u}'_\eta$ is concerned, the choice $\mathbf{v}_\eta=\mathbf{u}'_\eta(t)$ in (2.13) yields

$$\|\mathbf{u}'_\eta(t)\|^2_{L^2(\Omega)}+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_\eta(t)|^2_{H^1(\Omega)}\le\|\mathbf{u}'_\eta(t)\|_{L^2(\Omega)}\big(\|\mathbf{f}(t)\|_{L^2(\Omega)}+\|\mathbf{u}_\eta(t)\|_{L^6(\Omega)}|\mathbf{u}_\eta(t)|_{W^{1,3}(\Omega)}\big)\,,$$

and (2.40) and (2.38) readily imply that

$$\|\mathbf{u}'_\eta\|_{L^2(\Omega\times]0,T^\star[)}\le c_6\,. \tag{2.41}$$

□

Remark 2.6. To avoid a multiplicity of notation, we choose for constant in (2.38):

$$C=\Big(\frac{\nu}{2S_3S_6}\Big)^2\,.$$

Then (2.21) holds for $\eta\le\eta_0$, η_0 defined by (2.17). □

Remark 2.7. The assumptions of Theorem 2.5, namely Ω convex and $\mathbf{f}$ in $L^2(\Omega\times]0,T[)^3$, also imply that the solution $(\mathbf{u},p)$ of (0.1–0.4) satisfies $\mathbf{u}$ in $L^\infty(0,T^\star;H^1(\Omega)^3)\cap L^2(0,T^\star;H^2(\Omega)^3)$, $\mathbf{u}'$ in $L^2(\Omega\times]0,T^\star[)^3$, p in $L^2(0,T^\star;H^1(\Omega))$ and of course the solution is unique. □

3. Error estimates for the solution of Step One

In this section, we suppose that the assumptions of either Theorem 2.3 or Theorem 2.5 are valid. To unify the notation, in the case of Theorem 2.5, we denote $T^\star$ by T. Under both sets of assumptions, we have $\mathbf{u}' \in L^2(\Omega\times]0,T[)^3$ at least.

Let P_η be an approximation operator satisfying (2.7–2.9). As mentioned in Remark 2.2, this operator is not unique, because it is constructed by correcting a regularization operator that can be chosen according to convenience. In the theorem below, we need to apply P_η to $\mathbf{u}'$ that belongs only to $L^2(\Omega)^3$; thus we must define a regularization operator on $L^2(\Omega)$. The reader will find in the Appendix a brief derivation of an extension of the Scott and Zhang operator [42], that we denote by R_η. It is defined for $v \in L^1(\Omega)$ and we can force the boundary value of $R_\eta(v)$ to vanish, so that $R_\eta \in \mathcal{L}(L^1(\Omega)^3; X_\eta)$. Now we define:

$$P_\eta(\mathbf{v}) = R_\eta(\mathbf{v}) + \sum_{\kappa\in\mathcal{T}_\eta} \mathbf{c}_\kappa b_\kappa\,, \tag{3.1}$$

where

$$\forall\kappa \in \mathcal{T}_\eta\,,\ \mathbf{c}_\kappa = \frac{1}{\int_\kappa b_\kappa \mathrm{d}\mathbf{x}} \int_\kappa (\mathbf{v} - R_\eta(\mathbf{v}))\mathrm{d}\mathbf{x}\,. \tag{3.2}$$

Then $P_\eta \in \mathcal{L}(L^1(\Omega)^3; X_\eta)$; it satisfies (2.7–2.9) and in addition, for each number $r \geq 1$, there exists a constant $\hat{c}$, independent of η and κ, such that for all $\mathbf{v} \in L^r(\Omega)^3$,

$$\forall\kappa \in \mathcal{T}_\eta\,,\ \|P_\eta(\mathbf{v}) - \mathbf{v}\|_{L^r(\kappa)} \leq \hat{c}\|R_\eta(\mathbf{v}) - \mathbf{v}\|_{L^r(\kappa)}\,. \tag{3.3}$$

Moreover, it follows from (3.2) that

$$\forall\kappa \in \mathcal{T}_\eta\,,\ \int_\kappa (P_\eta(\mathbf{v}) - \mathbf{v})\mathrm{d}\mathbf{x} = 0\,. \tag{3.4}$$

This readily implies that, for a constant C independent of η

$$\forall\mathbf{v} \in L^2(\Omega)^3\,,\ \|P_\eta(\mathbf{v}) - \mathbf{v}\|_{H^{-1}(\Omega)} \leq C\eta\|P_\eta(\mathbf{v}) - \mathbf{v}\|_{L^2(\Omega)}\,. \tag{3.5}$$

Theorem 3.1. *Suppose that the assumptions of Theorems 2.3 or 2.5 are valid and let P_η be defined by (3.1) and (3.2). Then if $\eta \leq \eta_0$, there exist three constants C_1, C_2 and C_3, independent of η, such that*

$$\begin{aligned}
&\|\mathbf{u}_\eta - P_\eta(\mathbf{u})\|^2_{L^\infty(0,T;L^2(\Omega)^3)} + \frac{\nu}{2}\|\mathbf{u}_\eta - P_\eta(\mathbf{u})\|^2_{L^2(0,T;H^1(\Omega)^3)}\\
&\quad\leq \exp\big((\frac{2}{7\nu} + \frac{C_1}{\nu^3})T\big)\big(\frac{7}{\nu}C_2\eta^2\|\mathbf{u}'\|^2_{L^2(\Omega\times]0,T[)}\\
&\qquad + (\frac{\nu}{7} + C_3)\|\mathbf{u} - P_\eta(\mathbf{u})\|^2_{L^2(0,T;H^1(\Omega)^3)} + \frac{21}{\nu}\|p - r_\eta(p)\|^2_{L^2(\Omega\times]0,T[)}\big)\,.
\end{aligned} \tag{3.6}$$

Proof. First note that

$$(P_\eta(\mathbf{u}))' = P_\eta(\mathbf{u}')\,,$$

because the same property is true for R_η owing to (A.2). Then we take the scalar product of (0.1) with a test function $\mathbf{v}_\eta \in V_\eta$ and we take the difference between the resulting equation and (2.13). By inserting $(P_\eta(\mathbf{u}))'$ in the first term and $P_\eta(\mathbf{u})$ in the other terms, choosing $\mathbf{v}_\eta = \mathbf{u}_\eta - P_\eta(\mathbf{u})$, applying (2.21) and using (2.10), we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_\eta - P_\eta(\mathbf{u})\|^2_{L^2(\Omega)} + \frac{3}{4}\nu|\mathbf{u}_\eta - P_\eta(\mathbf{u})|^2_{H^1(\Omega)} \\
&\quad\le \big(\|\mathbf{u}' - P_\eta(\mathbf{u}')\|_{H^{-1}(\Omega)} + \nu|\mathbf{u} - P_\eta(\mathbf{u})|_{H^1(\Omega)} + \sqrt{3}\|p - r_\eta(p)\|_{L^2(\Omega)}\big)|\mathbf{u}_\eta - P_\eta(\mathbf{u})|_{H^1(\Omega)} \\
&\quad\quad + ((\mathbf{u} - P_\eta(\mathbf{u}))\cdot\nabla\,\mathbf{u}, \mathbf{u}_\eta - P_\eta(\mathbf{u})) - ((\mathbf{u}_\eta - P_\eta(\mathbf{u}))\cdot\nabla\,P_\eta(\mathbf{u}), \mathbf{u}_\eta - P_\eta(\mathbf{u})) \\
&\quad\quad + (P_\eta(\mathbf{u})\cdot\nabla(\mathbf{u} - P_\eta(\mathbf{u})), \mathbf{u}_\eta - P_\eta(\mathbf{u}))\,. \qquad (3.7)
\end{aligned}$$

Let us bound the non-linear terms in (3.7). Since $\mathbf{u} \in L^\infty(0,T;H^1(\Omega)^3)$ in view of Remarks 2.4 and 2.7, we set

$$c_1 = \|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^3)}\,.$$

Therefore, for any $\varepsilon_1 > 0$ and $\delta_1 > 0$, we have

$$\begin{aligned}
|((\mathbf{u} - P_\eta(\mathbf{u}))\cdot\nabla\,\mathbf{u}, \mathbf{u}_\eta - P_\eta(\mathbf{u}))| &\le c_1 S_6^{3/2}|\mathbf{u} - P_\eta(\mathbf{u})|_{H^1(\Omega)}\|\mathbf{u}_\eta - P_\eta(\mathbf{u})\|^{1/2}_{L^2(\Omega)}|\mathbf{u}_\eta - P_\eta(\mathbf{u})|^{1/2}_{H^1(\Omega)} \\
&\le \frac{c_1}{2}S_6^{3/2}\big(\frac{1}{\varepsilon_1}|\mathbf{u} - P_\eta(\mathbf{u})|^2_{H^1(\Omega)} + \frac{\varepsilon_1}{2}(\delta_1|\mathbf{u}_\eta - P_\eta(\mathbf{u})|^2_{H^1(\Omega)} + \frac{1}{\delta_1}\|\mathbf{u}_\eta - P_\eta(\mathbf{u})\|^2_{L^2(\Omega)})\big)\,.
\end{aligned}$$

Similarly, set

$$c_2 = \|P_\eta(\mathbf{u})\|_{L^\infty(0,T;H^1(\Omega)^3)}\,,$$

which is also bounded in view of (2.9) with $k = 0$ and $r = 2$. Then

$$\begin{aligned}
&|((\mathbf{u}_\eta - P_\eta(\mathbf{u}))\cdot\nabla\,P_\eta(\mathbf{u}), \mathbf{u}_\eta - P_\eta(\mathbf{u}))| \\
&\le \frac{c_2}{2}S_6^{3/2}\big(\frac{1}{\varepsilon_2}|\mathbf{u}_\eta - P_\eta(\mathbf{u})|^2_{H^1(\Omega)} + \frac{\varepsilon_2}{2}(\delta_2|\mathbf{u}_\eta - P_\eta(\mathbf{u})|^2_{H^1(\Omega)} + \frac{1}{\delta_2}\|\mathbf{u}_\eta - P_\eta(\mathbf{u})\|^2_{L^2(\Omega)})\big)\,,
\end{aligned}$$

$$\begin{aligned}
&|(P_\eta(\mathbf{u})\cdot\nabla(\mathbf{u} - P_\eta(\mathbf{u})), \mathbf{u}_\eta - P_\eta(\mathbf{u}))| \\
&\le \frac{c_2}{2}S_6^{3/2}\big(\frac{1}{\varepsilon_3}|\mathbf{u} - P_\eta(\mathbf{u})|^2_{H^1(\Omega)} + \frac{\varepsilon_3}{2}(\delta_3|\mathbf{u}_\eta - P_\eta(\mathbf{u})|^2_{H^1(\Omega)} + \frac{1}{\delta_3}\|\mathbf{u}_\eta - P_\eta(\mathbf{u})\|^2_{L^2(\Omega)})\big)\,.
\end{aligned}$$

