June, 1981 LIDS-P-1101

TWO INTERFERING QUEUES IN PACKET-RADIO NETWORKS

M. Sidi and A. Segall

Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa, Israel

ABSTRACT

We consider several classes of interfering queues that appear in packet-radio networks. We analyse the class of systems where one of the queues is given full priority and obtain an expression for the joint probability distribution of the queue lengths. For ALOHA-type systems with two symmetric queues we calculate the average packet waiting time and queue lengths and for symmetric systems with an arbitrary number of subscribers we develop a method to approximate these quantities. The approximation turns out to be close to the analysis and simulation results.

The work of A. Segall was conducted on a consulting agreement with the Laboratory for Information and Decision Systems at MIT, Cambridge, Mass., U.S.A., with partial support provided by the Advanced Research Project Agency of the US Department of Defense (monitored by ONR) under contract No. N00014-75-C-1183.

This paper has been submitted to the IEEE Transactions on Communications.

1. Introduction

The present study was motivated by the problem of investigating the behavior of random-multiple-access systems and of packet-radio networks. These systems are characterized by the fact that a number of radio stations exchange digital information by using a distributed random access algorithm on a common radio channel. In such situations, whenever a given station attempts transmission of a packet to another station, the attempt may be unsuccessful, in which case the packet must be retransmitted. In addition to channel noise, unsuccessful transmissions occur because of interference from another station trying to send a packet over the common channel at the same time or by the fact that the intended receiver is itself in transmission mode, in which case it is not able to detect incoming packets. The fact that the activity at one node affects the behavior of the queue at other nearby stations gives rise to statistical dependence between the queues at the nodes.

Unfortunately, in general, the queue length statistical dependence is quite complicated and there is little hope to obtain explicit analytical results for general topology networks. The purpose of this paper is to present several analytic as well as approximation results for certain classes of interfering queues. We assume throughout the paper that all packets have equal length and that the time is divided in slots corresponding to the transmission time of a packet. A station may start packet transmissions only at the beginning of a slot and the distances between stations are assumed to be such that propagation delay is negligible. Also, we neglect channel noise and assume no channel errors.

Because of the difficulty in the analysis of dependent queues of the type introduced above, even the case of two queues cannot be treated analytically in the general case. However, we consider here two classes of systems with two dependent queues where such results can be obtained. The first is the case when the length of one of the two queues is not allowed to decrease, unless the other queue is empty. A two node ALOHA system where one of the nodes is given full priority is an example for such a situation; other examples are given in Sec. 2. No other restrictions are necessary in order to allow for analytical solution of this class; in particular, the inputs may have arbitrary distributions and need not be independent processes. For this class of problems we present a general method for deriving the generating functions of the queue lengths and of the average delay times. Then these general results are applied to three special cases of packet-radio networks that can be shown to belong to the considered class of systems.

The second class of problems for which we can obtain explicit analytical results is the case of a two node symmetric ALOHA system. For this situation we cannot obtain the queue length probability distribution (or generating function), but we give a method for calculating the average queue length and hence the average time delay. The results are given in Sec. 3.

Since, as said before, explicit analytical results are hard to obtain for more general situations, another way to approach the problem is to obtain good approximations. An approximation method applicable to a symmetric ALOHA-type system with arbitrary number of stations is introduced in Sec. 4. We obtain there the approximate average queueing delay in such systems and compare this with the exact result for two node networks and with simulation results for larger networks.

Discrete time systems involving interfering queues have been rarely treated in the literature. In [2] a loop system using Asynchronous Time-Division Multiplexing has been analysed. In this system user i may use a slot for transmission only if all users $1,2,\ldots,i-1$ have nothing to transmit. In [3], the author investigates the case of two queues in tandem where each queue always attempts transmission provided it has a message in the buffer. Both systems considered in [2] (for two users) and [3] are cases that belong to the first general class of systems considered in the present paper. In [4] another system of two interfering queues is considered, whereby only one event (i.e. an arrival or departure) may occur during a given slot.

This system has been shown to have the product form solution. Finally, we may mention [5] and [6], where a slotted ALOHA network with finite number of users has been examined and a method was suggested for obtaining an approximate solution for this system.

2. Analytical Results

In this section we consider a class of discrete-time queueing systems consisting of two queues with the following properties : packets arrive randomly at the queues from two sources, that in general may be correlated. Let $A_1(t)$ and $A_2(t)$ be the number of packets entering node 1 and node 2 from their corresponding sources in the time interval (t, t+1]. The input process $[(A_1(t), A_2(t)]$ is assumed to be a sequence of independent identically distributed random vectors with integer-valued elements. Let

$$a(i,j) = \mathrm{Prob}(A_1(t)=i, A_2(t)=j) \ ; \ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a(i,j) = 1 \ . \tag{1}$$

and

$$F(x,y) = E[x^{A1(t)} y^{A2(t)}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a(i,j) x^i y^i \tag{2}$$

We assume that $F(x,y)$ cannot be x-independent, namely that packets arrive at the first queue with nonzero probability and that the queues have infinite buffers.

Next, we describe the departure processes. It is assumed that no more than one packet may leave each queue in any given time slot and the combined departure process is taken to be as follows : When both queues are empty, no departures may occur (packets arriving during a given slot may depart only in the next one). When only one of the queues is nonempty, a departure from that queue may occur and the packet may be transferred either to the outside of the system or to the other queue. We denote by p_{01}^0, p_{01}^1 the respective probabilities when the nonempty queue is queue 2 and by p_{10}^0, p_{10}^1 the corresponding probabilities when the nonempty queue is queue 1. Clearly $1 - p_{01}^0 - p_{01}^1$ is the probability that no departure occurs from the nonempty queue 2. The specific class of dependent queues considered in this section is characterized by an assumption on departures when both queues are nonempty. For this case it is assumed that a departure may occur only from queue 2. We denote by p_{11}^0, p_{11}^1 the probabilities that the departing packet leaves the system or joins the other queue (queue 1) respectively.

Consider the steady-state joint generating function of the queue lengths :

$$G(x,y) = \lim_{t\to\infty} E\left[x^{L_1(t)} y^{L_2(t)}\right] \tag{3}$$

where $L_1(t)$, $L_2(t)$ are the queue lengths at time t at nodes 1 and 2 respectively, and where we assume that the Markov chain $[L_1(t), L_2(t)]$ is ergodic, namely $G(0,0)>0$. For the class under consideration we can compute the function $G(x,y)$ and in the Appendix it is shown that $G(x,y)$ has the following form :

$$G(x,y) = F(x,y) \bullet \frac{b(x,y)G(x,0) + c(x,y)G(0,y) + d(x,y)G(0,0)}{x \cdot e(x,y)} \tag{4}$$

where the functions $b(x,y)$, $c(x,y)$, $d(x,y)$, $e(x,y)$, $G(y,0)$, $G(0,y)$ and the constant $G(0,0)$ are defined in the Appendix.

This general form can be made more explicit for certain special cases. We next consider several examples of two node packet networks, where it turns out that the general assumptions, given earlier, characterizing the class of two dependent queues indeed hold. The networks under consideration are given in Fig. 1. In all cases the nodes share a common radio channel and are equipped with radio transmitter devices and in systems 2 and 3, node 2 has also a receiving device. Node 2 can either transmit or receive, but not simultaneously. The circle in Fig. 1 represents a station equipped with a radio receiver that receives packets correctly provided that there is no interference. Finally, instantaneous feedback to the transmitter is assumed, meaning that a transmitter knows at the end of the slot if the packet has been received correctly.

In all three systems of Fig. 1 node 2 is assumed to have full access capability to the common channel. This means that it always transmit a packet when its buffer is not empty, while if its buffer is empty the node does not transmit and in systems 2 and 3 it is able to receive packets transmitted by node 1. Node 1 has only partial access capability to the channel and its transmission policy is randomized as follows :

At the beginning of each slot for which its own buffer is nonempty, node 1 tosses a coin with probability of success p, independently of any other event in the system and in case of success the node attempts to transmit the packet at the head of the queue. Both nodes are able to detect at the end of the slot if their transmissions were successful. At any node, if the transmission is not successful, either because the packet was sent to the other node while that node was not ready to receive it, or because of interference with a packet transmitted by the other node, the transmitter repeats the procedure described above.

Since node 2 has full access capability to the channel and node 2 cannot receive and transmit packets at the same time, it is clear that in all cases no packets can leave node 1 whenever the queue at node 2 is nonempty and therefore all cases of Fig. 1 belong to the class of queues considered earlier in this section.

We now turn to calculate the parameters $\{p_{ij}^k, 0 \leq i,j,k \leq 1, i+j>0\}$ in each of these three systems. System 1 depicted in Fig. 1(a) represents a two node non-symmetric ALOHA network, where both nodes send their packets to the station. Since no packets are sent from one node to the other we have $p_{10}^1 = p_{01}^1 = p_{11}^1 = 0$. When one of the nodes has packets to transmit while the other is empty, any attempted transmission is successful. Since node 2 transmits with probability 1 and node 1 with probability p, we have $p_{10}^0 = p$; $p_{01}^0 = 1$. When both nodes have nonempty queues, successful transmission occurs at node 2 whenever node 1 does not attempt transmission and therefore[1] $p_{11}^0 = \bar{p}$. System 2, depicted in Fig. 1(b) represents a situation of two tandem nodes where the station that is the "sink" for the packets transmitted by node 2, is out of the transmission range of node 1. Therefore node 1 cannot interfere with the transmissions of node 2. However node 2 does interfere with the transmissions of node 1 since when it is transmitting, it does not accept packets transmitted by node 1. Consequently $p_{10}^1 = p$; $p_{01}^0 = p_{11}^0 = 1$; $p_{10}^0 = p_{01}^1 = p_{11}^1 = 0$. System 3, depicted in Fig. 1(c) differs from system 2 only in that the station is in the transmission range of node 1, therefore node 1 does interfere with the transmissions of node 2 in this case, and therefore the parameters are $p_{10}^1 = p$; $p_{01}^0 = 1$; $p_{11}^0 = \bar{p}$; $p_{10}^0 = p_{01}^1 = p_{11}^1 = 0$.

Numerical Results

Although the results of this section hold for general input processes, the equations become much simpler when one considers independent Bernoulli processes. In this case we have

$$F(x,y) = (xr_1 + \bar{r}_1)(yr_2 + \bar{r}_2) \tag{5}$$

where r_1, r_2 are the input rates. For this example we calculate for each of the three systems the average delays (in units of slots) T_1, T_2 at nodes 1 and 2 respectively and the total average delay T in the network. This is done by first calculating the average queue lengths at the nodes and then applying Little's Theorem [7]. After straightforward but tedious algebra the following results are obtained :

System 1 (Fig. 1(a))

$$T_1 = 1 + \frac{(\bar{p})^2 + r_2 p}{p(\bar{p}-r_2) - r_1\bar{p}} + \frac{r_1 r_2 p\bar{p}}{(\bar{p}-r_2)^2[p(\bar{p}-r_2)-r_1\bar{p}]} \tag{6}$$

$$T_2 = 1 + \frac{r_1\bar{p}}{(\bar{p}-r_2)^2} \tag{7}$$

and

$$T = \frac{r_1 T_1 + r_2 T_2}{r_1 + r_2} \tag{8}$$

where these equations hold for $p(\bar{p}-r_2) > r_1\bar{p}$ which is the ergodicity condition in this system.

