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Two inequalities for the associated Pollaczek polynomials

Min-Jie Luo[1,2*] and Ravinder Krishna Raina[3]

*Correspondence: mathwinnie@live.com; mathwinnie@dhu.edu.cn
[1]Department of Mathematics, Donghua University, Shanghai, People's Republic of China
[2]Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, People's Republic of China
Full list of author information is available at the end of the article

Abstract

In this paper, we prove two concise inequalities for the associated Pollaczek polynomials. The first one is obtained by using Askey's theorem (SIAM J. Math. Anal. 2:340–346, 1971) on orthogonal expansions with positive coefficients. The second one is proved by using a triple integral representation due to the authors (Integral Transforms Spec. Funct. 30:893–919, 2019). In the concluding section, we briefly point out some useful variations and known cases of our inequalities.

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1 Introduction and main result

The associated Pollaczek polynomials $P_n^{\lambda}(\cos\theta;a,b,c)$ can be defined by (see, e.g., [13, 14] and [17])

$$\sum_{n=0}^{\infty} P_n^{\lambda}(\cos\theta;a,b,c)t^n = \left(1-te^{-i\theta}\right)^{-\lambda-i\Phi}\left(1-te^{i\theta}\right)^{-\lambda+i\Phi} \times F_1\left[c,1-\lambda-i\Phi,1-\lambda+i\Phi;c+1;te^{-i\theta},te^{i\theta}\right], \tag{1.1}$$

where $|t|<1$,

$$\Phi \equiv \Phi(\theta) := \frac{a\cos\theta+b}{\sin\theta}, \tag{1.2}$$

and F_1 is the Appell hypergeometric function defined by ([18, p. 53, Eq. (4)])

$$F_1[\alpha,\beta_1,\beta_2;\gamma;x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \quad \max\left\{|x|,|y|\right\}<1.$$

Here the Pochhammer symbol $(\lambda)_n$ is defined (for $\lambda\in\mathbb{C}$) by

$$(\lambda)_n := \begin{cases} 1 & (n=0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}:=\{1,2,\ldots\}). \end{cases}$$

When $c = 0$, we may obtain from (1.1) the generating function of the Pollaczek polynomials $P_n^\lambda(\cos\theta; a, b)$:

$$\sum_{n=0}^{\infty} P_n^\lambda(\cos\theta; a, b)t^n = \left(1 - te^{-\mathrm{i}\theta}\right)^{-\lambda-\mathrm{i}\Phi}\left(1 - te^{\mathrm{i}\theta}\right)^{-\lambda+\mathrm{i}\Phi}. \tag{1.3}$$

Conventionally, the polynomials obtained by setting $\lambda = \frac{1}{2}$ in $P_n^\lambda(\cos\theta; a, b)$ are simply denoted by $P_n(\cos\theta; a, b)$. For classical results on Pollaczek polynomials, we refer the interested reader to [6, 7, 10], and [16]. Furthermore, readers are encouraged to see [13, 14] and [9] for the latest development in this area.

Despite some important results, we still know little about the associated Pollaczek polynomials, especially about the inequalities satisfied by this class of polynomials. In this paper, our aim is to establish two interesting inequalities for the associated Pollaczek polynomials.

The first inequality is given by the following:

Theorem 1.1 *For $a, b, c \geq 0$, $\lambda > 0$, and $x \in [-1, 1]$, we have*

$$\left|P_n^\lambda(x; a, b, c)\right| \leq P_n^\lambda(1; a, b, c). \tag{1.4}$$

As we will show in the following sections, Theorem 1.1 is obtained by using an important result from Askey's work on orthogonal expansions with positive coefficients. Theorem 1.2 is obtained by using a completely different method.

Theorem 1.2 *For $\theta \in (0, \pi)$, $c > 0$, $\lambda > 0$, and $c + 2\lambda > 1$, we have*

$$\left|P_n^\lambda(\cos\theta; a, b, c)\right| \leq \frac{[\Gamma(\lambda)]^2}{|\Gamma(\lambda + \mathrm{i}\Phi)|^2} \frac{(c + 2\lambda)_n(2c + 2\lambda)_n}{(c + 1)_n n!}(1 + 2\sin\theta)^n, \tag{1.5}$$

where Φ is given by (1.2).

We denote by $P_n(\cos\theta; a, b, c)$ the polynomials obtained by letting $\lambda = \frac{1}{2}$ in $P_n^\lambda(\cos\theta; a, b, c)$. Then, using Theorem 1.2, we have

$$\operatorname{sech}(\pi\Phi)\left|P_n(\cos\theta; a, b, c)\right| \leq \frac{(2c + 1)_n}{n!}(1 + 2\sin\theta)^n,$$

where we have used the relation that $|\Gamma(\frac{1}{2} + \mathrm{i}y)|^2 = \pi\operatorname{sech}(\pi y)$ (see [16, p. 137, Eq. (5.4.4)]).

2 Key lemmas

In this section, we present some useful lemmas used in our proofs. The first result due to Askey [3] gives a sufficient condition for writing a set of orthogonal polynomials as a linear combination of a second set of orthogonal polynomials with nonnegative coefficients. We will use it in the proof of Theorem 1.1.