The linear terms are easily bounded; in particular, the fact that $\mathbf{u}'$ is in $L^2(\Omega\times]0,T[)^3$, (3.3), (3.5) and (A.5) with $p = 2$ yield

$$\|\mathbf{u}' - P_\eta(\mathbf{u}')\|_{L^2(0,T;H^{-1}(\Omega)^3)} \le c_3\eta\|\mathbf{u}'\|_{L^2(\Omega\times]0,T[)}\,.$$

Then (3.6) follows readily by substituting these inequalities with a suitable choice of parameters ε_i and δ_i into (3.7) and applying Gronwall's Lemma. □

Corollary 3.2. *In addition to the assumptions of Theorem 3.1, suppose that* $(\mathbf{u},p) \in L^2(0,T;H^2(\Omega)^3) \times L^2(0,T;H^1(\Omega))$. *Then, if* $\eta \le \eta_0$, *there exists a constant* $C(\mathbf{u},p,\nu)$ *independent of* η, *such that*

$$\|\mathbf{u}_\eta - \mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega)^3)} + \frac{\nu}{2}\|\mathbf{u}_\eta - \mathbf{u}\|^2_{L^2(0,T;H^1(\Omega)^3)} \le C(\mathbf{u},p,\nu)\eta^2\,. \qquad (3.8)$$

Remark 3.3. If the assumptions of Theorem 2.5 hold then $(\mathbf{u},p)$ has the above regularity provided $\mathbf{f} \in L^2(\Omega\times]0,T[)^3$. If the assumptions of Theorem 2.3 hold, the same conclusion is valid if in addition Ω is convex. □

Remark 3.4. The advantage of Theorem 2.3 versus Theorem 2.5 is that it does not require the uniformity assumption (2.27) on the triangulation. For this reason, we shall not use Theorem 2.5 in the sequel. □

4. Some error estimates for the Stokes problem

The error estimate of order two in $L^2(\Omega\times]0,T[)^3)$, that we shall derive in the next section, is based on a duality argument for the transient Stokes problem:

$$\frac{\partial}{\partial t}\mathbf{v}(\mathbf{x},t) - \nu\,\Delta\,\mathbf{v}(\mathbf{x},t) + \nabla\, q(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t) \quad \text{in } \Omega\times]0,T]\,, \tag{4.1}$$

$$\operatorname{div}\mathbf{v}(\mathbf{x},t) = 0 \quad \text{in } \Omega\times]0,T]\,, \tag{4.2}$$

$$\mathbf{v}(\mathbf{x},t) = \mathbf{0} \quad \text{on } \partial\Omega\times]0,T]\,, \tag{4.3}$$

$$\mathbf{v}(\mathbf{x},0) = \mathbf{0} \quad \text{in } \Omega\,. \tag{4.4}$$

The first lemma recalls the regularity of its solution.

Lemma 4.1. *If $\mathbf{g}\in L^2(\Omega\times]0,T[)^3$, then the solution $(\mathbf{v},q)$ of the Stokes problem (4.1–4.4) belongs to $L^2(0,T;H^{3/2}(\Omega)^3)\times L^2(0,T;H^{1/2}(\Omega))$, $\mathbf{v}$ belongs to $L^\infty(0,T;H^1(\Omega)^3)$ and $\mathbf{v}'$ to $L^2(\Omega\times]0,T[)^3$, with continuous dependence on $\mathbf{g}$. If Ω is convex, then $(\mathbf{v},q)\in L^2(0,T;H^2(\Omega)^3)\times L^2(0,T;H^1(\Omega))$. If in addition, $\mathbf{g}'\in L^2(0,T;H^{-1}(\Omega)^3)$, $\mathbf{g}\in L^r(0,T;L^2(\Omega)^3)$ for some number $r\geq 2$ and $\mathbf{g}(0)\in L^2(\Omega)^3$, then $(\mathbf{v},q)\in L^r(0,T;H^2(\Omega)^3)\times L^r(0,T;H^1(\Omega))$ and $\mathbf{v}'\in L^r(0,T;L^2(\Omega)^3)$ with continuous dependence on $\mathbf{g}$. Finally, without convexity assumption, if $\mathbf{g}\in H^1(0,T;H^{-1}(\Omega)^3)$ and $\mathbf{g}(0)\in L^2(\Omega)^3$, then $\mathbf{v}'\in L^\infty(0,T;L^2(\Omega)^3)\cap L^2(0,T;H^1(\Omega)^3)$.*

Proof. Assume that $\mathbf{g}\in L^2(\Omega\times]0,T[)^3$ and consider a Galerkin discretization of (4.1) in a suitable finite-dimensional subspace of V, say V_m. Let $\mathbf{v}_m$ be the Galerkin solution and multiply the discrete equation by $\mathbf{v}'_m$. We obtain

$$\|\mathbf{v}'_m\|^2_{L^2(\Omega\times]0,T[)} + \nu\|\mathbf{v}_m\|^2_{L^\infty(0,T;H^1(\Omega)^3)} \leq \|\mathbf{g}\|^2_{L^2(\Omega\times]0,T[)}\,.$$

This uniform estimate shows that in the limit, $\mathbf{v}'$ belongs to $L^2(\Omega\times]0,T[)^3$ and

$$\|\mathbf{v}'\|^2_{L^2(\Omega\times]0,T[)} + \nu\|\mathbf{v}\|^2_{L^\infty(0,T;H^1(\Omega)^3)} \leq \|\mathbf{g}\|^2_{L^2(\Omega\times]0,T[)}\,. \tag{4.5}$$

Therefore, almost everywhere in $]0,T[$, the pair $(\mathbf{v}(t),q(t))$ is the solution of the Stokes problem

$$-\nu\Delta\,\mathbf{v}(t) + \nabla\, q(t) = \mathbf{g}(t) - \mathbf{v}'(t) \text{ in } \Omega\,,$$

$$\operatorname{div}\mathbf{v}(t) = 0 \text{ in } \Omega\,,$$
$$\mathbf{v}(t) = \mathbf{0} \text{ on } \partial\Omega\,,$$

where t is a parameter as in Remark 2.4. Depending on the hypotheses made on Ω (Lipschitz or convex), Theorem 0.8 implies that $\sqrt{\nu}\mathbf{v}$ is bounded in $L^2(0,T;H^{3/2}(\Omega)^3)$ or $L^2(0,T;H^2(\Omega)^3)$ and q is bounded in $L^2(0,T;H^{1/2}(\Omega))$ or $L^2(0,T;H^1(\Omega))$.

Finally, if in addition $\mathbf{g}'\in L^2(0,T;H^{-1}(\Omega)^3)$ and $\mathbf{g}(0)\in L^2(\Omega)^3$, then $\mathbf{g}$ satisfies in particular the regularity assumptions of the data of Theorem 2.3. Since the problem is linear, the analogue of (2.24) holds in $[0,T]$:

$$\|\mathbf{v}'_m\|^2_{L^\infty(0,T;L^2(\Omega)^3)} \leq \|\mathbf{g}(0)\|^2_{L^2(\Omega)} + \frac{1}{2\nu}\|\mathbf{g}'\|^2_{L^2(0,T;H^{-1}(\Omega)^3)}\,. \tag{4.6}$$

Thus, in the limit $\mathbf{v}'$ belongs to $L^\infty(0,T;L^2(\Omega)^3)$ and when $\mathbf{g}$ belongs to $L^r(0,T;L^2(\Omega)^3)$ for some number $r\geq 2$, the above interpretation of $(\mathbf{v}(t),q(t))$ gives the desired regularity.

Similarly, if $\mathbf{g} \in H^1(0,T;H^{-1}(\Omega)^3)$ and $\mathbf{g}(0) \in L^2(\Omega)^3$, the last conclusion holds owing to:

$$\nu\|\mathbf{v}'_m\|^2_{L^2(0,T;H^1(\Omega)^3)} \le \|\mathbf{g}(0)\|^2_{L^2(\Omega)} + \frac{1}{\nu}\|\mathbf{g}'\|^2_{L^2(0,T;H^{-1}(\Omega)^3)}\,.$$

□

We discretize (4.1–4.4) by the scheme of Step One without the non-linear term: Find $(\mathbf{v}_\eta, q_\eta)$ with values in $X_\eta \times M_\eta$ for each $t \in [0,T]$, solution of

$$\forall \mathbf{w}_\eta \in X_\eta\ ,\ (\mathbf{v}'_\eta(t), \mathbf{w}_\eta) + \nu(\nabla \mathbf{v}_\eta(t), \nabla \mathbf{w}_\eta) - (q_\eta(t), \operatorname{div} \mathbf{w}_\eta) = \langle \mathbf{g}(t), \mathbf{w}_\eta\rangle \quad \text{in }]0,T]\,, \tag{4.7}$$

$$\forall \lambda_\eta \in M_\eta\ ,\ (\lambda_\eta, \operatorname{div} \mathbf{v}_\eta(t)) = 0 \quad \text{in }]0,T]\,, \tag{4.8}$$

$$\mathbf{v}_\eta(\mathbf{x},0) = \mathbf{0} \quad \text{in } \Omega\,. \tag{4.9}$$

As this is a linear problem, it has a unique solution over the interval $[0,T]$, and it satisfies the following error estimate (*cf.* for instance [43]):

Lemma 4.2. *Let* $\mathbf{g} \in L^2(\Omega\times]0,T[)^3$ *and suppose* Ω *is convex. Let* $(\mathbf{v}, q)$ *and* $(\mathbf{v}_\eta, q_\eta)$ *be the respective solutions of (4.1–4.4) and (4.7–4.9). There exists a constant* C*, independent of* η *such that*

$$\|\mathbf{v}_\eta - \mathbf{v}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \sqrt{\nu}\|\mathbf{v}_\eta - \mathbf{v}\|_{L^2(0,T;H^1(\Omega)^3)} \le C\eta\|\mathbf{g}\|_{L^2(\Omega\times]0,T[)}\,. \tag{4.10}$$

The next theorem establishes that the error satisfies an estimate of order two in $L^2(\Omega\times]0,T[)^3$.

