

# Two Laplacians for the distance matrix of a graph

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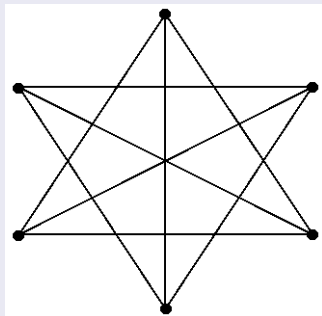
# 1. Adjacency related matrices

## Adjacency matrix

- For a graph  $G = (V, E)$  on  $n$  vertices, the **adjacency matrix**  $A = A(G)$  is the  $0$ - $1$   $n \times n$ -matrix indexed by the vertices of  $G$  and defined by  $a_{i,j} = 1$  if and only if  $ij \in E$
- The **(adjacency) spectrum**  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  of  $G$ , with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , is the  $A$ 's spectrum

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$A$ -spectrum :  $(3, 1, 0, 0, -2, -2)$



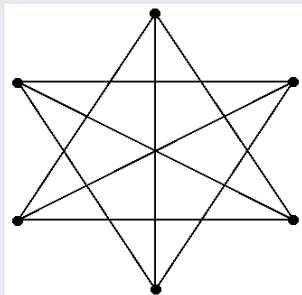
# 1. Adjacency related matrices

## Laplacian matrix

- The **Laplacian** of  $G$  is defined by  $L = L(G) = Deg - A$ , where  $Deg$  is the diagonal matrix whose diagonal entries are the degrees in  $G$ , and  $A$  the adjacency matrix of  $G$
- The **Laplacian spectrum**  $(\mu_1, \mu_2, \dots, \mu_n)$  of  $G$ , with  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$ , is the  $L$ 's spectrum

$$L = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 & 0 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ -1 & 0 & 3 & 0 & -1 & -1 \\ -1 & -1 & 0 & 3 & 0 & -1 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ 0 & -1 & -1 & -1 & 0 & 3 \end{bmatrix}$$

$L$ -spectrum :  $(5, 5, 3, 3, 2, 0)$



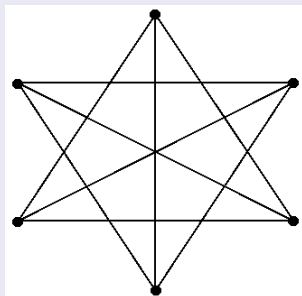
# 1. Adjacency related matrices

## Signless Laplacian matrix

- The **signless Laplacian** of  $G$  is defined by  $Q = Q(G) = Deg + A$
- The **Laplacian spectrum**  $(q_1, q_2, \dots, q_n)$  of  $G$ , with  $q_1 \geq q_2 \geq \dots \geq q_n$ , is the  $Q$ 's spectrum

$$Q = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 \\ 1 & 1 & 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 3 \end{bmatrix}$$

$Q$ -spectrum : (6, 4, 3, 3, 1, 1)



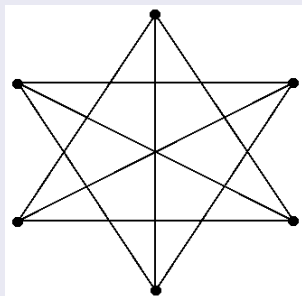
## 2. Distance matrix

### Definition

- In a connected graph  $G$  the **distance**  $d(i, j) = d_G(i, j)$  is the length of a shortest path between  $i$  and  $j$
- The **distance matrix**  $\mathcal{D} = \mathcal{D}(G)$  of a connected graph  $G$  is the  $n \times n$ -matrix indexed by the vertices of  $G$  and where  $\mathcal{D}_{i,j} = d(i, j)$
- The **distance spectrum** or  **$\mathcal{D}$ -spectrum** is denoted by  $(\partial_1, \partial_2, \dots, \partial_n)$  with  $\partial_1 \geq \partial_2 \geq \dots \geq \partial_n$

$$\mathcal{D} = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

$\mathcal{D}$ -spectrum :  $(7, 0, 0, -2, -2, -3)$



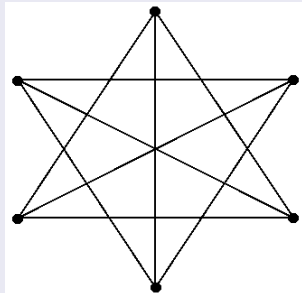
### 3. Distance Laplacian

#### Definition

- The **transmission** of a vertex  $i$  is the sum of all the distances from  $i$  to all other vertices  $t_i = \sum_{j \in V} d(i, j)$ .
- The **distance Laplacian matrix** of  $G$  is defined by  $\mathcal{D}^L = Tr - \mathcal{D}$ , where  $Tr$  is the diagonal matrix whose diagonal entries are the transmissions in  $G$
- The **distance Laplacian spectrum** or  **$\mathcal{D}^L$ -spectrum** is denoted by  $(\partial_1^L, \partial_2^L, \dots, \partial_n^L)$  with  $\partial_1^L \geq \partial_2^L \geq \dots \geq \partial_n^L = 0$