Let $p_n(x)$ be defined by

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$, α_{n-1} real, $\beta_n > 0$, $n = 1, 2, \dots$, and the polynomials are normalized to be monic (i.e., the leading coefficients of the polynomials are one).

Similarly, let $p_n(x)$ be defined by

$$xq_n(x) = q_{n+1}(x) + \gamma_n q_n(x) + \delta_n q_{n-1}(x), \quad n = 0, 1, 2, \dots, \tag{2.2}$$

where $q_{-1}(x) = 0$, $q_0(x) = 1$, γ_{n-1} real, $\delta_n > 0$, $n = 1, 2, \dots$. Then Askey's result can be stated as follows.

Lemma 2.1 ([3, p. 341]) *Let $p_n(x)$ and $q_n(x)$ be defined by (2.1) and (2.2) and set*

$$q_n(x) = \sum_{k=0}^{n} a(k,n) p_k(x).$$

Then $a(k,n) \geq 0$ if

$$\alpha_k \geq \gamma_n, \quad k = 0, 1, \dots, n, n = 0, 1, \dots, \tag{2.3}$$

$$\beta_k \geq \delta_n, \quad k = 0, 1, \dots, n, n = 0, 1, \dots. \tag{2.4}$$

To use this lemma, we require three-term recurrence relations for the normalized (monic) associated Pollaczek polynomials and some of their particular cases. Let $\hat{P}_n^\lambda(x; a, b, c)$ denote the normalized associated Pollaczek polynomials. We have ([6, p. 185])

$$\hat{P}_n^\lambda(x; a, b, c) = \frac{(c+1)_n}{2^n(\lambda + a + c)_n} P_n^\lambda(x; a, b, c),$$

from which we have the following three-term recurrence relation ([6, p. 185, Eq. (5.9)]):

$$\begin{aligned} x\hat{P}_n^\lambda(x; a, b, c) = \hat{P}_{n+1}^\lambda(x; a, b, c) &- \frac{b}{n + \lambda + a + c} \tilde{P}_n^\lambda(x; a, b, c) \\ &+ \frac{(n+c)(n+2\lambda+c-1)}{4(n+\lambda+a+c)(n+\lambda+a+c-1)} \hat{P}_{n-1}^\lambda(x; a, b, c). \end{aligned} \tag{2.5}$$

When $a = b = 0$, (2.5) reduces to the three-term recurrence relation of the normalized *associated ultraspherical (Gegenbauer) polynomials* $\hat{C}_n^\lambda(x, c)$ given by

$$x\hat{C}_n^\lambda(x, c) = \hat{C}_{n+1}^\lambda(x, c) + \frac{(n+c)(n+2\lambda+c-1)}{4(n+\lambda+c)(n+\lambda+c-1)} \hat{C}_{n-1}^\lambda(x, c). \tag{2.6}$$

It was also proved in [3, p. 345] that

$$\left|\hat{C}_n^\lambda(x, c)\right| \leq \hat{C}_n^\lambda(1, c), \quad x \in [-1, 1], \lambda > 0, c \geq 0. \tag{2.7}$$

Following Carlson [5, p. 52, Def. 3.11-1], we define the *Euler measure* $\mathfrak{m}_\alpha$ on $\mathbb{R}_+$ by

$$d\mathfrak{m}_\alpha(u) := \frac{1}{\Gamma(\alpha)} u^{\alpha-1} e^{-u} \, du \quad (\alpha > 0).$$

It is not difficult to verify that $\mathfrak{m}_\alpha(\mathbb{R}_+) = 1$. For $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, we define

$$\mathrm{d}\mu_{\alpha,\beta}(t) := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t, \tag{2.8}$$

which is a particular case of the *Dirichlet measure* (see [5, p. 59]) and satisfies $\mu_{\alpha,\beta}([0,1]) = 1$. Throughout this paper, we let

$$\mathrm{d}\Pi(t,u_1,u_2) := \mathrm{d}\mu_{c,\lambda+\mathrm{i}\Phi}(t)\,\mathrm{d}\mathfrak{m}_{c+2\lambda-1}(u_1)\,\mathrm{d}\mathfrak{m}_1(u_2). \tag{2.9}$$

The following Lemma 2.2 is proved in [13] and is essential in the proof of Theorem 1.2.

Lemma 2.2 *For* $c > 0$, $\lambda > 0$, $c + 2\lambda > 1$, *we have*

$$\begin{aligned} P_n^{\lambda}(\cos\theta; a, b, c) = {} & \frac{(c+2\lambda)_n}{n!} \mathrm{e}^{\mathrm{i}n\theta} \\ & \times \int_{(0,1)\times\mathbb{R}_+^2} {}_2F_2\left[\begin{matrix} -n, c+\lambda+\mathrm{i}\Phi \\ c+1, c+2\lambda \end{matrix}; tu_1 + u_2\big(1-\mathrm{e}^{-2\mathrm{i}\theta}\big)\right] \mathrm{d}\Pi(t,u_1,u_2), \end{aligned} \tag{2.10}$$

where ${}_2F_2$ *is the generalized hypergeometric function defined by*

$${}_2F_2\left[\begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!} \quad (z \in \mathbb{C}). \tag{2.11}$$

In addition to this result, the proof of Theorem 1.2 heavily relies on the properties of hypergeometric functions and Laguerre polynomials. The following lemma giving an Eulerian-type integral representation for ${}_2F_2$-function enables us to handle appropriately the ${}_2F_2$-function occurring in (2.10).