Theorem 4.3. *We retain the assumptions and notation of Lemma 4.2. There exists a constant* C*, independent of* η *such that*

$$\|\mathbf{v}_\eta - \mathbf{v}\|_{L^2(\Omega\times]0,T[)} \le C\eta^2\|\mathbf{g}\|_{L^2(\Omega\times]0,T[)}\,. \tag{4.11}$$

Proof. Following [22], we use the following parabolic duality argument: for any $t \in]0,T]$, let $(\mathbf{w}, \lambda)$ be the solution of the backward Stokes system:

$$\frac{\partial}{\partial t}\mathbf{w} + \nu\,\Delta\,\mathbf{w} - \nabla\,\lambda = \mathbf{v}_\eta - \mathbf{v} \quad \text{in } \Omega\times]0,t]\,, \tag{4.12}$$

$$\operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega\times]0,t]\,,$$
$$\mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega\times]0,t]\,,$$
$$\mathbf{w}(\mathbf{x},t) = \mathbf{0} \quad \text{in } \Omega\,.$$

As $\mathbf{v}_\eta - \mathbf{v}$ belongs to $L^2(\Omega\times]0,T[)^3$ and Ω is convex, Lemma 4.1 implies that $(\mathbf{w}, \lambda) \in L^2(0,t;H^2(\Omega)^3) \times L^2(0,t;H^1(\Omega))$, $\mathbf{w}' \in L^2(\Omega\times]0,t[)^3$ and

$$\|\mathbf{w}'\|_{L^2(\Omega\times]0,t[)} + \|\mathbf{w}\|_{L^2(0,t;H^2(\Omega)^3)} + \|\lambda\|_{L^2(0,t;H^1(\Omega))} \le c_1\|\mathbf{v}_\eta - \mathbf{v}\|_{L^2(\Omega\times]0,t[)}\,. \tag{4.13}$$

Now, on one hand, we take the scalar product of (4.1) with a test function $\mathbf{z}_\eta \in V_\eta$ and we take the difference between the resulting equation and (4.7). This gives

$$\forall \mathbf{z}_\eta \in V_\eta\,,\ (\mathbf{v}'_\eta - \mathbf{v}', \mathbf{z}_\eta) + \nu(\nabla(\mathbf{v}_\eta - \mathbf{v}), \nabla\,\mathbf{z}_\eta) = -(q - r_\eta(q), \operatorname{div}\mathbf{z}_\eta)\,. \tag{4.14}$$

On the other hand, we multiply (4.12) by $\mathbf{v}_\eta - \mathbf{v}$ and we obtain for any $\mathbf{z}_\eta \in V_\eta$:

$$\begin{aligned}\|\mathbf{v}_\eta - \mathbf{v}\|^2_{L^2(\Omega)} &= (\mathbf{w}', \mathbf{v}_\eta - \mathbf{v}) - \nu(\nabla \mathbf{w}, \nabla(\mathbf{v}_\eta - \mathbf{v})) + (\lambda, \operatorname{div}(\mathbf{v}_\eta - \mathbf{v}))\\ &= (\mathbf{w}' - \mathbf{z}'_\eta, \mathbf{v}_\eta - \mathbf{v}) - \nu(\nabla(\mathbf{w} - \mathbf{z}_\eta), \nabla(\mathbf{v}_\eta - \mathbf{v}))\\ &\quad + (\lambda - r_\eta(\lambda), \operatorname{div}(\mathbf{v}_\eta - \mathbf{v})) + (\mathbf{z}'_\eta, \mathbf{v}_\eta - \mathbf{v}) - \nu(\nabla \mathbf{z}_\eta, \nabla(\mathbf{v}_\eta - \mathbf{v}))\,.\end{aligned} \tag{4.15}$$

Using (4.14), we have the identity

$$\begin{aligned}(\mathbf{z}'_\eta, \mathbf{v}_\eta - \mathbf{v}) &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{z}_\eta, \mathbf{v}_\eta - \mathbf{v}) - (\mathbf{z}_\eta, \mathbf{v}'_\eta - \mathbf{v}')\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{z}_\eta, \mathbf{v}_\eta - \mathbf{v}) + \nu(\nabla \mathbf{z}_\eta, \nabla(\mathbf{v}_\eta - \mathbf{v})) + (q - r_\eta(q), \operatorname{div}(\mathbf{z}_\eta - \mathbf{w}))\,.\end{aligned}$$

Therefore, for all $\mathbf{z}_\eta \in V_\eta$,

$$\begin{aligned}\|\mathbf{v}_\eta - \mathbf{v}\|^2_{L^2(\Omega)} &= (\mathbf{w}' - \mathbf{z}'_\eta, \mathbf{v}_\eta - \mathbf{v}) - \nu(\nabla(\mathbf{w} - \mathbf{z}_\eta), \nabla(\mathbf{v}_\eta - \mathbf{v})) + (\lambda - r_\eta(\lambda), \operatorname{div}(\mathbf{v}_\eta - \mathbf{v}))\\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{z}_\eta, \mathbf{v}_\eta - \mathbf{v}) + (q - r_\eta(q), \operatorname{div}(\mathbf{z}_\eta - \mathbf{w}))\,.\end{aligned}$$

By choosing $\mathbf{z}_\eta = P_\eta(\mathbf{w})$, integrating both sides of this equation from 0 to t and applying (3.5), (2.5) and (2.9), the initial condition for $\mathbf{v}$, $\mathbf{v}_\eta$ and the final condition for $\mathbf{w}$, we find

$$\begin{aligned}\|\mathbf{v}_\eta - \mathbf{v}\|^2_{L^2(\Omega\times]0,t[)} &\le \int_0^t |\mathbf{v}_\eta - \mathbf{v}|_{H^1(\Omega)}\big(\|\mathbf{w}' - P_\eta(\mathbf{w}')\|_{H^{-1}(\Omega)} + \nu|\mathbf{w} - P_\eta(\mathbf{w})|_{H^1(\Omega)} + \sqrt{3}\|\lambda - r_\eta(\lambda)\|_{L^2(\Omega)}\big)\mathrm{d}\tau\\ &\quad + \sqrt{3}\int_0^t \|q - r_\eta(q)\|_{L^2(\Omega)}|\mathbf{w} - P_\eta(\mathbf{w})|_{H^1(\Omega)}\mathrm{d}\tau\\ &\le c_2\eta \int_0^t \big[|\mathbf{v}_\eta - \mathbf{v}|_{H^1(\Omega)}(\|\mathbf{w}'\|_{L^2(\Omega)} + |\mathbf{w}|_{H^2(\Omega)} + |\lambda|_{H^1(\Omega)}) + \|q - r_\eta(q)\|_{L^2(\Omega)}|\mathbf{w}|_{H^2(\Omega)}\big]\mathrm{d}\tau\,.\end{aligned}$$

Then (4.11) follows from (4.13), (4.10) and (2.5). □

Finally, the error also satisfies an estimate in $L^\infty(0,T;H^1(\Omega)^3)$. The proof uses the following variant of the Stokes projection (*cf.* [22]): for any pair $(\mathbf{u},p) \in V \times L^2_0(\Omega)$, $S_\eta(\mathbf{u}) \in V_\eta$ is defined by

$$\forall \mathbf{v}_\eta \in V_\eta\,,\ \nu(\nabla(S_\eta(\mathbf{u}) - \mathbf{u}), \nabla \mathbf{v}_\eta) = -(p, \operatorname{div} \mathbf{v}_\eta)\,. \tag{4.16}$$

Clearly $S_\eta(\mathbf{u})$ is uniquely defined by $(\mathbf{u},p)$ (to simplify the notation, we do not indicate the dependence on p). In addition, it satisfies the following error bounds. We skip the proof, because it is the same as for the standard Stokes projection.

Lemma 4.4. *Let the pair $(\mathbf{u},p)$ be given in $V \times L^2_0(\Omega)$. Then $S_\eta(\mathbf{u})$ defined by (4.16) satisfies:*

$$|S_\eta(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} \le 2|P_\eta(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + \frac{\sqrt{3}}{\nu}\|r_\eta(p) - p\|_{L^2(\Omega)}\,. \tag{4.17}$$

If in addition, Ω is convex, there exists a constant C, independent of η, such that

$$\|S_\eta(\mathbf{u}) - \mathbf{u}\|_{L^2(\Omega)} \le C\eta\big(|S_\eta(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + \|r_\eta(p) - p\|_{L^2(\Omega)}\big)\,. \tag{4.18}$$

Lemma 4.5. *In addition to the hypotheses of Lemma 4.2, suppose that* $\mathbf{g}$ *is in* $L^\infty(0,T;L^2(\Omega)^3)$, $\mathbf{g}' \in L^2(0,T;H^{-1}(\Omega)^3)$ *and* q' *in* $L^2(\Omega\times]0,T[)$. *Then there exists a constant* C, *independent of* η, *such that*

$$\|\mathbf{v}'_\eta - \mathbf{v}'\|_{L^2(\Omega\times]0,T[)} + \sqrt{\nu}\|\mathbf{v}_\eta - \mathbf{v}\|_{L^\infty(0,T;H^1(\Omega)^3)} \\ \leq C\eta\big(\|\mathbf{g}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \|\mathbf{g}'\|_{L^2(0,T;H^{-1}(\Omega)^3)} + \|q'\|_{L^2(\Omega\times]0,T[)}\big)\,. \tag{4.19}$$

Proof. First note that, in view of Lemma 4.1 and the above assumptions, $q \in \mathcal{C}^0([0,T];L^2(\Omega))$. Hence $q(0)$ is well-defined, and since, again by Lemma 4.1, $q \in L^\infty(0,T;H^1(\Omega))$, then $q(0)$ belongs to $H^1(\Omega)$.