In Fig. 2,3,4 we plot T_1, T_2 and T respectively versus p, the transmission probability at node 1, for $r_1 = 0.1$ and r_2 ranges from 0.01 to 0.4. As expected the average delay at node 2 increases when p increases since then its transmissions are more frequently interfered with transmissions from node 1. More interesting is the behavior of the average delay at node 1. Here p has some value for which T_1 is minimized (for given r_1 and r_2). When p either increases or decreases from this value T_1 increases. The reason is that when p becomes small, node 1 attempts to transmit relatively rarely, so its queue increases. When p becomes large, then node 1 attempts to transmit more frequently, thus interfering with the transmissions of node 2, and the queue lengths at both nodes are large. As we see from Fig. 4 the parameter p is a very critical design parameter of this system and for given values for r_1 and r_2, there exists an optimal p, denoted by p^*, that minimizes the total average delay in the network. In Fig. 5 p^* is plotted versus r_1 for various values of r_2. Notice that p^* is much less sensitive to changes in r_1 than to changes in r_2. When r_2 is small, then p^* should be large for all values of r_1, since interference between transmissions is rare. When r_2 increases, p^* should decrease in order to reduce the interference. Finally, in Fig. 6, T_{min} the minimum total average delay is plotted versus γ the total throughput of the system, when $r_1 = r_2 = r$ (clearly $\gamma = 2r$).

System 2 (Fig. 1(b))

The average delays are :

$$T_1 = 1 + \frac{r_1 p + \bar{r}_2(1-p\bar{r}_2)}{\bar{r}_2[p(1-r_1-r_2)-r_1]} \tag{9}$$

$$T_2 = \frac{1}{r_1+r_2}\left(r_2 + \frac{r_1}{1-r_2}\right) \tag{10}$$

and

$$T = \frac{r_1}{r_1+r_2} T_1 + T_2 \tag{11}$$

where these equations hold for $p(1-r_1-r_2) > r_1$ which is the ergodicity condition for this system.

In this system node 1 does not interfere with the transmissions of node 2. Therefore, it is optimal to always attempt transmission at node 1 as well, namely to take $p = 1$. Eq. (9) indeed shows that T_1 is monotonically decreasing when p increases, and it achieves its minimum value for $p = 1$. In Fig. 7,8 T_1 and T are plotted respectively versus r_1 for $p = 1$ when r_2 ranges from 0.01 to 0.8. T_2 is not plotted here since it takes values from 1 to 1.5 only when $0.01 \le r_2 \le 0.8$, $0 < r_1 < 0.5$. From Fig. 7 we see, as expected, that T_1 increases when the arrival rate r_1 to node 1 increases. When r_2 increases, then the time spent by node 2 in the transmitting mode increases. Therefore, packets transmitted by node 1 are rarely received by node 2 and the average time delay T_1 at node 1, increases in this case as seen in Fig. 7.

System 3 (Fig. 1(c))

$$T_1 = 1 + \frac{p(r_1+r_2\bar{r}_2) + (\bar{p}-r_2)(\bar{p})^2}{[p(\bar{p}-r_2)-r_1](\bar{p}-r_2)} - \frac{p[r_1+r_2(r_1+\bar{r}_1\bar{r}_2)]}{[\bar{r}_1(\bar{r}_2)^2-p(1-\bar{r}_1r_2)](\bar{p}-r_2)} \tag{12}$$

$$T_2 = \frac{1}{r_1+r_2}\left\{\frac{r_1+r_2\bar{r}_2}{\bar{p}-r_2} - \frac{p[r_1+r_2(r_1+\bar{r}_1\bar{r}_2)][p(\bar{p}-r_2)-r_1]}{(\bar{p}-r_2)[\bar{r}_1(\bar{r}_2)^2-p(1-\bar{r}_1r_2)]}\right\} \tag{13}$$

and

$$T = \frac{r_1}{r_1+r_2}T_1 + T_2 \tag{14}$$

where these equations hold for $p(\bar{p}-r_2) - r_1 > 0$ which is the ergodicity condition for this system. Similarily to System 1, T_1, T_2 and T are plotted in Fig. 9,10,11 for System 3. The behavior of the average delay in this system is very similar to that of System 1, therefore we will not give here more detailed explanations.

3. Symmetric Two-Node ALOHA Network Delay Analysis

In this section a two node symmetric ALOHA network is considered. This network is similar to System 1 with the following modifications. Here node 2 uses the same channel access scheme as node 1, i.e. at the beginning of each slot, if its buffer is not empty, node 2 tosses a coin, independently from any other event in the system, with probability of success p. According to the outcome, the node either transmits or remains silent during the current slot. In addition, it is assumed that the arrival processes to the nodes are independent Bernoulli processes with equal rates, denoted by $0<r<1/4$. Therefore we have for this system $F(x,y) = (xr+\bar{r})(yr+\bar{r})$. By the same method used in the Appendix it is easy to see that for the current system we have :

$$G(x,y)=F(x,y)p\frac{[x(y-1)-p(2xy-x-y)]G(x,0)+[y(x-1)-p(2xy-x-y)]G(0,y)+p[2xy-x-y]G(0,0)}{xy-F(x,y)[(x+y)p\bar{p}+xy(p^2+(\bar{p})^2)]} \tag{15}$$

In this case we cannot obtain an explicit form for $G(x,0)$, $G(0,y)$, $G(0,0)$ and hence for $G(x,y)$. However we can exploit the symmetry to obtain an expression for the average delay in the system. If we denote by $G_1(x,y)$, $G_2(x,y)$ the derivative of $G(x,y)$ with respect to x and y respectively, we clearly have $G_1(1,1) = G_2(1,1)$ and $G_1(1,0) = G_2(0,1)$. Then from (15) we obtain

$$G_1(1,1) = r + \frac{r[p+(\bar{p})^2] - p^2G_1(1,0)}{p\bar{p} - r} \tag{16}$$

and

$$\frac{d}{dx}[G(x,x)]\bigg|_{x=1} = 2r + \frac{G_1(1,0)p(1-2p)}{p\bar{p} - r} + \frac{r^2+2r-4rp\bar{p}}{2(p\bar{p} - r)} \tag{17}$$

for $p\bar{p} > r$.

Now, if we use the fact that

$$\frac{d}{dx}[G(x,x)]\bigg|_{x=1} = G_1(1,1) + G_2(1,1) = 2G_1(1,1) \tag{18}$$

we can solve for $G_1(1,0)$ and hence for $G_1(1,1)$ and obtain

$$G_1(1,1) = G_2(1,1) = r + r\frac{(\bar{p})^2 + \frac{1}{2}rp}{p\bar{p} - r} \tag{19}$$

Therefore, applying Little's Theorem we obtain the average system delay

$$T = \frac{G_1(1,1)}{r} = 1 + \frac{(\bar{p})^2 + \frac{1}{2}rp}{p\bar{p} - r} \quad \text{for} \quad p\bar{p} > r . \tag{20}$$

From (20) it is found that $p^* = 1-\{0.5r + [0.5r(1-r+0.5r^2)]^{1/2}\}/(1-0.5r)$ minimizes T for $0<r<0.25$. In Fig. 6 the minimum total average delay T_{min} is plotted versus γ, the total throughput of this system. Comparing the curves in Fig. 6 it is clear that the non-symmetric access scheme used in system 1 provides very slight better performance than the symmetric access scheme, when the arrival rates into the nodes are equal (the difference in the minimum delay is less than 3.5% in the range $0<\gamma<0.5$). However the former scheme is unfair, giving priority to queue 2, although the arrival rates into the two queues are the same.

Finally we may mention that the method presented in this section for calculating average delay times without obtaining explicitly the generating functions, can be used in many other symmetric (i.e. $G(x,y) = G(y,x)$) two node systems. Specifically, we can easily obtain average delay time in the symmetric two nodes ALOHA network for general arrival processes into the nodes for which $F(x,y) = F(y,x)$.

4. Symmetric ALOHA Network with M Nodes : Approximate delay

Consider an ALOHA network consisting of M nodes, that share a common radio channel. Assume that all nodes in the network use the same channel access scheme as the two nodes in the previous section, i.e. each node tosses independently a coin with the same probability of success p at the beginning of each slot when its queue is nonempty. Also assume that the arrival process at each node is a Bernoulli process, independent from node to node, with rate r. For this case we cannot obtain the exact average delay for $M>2$, and therefore we must consider approximations. The approximation method proposed here is the following. Each node i in the network may at any time be in one of M possible situations. Situation j refers to the case where $j-1$ nodes other than i have nonempty queues, while the other $M-j$ nodes (not including i) are empty. The approximation considered here consists of assuming that in steady state, while in situation j, node i behaves as a discrete M/M/1 queue with arrival parameter r and departure parameter $p(1-p)^{j-1}$. If the transitions between the various situations are neglected then the average number of packets at node i denoted by L_i is :

$$L_i = \sum_{j=1}^{M} \theta_j \frac{r(1-r)}{p(1-p)^{j-1}-r} \tag{21}$$

for $i=1,2,\ldots,M$ and $r<p(1-p)^{M-1}$, where θ_j is the probability of being in situation j. We approximate θ_j as

$$\theta_j = (1 - \frac{r}{p})^{M-j}(\frac{r}{p})^{j-1} \qquad j = 1,2,\ldots,M \tag{22}$$

where $\frac{r}{p}$ approximates the probability that a node has packets ready for transmission and we assume independence between the nodes. Using (21) and (22) and applying Little's Theorem we find that the approximate total average delay T_{ap} is given by :

$$T_{ap} = \sum_{j=1}^{M} (\frac{r}{p})^{j-1}(1 - \frac{r}{p})^{M-j} \frac{1-r}{p(1-p)^{j-1}-r} \tag{23}$$

The general formula (23) can be specialized to the case M=2, and for this case we can compare the approximate and exact results. For M=2 we have from (20)

$$T_{analysis} = 1 + \frac{(\bar{p})^2 + \frac{1}{2} rp}{p\bar{p} - r} \tag{24}$$

and from (23)

$$T_{ap} = 1 + \frac{(\bar{p})^2 + rp}{p\bar{p} - r} \tag{25}$$

T_{ap} and $T_{analysis}$ are plotted versus p in Fig. 12, for various values of r. It can be seen that although we have used a simple approximation, it is quite close to the exact values. We also compare this approximation versus simulation results, for networks having three and four nodes. This comparison is plotted in Fig. 13 and 14 and shows again good results.