$$\mathcal{D}^L = \begin{bmatrix} 7 & -2 & -1 & -1 & -1 & -2 \\ -2 & 7 & -2 & -1 & -1 & -1 \\ -1 & -2 & 7 & -2 & -1 & -1 \\ -1 & -1 & -2 & 7 & -2 & -1 \\ -1 & -1 & -1 & -2 & 7 & -2 \\ -2 & -1 & -1 & -1 & -2 & 7 \end{bmatrix}$$

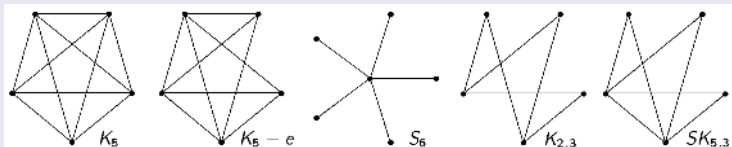
$\mathcal{D}^L$ -spectrum : (10, 9, 9, 7, 7, 0)



### 3. Distance Laplacian

#### Examples of distance Laplacian spectra

- The complete graph  $K_n : (n^{(n-1)}, 0)$  (also the Laplacian spectrum)
- The complement of an edge  $K_n - e : (n+2, n^{(n-2)}, 0)$
- The star  $S_n : (2n-1^{(n-2)}, n, 0)$
- The complete bipartite graph  $K_{a,b} : (2n-a^{(b-1)}, 2n-b^{(a-1)}, n, 0)$
- The complete split graph  $SK_{n,\alpha} : (n+\alpha^{\alpha-1}, n^{n-\alpha}, 0)$





### 3. Distance Laplacian

#### Properties

- For any connected graph  $\partial_n^L = 0$  (with multiplicity  $m(0) = 1$ )
- $m(\partial_1^L) \leq n - 1$ , equality holds only for  $K_n$
- Among trees  $\partial_1^L \geq 2n - 1$ , equality holds only for  $S_n$
- For  $L$ -spectra :

$$\mu_1(G) \geq \mu_1(G - e) \geq \mu_2(G) \geq \mu_2(G - e) \geq \dots \geq \mu_n(G) = \mu_n(G - e) = 0$$

There is no similar result for  $\mathcal{D}^L$ -spectra

- The  $\mathcal{D}^L$ -spectra of  $P_6$  and  $C_6$  are  $(21.3929, 15, 12.8532, 11, 9.7539, 0)$  and  $(13, 13, 10, 9, 9, 0)$ , respectively
- If  $e$  is an edge in  $G$  such that  $G - e$  is connected, then  $\partial_i^L(G - e) \geq \partial_i^L(G)$ , for  $i = 1, 2, \dots, n$
- $\partial_i^L(G) \geq \partial_i^L(K_n) = n$ , for  $i = 1, 2, \dots, n - 1$

### 3. Distance Laplacian

#### Transmission regular graphs

A connected graph  $G$  is  $k$ -**transmission regular** if  $t_i = k$ , for  $i = 1, 2, \dots, n$

If  $G$  is  $k$ -transmission regular with  $\mathcal{D}$ -spectrum  $(\partial_1, \partial_2, \dots, \partial_n)$ , then

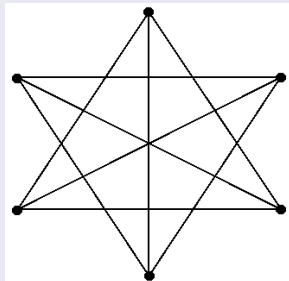
$(k - \partial_n, \dots, k - \partial_1)$  is the  $\mathcal{D}^L$ -spectrum of  $G$