Now, taking the difference between (4.1) and (4.7) multiplied by a test function $\mathbf{w}_\eta \in V_\eta$, inserting $S_\eta(\mathbf{v})$ defined by $(\mathbf{v},q)$, choosing $\mathbf{w}_\eta = \mathbf{v}'_\eta - S_\eta(\mathbf{v})'$, and observing that here again $S_\eta(\mathbf{v})' = S_\eta(\mathbf{v}')$, we obtain:

$$\|\mathbf{v}'_\eta - S_\eta(\mathbf{v}')\|^2_{L^2(\Omega)} + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}_\eta - S_\eta(\mathbf{v})|^2_{H^1(\Omega)} \leq \|\mathbf{v}' - S_\eta(\mathbf{v}')\|_{L^2(\Omega)}\|\mathbf{v}'_\eta - S_\eta(\mathbf{v}')\|_{L^2(\Omega)}\,.$$

Therefore

$$\|\mathbf{v}'_\eta - S_\eta(\mathbf{v}')\|^2_{L^2(0,t;L^2(\Omega)^3)} + \nu|(\mathbf{v}_\eta - S_\eta(\mathbf{v}))(t)|^2_{H^1(\Omega)} \\ \leq \|\mathbf{v}' - S_\eta(\mathbf{v}')\|^2_{L^2(0,t;L^2(\Omega)^3)} + \nu|\mathbf{v}_\eta(0) - S_\eta(\mathbf{v})(0)|^2_{H^1(\Omega)}\,. \tag{4.20}$$

But (4.18) applied to $\mathbf{v}'$ gives a.e. in $]0,T[$:

$$\|\mathbf{v}' - S_\eta(\mathbf{v}')\|_{L^2(\Omega)} \leq C\eta\big(|\mathbf{v}'|_{H^1(\Omega)} + \|q'\|_{L^2(\Omega)}\big)\,,$$

and (4.16) gives immediately

$$|\mathbf{v}_\eta(0) - S_\eta(\mathbf{v})(0)|_{H^1(\Omega)} \leq \frac{\sqrt{3}}{\nu}\|r_\eta(q(0)) - q(0)\|_{L^2(\Omega)} \leq C\eta|q(0)|_{H^1(\Omega)}\,,$$

with another constant C independent of η. Then (4.19) follows by substituting these bounds into (4.20). ☐

The following lemma gives a sufficient condition for q' in $L^2(\Omega\times]0,T[)$.

Lemma 4.6. *If* $\mathbf{g} \in H^1(0,T;L^2(\Omega)^3)$ *and* $\mathbf{g}(0) \in V$, *then* $q' \in L^2(0,T;H^{1/2}(\Omega)/\mathbb{R})$. *If in addition,* Ω *is convex, then* $q' \in L^2(0,T;H^1(\Omega)/\mathbb{R})$.

Proof. As in Lemma 4.1, we consider the Galerkin discretization of (4.1–4.4), but here we define it in the basis of the eigenfunctions of the Stokes operator: $\mathbf{w}_i \in V$,

$$-\Delta\,\mathbf{w}_i + \nabla\,\pi_i = \lambda_i\mathbf{w}_i\,.$$

This basis is orthonormal in $L^2(\Omega)^3$ and orthogonal in $H^1(\Omega)^3$. Then for any $t \in [0,T]$, we have:

$$(\mathbf{v}'_m(t),\mathbf{w}_i) + \nu(\nabla\,\mathbf{v}_m(t),\nabla\,\mathbf{w}_i) = (\mathbf{g}(t),\mathbf{w}_i)\ ,\ 1 \leq i \leq m\,. \tag{4.21}$$

In particular, at time $t = 0$, since $\mathbf{v}_m(0) = \mathbf{0}$, we see that $\mathbf{v}'_m(0) = \mathcal{P}_m\mathbf{g}(0)$, the orthogonal projection of $\mathbf{g}(0)$ onto V_m for the L^2 norm. Then on one hand,

$$\|\mathbf{v}'_m(0)\|_{L^2(\Omega)} \leq \|\mathbf{g}(0)\|_{L^2(\Omega)}\,,$$

and on the other hand,

$$\|\nabla\,\mathbf{v}'_m(0)\|^2_{L^2(\Omega)} = \sum_{i=1}^m(\mathbf{v}'_m(0),\mathbf{w}_i)^2\|\nabla\,\mathbf{w}_i\|^2_{L^2(\Omega)} = \sum_{i=1}^m(\mathbf{g}(0),\mathbf{w}_i)^2\lambda_i \leq \|\nabla\,\mathbf{g}(0)\|^2_{L^2(\Omega)}\,,$$

since $\mathbf{g}(0) \in V$. Therefore,

$$|\mathbf{v}'_m(0)|_{H^1(\Omega)} \le |\mathbf{g}(0)|_{H^1(\Omega)} . \tag{4.22}$$

Hence differentiating (4.21) with respect to t, we obtain

$$\|\mathbf{v}''_m(t)\|^2_{L^2(\Omega)} + \nu \frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{v}'_m(t)|^2_{H^1(\Omega)} \le \|\mathbf{g}'(t)\|^2_{L^2(\Omega)} ,$$

and next integrating this last equation with respect to t and using (4.22):

$$\int_0^t \|\mathbf{v}''_m(s)\|^2_{L^2(\Omega)} \mathrm{d}s + \nu |\mathbf{v}'_m(t)|^2_{H^1(\Omega)} \le \nu |\mathbf{g}(0)|^2_{H^1(\Omega)} + \int_0^t \|\mathbf{g}'(s)\|^2_{L^2(\Omega)} \mathrm{d}s . \tag{4.23}$$

Passing to the limit with respect to m, (4.23) implies that $\mathbf{v}'' \in L^2(\Omega\times]0,T[)^3$, $\mathbf{v}' \in L^\infty(0,T;H^1_0(\Omega)^3)$ and $\mathbf{v}'(0) = \mathbf{g}(0)$. Thus, for almost every t, $\mathbf{v}'(t)$ is the solution of the Stokes problem (0.34), (0.35) with data $\mathbf{g}' - \mathbf{v}''$ instead of $\mathbf{g}$. Since $\mathbf{g}' - \mathbf{v}'' \in L^2(\Omega\times]0,T[)^3$, the lemma follows from Theorem 0.8. □

Remark 4.7. The result of Lemma 4.6 is stronger than what is really needed, namely $q' \in L^2(\Omega\times]0,T[)$, but so far we do not know what minimal conditions guarantee exactly this regularity. □

5. An error estimate of order two in $L^2(\Omega\times]0,T[)^3$

We are going to derive an estimate of order two for the L^2 norm of $\mathbf{u}_\eta - \mathbf{u}$ with $\mathbf{u}_\eta$ introduced in Section 2. As in [22], we split the error into a linear contribution and a non-linear one. The linear contribution, which is the discrete solution of the Stokes part of (0.1), is estimated by Theorem 4.3. Then we prove a "superconvergence" result for the error of the non-linear part. More precisely, let $\mathbf{v}_\eta \in V_\eta$ be defined by

$$\forall \mathbf{w}_\eta \in V_\eta \ , \ (\mathbf{v}'_\eta(t), \mathbf{w}_\eta) + \nu(\nabla \mathbf{v}_\eta(t), \nabla \mathbf{w}_\eta) = \langle \mathbf{f}(t), \mathbf{w}_\eta \rangle - (\mathbf{u}(t) \cdot \nabla \mathbf{u}(t), \mathbf{w}_\eta) \quad \text{in }]0,T] , \tag{5.1}$$

$$\mathbf{v}_\eta(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega . \tag{5.2}$$

Then $\mathbf{v}_\eta$ satisfies (4.7) with data $\mathbf{g} = \mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}$ and Theorem 4.3 has the following corollary.

Corollary 5.1. *In addition to the assumptions of Theorem 2.3, suppose that* Ω *is convex and* $\mathbf{f} \in L^2(\Omega\times]0,T[)^3$. *Then*

$$\|\mathbf{v}_\eta - \mathbf{u}\|_{L^2(\Omega\times]0,T[)} \le C\eta^2 \big(\|\mathbf{f}\|_{L^2(\Omega\times]0,T[)} + S_6 \|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^3)} \|\mathbf{u}\|_{L^2(0,T;W^{1,3}(\Omega)^3)} \big) , \tag{5.3}$$

where C *is the constant of Theorem 4.3.*

Similarly, Lemma 4.5 has the following consequence.

Corollary 5.2. *In addition to the assumptions of Theorem 2.3, suppose that* Ω *is convex,* $\mathbf{f} \in L^\infty(0,T;L^2(\Omega)^3)$ *and* $p' \in L^2(\Omega\times]0,T[)$. *Then, there exists a constant* C*, independent of* η*, such that*

$$\|\mathbf{v}_\eta - \mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^3)} \le C\eta \big(\|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \|\mathbf{f}'\|_{L^2(0,T;H^{-1}(\Omega)^3)} + \|p'\|_{L^2(\Omega\times]0,T[)} \big) . \tag{5.4}$$

Remark 5.3. Lemma 4.6 implies that if $\mathbf{f} \in H^1(0,T;L^2(\Omega)^3)$ and $\mathbf{f}(0) \in V$, then $p' \in L^2(0,T;H^{1/2}(\Omega)/\mathbb{R})$. Indeed, under these assumptions, we easily prove that $\mathbf{u} \cdot \nabla \mathbf{u} \in H^1(0,T;L^2(\Omega)^3)$. Since $\mathbf{u} \cdot \nabla \mathbf{u}(0) = 0$, the assumptions of Lemma 4.6 are satisfied. Similarly, if in addition, Ω is convex, then $p' \in L^2(0,T;H^1(\Omega)/\mathbb{R})$. □

In view of (5.3), it remains to derive a sharp bound for $\mathbf{u}_\eta - \mathbf{v}_\eta$ in $L^2(\Omega\times]0,T[)^3$. First, we observe that, owing to Remark 2.4, the assumptions of Corollary 5.2 imply that:

$$\mathbf{u} \in L^\infty(0,T;H^2(\Omega)^3)\,. \tag{5.5}$$

Theorem 5.4. *Under the assumptions of Corollary 5.2, and if $\eta \le \eta_1$, where*

$$\eta_1 = \min\big(\eta_0, \frac{1}{3}\frac{\nu^2}{\tilde{C}^2\hat{C}^2}\big)\,, \tag{5.6}$$