References

[1] H. Kobayashi and A.G. Konheim, Queueing Models for Computer Communications System Analysis, IEEE Trans. Communications, Vol. COM-25, pp.2-28, Jan. 1977.

[2] A.G. Konheim and B. Meister, Service in a Loop System, J. Ass. Comput. Mach., Vol. 19, pp. 92-108, Jan. 1972.

[3] J.A. Morrison, Two Discrete-Time Queues in Tandem, IEEE Trans. Communications, Vol. COM-27, pp. 563-573, March 1979.

[4] L. Kleinrock and Y. Yemini, Interfering Queueing Processes in Packet-Switched Broadcast Communication, IFIP Congress, Tokyo, 1980.

[5] A. Ephremides and T.N. Saadawi, The Analysis of Interacting Buffered Terminals in a Random Access Channel, preprint.

[6] T.N. Saadawi and A. Ephremides, Analysis, Stability and Optimization of Slotted ALOHA with Finite Number of Buffered Users, preprint.

[7] J.D.C. Little, A Proof for the Queueing Formula $L = \lambda W$, Op. Res. 9, pp. 383-387, 1961.

[8] E.T. Copson, Theory of Functions of a Complex Variable, London : Oxford University Press, 1948.

Footnotes

1. We use the notation $\bar{v} = 1-v$.

2. The subscripts 1 and 2 correspond to the derivative of the function with respect to the first and second variables respectively.

Appendix A

In order to describe the behavior of the queue lengths, we need several definitions. Let $D_{01}^{0}(t)$ be a binary-valued random variable that takes value 1 if $L_1(t) = 0$, $L_2(t) > 0$ ($L_1(t)$ and $L_2(t)$ are the queue lengths at time t at nodes 1 and 2 respectively), a departure occurs (from queue 2) and the packet leaves the system. Similarly $D_{01}^{1}(t)$ takes value 1 under the same conditions, except that the packet is transferred to queue 1. In a similar way we define the binary-valued variables $D_{10}^{0}(t)$, $D_{10}^{1}(t)$ for the case that $L_1(t)>0$, $L_2(t)=0$ and $D_{11}^{0}(t)$, $D_{11}^{1}(t)$ for the situation when both queues are nonempty and departure occurs from queue 2. Then the recursive equations for $L_i(t)$ are :

$$L_1(t+1) = \begin{cases} A_1(t) & \text{if} \quad L_1(t)=0,\ L_2(t)=0 \\ A_1(t)+L_1(t)-D_{10}^{0}(t)-D_{10}^{1}(t) & \text{if} \quad L_1(t)>0,\ L_2(t)=0 \\ A_1(t)+D_{01}^{1}(t) & \text{if} \quad L_1(t)=0,\ L_2(t)>0 \\ A_1(t)+L_1(t)+D_{11}^{1}(t) & \text{if} \quad L_1(t)>0,\ L_2(t)>0 \end{cases} \tag{A1}$$

$$L_2(t+1) = \begin{cases} A_2(t) & \text{if} \quad L_1(t)=0,\ L_2(t)=0 \\ A_2(t)+D_{10}^{1}(t) & \text{if} \quad L_1(t)>0,\ L_2(t)=0 \\ A_2(t)+L_2(t)-D_{01}^{0}(t)-D_{01}^{1}(t) & \text{if} \quad L_1(t)=0,\ L_2(t)>0 \\ A_2(t)+L_2(t)-D_{11}^{0}(t)-D_{11}^{1}(t) & \text{if} \quad L_1(t)>0,\ L_2(t)>0 \end{cases} \tag{A2}$$

From (A1), (A2) and (3) we have for $t\to\infty$ that

$$\begin{aligned} G(x,y) = F(x,y)\{G(0,0)&+[G(x,0)-G(0,0)][x^{-1}p_{10}^{0}+x^{-1}yp_{10}^{1}+\overline{(p_{10}^{0}+p_{10}^{1})}] + \\ &+ [G(0,y)-G(0,0)][y^{-1}p_{01}^{0}+y^{-1}xp_{01}^{1} + \overline{(p_{01}^{0}+p_{01}^{1})}] + \\ &+ [G(x,y)-G(x,0)-G(0,y)+G(0,0)][y^{-1}p_{11}^{0}+y^{-1}xp_{11}^{1} + \overline{(p_{11}^{0}+p_{11}^{1})}]\} \end{aligned} \tag{A3}$$

(Remember that $\bar{v}$ denotes $1-v$).

Arranging (A3) we obtain

$$G(x,y) = F(x,y)\,\frac{b(x,y)G(x,0)+c(x,y)G(0,y)+d(x,y)G(0,0)}{x\;e(x,y)} \tag{A4}$$

where

$$b(x,y) = y(p^0_{10}+yp^1_{10}) - x(p^0_{11}+xp^1_{11})+xy(p^0_{11}+p^1_{11}-p^0_{10}-p^1_{10}) \tag{A5}$$

$$c(x,y) = x[p^0_{01}-p^0_{11}+x(p^1_{01}-p^1_{11})] + xy(p^0_{11}+p^1_{11}-p^0_{01}-p^1_{01}) \tag{A6}$$

$$d(x,y) =-y(p^0_{10}+yp^1_{10})+x[p^0_{11}-p^0_{01}+x(p^1_{11}-p^1_{01})] + xy(p^0_{10}+p^1_{10}+p^0_{01}+p^1_{01}-p^0_{11}-p^1_{11})$$

$$e(x,y) = y-F(x,y)[p^0_{11}+xp^1_{11}+y(\overline{p^0_{11}+p^1_{11}})] \tag{A8}$$

From (A4) we see that the steady-state generating functions for the queues' lengths are :

For queue 1 :

$$G(x,1) = F(x,1)\,\frac{b(x,1)G(x,0)+c(x,1)G(0,1)+d(x,1)G(0,0)}{x\;e(x,1)} \tag{A9}$$

For queue 2 :

$$G(1,y) = F(1,y)\,\frac{b(1,y)G(1,0)+c(1,y)G(0,y)+d(1,y)G(0,0)}{e(1,y)} \tag{A10}$$

We still have to determine the functions $G(x,0)$, $G(0,y)$ and the constant $G(0,0)$. To find $G(0,y)$ let $x\to 0$ in (A3). Then

$$\begin{aligned} G(0,y) = F(0,y)\{&G(0,0)+G_1(0,0)(p^0_{10}+yp^1_{10}) + \\ &+ [G(0,y)-G(0,0)][y^{-1}p^0_{01}+ (\overline{p^0_{01}+p^1_{01}})]\} \end{aligned} \tag{A11}$$

where $G_1(x,y)$ is the derivative of $G(x,y)$ with respect to x.

From (A11) we obtain

$$G(0,y) = F(0,y)\,\frac{(p^0_{01}+p^1_{01}-y^{-1}p^0_{01})G(0,0)+(p^0_{10}+yp^1_{10})G_1(0,0)}{1 - F(0,y)[y^{-1}p^0_{01}+(\overline{p^0_{01}+p^1_{01}})]} \tag{A12}$$

Before proceeding we need to prove the following Lemma.

Lemma

For $|x|<1$ the equation

$$e(x,y) = 0$$

where $e(x,y)$ is the expression of (A8) has a unique solution within the unit circle $|y|=1$.

Proof

Since $F(x,y)$ is not independent of x, then there exists $a(i,j)>0$ for some i and some $j>0$. Therefore for $|x|<1$ and $|y| = 1$ we have

$$|F(x,y)[p_{11}^0+p_{11}^1+y\ \overline{(p_{11}^0+p_{11}^1)}]| \leqslant$$

$$\leqslant\ |F(x,y)| = |\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a(i,j)x^i y^j| \leqslant \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a(i,j)|x|^i < \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (a(i,j)=1=|y| \qquad \text{(A13)}$$

Hence, applying Rouchês' theorem [8], $e(x,y)$ has exactly one zero within $|y|=1$ for $|x| < 1$.

Q.E.D.

Now let t be the solution of

$$F(0,t)[t^{-1}p_{01}^0 + \overline{(p_{01}^0+p_{01}^1)}] = 1 \qquad \text{(A14)}$$

in the unit circle. Using the Lemma for $x=0$, it is clear that t is unique. Then from the analyticity of $G(0,y)$ for $|y|< 1$ it follows that

$$G_1(0,0) = \frac{t^{-1}p_{01}^0-p_{01}^0-p_{01}^1}{p_{10}^0+tp_{10}^1}\,G(0,0)\quad . \qquad \text{(A15)}$$

Substituting (A15) in (A12), $G(0,y)$ is determined up to the constant $G(0,0)$. To find $G(x,0)$ let $f=f(x)$, be the solution for $|x| < 1$ of

$$e(x,f) = 0 \qquad \text{(A16)}$$

in the circle $|f| = 1$. In the lemma it was proved that such a solution f is unique.

Then again from the analyticity of $G(x,y)$ for $|x| < 1$ it follows that

$$G(x,0) = -\frac{c(x,f)G(0,f)+d(x,f)G(0,0)}{b(x,f)} \tag{A17}$$

Substituting (A12) in (A17), the function $G(x,0)$ is determined up to the constant $G(0,0)$. To find $G(0,0)$ let $x\to 1$ in (A9) and $y\to 1$ in (A10) and use the normalization condition $G(1,1) = 1$ to obtain[2] :

$$e_1(1,1)=b_1(1,1)G(1,0)+c_1(1,1)G(0,1)+d_1(1,1)G(0,0) \tag{A18}$$

and

$$e_2(1,1)=b_2(1,1)G(1,0)+c_2(1,1)G(0,1)+d_2(1,1)G(0,0) \tag{A19}$$

and from (A12) we have

$$G(0,1) = F(0,1)\,\frac{p_{01}^{1}G(0,0)+(p_{10}^{0}+p_{10}^{1})G_1(0,0)}{1 - F(0,1)\overline{p_{01}^{1}}} \tag{A20}$$

Solving (A15), (A18),(A19) and (A20), we obtain $G(0,0)$. The ergodicity condition is that $G(0,0) > 0$.