Lemma 2.3 ([16, p. 408, Eq. (16.5.2)]) *For* $\Re(b_2) > \Re(a_2) > 0$, *we have*

$${}_2F_2\left[\begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix}; z\right] = \frac{\Gamma(b_2)}{\Gamma(a_2)\Gamma(b_2-a_2)} \int_0^1 t^{a_2-1}(1-t)^{b_2-a_2-1} {}_1F_1\left[\begin{matrix} a_1 \\ b_1 \end{matrix}; zt\right] \mathrm{d}t, \tag{2.12}$$

where ${}_2F_2$ *is defined by* (2.11), *and* ${}_1F_1$ *is the confluent hypergeometric function* (*see, e.g.,* [18, p. 36, Eq. (3)]).

Also, we will frequently use the following version of the Chu–Vandermonde identity (see, e.g., [18, p. 31]).

Lemma 2.4

$$(x+y)_n = \sum_{\ell=0}^{n} \binom{n}{\ell} (x)_\ell (y)_{n-\ell}. \tag{2.13}$$

For the Laguerre polynomials defined by [16, p. 443, Eq. (18.5.12)]

$$L_n^{(\alpha)}(x) := \frac{(\alpha+1)_n}{n!} {}_1F_1\left[\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right], \tag{2.14}$$

Eqs. (2.15) and (2.16) given further by Lemma 2.5 are known as the addition and multiplication theorems for the Laguerre polynomials, respectively, and would be required in the proof of our inequality (1.5).

Lemma 2.5 (see [16, p. 460, Eq. (18.18.12)] and [16, p. 461, Eq. (18.18.38)])

$$L_n^{(\alpha+\beta+1)}(x+y)=\sum_{\ell=0}^{n}L_\ell^{(\alpha)}(x)L_{n-\ell}^{(\beta)}(y), \tag{2.15}$$

$$\frac{L_n^{(\alpha)}(\lambda x)}{L_n^{(\alpha)}(0)}=\sum_{\ell=0}^{n}\binom{n}{\ell}\lambda^{\ell}(1-\lambda)^{n-\ell}\frac{L_\ell^{(\alpha)}(x)}{L_\ell^{(\alpha)}(0)}. \tag{2.16}$$

The Laguerre polynomials satisfy the following well-known important inequality (see [1, p. 786, Eq. (22.14.13)]):

$$\left|L_n^{(\alpha)}(x)\right|\le\frac{(\alpha+1)_n}{n!}\mathrm{e}^{x/2}\quad(\alpha\ge 0,x\ge 0), \tag{2.17}$$

In 1997, Love [12] published several inequalities for the Laguerre polynomials $L_n^{(\alpha)}(x)$ (with complex α) and for the Laguerre functions $L_\nu^{(\mu)}(x)$ (with complex μ and ν).

Although inequality (2.17) is quite elegant, the involved exponential factor $\mathrm{e}^{x/2}$ can make an integral to be divergent. The same exponential factor occurs in Love's generalization of (2.17) (see [12, p. 295, Theorem 1]). So we need a particular bound for Laguerre polynomials. To remedy such a situation, we denote by $\sigma_n^{(\alpha)}$ the Cesáro mean defined by

$$\sigma_n^{(\alpha)}\left(\sum_{n=1}^{\infty}a_n\right)=\frac{n!}{(\alpha+1)_n}\sum_{k=0}^{n}\frac{(\alpha+1)_{n-k}}{(n-k)!}a_k.$$

Lemma 2.6 ([11, p. 532, Eq. (10)]) *For $\alpha\ge-\frac{1}{2}$, $x\ge 0$, and $n=0,1,\ldots$, we have*

$$\left|L_n^{(\alpha)}(x)\right|\le\frac{(\alpha+1)_n}{n!}\sigma_n^{(\alpha)}\left(\mathrm{e}^{x}\right), \tag{2.18}$$

where

$$\sigma_n^{(\alpha)}\left(\mathrm{e}^{x}\right)=\frac{n!}{(\alpha+1)_n}\sum_{k=0}^{n}\frac{(\alpha+1)_{n-k}}{(n-k)!}\frac{x^k}{k!}.$$

Estimate (2.18) is better than (2.17) for large x.

3 Proof of Theorem 1.1

In view of the range of the parameter λ, the proof is divided into two parts. We first prove inequality (1.4) for $0<\lambda<1$. We then consider the case $\lambda\ge 1$.

When $0<\lambda<1$, our aim is to show that the normalized associated Pollaczek polynomials can be written as linear combinations of some normalized associated ultraspherical polynomials with nonnegative coefficients. More precisely, we want to prove that

$$\hat{P}_n^{\lambda}(x;a,b,c)=\sum_{k=0}^{n}a(k,n)\hat{C}_n^{\lambda}(x,c),\quad\text{where } a(k,n)\ge 0. \tag{3.1}$$

By comparing (2.1) with (2.6) we have

$$\alpha_n = 0 \quad \text{and} \quad \beta_n = \frac{(n+c)(n+2\lambda+c-1)}{4(n+\lambda+c)(n+\lambda+c-1)},$$

where $\beta_n > 0$ for $n = 1, 2, \dots$. Similarly, comparing (2.2) with (2.5) gives

$$\gamma_n = -\frac{b}{n+\lambda+a+c} \quad \text{and} \quad \delta_n = \frac{(n+c)(n+2\lambda+c-1)}{4(n+\lambda+a+c)(n+\lambda+a+c-1)},$$

where $\delta_n > 0$ for $n = 1, 2, \dots$. Obviously, we have $\beta_n \geq \delta_n$ because $a \geq 0$. We also have

$$\alpha_k \geq \gamma_n, \quad k = 0, 1, \dots, n, n = 0, 1, \dots,$$

since $a, b, c \geq 0$ and $0 < \lambda < 1$. So condition (2.3) of Lemma 2.1 is satisfied.