$\hat{C}$ is the constant of (2.14), η_0 the constant of (2.17) and

$$\tilde{C} = \|\mathbf{v}_\eta\|_{L^\infty(0,T;H^1(\Omega)^3)}\,,$$

there exists a constant C, that depends on $\|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega)^3)}$, $\|\mathbf{f}'\|_{L^2(0,T;H^{-1}(\Omega)^3)}$ and $\|p'\|_{L^2(\Omega\times]0,T[)}$, but not on η, such that

$$\|\mathbf{v}_\eta - \mathbf{u}_\eta\|_{L^\infty(0,T;L^2(\Omega)^3)} + \sqrt{\nu}\|\mathbf{v}_\eta - \mathbf{u}_\eta\|_{L^2(0,T;H^1(\Omega)^3)} \le C\eta^2\,. \tag{5.7}$$

Proof. Subtracting (2.13) from (5.1), we have for all $\mathbf{w}_\eta \in V_\eta$:

$$\begin{aligned}(\mathbf{v}'_\eta - \mathbf{u}'_\eta, \mathbf{w}_\eta) + \nu(\nabla(\mathbf{v}_\eta - \mathbf{u}_\eta), \nabla\,\mathbf{w}_\eta) &= ((\mathbf{u}_\eta - \mathbf{v}_\eta)\cdot\nabla\,\mathbf{u}_\eta, \mathbf{w}_\eta) + (\mathbf{v}_\eta\cdot\nabla(\mathbf{u}_\eta - \mathbf{v}_\eta), \mathbf{w}_\eta)\\ &+ ((\mathbf{v}_\eta - \mathbf{u})\cdot\nabla(\mathbf{v}_\eta - \mathbf{u}), \mathbf{w}_\eta) + ((\mathbf{v}_\eta - \mathbf{u})\cdot\nabla\,\mathbf{u}, \mathbf{w}_\eta) + (\mathbf{u}\cdot\nabla(\mathbf{v}_\eta - \mathbf{u}), \mathbf{w}_\eta)\,.\end{aligned} \tag{5.8}$$

Let us bound the right-hand side of (5.8) taking $\mathbf{w}_\eta = \mathbf{v}_\eta - \mathbf{u}_\eta$. The first term is absorbed by the left-hand side; indeed, set

$$c_1 = \|\mathbf{u}_\eta\|_{L^\infty(0,T;H^1(\Omega)^3)}\,,$$

then as in the proof of Theorem 3.1, we have

$$|((\mathbf{u}_\eta - \mathbf{v}_\eta)\cdot\nabla\,\mathbf{u}_\eta, \mathbf{v}_\eta - \mathbf{u}_\eta)| \le \frac{c_1}{2}S_6^{3/2}\big(\frac{1}{\varepsilon}|\mathbf{v}_\eta - \mathbf{u}_\eta|^2_{H^1(\Omega)} + \frac{\varepsilon}{2}(\delta|\mathbf{v}_\eta - \mathbf{u}_\eta|^2_{H^1(\Omega)} + \frac{1}{\delta}\|\mathbf{v}_\eta - \mathbf{u}_\eta\|^2_{L^2(\Omega)})\big)\,.$$

The second term is also absorbed by the left-hand side, in view of (2.14) and the fact that $\mathbf{v}_\eta$ is bounded in $L^\infty(0,T;H^1(\Omega)^3)$; indeed

$$|(\mathbf{v}_\eta\cdot\nabla(\mathbf{u}_\eta - \mathbf{v}_\eta), \mathbf{v}_\eta - \mathbf{u}_\eta)| \le \frac{\sqrt{3}}{2}\eta^{1/2}\tilde{C}\hat{C}|\mathbf{v}_\eta - \mathbf{u}_\eta|^2_{H^1(\Omega)} \le \frac{\nu}{2}|\mathbf{v}_\eta - \mathbf{u}_\eta|^2_{H^1(\Omega)}\,,$$

owing to (5.6). The third term is bounded by virtue of Lemma 4.2 and Corollary 5.2:

$$\begin{aligned}|((\mathbf{v}_\eta - \mathbf{u})\cdot\nabla(\mathbf{v}_\eta - \mathbf{u}), \mathbf{v}_\eta - \mathbf{u}_\eta)| &\le S_3S_6\|\mathbf{v}_\eta - \mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^3)}|\mathbf{v}_\eta - \mathbf{u}|_{H^1(\Omega)}|\mathbf{v}_\eta - \mathbf{u}_\eta|_{H^1(\Omega)}\\ &\le \eta^2C(\mathbf{f},\mathbf{f}',p')S_3S_6|\mathbf{v}_\eta - \mathbf{u}_\eta|_{H^1(\Omega)}\,.\end{aligned}$$

The fourth term is bounded by virtue of Corollary 5.1 and (5.5); set

$$c_2 = \|\mathbf{u}\|_{L^\infty(0,T;W^{1,3}(\Omega)^3)}\,, \tag{5.9}$$

then

$$|((\mathbf{v}_\eta - \mathbf{u})\cdot\nabla\,\mathbf{u}, \mathbf{v}_\eta - \mathbf{u}_\eta)| \le c_2S_6\|\mathbf{v}_\eta - \mathbf{u}\|_{L^2(\Omega\times]0,T[)}|\mathbf{v}_\eta - \mathbf{u}_\eta|_{H^1(\Omega)} \le \eta^2C(\mathbf{f},\mathbf{f}')c_2S_6|\mathbf{v}_\eta - \mathbf{u}_\eta|_{H^1(\Omega)}\,.$$

Finally, the last term is bounded by Green's formula, Corollary 5.1 and (5.5); set:

$$c_3 = \|\mathbf{u}\|_{L^\infty(\Omega\times]0,T[)^3)}\,,$$

then

$$|(\mathbf{u}\cdot\nabla(\mathbf{v}_\eta-\mathbf{u}),\mathbf{v}_\eta-\mathbf{u}_\eta)| = |(\mathbf{u}\cdot\nabla(\mathbf{v}_\eta-\mathbf{u}_\eta),\mathbf{v}_\eta-\mathbf{u})| \le \eta^2 c_3 C(\mathbf{f},\mathbf{f}')|\mathbf{v}_\eta-\mathbf{u}_\eta|_{H^1(\Omega)}\,.$$

Then (5.7) follows from these bounds with a suitable choice of parameters ε and δ. □

Corollary 5.1 and Theorem 5.4 imply:

Corollary 5.5. *Under the assumptions of Theorem 5.4, there exists a constant C that depends on $\|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega)^3)}$, $\|\mathbf{f}'\|_{L^2(0,T;H^{-1}(\Omega)^3)}$ and $\|p'\|_{L^2(\Omega\times]0,T[)}$, but not on η, such that*

$$\|\mathbf{u}-\mathbf{u}_\eta\|_{L^2(\Omega\times]0,T[)} \le C\eta^2\,. \tag{5.10}$$

6. An estimate for the pressure

The results of the preceding section allow one to establish an error estimate for the pressure. We start with a general bound.

Lemma 6.1. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_\eta,p_\eta)$ be the respective solutions of (0.1–0.4) and (0.12–0.14). Under the assumptions of Theorem 2.3, we have*

$$\begin{aligned}\|p_\eta - r_\eta(p)\|_{L^2(\Omega\times]0,T[)} \le \frac{1}{\beta^*}\big(\sqrt{3}\|p-r_\eta(p)\|_{L^2(\Omega\times]0,T[)} + \|\mathbf{u}'-\mathbf{u}'_\eta\|_{L^2(0,T;H^{-1}(\Omega)^3)} + \nu\|\mathbf{u}-P_\eta(\mathbf{u})\|_{L^2(0,T;H^1(\Omega)^3)}\\ + S_6(\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^3)}\|\mathbf{u}-\mathbf{u}_\eta\|_{L^2(0,T;L^3(\Omega)^3)} + \|\mathbf{u}_\eta\|_{L^\infty(0,T;L^3(\Omega)^3)}\|\mathbf{u}-\mathbf{u}_\eta\|_{L^2(0,T;H^1(\Omega)^3)})\big)\,,\end{aligned} \tag{6.1}$$

where β^* is the constant of the inf-sup condition (2.6).

Proof. Taking the difference between (0.1) and (0.12) multiplied by a test function $\mathbf{w}_\eta \in X_\eta$ and inserting $r_\eta(p)$, we obtain

$$(r_\eta(p)-p_\eta,\operatorname{div}\mathbf{w}_\eta) = (\mathbf{u}'-\mathbf{u}'_\eta,\mathbf{w}_\eta) + \nu(\nabla(\mathbf{u}-\mathbf{u}_\eta),\nabla\mathbf{w}_\eta) + (\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{u}_\eta\cdot\nabla\mathbf{u}_\eta,\mathbf{w}_\eta) + (r_\eta(p)-p,\operatorname{div}\mathbf{w}_\eta)\,. \tag{6.2}$$

Owing to the inf-sup condition (2.6), there exists a function $\mathbf{w}_\eta \in V_\eta^\perp$, the discrete analogue of $V^\perp$, such that

$$(r_\eta(p)-p_\eta,\operatorname{div}\mathbf{w}_\eta) = \|r_\eta(p)-p_\eta\|^2_{L^2(\Omega)}\ ,\ |\mathbf{w}_\eta|_{H^1(\Omega)} \le \frac{1}{\beta^*}\|r_\eta(p)-p_\eta\|_{L^2(\Omega)}\,.$$

Then (6.1) readily follows by substituting this function into (6.2) and bounding the non-linear term in the right-hand side by

$$|(\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{u}_\eta\cdot\nabla\mathbf{u}_\eta,\mathbf{w}_\eta)| \le \big(\|\mathbf{u}-\mathbf{u}_\eta\|_{L^3(\Omega)}|\mathbf{u}|_{H^1(\Omega)} + \|\mathbf{u}_\eta\|_{L^3(\Omega)}|\mathbf{u}-\mathbf{u}_\eta|_{H^1(\Omega)}\big)\|\mathbf{w}_\eta\|_{L^6(\Omega)}\,.$$

□

Clearly, the difficulty lies in estimating $\mathbf{u}'-\mathbf{u}'_\eta$ in $L^2(0,T;H^{-1}(\Omega)^3)$. Unfortunately, exploiting the weaker norm $H^{-1}(\Omega)$ is not easy and we shall evaluate this quantity in $L^2(\Omega\times]0,T[)^3$. This estimate is proven assuming the triangulation satisfies a milder regularity property than uniform regularity (2.27): in addition to (2.1), there exists a constant $\tilde\tau$ independent of η such that

$$\rho_{\min} \ge \tilde\tau\,\eta^{1+1/3},\ \text{where}\ \rho_{\min} = \inf_{\kappa\in\mathcal{T}_\eta}\rho_\kappa\,. \tag{6.3}$$

More precisely, this assumption is used in proving that $\mathbf{u}_\eta$ is bounded in $L^\infty(0,T;W^{1,3}(\Omega)^3)$.