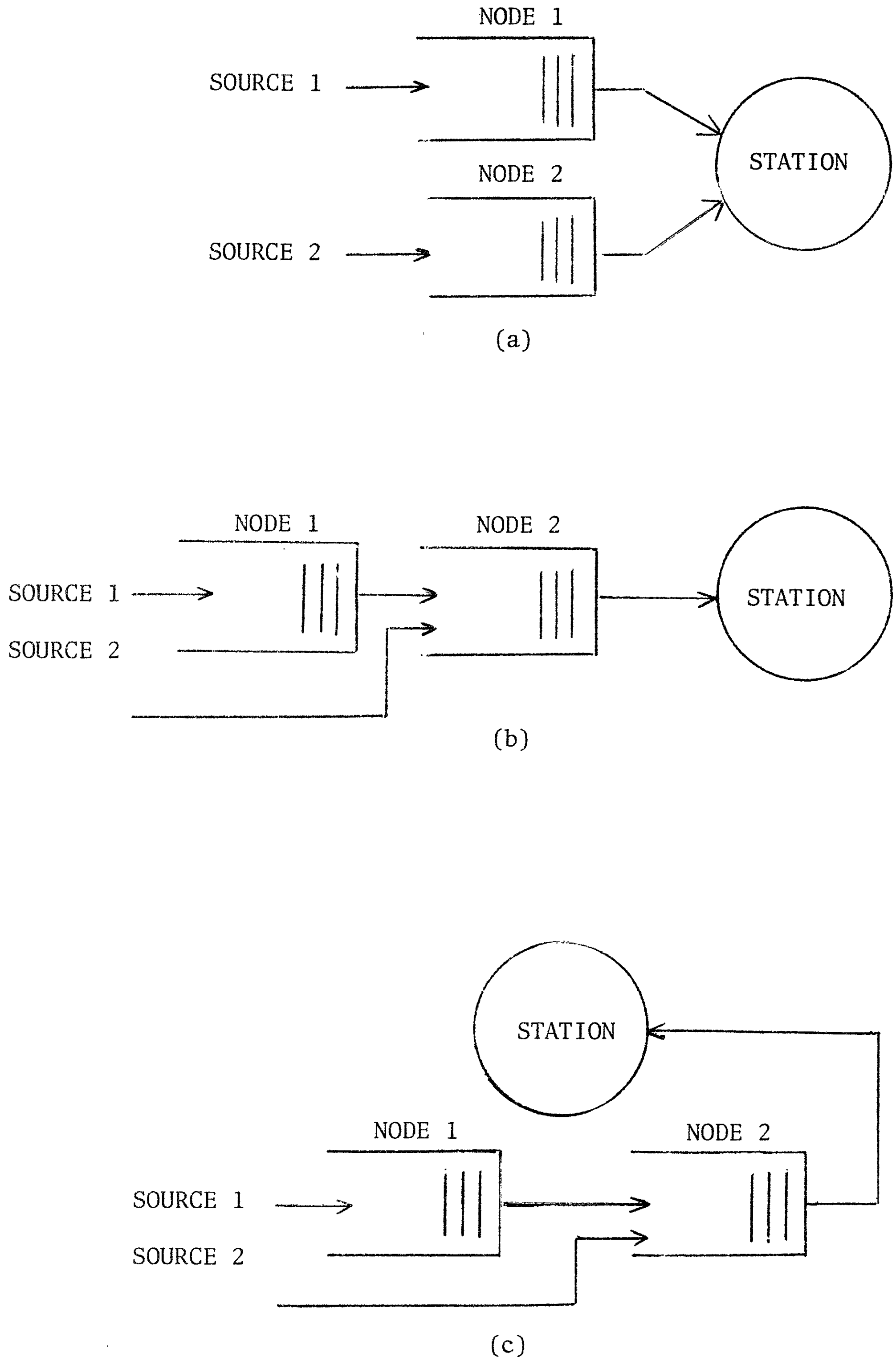

Fig. 1 - Two-node networks. (a) System 1: non-symmetric ALOHA network.
(b) System 2: tandem network; no interference at the station.
(c) System 3: tandem network; interference at the station.

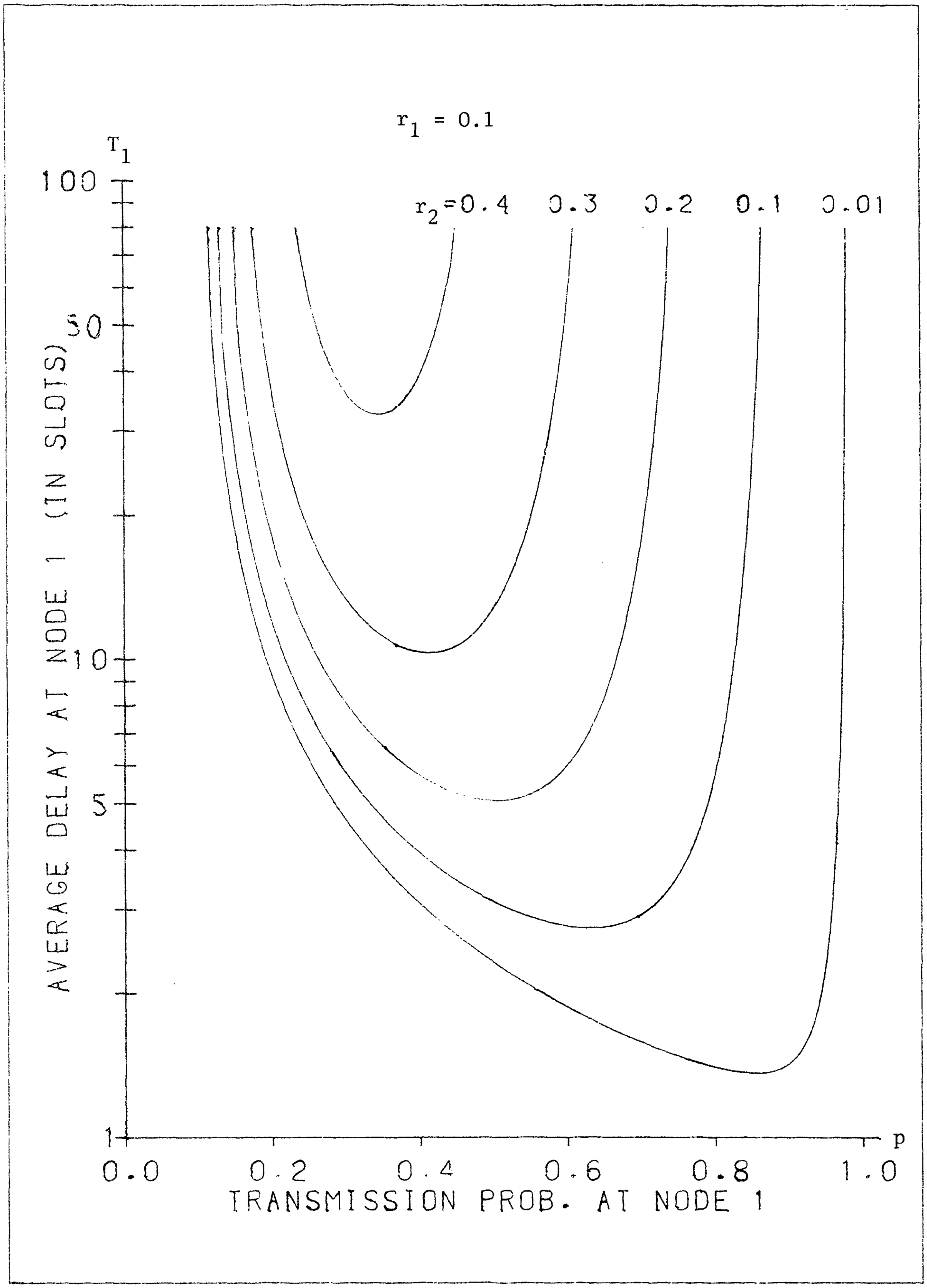

Fig. 2 - System 1: T_1 versus p.

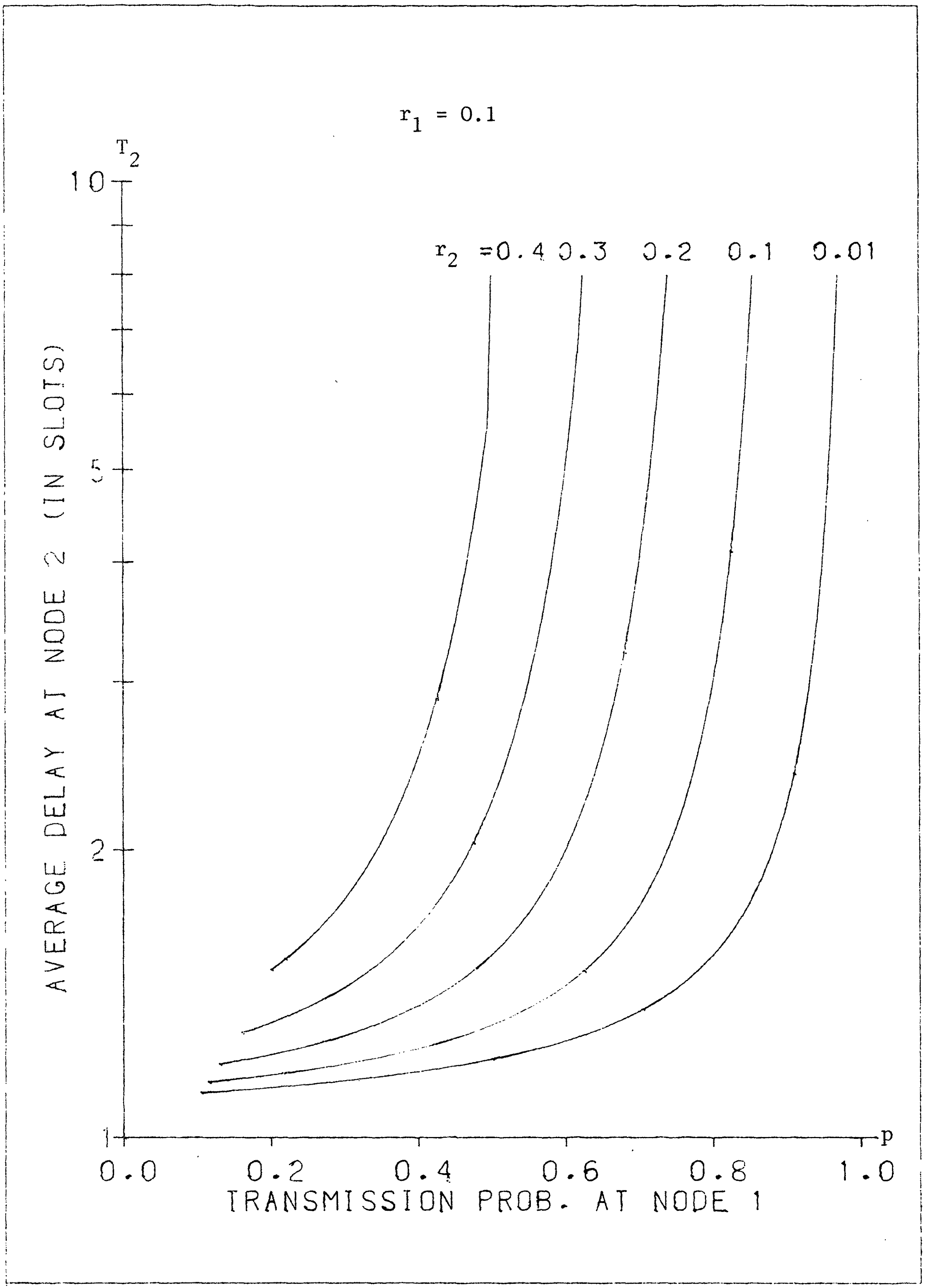

Fig. 3 - System 1: T_2 versus p.