To prove that $\beta_k \geq \delta_n$, $k = 0, 1, \dots, n$, $n = 0, 1, \dots$, we need to show that

$$f(x) := \frac{x(x+2\lambda-1)}{4(x+\lambda)(x+\lambda-1)}$$

is a decreasing function on $(1-\lambda, \infty)$. By taking the logarithmic derivative with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\log f(x) = \frac{1}{x} + \frac{1}{x+2\lambda-1} - \frac{1}{x+\lambda} - \frac{1}{x+\lambda-1}.$$

The condition under which $\frac{\mathrm{d}}{\mathrm{d}x}\log f(x) < 0$ is obtained by observing that

$$\begin{aligned} \frac{1}{x} + \frac{1}{x+2\lambda-1} < \frac{1}{x+\lambda} + \frac{1}{x+\lambda-1} \quad &\Leftrightarrow \quad (x+\lambda)(x+\lambda-1) < x(x+2\lambda-1) \\ &\Leftrightarrow \quad \lambda(\lambda-1) < 0. \end{aligned}$$

So f decreases on $(1-\lambda, \infty)$ for $0 < \lambda < 1$. Noting that $\beta_n = f(n+c)$ $(c \geq 0)$, we have

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq \delta_n,$$

which completes the verification of condition (2.4). Then expansion (3.1) follows from Lemma 2.1.

Since $a(k,n) \geq 0$, by inequality (2.7) we have

$$\left|\hat{P}_n^\lambda(x;a,b,c)\right| \leq \sum_{k=0}^{n} a(k,n)\left|\hat{C}_n^\lambda(x,c)\right| \leq \sum_{k=0}^{n} a(k,n)\hat{C}_n^\lambda(1,c) = \hat{P}_n^\lambda(1;a,b,c).$$

This completes the proof of the case $0 < \lambda < 1$.

Next, we consider the case $\lambda \geq 1$. Note that for $\lambda = 1$ in (2.6), it becomes

$$x\hat{C}_n^1(x,c) = \hat{C}_{n+1}^1(x,c) + \frac{1}{4}\hat{C}_{n-1}^1(x,c),$$

which still defines a set of (particular) associated ultraspherical polynomials. Let us demonstrate that

$$\hat{P}_n^{\lambda}(x;a,b,c)=\sum_{k=0}^{n}\bar{a}(k,n)\hat{C}_n^{1}(x,c),\quad \text{where } \bar{a}(k,n)\geq 0. \tag{3.2}$$

As mentioned before, condition (2.3) is straightforwardly satisfied. To show that $\frac{1}{4}=\beta_k\geq\delta_n$, we only need to require that

$$\frac{1}{4}\geq\frac{(n+c)(n+2\lambda+c-1)}{4(n+\lambda+c)(n+\lambda+c-1)}\quad(\geq\delta_n).$$

After little computation, the just mentioned inequality can be simplified to to $\lambda(\lambda-1)\geq 0$, which suggests the condition $\lambda\geq 1$. This validates expansion (3.2).

Finally, we have

$$\left|\hat{P}_n^{\lambda}(x;a,b,c)\right|\leq\sum_{k=0}^{n}\bar{a}(k,n)\left|\hat{C}_n^{1}(x,c)\right|\leq\sum_{k=0}^{n}\bar{a}(k,n)\hat{C}_n^{1}(1,c)=\hat{P}_n^{\lambda}(1;a,b,c).$$

4 Proof of Theorem 1.2

To establish inequality (1.5), we first let $a_1=-n$, $a_2=c+\lambda+\mathrm{i}\Phi$, $b_1=c+1$, and $b_2=c+2\lambda$ in (2.12), and then in view of the defining expression (2.14) of the Laguerre polynomials, we obtain

$$\begin{aligned}{}_2F_2\left[\begin{matrix}-n,c+\lambda+\mathrm{i}\Phi\\ c+1,c+2\lambda\end{matrix};z\right]&=\frac{n!}{(c+1)_n}\frac{\Gamma(c+2\lambda)}{\Gamma(c+\lambda+\mathrm{i}\Phi)\Gamma(\lambda-\mathrm{i}\Phi)}\\&\quad\times\int_0^1 t^{c+\lambda+\mathrm{i}\Phi-1}(1-t)^{\lambda-\mathrm{i}\Phi-1}L_n^{(c)}(zt)\,\mathrm{d}t\end{aligned}$$

for $c>0$ and $\lambda>0$. Substituting this expression of ${}_2F_2$ into (2.10) and simplifying the resulting equation by using (2.8), we obtain