Lemma 6.2. *Under the assumptions of Theorem 5.4 and if $\mathcal{T}_\eta$ satisfies (6.3), there exists a constant C that depends on $\|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega)^3)}$, $\|\mathbf{f}'\|_{L^2(0,T;H^{-1}(\Omega)^3)}$ and $\|p'\|_{L^2(\Omega\times]0,T[)}$, but not on η, such that*

$$\|\mathbf{u}_\eta\|_{L^\infty(0,T;W^{1,3}(\Omega)^3)} \le C\,. \tag{6.4}$$

Proof. We write

$$|\mathbf{u}_\eta|_{W^{1,3}(\Omega)} \le |\mathbf{u}_\eta - \mathbf{v}_\eta|_{W^{1,3}(\Omega)} + |\mathbf{v}_\eta - P_\eta(\mathbf{u})|_{W^{1,3}(\Omega)} + |P_\eta(\mathbf{u}) - \mathbf{u}|_{W^{1,3}(\Omega)} + |\mathbf{u}|_{W^{1,3}(\Omega)}\,.$$

Consider a tetrahedron κ. Since $\mathbf{u}_\eta - \mathbf{v}_\eta$ belongs to a finite-dimensional space on the reference tetrahedron $\hat{\kappa}$ where all norms are equivalent, we can write:

$$|\mathbf{u}_\eta - \mathbf{v}_\eta|_{W^{1,3}(\kappa)} \le C|\kappa|^{1/3}\frac{1}{\rho_\kappa}\|\hat{\mathbf{u}}_\eta - \hat{\mathbf{v}}_\eta\|_{L^2(\hat{\kappa})}\,,$$

where C denotes various constants independent of η; reverting to κ, this becomes

$$|\mathbf{u}_\eta - \mathbf{v}_\eta|_{W^{1,3}(\kappa)} \le C|\kappa|^{-1/6}\frac{1}{\rho_\kappa}\|\mathbf{u}_\eta - \mathbf{v}_\eta\|_{L^2(\kappa)}\,.$$

Summing over all $\kappa \in \mathcal{T}_\eta$, applying Jensen's inequality and the regularity of $\mathcal{T}_\eta$, we obtain the inverse inequality:

$$|\mathbf{u}_\eta - \mathbf{v}_\eta|_{W^{1,3}(\Omega)} \le C\frac{1}{\rho_{\min}^{3/2}}\|\mathbf{u}_\eta - \mathbf{v}_\eta\|_{L^2(\Omega)}\,.$$

Similarly,

$$|\mathbf{v}_\eta - P_\eta(\mathbf{u})|_{W^{1,3}(\Omega)} \le C\frac{1}{\rho_{\min}^{1/2}}|\mathbf{v}_\eta - P_\eta(\mathbf{u})|_{H^1(\Omega)}\,.$$

Hence

$$\begin{aligned}|\mathbf{u}_\eta|_{W^{1,3}(\Omega)} \le C\big(&\rho_{\min}^{-3/2}\|\mathbf{u}_\eta - \mathbf{v}_\eta\|_{L^2(\Omega)} + \rho_{\min}^{-1/2}|\mathbf{v}_\eta - \mathbf{u}|_{H^1(\Omega)} + \rho_{\min}^{-1/2}|P_\eta(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)}\big)\\ &+ |P_\eta(\mathbf{u}) - \mathbf{u}|_{W^{1,3}(\Omega)} + |\mathbf{u}|_{W^{1,3}(\Omega)}\,.\end{aligned} \tag{6.5}$$

Then (6.3), (5.7), (5.4) and (2.9) imply

$$|\mathbf{u}_\eta|_{W^{1,3}(\Omega)} \le C_1(\mathbf{f}, \mathbf{f}', p', \tilde{\tau}) + C_2|\mathbf{u}|_{W^{1,3}(\Omega)}\,.$$

□

Remark 6.3. Observe that (6.3) is less restrictive than (2.27). Its use is made possible here because the negative exponents of $\rho_{\min}$ in (6.5) are balanced by error terms of higher order. This is not the case in the proof of Theorem 2.5 where the denominator and numerator in (2.35) are of the same order. □

Lemma 6.4. *Under the assumptions of Theorem 5.4 and if $\mathcal{T}_\eta$ satisfies (6.3), there exists a constant C that depends on $\|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega)^3)}$, $\|\mathbf{f}'\|_{L^2(0,T;H^{-1}(\Omega)^3)}$ and $\|p'\|_{L^2(\Omega\times]0,T[)}$, but not on η, such that*

$$\|\mathbf{u}' - \mathbf{u}_\eta'\|_{L^2(\Omega\times]0,T[)} + \sqrt{\nu}\|\mathbf{u} - \mathbf{u}_\eta\|_{L^\infty(0,T;H^1(\Omega)^3)} \le C\eta\,. \tag{6.6}$$

Proof. The proof is similar to that of Lemma 4.5, except that here we have to find a bound in $L^2(\Omega\times]0,T[)$ for the difference of the non-linear terms. Indeed, we have

$$\frac{1}{2}\|\mathbf{u}'_\eta - S_\eta(\mathbf{u}')\|^2_{L^2(\Omega)} + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_\eta - S_\eta(\mathbf{u})|^2_{H^1(\Omega)} \leq \|\mathbf{u}' - S_\eta(\mathbf{u}')\|^2_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla\,\mathbf{u} - \mathbf{u}_\eta\cdot\nabla\,\mathbf{u}_\eta\|^2_{L^2(\Omega)}\,.$$

Now,

$$\mathbf{u}\cdot\nabla\,\mathbf{u} - \mathbf{u}_\eta\cdot\nabla\,\mathbf{u}_\eta = \mathbf{u}\cdot\nabla(\mathbf{u}-\mathbf{u}_\eta) + (\mathbf{u}-\mathbf{u}_\eta)\nabla\,\mathbf{u}_\eta\,.$$

First

$$\|\mathbf{u}\cdot\nabla(\mathbf{u}-\mathbf{u}_\eta)\|_{L^2(\Omega)} \leq \|\mathbf{u}\|_{L^\infty(\Omega)}|\mathbf{u}-\mathbf{u}_\eta|_{H^1(\Omega)} \leq c_1|\mathbf{u}-\mathbf{u}_\eta|_{H^1(\Omega)}\,.$$

Next, we write

$$\|(\mathbf{u}-\mathbf{u}_\eta)\nabla\,\mathbf{u}_\eta\|_{L^2(\Omega)} \leq \|\mathbf{u}-\mathbf{u}_\eta\|_{L^6(\Omega)}|\mathbf{u}_\eta|_{W^{1,3}(\Omega)} \leq S_6 C|\mathbf{u}-\mathbf{u}_\eta|_{H^1(\Omega)}\,,$$

where C is the constant of Lemma 6.2. Therefore

$$\|\mathbf{u}'_\eta - S_\eta(\mathbf{u}')\|^2_{L^2(\Omega)} + \nu\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_\eta - S_\eta(\mathbf{u})|^2_{H^1(\Omega)} \leq 2\|\mathbf{u}' - S_\eta(\mathbf{u}')\|^2_{L^2(\Omega)} + c_2|\mathbf{u}-\mathbf{u}_\eta|^2_{H^1(\Omega)}\,.$$

Then the proof finishes as in Lemma 4.5. □

From these three lemmas, we easily derive an estimate of order one for the pressure.

Theorem 6.5. *Under the assumptions of Theorem 5.4 and if $\mathcal{T}_\eta$ satisfies (6.3), there exists a constant C, independent of η such that*

$$\|p - p_\eta\|_{L^2(\Omega\times]0,T[)} \leq C\eta\,. \tag{6.7}$$

7. Two-grid algorithm

Let us recall the two-grid algorithm described in the Introduction.

- Find $(\mathbf{u}_H, p_H) \in X_H \times M_H$ for each $t \in [0,T]$, solution of (0.12–0.14):

$$\forall \mathbf{v}_H \in X_H\;,\;(\mathbf{u}'_H,\mathbf{v}_H) + \nu(\nabla\,\mathbf{u}_H,\nabla\,\mathbf{v}_H) + (\mathbf{u}_H\cdot\nabla\,\mathbf{u}_H,\mathbf{v}_H) - (p_H,\operatorname{div}\mathbf{v}_H) = \langle\mathbf{f},\mathbf{v}_H\rangle\,,$$

$$\forall q_H \in M_H\;,\;(q_H,\operatorname{div}\mathbf{u}_H) = 0\,,$$
$$\mathbf{u}_H(\mathbf{x},0) = \mathbf{0}\quad\text{in }\Omega\,.$$

- Find $(\mathbf{u}_h, p_h) \in X_h \times M_h$ for each $t \in [0,T]$, solution of (0.15–0.17):

$$\forall \mathbf{v}_h \in X_h\;,\;(\mathbf{u}'_h,\mathbf{v}_h) + \nu(\nabla\,\mathbf{u}_h,\nabla\,\mathbf{v}_h) + (\mathbf{u}_H\cdot\nabla\,\mathbf{u}_h,\mathbf{v}_h) - (p_h,\operatorname{div}\mathbf{v}_h) = \langle\mathbf{f},\mathbf{v}_h\rangle\,,$$

$$\forall q_h \in M_h\;,\;(q_h,\operatorname{div}\mathbf{u}_h) = 0\,,$$
$$\mathbf{u}_h(\mathbf{x},0) = \mathbf{0}\quad\text{in }\Omega\,.$$

We retain the assumptions of Theorem 2.3. Then the function $\mathbf{u}_H$ exists on the interval $[0,T]$, and since (0.15) is a system of linear differential equations with smooth enough coefficients, it has a unique solution over the whole interval $[0,T]$. Therefore we can estimate directly the error of Step Two, but as we wish to use the bound (5.10) for $\mathbf{u} - \mathbf{u}_{\eta,\eta=H}$, we shall need the hypotheses of Theorem 5.4.

Theorem 7.1. *Under the assumptions of Theorem 5.4, the solution $(\mathbf{u}_h, p_h)$ of (0.15–0.17) satisfies the error bound:*

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \sqrt{\nu}\|\mathbf{u}_h - \mathbf{u}\|_{L^2(0,T;H^1(\Omega)^3)} \leq C(H^2 + h)\,, \tag{7.1}$$

with a constant C independent of h and H.