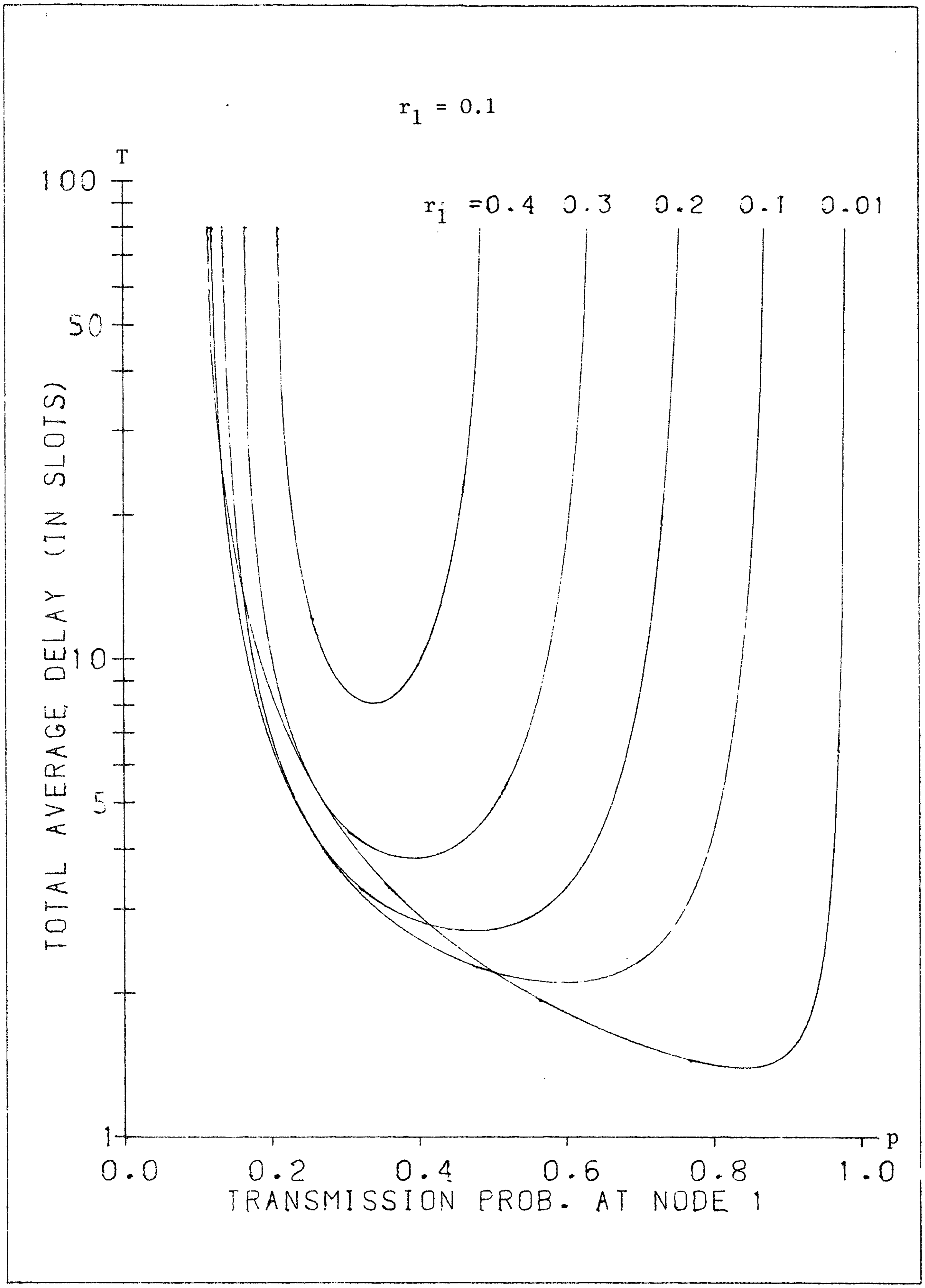

Fig. 4 - System 1: T versus p.

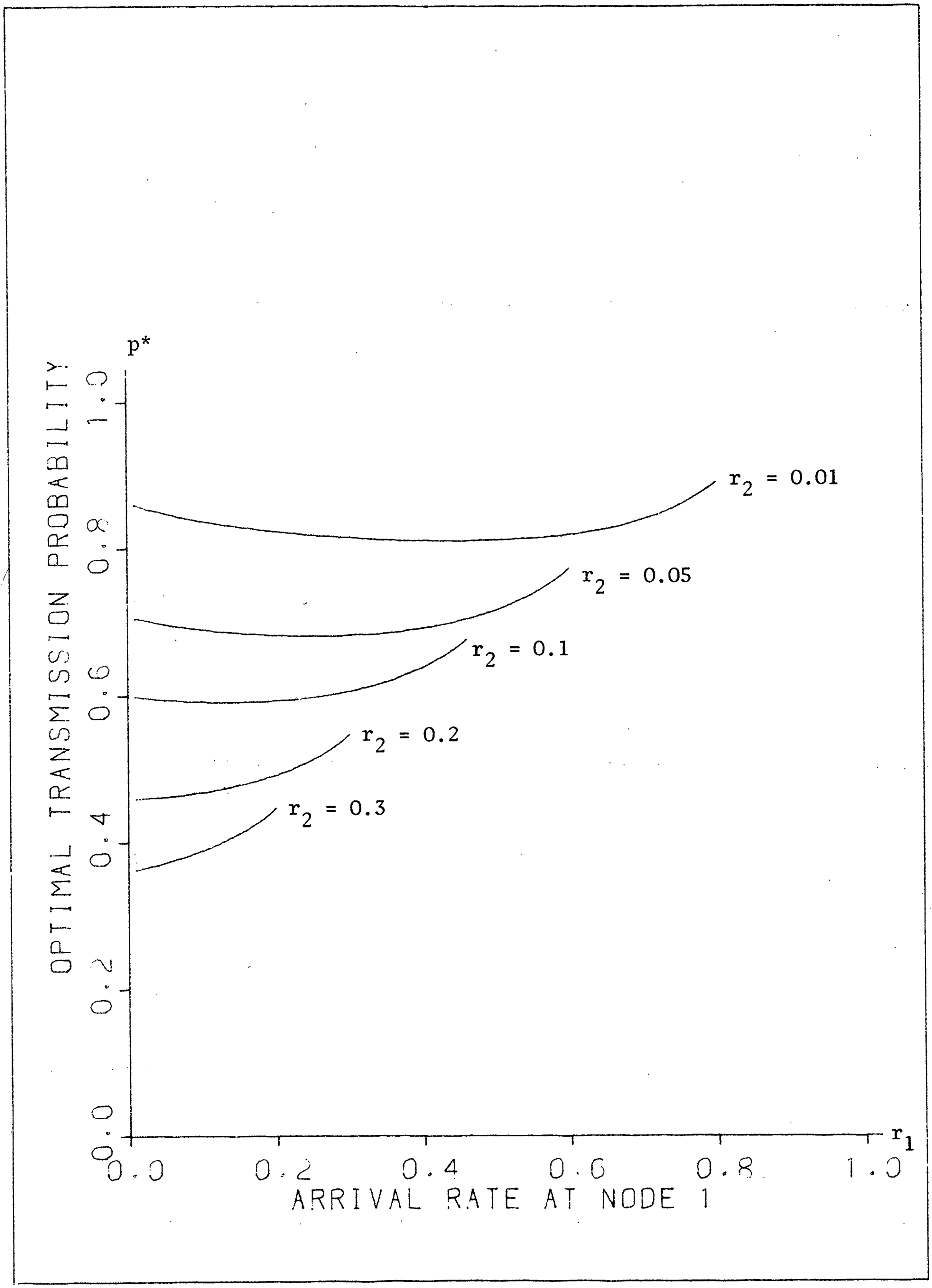

Fig. 5 - System 1: p* versus r_1.

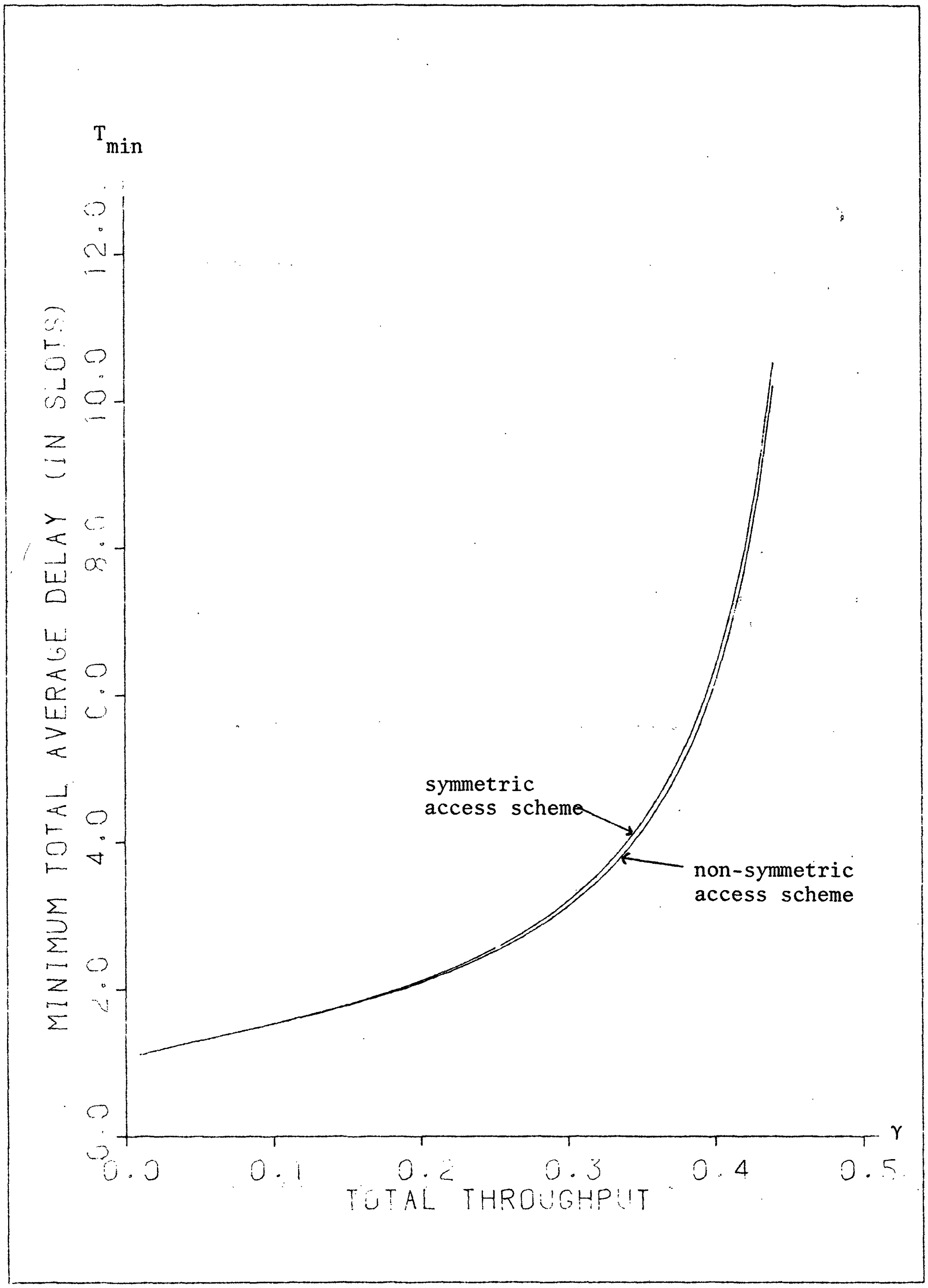

Fig. 6 - T_{min} versus γ for non-symmetric and symmetric access schemes.