$$\begin{aligned}&P_n^{\lambda}(\cos\theta;a,b,c)\\&\quad=\frac{(c+2\lambda)_n}{(c+1)_n}\mathrm{e}^{\mathrm{i}n\theta}\\&\qquad\times\int_{(0,1)\times\mathbb{R}_+^2}\left[\int_0^1 L_n^{(c)}\big(tu_1u_3+u_2u_3\big(1-\mathrm{e}^{-2\mathrm{i}\theta}\big)\big)\,\mathrm{d}\mu_{c+\lambda+\mathrm{i}\Phi,\lambda-\mathrm{i}\Phi}(u_3)\right]\mathrm{d}\Pi(t,u_1,u_2),\end{aligned}\tag{4.1}$$

where $\mathrm{d}\Pi(t,u_1,u_2)$ is given by (2.9). This assertion means that the associated Pollaczek polynomials can alternatively be obtained by integrating the Laguerre polynomials.

Next, we choose $\alpha=c$, $\lambda=u_3$, and $x=tu_1+u_2(1-\mathrm{e}^{-2\mathrm{i}\theta})$ in (2.16), so that the Laguerre polynomials involved in (4.1) can be rewritten as

$$\begin{aligned}L_n^{(c)}\big(tu_1u_3+u_2u_3\big(1-\mathrm{e}^{-2\mathrm{i}\theta}\big)\big)&=\sum_{\ell=0}^{n}\binom{n}{\ell}u_3^{\ell}(1-u_3)^{n-\ell}\frac{L_n^{(c)}(0)}{L_\ell^{(c)}(0)}\\&\quad\times L_\ell^{(c)}\big(tu_1+u_2\big(1-\mathrm{e}^{-2\mathrm{i}\theta}\big)\big).\end{aligned}\tag{4.2}$$

Let us assume that $c = c_1 + c_2 + 1$ ($c_2 > 0$, $c_1 + 1 > 0$) for convenience. Then by using (2.15) we get

$$\begin{aligned} L_\ell^{(c)}\bigl(tu_1 + u_2\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)\bigr) &= L_\ell^{(c_2+c_1+1)}\bigl(u_2\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)+tu_1\bigr) \\ &= \sum_{m=0}^{\ell} L_m^{(c_2)}\bigl(u_2\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)\bigr)L_{\ell-m}^{(c_1)}(tu_1). \end{aligned} \tag{4.3}$$

Further, applying (2.16) to $L_m^{(c_2)}(u_2(1-\mathrm{e}^{-2\mathrm{i}\theta}))$, we have

$$L_m^{(c_2)}\bigl(u_2\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)\bigr) = \sum_{k=0}^{m}\binom{m}{k}\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)^k\mathrm{e}^{-2\mathrm{i}(m-k)\theta}\frac{L_m^{(c_2)}(0)}{L_k^{(c_2)}(0)}L_k^{(c_2)}(u_2), \tag{4.4}$$

and hence combining equations (4.2), (4.3), and (4.4) suitably, we obtain

$$\begin{aligned} &L_n^{(c)}\bigl(tu_1u_3 + u_2u_3\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)\bigr) \\ &\quad= \sum_{\ell=0}^{n}\binom{n}{\ell}u_3^\ell(1-u_3)^{n-\ell}\frac{L_n^{(c)}(0)}{L_\ell^{(c)}(0)} \\ &\qquad\times\sum_{m=0}^{\ell}L_{\ell-m}^{(c_1)}(tu_1)\sum_{k=0}^{m}\binom{m}{k}\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)^k\mathrm{e}^{-2\mathrm{i}(m-k)\theta}\frac{L_m^{(c_2)}(0)}{L_k^{(c_2)}(0)}L_k^{(c_2)}(u_2). \end{aligned} \tag{4.5}$$

From (4.5) we have

$$\begin{aligned} &\int_0^1 L_n^{(c)}\bigl(tu_1u_3 + u_2u_3\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)\bigr)\,\mathrm{d}\mu_{c+\lambda+\mathrm{i}\Phi,\lambda-\mathrm{i}\Phi}(u_3) \\ &\quad= \sum_{\ell=0}^{n}\binom{n}{\ell}\int_0^1 u_3^\ell(1-u_3)^{n-\ell}\,\mathrm{d}\mu_{c+\lambda+\mathrm{i}\Phi,\lambda-\mathrm{i}\Phi}(u_3)\frac{L_n^{(c)}(0)}{L_\ell^{(c)}(0)} \\ &\qquad\times\sum_{m=0}^{\ell}L_{\ell-m}^{(c_1)}(tu_1)\sum_{k=0}^{m}\binom{m}{k}\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)^k\mathrm{e}^{-2\mathrm{i}(m-k)\theta}\frac{L_m^{(c_2)}(0)}{L_k^{(c_2)}(0)}L_k^{(c_2)}(u_2) \\ &\quad= \frac{(c+1)_n}{(c+2\lambda)_n n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(c+\lambda+\mathrm{i}\Phi)_\ell(\lambda-\mathrm{i}\Phi)_{n-\ell}}{L_\ell^{(c)}(0)} \\ &\qquad\times\sum_{m=0}^{\ell}L_{\ell-m}^{(c_1)}(tu_1)\sum_{k=0}^{m}\binom{m}{k}\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)^k\mathrm{e}^{-2\mathrm{i}(m-k)\theta}\frac{L_m^{(c_2)}(0)}{L_k^{(c_2)}(0)}L_k^{(c_2)}(u_2). \end{aligned}$$