Proof. Taking the difference between (0.1) multiplied by a test function $\mathbf{v}_h \in V_h$ and (0.15), inserting $P_h(\mathbf{u}) \in V_h$ and $r_h(p)$, and choosing $\mathbf{v}_h = \mathbf{u}_h - P_h(\mathbf{u})$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_h - P_h(\mathbf{u})\|^2_{L^2(\Omega)} &+ \nu|\mathbf{u}_h - P_h(\mathbf{u})|^2_{H^1(\Omega)}\\ &= \nu(\nabla(\mathbf{u} - P_h(\mathbf{u})), \nabla(\mathbf{u}_h - P_h(\mathbf{u}))) + (r_h(p) - p, \mathrm{div}(\mathbf{u}_h - P_h(\mathbf{u})))\\ &\quad + (\mathbf{u}' - P_h(\mathbf{u}'), \mathbf{u}_h - P_h(\mathbf{u})) + ((\mathbf{u}_H - \mathbf{u})\cdot\nabla(\mathbf{u}_h - P_h(\mathbf{u})), \mathbf{u}_h - P_h(\mathbf{u}))\\ &\quad + (\mathbf{u}_H \cdot \nabla(P_h(\mathbf{u}) - \mathbf{u}), \mathbf{u}_h - P_h(\mathbf{u})) + ((\mathbf{u}_H - \mathbf{u})\cdot\nabla\mathbf{u}, \mathbf{u}_h - P_h(\mathbf{u}))\,.\end{aligned} \tag{7.2}$$

Let us estimate the non-linear terms in the right-hand side. Comparing with (1.7), we see that the first term appears here because $\mathrm{div}\,\mathbf{u}_H \neq 0$. But it can be absorbed by the left-hand side:

$$|((\mathbf{u}_H - \mathbf{u})\cdot\nabla(\mathbf{u}_h - P_h(\mathbf{u})), \mathbf{u}_h - P_h(\mathbf{u}))| \le S_6^{3/2}|\mathbf{u}_H - \mathbf{u}|_{H^1(\Omega)}|\mathbf{u}_h - P_h(\mathbf{u})|^{3/2}_{H^1(\Omega)}\|\mathbf{u}_h - P_h(\mathbf{u})\|^{1/2}_{L^2(\Omega)}\,;$$

and according to Theorem 2.3,

$$\|\mathbf{u}_H - \mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^3)} \le c_1\,. \tag{7.3}$$

Therefore

$$\begin{aligned}2|((\mathbf{u}_H - \mathbf{u})\cdot\nabla(\mathbf{u}_h - P_h(\mathbf{u})), \mathbf{u}_h - P_h(\mathbf{u}))| \le c_1 S_6^{3/2}\big(&\frac{1}{\varepsilon_1}|\mathbf{u}_h - P_h(\mathbf{u})|^2_{H^1(\Omega)}\\ &+ \frac{\varepsilon_1}{2}(\delta|\mathbf{u}_h - P_h(\mathbf{u})|^2_{H^1(\Omega)} + \frac{1}{\delta}\|\mathbf{u}_h - P_h(\mathbf{u})\|^2_{L^2(\Omega)})\big)\,.\end{aligned}$$

Next, the analogue of (1.8) holds; thus setting

$$c_2 = \|\mathbf{u}_H\|_{L^\infty(0,T;L^3(\Omega)^3)}\,,$$

we obtain the analogue of (1.9)

$$2|(\mathbf{u}_H\cdot\nabla(P_h(\mathbf{u}) - \mathbf{u}), \mathbf{u}_h - P_h(\mathbf{u}))| \le c_2 S_6(\varepsilon_2|\mathbf{u}_h - P_h(\mathbf{u})|^2_{H^1(\Omega)} + \frac{1}{\varepsilon_2}|P_h(\mathbf{u}) - \mathbf{u}|^2_{H^1(\Omega)})\,.$$

Finally, since the assumptions of Theorem 5.4 are strong enough, we have the analogue of (1.14) and we set

$$c_3 = \|\mathbf{u}\|_{L^\infty(0,T;W^{1,3}(\Omega)^3)}\,,$$

then

$$2|((\mathbf{u}_H - \mathbf{u})\cdot\nabla\mathbf{u}, \mathbf{u}_h - P_h(\mathbf{u}))| \le c_3 S_6(\varepsilon_3|\mathbf{u}_h - P_h(\mathbf{u})|^2_{H^1(\Omega)} + \frac{1}{\varepsilon_3}\|\mathbf{u}_H - \mathbf{u}\|^2_{L^2(\Omega)})\,.$$

The linear terms are bounded as in Theorem 3.1. Then collecting these inequalities, substituting into (7.2), applying Corollary 5.5 with $\eta = H$, choosing suitably the parameters ε_i and δ, and integrating over $[0, T]$, we readily derive (7.1). ☐

Thus, if $h = H^2$, then

$$\|\mathbf{u}_h - \mathbf{u}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \sqrt{\nu}\|\mathbf{u}_h - \mathbf{u}\|_{L^2(0,T;H^1(\Omega)^3)} = O(h)\,.$$

Remark 7.2. We could have improved (7.3) by applying (6.6), and it would have led to smaller constants. But (6.6) is proven under the assumption that $\mathcal{T}_\eta$ satisfies (6.3) and we wish to avoid this restriction. ☐

Finally, we consider the error of the pressure. As in Section 6, the pressure satisfies the following bound.

Lemma 7.3. *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the respective solutions of (0.1–0.4) and (0.15–0.17). Under the assumptions of Theorem 2.3, and if in addition* $\mathbf{f} \in L^\infty(0,T; L^{3/2}(\Omega)^3)$*, we have*

$$\begin{aligned}\|p_h - r_h(p)\|_{L^2(\Omega\times]0,T[)} \le \frac{1}{\beta^\star}\big(\sqrt{3}\|p - r_h(p)\|_{L^2(\Omega\times]0,T[)} + \|\mathbf{u}' - \mathbf{u}_h'\|_{L^2(0,T;H^{-1}(\Omega)^3)}\\ + \nu\|\mathbf{u} - P_h(\mathbf{u})\|_{L^2(0,T;H^1(\Omega)^3)} + S_6\|\mathbf{u}\|_{L^\infty(0,T;W^{1,3}(\Omega)^3)}\|\mathbf{u} - \mathbf{u}_H\|_{L^2(\Omega\times]0,T[)}\\ + S_6\|\mathbf{u} - \mathbf{u}_h\|_{L^2(0,T;H^1(\Omega)^3)}\big(\|\mathbf{u} - \mathbf{u}_H\|_{L^\infty(0,T;L^3(\Omega)^3)} + \|\mathbf{u}\|_{L^\infty(0,T;L^3(\Omega)^3)}\big)\big)\,.\end{aligned} \tag{7.4}$$

Proof. The only difference with the proof of Lemma 6.1 concerns the non-linear term. Here we write

$$|(\mathbf{u}\cdot\nabla\,\mathbf{u} - \mathbf{u}_H\cdot\nabla\,\mathbf{u}_h, \mathbf{w}_h)| \le \big(\|\mathbf{u} - \mathbf{u}_H\|_{L^2(\Omega)}|\mathbf{u}|_{W^{1,3}(\Omega)} + |\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)}(\|\mathbf{u} - \mathbf{u}_H\|_{L^3(\Omega)} + \|\mathbf{u}\|_{L^3(\Omega)})\big)\|\mathbf{w}_h\|_{L^6(\Omega)},$$

whence (7.4) □

Therefore, here again, we must derive an estimate for $\|\mathbf{u}' - \mathbf{u}_h'\|_{L^2(0,T;H^{-1}(\Omega)^3)}$, and we derive it in $L^2(\Omega\times]0,T[)^3$ because the norm of $H^{-1}(\Omega)$ does not appear to bring any improvement. We write the proof for a uniformly regular triangulation, *i.e.* satisfying (2.27).

Lemma 7.4. *Under the assumptions of Theorem 5.4 and if* $\mathcal{T}_h$ *satisfies (2.27), we have*

$$\|\mathbf{u}' - \mathbf{u}_h'\|_{L^2(\Omega\times]0,T[)} + \sqrt{\nu}\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(0,T;H^1(\Omega)^3)} \le C(h + h^{1/2}H + H^{3/2} + H^2)\,, \tag{7.5}$$

with a constant C *independent of* h *and* H*.*

Proof. The proof is similar to that of Lemma 6.4, except for the treatment of the non-linear term. We write:

$$\mathbf{u}_H\cdot\nabla\,\mathbf{u}_h - \mathbf{u}\cdot\nabla\,\mathbf{u} = (\mathbf{u}_H - \mathbf{u})\cdot\nabla(\mathbf{u}_h - \mathbf{u}) + (\mathbf{u}_H - \mathbf{u})\cdot\nabla\,\mathbf{u} + \mathbf{u}\cdot\nabla(\mathbf{u}_h - \mathbf{u})\,.$$

The worst term is the second one and it accounts for the term $H^{3/2}$ in (7.5). Setting

$$c_1 = \|\mathbf{u}\|_{L^\infty(0,T;W^{1,6}(\Omega)^3)}\,,$$

we have

$$\|(\mathbf{u}_H - \mathbf{u})\cdot\nabla\,\mathbf{u}\|_{L^2(\Omega)} \le c_1\|\mathbf{u}_H - \mathbf{u}\|_{L^3(\Omega)}\,. \tag{7.6}$$

As in Section 1, Corollaries 3.2 and 5.5 give

$$\|\mathbf{u}_H - \mathbf{u}\|_{L^3(\Omega)} \le c_2 H^{3/2}\,.$$

Thus

$$\|(\mathbf{u}_H - \mathbf{u})\cdot\nabla\,\mathbf{u}\|_{L^2(\Omega)} \le c_1 c_2 H^{3/2}\,.$$

Next, setting

$$c_3 = \|\mathbf{u}\|_{L^\infty(\Omega\times]0,T[)}\,,$$

and applying (7.1), we find

$$\|\mathbf{u}\cdot\nabla(\mathbf{u}_h - \mathbf{u})\|_{L^2(\Omega)} \le c_3|\mathbf{u}_h - \mathbf{u}|_{H^1(\Omega)} \le c_4(H^2 + h)\,.$$

Finally, applying Lemma 6.4, an inverse inequality and (7.1), we derive

$$\|(\mathbf{u}_H - \mathbf{u})\cdot\nabla(\mathbf{u}_h - \mathbf{u})\|_{L^2(\Omega)} \le \|\mathbf{u}_H - \mathbf{u}\|_{L^6(\Omega)}|\mathbf{u}_h - \mathbf{u}|_{W^{1,3}(\Omega)} \le c_5 H(h^{1/2} + H)\,.$$

Hence

$$\|\mathbf{u}_H \cdot \nabla \mathbf{u}_h - \mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2(\Omega\times]0,T[)} \leq c_2 H^{3/2} + c_4(H^2 + h) + c_5 H(h^{1/2} + H). \tag{7.7}$$

Then (7.5) follows readily from (7.7) and the argument of Lemma 4.5. □

These two lemmas yield immediately the following theorem.