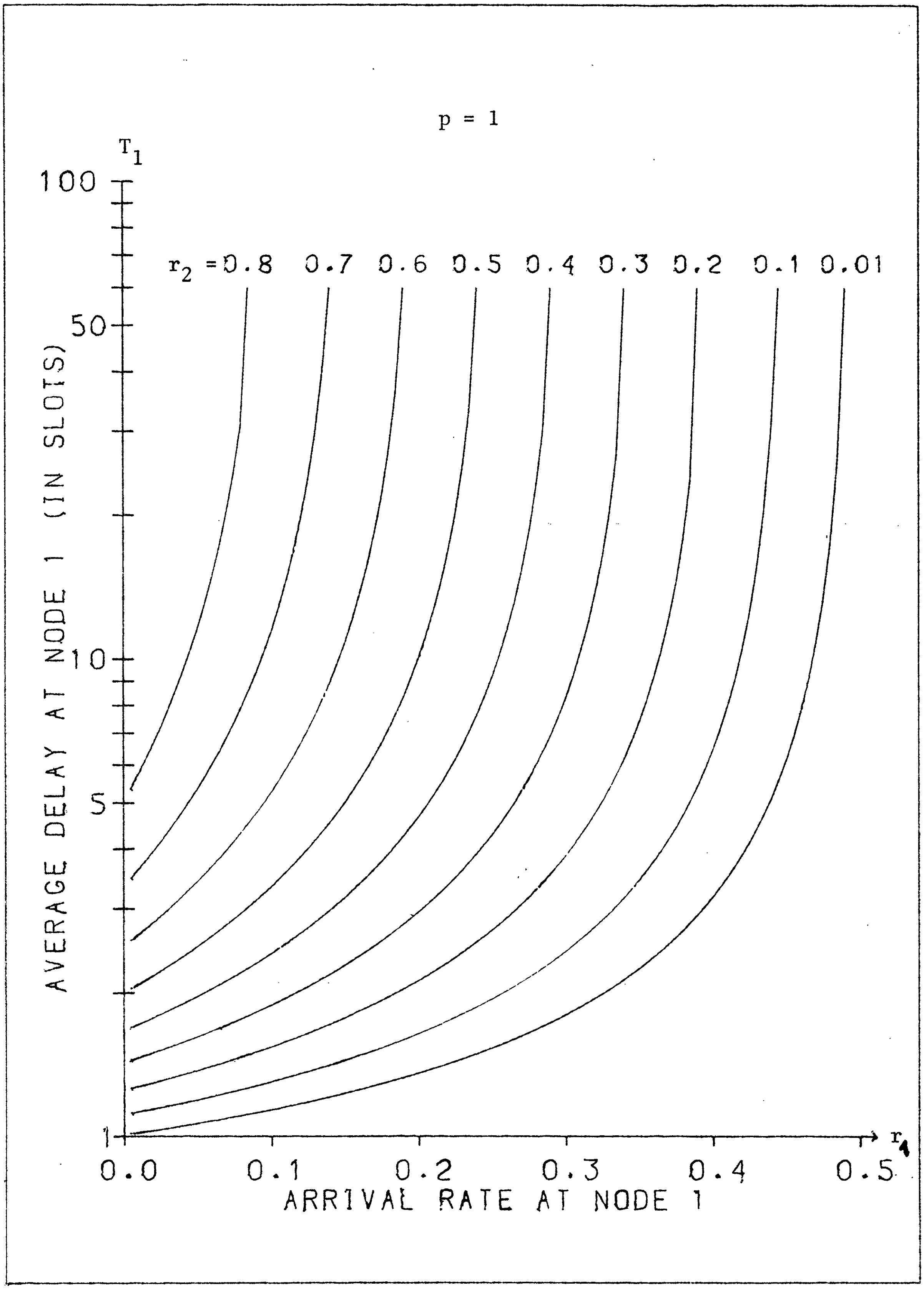

Fig. 7 - System 2: T_1 versus r_1.

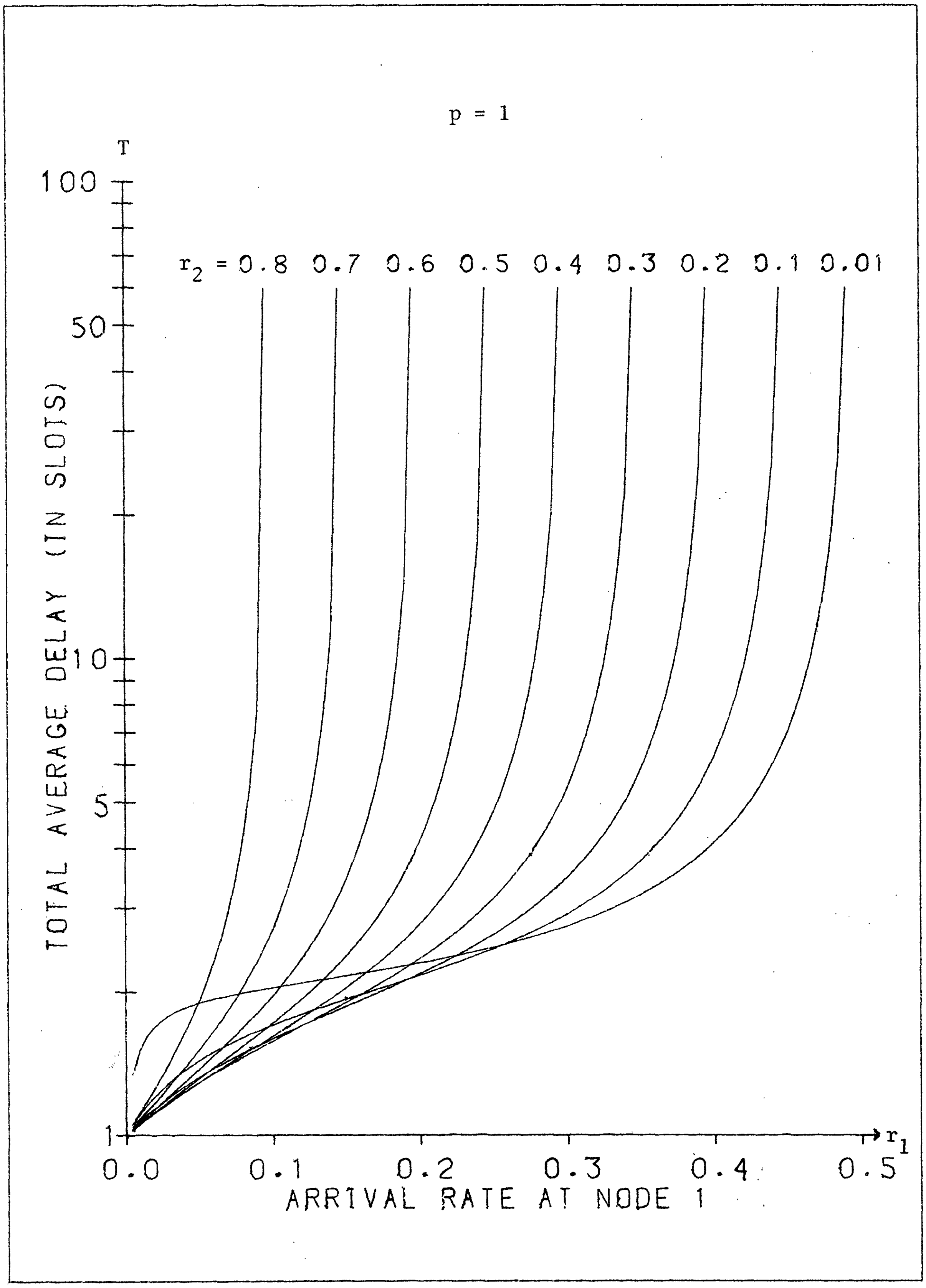

Fig. 8 - System 2: T versus r_1.