Thus

$$\begin{aligned} &P_n^\lambda(\cos\theta;a,b,c) \\ &\quad= \frac{\mathrm{e}^{\mathrm{i}n\theta}}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(c+\lambda+\mathrm{i}\Phi)_\ell(\lambda-\mathrm{i}\Phi)_{n-\ell}}{L_\ell^{(c)}(0)} \\ &\qquad\times\int_{(0,1)\times\mathbb{R}_+^2}\Biggl[\sum_{m=0}^{\ell}L_{\ell-m}^{(c_1)}(tu_1) \\ &\qquad\times\sum_{k=0}^{m}\binom{m}{k}\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)^k\mathrm{e}^{-2\mathrm{i}(m-k)\theta}\frac{L_m^{(c_2)}(0)}{L_k^{(c_2)}(0)}L_k^{(c_2)}(u_2)\Biggr]\,\mathrm{d}\Pi(t,u_1,u_2). \end{aligned}$$

Now we need to carry out some evaluations to obtain a more accurate estimate for the associated Pollaczek polynomials. It is easy to observe from [8, p. 810, Eq. (11)] that

$$\int_{\mathbb{R}_+} L_k^{(c_2)}(u_2)\,\mathrm{d}\mathbf{m}_1(u_2) = \int_{\mathbb{R}_+} \mathrm{e}^{-u_2} L_k^{(c_2)}(u_2)\,\mathrm{d}u_2 = \frac{(c_2)_k}{k!}.$$

Then we have

$$\begin{aligned} P_n^{\lambda}(\cos\theta;a,b,c) &= \frac{\mathrm{e}^{\mathrm{i}n\theta}}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(c+\lambda+\mathrm{i}\Phi)_\ell(\lambda-\mathrm{i}\Phi)_{n-\ell}}{L_\ell^{(c)}(0)} \\ &\quad\times\sum_{m=0}^{\ell}\frac{(c_2+1)_m}{m!}\int_{(0,1)\times\mathbb{R}_+} L_{\ell-m}^{(c_1)}(tu_1)\,\mathrm{d}\mu_{c,\lambda+\mathrm{i}\Phi}(t)\,\mathrm{d}\mathfrak{m}_{c+2\lambda-1}(u_1) \\ &\quad\times\sum_{k=0}^{m}\binom{m}{k}\bigl(1-\mathrm{e}^{-2\mathrm{i}\theta}\bigr)^k\mathrm{e}^{-2\mathrm{i}(m-k)\theta}\frac{(c_2)_k}{(c_2+1)_k}. \end{aligned}$$

Therefore it follows that

$$\begin{aligned} &\bigl|P_n^{\lambda}(\cos\theta;a,b,c)\bigr| \\ &\quad\le\frac{1}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{|(c+\lambda+\mathrm{i}\Phi)_\ell||(\lambda-\mathrm{i}\Phi)_{n-\ell}|}{L_\ell^{(c)}(0)} \\ &\quad\quad\times\sum_{m=0}^{\ell}\frac{(c_2+1)_m}{m!}\biggl|\int_{(0,1)\times\mathbb{R}_+} L_{\ell-m}^{(c_1)}(tu_1)\,\mathrm{d}\mu_{c,\lambda+\mathrm{i}\Phi}(t)\,\mathrm{d}\mathfrak{m}_{c+2\lambda-1}(u_1)\biggr| \\ &\quad\quad\times\sum_{k=0}^{m}\binom{m}{k}(2\sin\theta)^k\frac{(c_2)_k}{(c_2+1)_k} \\ &\quad\le\frac{1}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{|(c+\lambda+\mathrm{i}\Phi)_\ell||(\lambda-\mathrm{i}\Phi)_{n-\ell}|}{L_\ell^{(c)}(0)}\sum_{m=0}^{\ell}\frac{(c_2+1)_m}{m!}(1+2\sin\theta)^m \\ &\quad\quad\times\biggl|\int_{(0,1)\times\mathbb{R}_+} L_{\ell-m}^{(c_1)}(tu_1)\,\mathrm{d}\mu_{c,\lambda+\mathrm{i}\Phi}(t)\,\mathrm{d}\mathfrak{m}_{c+2\lambda-1}(u_1)\biggr|. \end{aligned}$$

To estimate

$$\biggl|\int_{(0,1)\times\mathbb{R}_+} L_{\ell-m}^{(c_1)}(tu_1)\,\mathrm{d}\mu_{c,\lambda+\mathrm{i}\Phi}(t)\,\mathrm{d}\mathfrak{m}_{c+2\lambda-1}(u_1)\biggr|,$$

we first define

$$H(t) := \int_{\mathbb{R}_+} L_{\ell-m}^{(c_1)}(tu_1)\,\mathrm{d}\mathbf{m}_{c+2\lambda-1}(u_1)$$

and observe that

$$\begin{aligned} \bigl|H(t)\bigr| &= \biggl|\int_{\mathbb{R}_+} L_{\ell-m}^{(c_1)}(tu_1)\,\mathrm{d}\mathbf{m}_{c+2\lambda-1}(u_1)\biggr| \\ &\le\frac{1}{\Gamma(c+2\lambda-1)}\int_{\mathbb{R}_+} u_1^{c+2\lambda-2}\mathrm{e}^{-u_1}\bigl|L_{\ell-m}^{(c_1)}(tu_1)\bigr|\,\mathrm{d}u_1. \end{aligned}$$