Theorem 7.5. *Under the assumptions of Lemma 7.4, we have*

$$\|p - p_h\|_{L^2(\Omega\times]0,T[)} \leq C(h + h^{1/2}H + H^{3/2} + H^2), \tag{7.8}$$

with a constant C independent of h and H.

Remark 7.6. As a consequence, if $h = H^2$, then

$$\|p - p_h\|_{L^2(\Omega\times]0,T[)} = O(h^{3/4}). \tag{7.9}$$

However, if $\mathbf{u} \in L^\infty(0,T; W^{1,\infty}(\Omega)^3)$, then (7.6) is replaced by

$$\|(\mathbf{u}_H - \mathbf{u}) \cdot \nabla \mathbf{u}\|_{L^2(\Omega)} \leq \|\mathbf{u}_H - \mathbf{u}\|_{L^2(\Omega)} |\mathbf{u}|_{W^{1,\infty}(\Omega)} \leq CH^2,$$

and (7.9) is replaced by:

$$\|p - p_h\|_{L^2(\Omega\times]0,T[)} = O(h).$$

□

Appendix

Let us describe briefly our extension of the Scott and Zhang operator to L^1 functions. The details of the proofs can be found in [42]. Here we denote the mesh-size by h and we assume that the triangulation is regular, *i.e.* it satisfies (2.1). Let $v \in L^1(\Omega)$; since we can take for $R_h(v)$ a polynomial of degree one in each tetrahedron, it suffices to regularize the "nodal values" of v on all the vertices of $\mathcal{T}_h$. Let $\mathbf{a}$ be any vertex of $\mathcal{T}_h$. If $\mathbf{a} \in \partial\Omega$, we set $R_h(v)(\mathbf{a}) = 0$. If $\mathbf{a}$ is an interior vertex of $\mathcal{T}_h$, we choose freely a tetrahedron, say $\kappa_\mathbf{a}$ with vertex $\mathbf{a}$. Let $\psi_\mathbf{a}$ be the dual basis function, piecewise $\mathbb{P}_1$, of the four Lagrange basis functions $\varphi_\mathbf{b}$ (also piecewise $\mathbb{P}_1$) that take the value one at the vertex $\mathbf{b}$ and zero at all other vertices:

$$\int_{\kappa_\mathbf{a}} \psi_\mathbf{a}(\mathbf{x}) \varphi_\mathbf{b}(\mathbf{x}) \mathrm{d}\mathbf{x} = \delta_{\mathbf{a},\mathbf{b}}. \tag{A.1}$$

We set

$$R_h(v)(\mathbf{a}) = \int_{\kappa_\mathbf{a}} v(\mathbf{x}) \psi_\mathbf{a}(\mathbf{x}) \mathrm{d}\mathbf{x},$$

i.e.

$$R_h(v)(\mathbf{x}) = \sum_{\mathbf{a}\in\mathcal{T}_h^{\mathrm{int}}} \Big(\int_{\kappa_\mathbf{a}} v(\mathbf{y}) \psi_\mathbf{a}(\mathbf{y}) \mathrm{d}\mathbf{y}\Big) \varphi_\mathbf{a}(\mathbf{x}), \tag{A.2}$$

where $\mathcal{T}_h^{\mathrm{int}}$ denotes the set of interior vertices of $\mathcal{T}_h$.

First observe that, if $v \in L^p(\Omega)$, for some $p \geq 1$, we have

$$|R_h(v)(\mathbf{a})| \leq C|\kappa_\mathbf{a}|^{-1/p} \|v\|_{L^p(\kappa_\mathbf{a})}, \tag{A.3}$$

where here and in the sequel, C denotes various constants independent of h. Consequently, for any tetrahedron κ, for $m = 0$ or 1, for any number $p \geq 1$ and for any function $v \in W^{m,p}(\kappa)$, we have, using the regularity (2.1) of $\mathcal{T}_h$:

$$|R_h(v)|_{W^{m,p}(\kappa)} \leq C h_\kappa^{-m} \|v\|_{L^p(D_\kappa)}\,, \tag{A.4}$$

where D_κ denotes the union of all the tetrahedra of $\mathcal{T}_h$ that share a vertex, an edge or a face with κ. Since Ω is a Lipschitz continuous polyhedron, D_κ is connected. As the triangulation is regular, the number of tetrahedra in a given D_κ is bounded by a constant independent of D_κ and h, and the number of occurrences of a given tetrahedron in all the D_κ is also bounded by a constant independent of κ and h. Therefore, (A.4) implies that R_h is stable in $L^p(\Omega)$ for any number $p \geq 1$:

$$\forall v \in L^p(\Omega)\,,\ \|R_h(v)\|_{L^p(\Omega)} \leq C\|v\|_{L^p(\Omega)}\,. \tag{A.5}$$

Next, observe that, in view of (A.1) and (A.2), R_h is a projection on the space

$$\Theta_h = \{v \in \mathcal{C}^0(\overline{\Omega})\,;\, \forall \kappa \in \mathcal{T}_h, v|_\kappa \in \mathbb{P}_1\,,\ v|_{\partial\Omega} = 0\}\ :$$

$$\forall v \in \Theta_h\,,\ R_h(v) = v\,.$$

Now, let v be function in $W^{k+1,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for $k = 0$ or 1 and a number $p \geq 1$, where the index zero means that the function has a zero trace on the boundary $\partial\Omega$. Note that the functions of $W^{1,1}(\Omega)$ have a trace. Let κ be a tetrahedron with no vertex on $\partial\Omega$; then by construction

$$\forall q \in \mathbb{P}_1\,,\ R_h(q)|_\kappa = q\,.$$

Therefore, for $m = 0$ or 1 and for all $q \in \mathbb{P}_1$, applying (A.4), we have

$$|R_h(v) - v|_{W^{m,p}(\kappa)} = |R_h(v-q) - (v-q)|_{W^{m,p}(\kappa)} \leq |v-q|_{W^{m,p}(\kappa)} + C h_\kappa^{-m} \|v-q\|_{L^p(D_\kappa)}\,.$$

As D_κ is connected, we can apply the argument of Dupont and Scott [13] that gives:

$$\inf_{q\in\mathbb{P}_1} (|v-q|_{W^{m,p}(\kappa)} + h_\kappa^{-m}\|v-q\|_{L^p(D_\kappa)}) \leq C h_\kappa^{k+1-m} |v|_{W^{k+1,p}(D_\kappa)}\,. \tag{A.6}$$

Hence

$$|R_h(v) - v|_{W^{m,p}(\kappa)} \leq C h_\kappa^{k+1-m} |v|_{W^{k+1,p}(D_\kappa)}\,. \tag{A.7}$$

If κ has at least one vertex on $\partial\Omega$, say $\mathbf{a}_0$, then D_κ contains at least one tetrahedron $\kappa_{\mathbf{a}_0}$ with a face, say F_0, containing $\mathbf{a}_0$ and lying on $\partial\Omega$. But the trace on $\partial\Omega$ of $v \in W_0^{1,1}(\Omega)$ is well-defined and is zero, therefore $R_h(v)$ satisfies trivially

$$0 = R_h(v)(\mathbf{a}_0) = \int_{F_0} v(\sigma)\psi_{\mathbf{a}_0}(\sigma)\mathrm{d}\sigma\,, \tag{A.8}$$

where $\psi_{\mathbf{a}_0}$ denotes the dual $\mathbb{P}_1$ basis function on F_0 of the three $\mathbb{P}_1$ Lagrange basis functions on F_0. Clearly, this is valid if κ has more than one vertex on $\partial\Omega$. In other words, for functions in $W_0^{1,1}(\Omega)$, $R_h(v)$ has the same degrees of freedom on $\partial\Omega$ as the Scott and Zhang operator, defined in [42]. Thus we can apply to it the results of this reference. On one hand, with the formulation (A.8),

$$\forall q \in \mathbb{P}_1\,,\ R_h(q)|_\kappa = q\,.$$

On the other hand, in view of (A.8), we have

$$|R_h(v)(\mathbf{a}_0)| \le C|\kappa_{\mathbf{a}_0}|^{-1/p} \left(\|v\|_{L^p(\kappa_{\mathbf{a}_0})} + h_{\kappa_{\mathbf{a}_0}} |v|_{W^{1,p}(\kappa_{\mathbf{a}_0})} \right) .$$

Therefore, combining this bound with (A.3), we obtain:

$$|R_h(v)|_{W^{1,p}(\kappa)} \le C h_\kappa^{-1} \left(\|v\|_{L^p(D_\kappa)} + h_\kappa |v|_{W^{1,p}(D_\kappa)} \right) .$$

Collecting these results, we find

$$|R_h(v) - v|_{W^{1,p}(\kappa)} = |R_h(v-q) - (v-q)|_{W^{1,p}(\kappa)} \le |v-q|_{W^{1,p}(\kappa)} + C h_\kappa^{-1} \left(\|v-q\|_{L^p(D_\kappa)} + h_\kappa |v-q|_{W^{1,p}(D_\kappa)} \right) .$$

Then (A.6) implies

$$|R_h(v) - v|_{W^{m,p}(\kappa)} \le C h_\kappa^{k+1-m} |v|_{W^{k+1,p}(D_\kappa)} , \tag{A.9}$$

which together with (A.7) yields for $m = 0$ or 1, $k = 0$ or 1 and any number $p \ge 1$:

$$\forall v \in W^{k+1,p}(\Omega) \cap W_0^{1,p}(\Omega) \,,\ |R_h(v) - v|_{W^{m,p}(\Omega)} \le C h^{k+1-m} |v|_{W^{k+1,p}(\Omega)} . \tag{A.10}$$

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