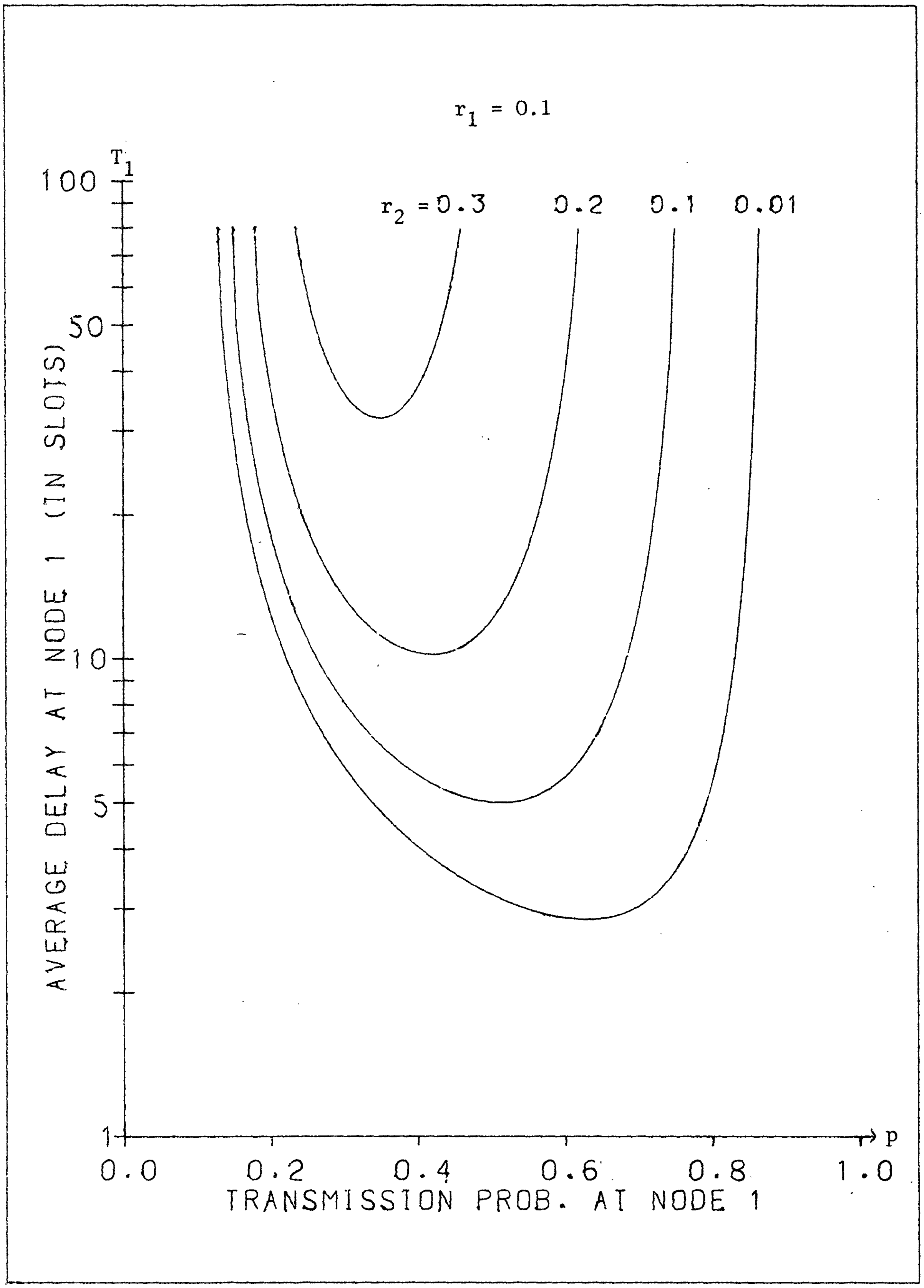

Fig. 9 - System 3: T_1 versus p.

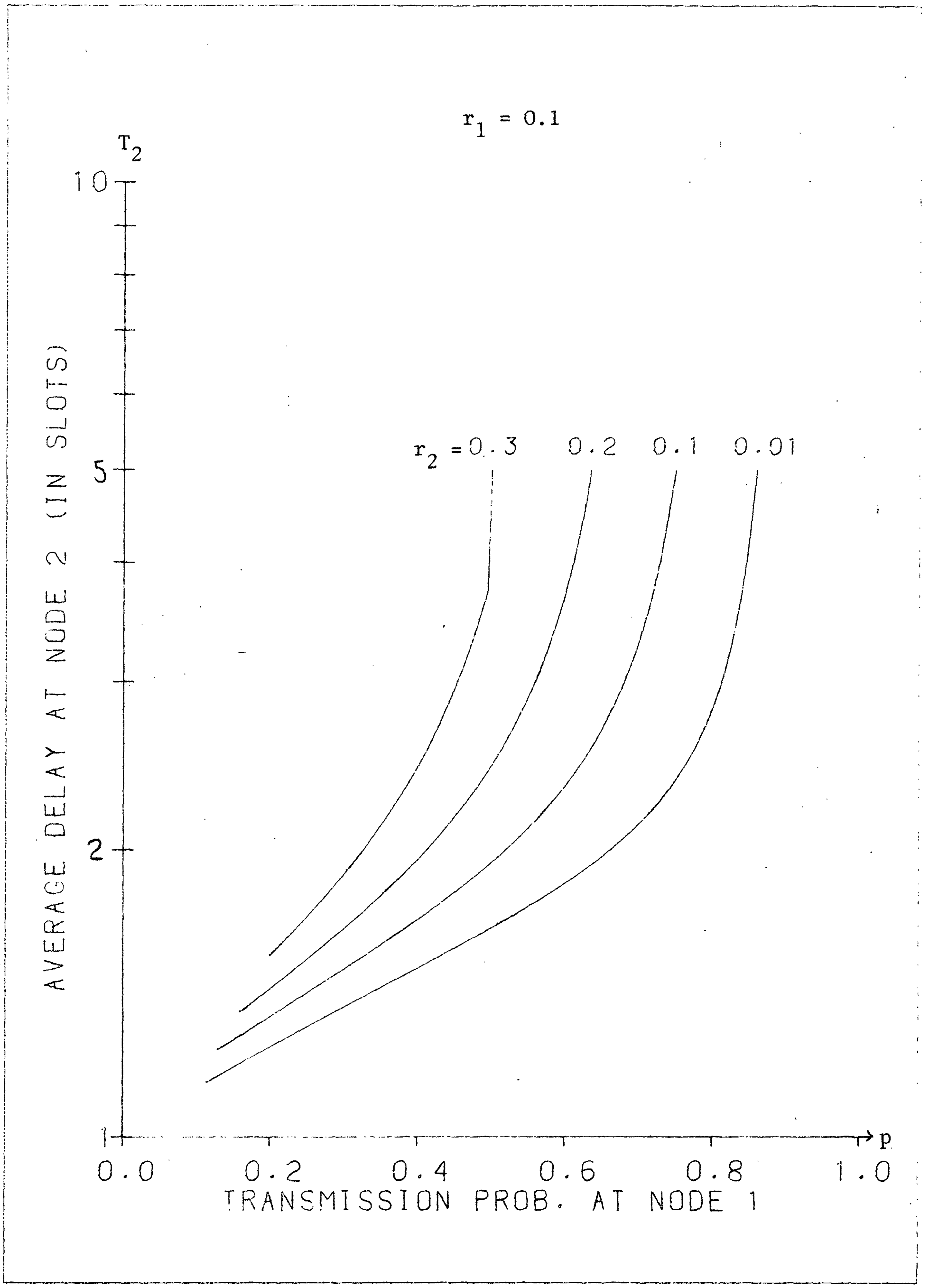

Fig. 10 - System 3: T_2 versus p.

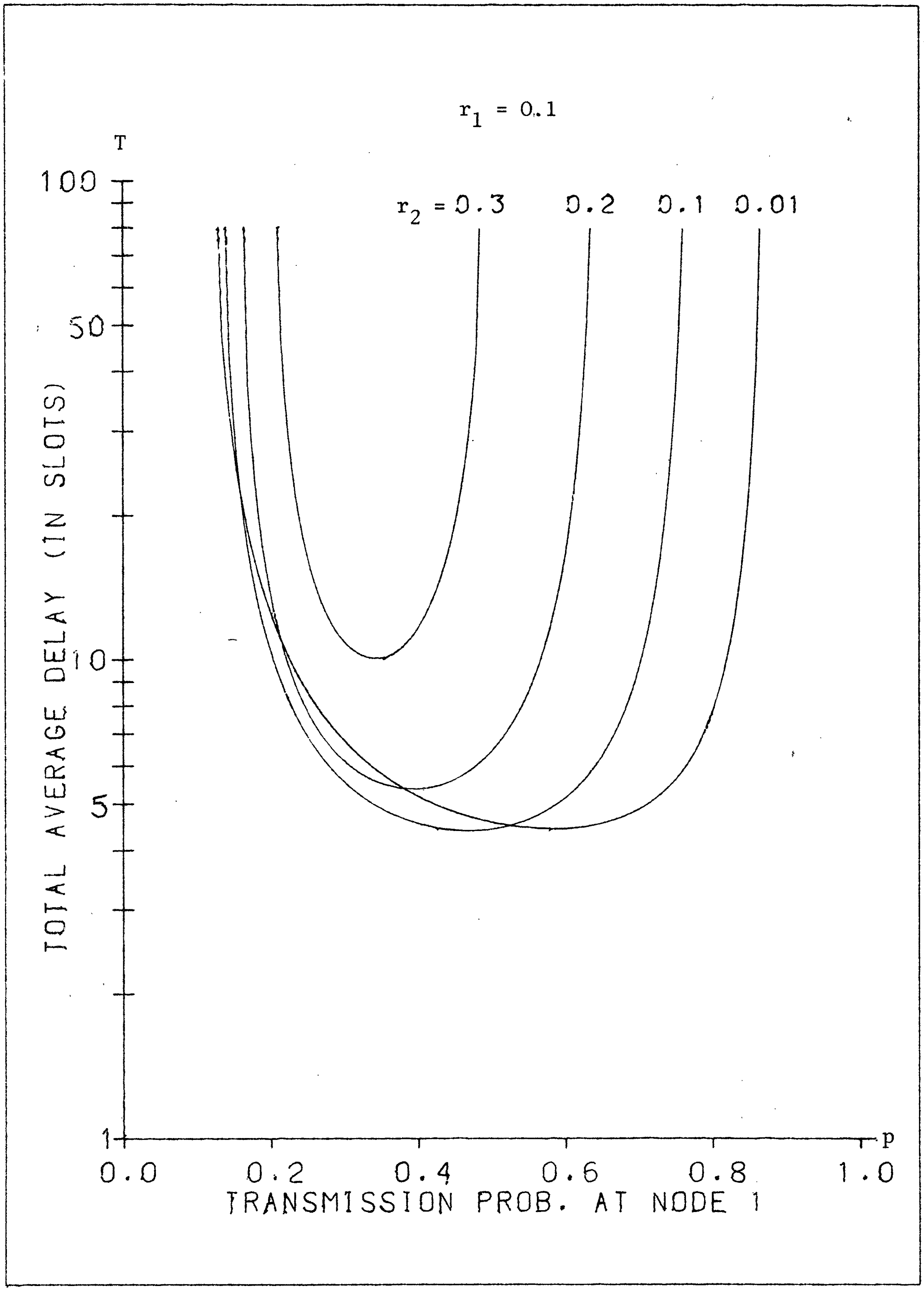

Fig. 11 - System 3: T versus p.