Using (2.18), we have

$$\begin{aligned}\left|H(t)\right| &\le \frac{1}{\Gamma(c+2\lambda-1)}\int_{\mathbb{R}_+} u_1^{c+2\lambda-2}\mathrm{e}^{-u_1}\sum_{k=0}^{\ell-m}\frac{(c_1+1)_{\ell-m-k}}{(\ell-m-k)!}\frac{(tu_1)^k}{k!}\,\mathrm{d}u_1\\ &= \sum_{k=0}^{\ell-m}\frac{(c_1+1)_{\ell-m-k}(c+2\lambda-1)_k}{(\ell-m-k)!}\frac{t^k}{k!}.\end{aligned}$$

Then

$$\begin{aligned}&\left|\int_{(0,1)\times\mathbb{R}_+} L_{\ell-m}^{(c_1)}(tu_1)\,\mathrm{d}\mu_{c,\lambda+\mathrm{i}\Phi}(t)\,\mathrm{d}\mathfrak{m}_{c+2\lambda-1}(u_1)\right|\\ &\quad= \left|\int_0^1 H(t)\,\mathrm{d}\mu_{c,\lambda+\mathrm{i}\Phi}(t)\right|\\ &\quad\le \left|\frac{\Gamma(c+\lambda+\mathrm{i}\Phi)}{\Gamma(c)\Gamma(\lambda+\mathrm{i}\Phi)}\right|\int_0^1\left|H(t)\right|t^{c-1}(1-t)^{\lambda-1}\,\mathrm{d}t\\ &\quad\le \left|\frac{\Gamma(c+\lambda+\mathrm{i}\Phi)}{\Gamma(c)\Gamma(\lambda+\mathrm{i}\Phi)}\right|\sum_{k=0}^{\ell-m}\frac{(c_1+1)_{\ell-m-k}(c+2\lambda-1)_k}{(\ell-m-k)!k!}\int_0^1 t^{c+k-1}(1-t)^{\lambda-1}\,\mathrm{d}t\\ &\quad= \frac{\Gamma(\lambda)}{\Gamma(c+\lambda)}\left|\frac{\Gamma(c+\lambda+\mathrm{i}\Phi)}{\Gamma(\lambda+\mathrm{i}\Phi)}\right|\sum_{k=0}^{\ell-m}\frac{(c_1+1)_{\ell-m-k}(c+2\lambda-1)_k}{(\ell-m-k)!k!}\frac{(c)_k}{(c+\lambda)_k}\\ &\quad\le \frac{\Gamma(\lambda)}{\Gamma(c+\lambda)}\left|\frac{\Gamma(c+\lambda+\mathrm{i}\Phi)}{\Gamma(\lambda+\mathrm{i}\Phi)}\right|\frac{1}{(\ell-m)!}\sum_{k=0}^{\ell-m}\binom{\ell-m}{k}(c_1+1)_{\ell-m-k}(c+2\lambda-1)_k\\ &\quad= \frac{\Gamma(\lambda)}{\Gamma(c+\lambda)}\left|\frac{\Gamma(c+\lambda+\mathrm{i}\Phi)}{\Gamma(\lambda+\mathrm{i}\Phi)}\right|\frac{(c+2\lambda+c_1)_{\ell-m}}{(\ell-m)!},\end{aligned}$$

where we have used the Chu–Vandermonde identity (2.13) to get the last equality.

Therefore we have

$$\begin{aligned}&\left|P_n^{\lambda}(\cos\theta;a,b,c)\right|\\ &\quad\le \frac{\Gamma(\lambda)}{\Gamma(c+\lambda)}\left|\frac{\Gamma(c+\lambda+\mathrm{i}\Phi)}{\Gamma(\lambda+\mathrm{i}\Phi)}\right|\frac{1}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{|(c+\lambda+\mathrm{i}\Phi)_\ell||(\lambda-\mathrm{i}\Phi)_{n-\ell}|}{L_\ell^{(c)}(0)}\\ &\qquad\times\sum_{m=0}^{\ell}\frac{(c_2+1)_m}{m!}\frac{(c+2\lambda+c_1)_{\ell-m}}{(\ell-m)!}(1+2\sin\theta)^m\\ &\quad\le \frac{\Gamma(\lambda)\Gamma(\lambda)}{|\Gamma(\lambda+\mathrm{i}\Phi)\Gamma(\lambda-\mathrm{i}\Phi)|}\frac{1}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(c+\lambda)_\ell(\lambda)_{n-\ell}}{L_\ell^{(c)}(0)}\\ &\qquad\times\sum_{m=0}^{\ell}\frac{(c_2+1)_m}{m!}\frac{(c+2\lambda+c_1)_{\ell-m}}{(\ell-m)!}(1+2\sin\theta)^m\\ &\quad\le \frac{\Gamma(\lambda)\Gamma(\lambda)}{|\Gamma(\lambda+\mathrm{i}\Phi)|^2}\frac{1}{n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(c+\lambda)_\ell(\lambda)_{n-\ell}}{(c+1)_\ell}(1+2\sin\theta)^\ell\\ &\qquad\times\sum_{m=0}^{\ell}\binom{\ell}{m}(c_2+1)_m(c+2\lambda+c_1)_{\ell-m}\end{aligned}$$

$$\le \frac{\Gamma(\lambda)\Gamma(\lambda)}{|\Gamma(\lambda+\mathrm{i}\Phi)|^2}\frac{1}{n!}(1+2\sin\theta)^n\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{(c+\lambda)_\ell(\lambda)_{n-\ell}}{(c+1)_\ell}(2c+2\lambda)_\ell$$

$$\le \frac{\Gamma(\lambda)\Gamma(\lambda)}{|\Gamma(\lambda+\mathrm{i}\Phi)|^2}\frac{(2c+2\lambda)_n}{(c+1)_n n!}(1+2\sin\theta)^n\sum_{\ell=0}^{n}\binom{n}{\ell}(c+\lambda)_\ell(\lambda)_{n-\ell}. \tag{4.6}$$

To verify the last inequality in (4.6), we prove that the sequence $\{(2c+2\lambda)_\ell/(c+1)_\ell\}$ is increasing for $\ell = 0, 1, \ldots, n$. Consider the function

$$f(x) := \log\frac{\Gamma(2c+2\lambda+x)}{\Gamma(c+1+x)}, \quad x \in [0,\infty).$$

Taking the derivative with respect to x, we obtain

$$f'(x) := \psi(2c+2\lambda+x) - \psi(c+1+x),$$

where $\psi(z)$ denotes the psi function (or digamma function). Since $c + 2\lambda > 1$, from the monotonicity of the psi function we have that $f'(x) > 0$. Thus

$$\frac{\Gamma(2c+2\lambda+x)}{\Gamma(c+1+x)},$$

as a function of x, is increasing on $[0,\infty)$. So the sequence

$$\frac{(2c+2\lambda)_\ell}{(c+1)_\ell} = \frac{\Gamma(c+1)}{\Gamma(2c+2\lambda)}\frac{\Gamma(2c+2\lambda+\ell)}{\Gamma(c+1+\ell)}$$

also increases on $\{0, 1, \ldots, n\}$.

The result (1.5) finally follows by the Chu–Vandermonde identity (2.13).

5 Remarks and observations

By letting $c = 0$ in Theorem 1.1 we obtain the following result:

Corollary 5.1 ([20, p. 4]) *For $a, b \ge 0$, $\lambda > 0$, and $x \in [-1,1]$, we have*

$$\left|P_n^\lambda(x;a,b)\right| \le P_n^\lambda(1;a,b). \tag{5.1}$$

Note that Yadav's inequality (5.1) generalizes Askey's inequalities obtained in [2] and [3].

(i) Note that inequality (5.1) can be equivalently written as

$$\left|P_n^\lambda(x;a,b)\right| \le L_n^{(2\lambda-1)}\bigl(-2(a+b)\bigr),$$

since $P_n^\lambda(1;a,b) = L_n^{(2\lambda-1)}(-2(a+b))$, where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials defined by (2.14). For the associated Pollaczek polynomials, we have ([19, p. 305])

$$P_n^\lambda(1;a,b,c) = L_n^{(2\lambda-1)}\bigl(-2(a+b);c\bigr),$$

where $L_n^{(\alpha)}(x;c)$ are *the associated Laguerre polynomials* (see, e.g., [4] and [19]). Therefore inequality (1.4) can be also expressed as

$$\left|P_n^\lambda(x;a,b,c)\right| \le L_n^{(2\lambda-1)}\bigl(-2(a+b);c\bigr). \tag{5.2}$$

Although these inequalities are quite elegant, they are actually somewhat difficult to use directly, especially for (5.2). The asymptotic behavior of $L_n^{(\alpha)}(x;c)$ for fixed $x>0$, although not applicable to our case, can be found in [4, p. 24, Eq. (2.16)].

(ii) Note that the associated ultraspherical (Gegenbauer) polynomials can be represented in terms of the associated Pollaczek polynomials as $C_n^\lambda(\cos\theta;c) = P_n^\lambda(\cos\theta;0,0,c)$. Hence Theorem 1.1 gives

$$\left|C_n^\lambda(\cos\theta;c)\right| \le \frac{(c+2\lambda)_n(2c+2\lambda)_n}{(c+1)_n n!}(1+2\sin\theta)^n,$$

which seems to be a new inequality for the associated ultraspherical polynomials.

(iii) Finally, we may mention that a more accurate (and much involved) upper bound than (2.18) for the Laguerre polynomials $L_n^{(\alpha)}(x)$ with $n\ge 2$ can be found in [15, p. 491, Theorem 1]. It is possible to obtain an improvement of Theorem 1.1, which may be not easy by using this inequality [15, p. 491, Theorem 1] because derivations would be quite complicated. However, we do not pursue it here but leave it as a worthwhile problem for the interested reader.

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Author details
[1]Department of Mathematics, Donghua University, Shanghai, People's Republic of China. [2]Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai, People's Republic of China. [3]*Present address:* M.P. University of Agriculture and Technology, Udaipur, India.

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