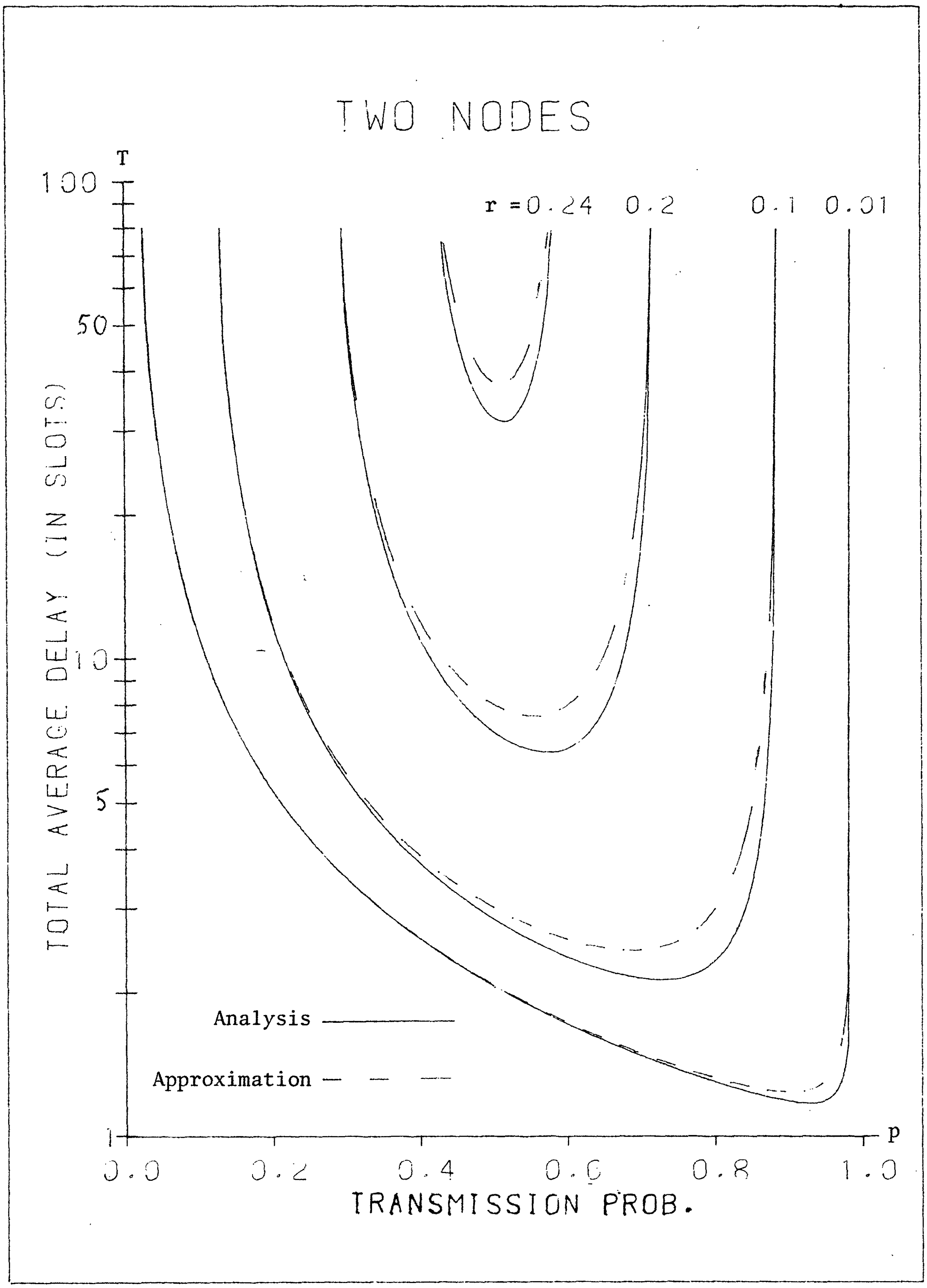

Fig. 12 - Comparison of analysis and approximation of two node symmetric ALOHA network.

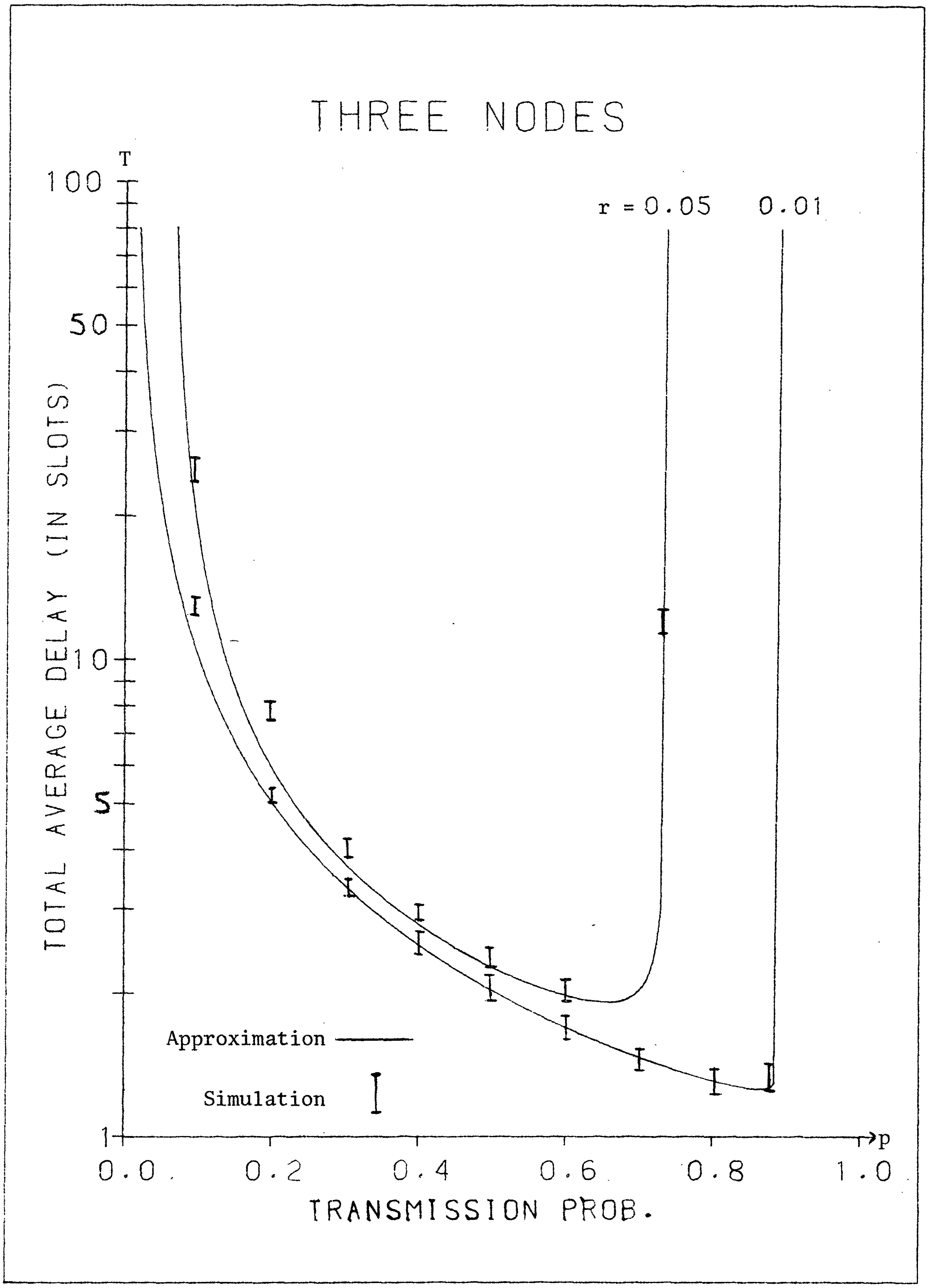

Fig. 13 - Comparison of approximation and simulation results for three-node symmetric ALOHA network.

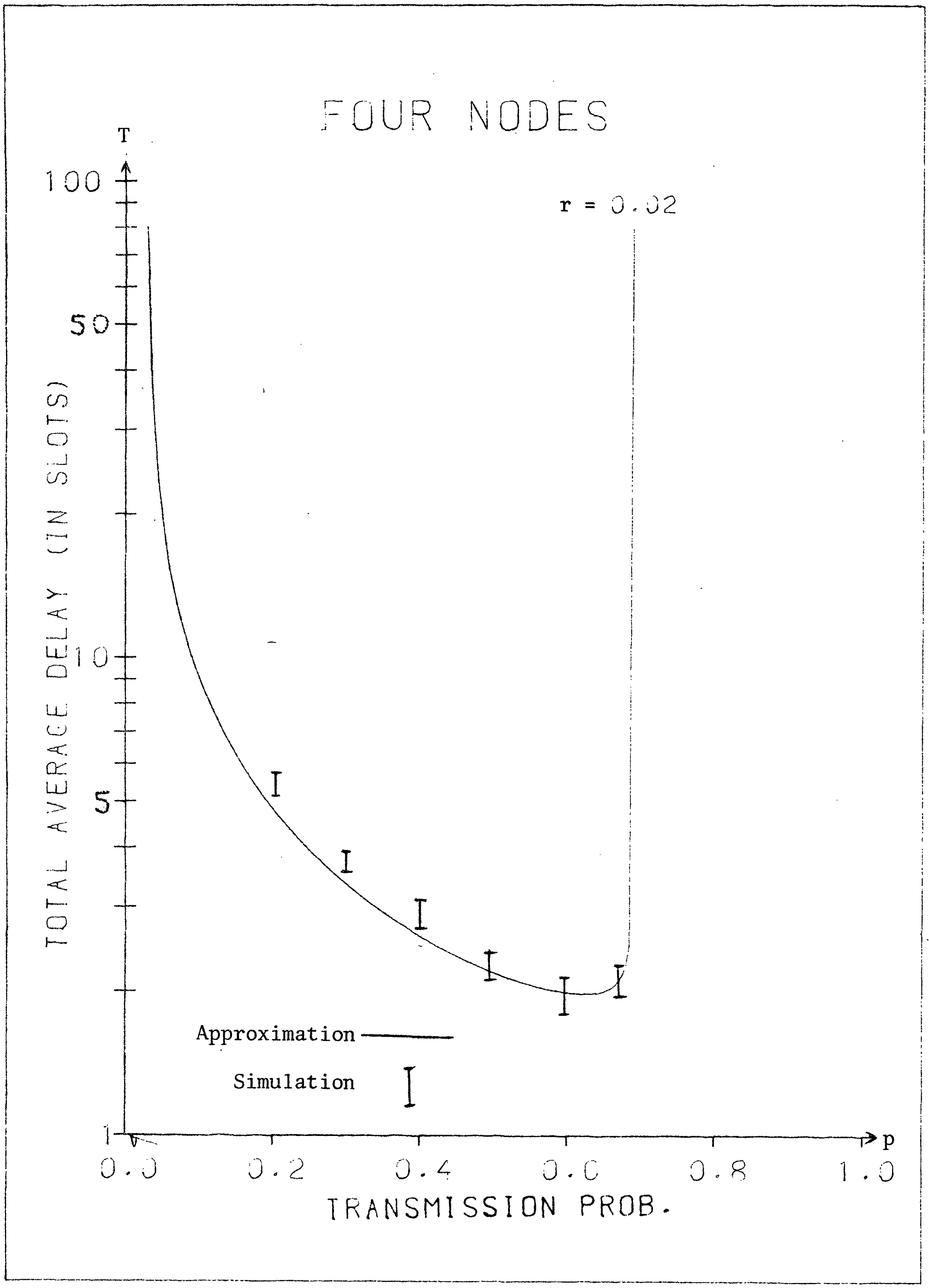

Fig. 14 - Comparison of approximation and simulation results of four-node symmetric ALOHA network.