Moreover, the eigenspaces are the same

A 7-transmission regular regular

$\mathcal{D}$ -spectrum :  $(7, 0, 0, -2, -2, -3)$

$\mathcal{D}^L$ -spectrum :  $(10, 9, 9, 7, 7, 0)$



### 3. Distance Laplacian

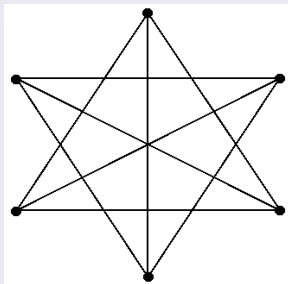
#### Graphs of diameter 2

Let  $G$  be a graph of diameter  $D = 2$ ,  $(\mu_1, \mu_2, \dots, \mu_n = 0)$  its  $L$ -spectrum and  $(\partial_1, \partial_2, \dots, \partial_n = 0)$  its  $\mathcal{D}^L$ -spectrum. Then  $\partial_i = 2n - \mu_{n-i}$ , for  $i = 1, 2, \dots, n - 1$ . Moreover, the  $L$ -eigenspaces and  $\mathcal{D}$ -eigenspaces coincide

A 7-transmission regular regular

$L$ -spectrum :  $(5, 5, 3, 3, 2, 0)$

$\mathcal{D}^L$ -spectrum :  $(10, 9, 9, 7, 7, 0)$



### 3. Distance Laplacian

#### Similarities with the algebraic connectivity

For the Laplacian  $L$  [Fiedler, 1973] :

- $\mu_{n-1} = 0$  if and only if  $G$  is disconnected
- The multiplicity of 0 in the  $L$ -spectrum of  $G$  equals the number of connected components of  $G$
- $\mu_{n-1}$  is called algebraic connectivity

For the distance Laplacian  $\mathcal{D}^L$  :

- $n$  is a  $\mathcal{D}^L$ -eigenvalue of  $G$  if and only if the complement  $\overline{G}$  is disconnected
- The multiplicity of  $n$  in the  $\mathcal{D}^L$ -spectrum of  $G$  is 1 less than the number of connected components of  $\overline{G}$

### 3. Distance Laplacian

#### Similarities with the algebraic connectivity

Corollaries :

- $\partial_1(G) \geq n$  with equality if and only if  $G \cong K_n$
- If  $G$  is bipartite and  $n$  is a distance Laplacian eigenvalue of  $G$ , then  $G$  is complete bipartite
- The star  $S_n$  is the only tree for which  $n$  is a distance Laplacian eigenvalue
- If the maximum degree  $\Delta = n - 1$ , then  $n$  is a  $\mathcal{D}^L$ -eigenvalue with multiplicity at least  $n_\Delta$  (number of vertices of degree  $\Delta$ )

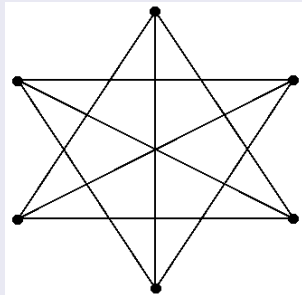
## 4. Distance signless Laplacian

### Definition

- The **transmission** of a vertex  $i$  is the sum of all the distances from  $i$  to all other vertices  $t_i = \sum_{j \in V} d(i, j)$ .
- The **distance Laplacian matrix** of  $G$  is defined by  $\mathcal{D}^Q = Tr + \mathcal{D}$ , where  $Tr$  is the diagonal matrix whose diagonal entries are the transmissions in  $G$
- The **distance Laplacian spectrum** or  $\mathcal{D}^Q$ -**spectrum** is denoted by  $(\partial_1^Q, \partial_2^Q, \dots, \partial_n^Q)$  with  $\partial_1^Q \geq \partial_2^Q \geq \dots \geq \partial_n^Q$

$$\mathcal{D}^Q = \begin{bmatrix} 7 & 2 & 1 & 1 & 1 & 2 \\ 2 & 7 & 2 & 1 & 1 & 1 \\ 1 & 2 & 7 & 2 & 1 & 1 \\ 1 & 1 & 2 & 7 & 2 & 1 \\ 1 & 1 & 1 & 2 & 7 & 2 \\ 2 & 1 & 1 & 1 & 2 & 7 \end{bmatrix}$$

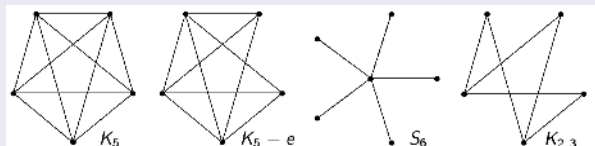
$\mathcal{D}^Q$ -spectrum : (14, 7, 7, 5, 5, 4)



## 4. Distance signless Laplacian

### Examples of distance signless Laplacian spectra

- For  $K_n$  :  $\left(2n - 2, n - 2^{(n-1)}\right)$  (also the signless Laplacian spectrum)
- For  $K_n - e$  :  $\left(\frac{3n-2 \pm \sqrt{(n-2)^2 + 16}}{2}, n - 2^{(n-2)}\right)$
- For  $S_n$  :  $\left(\frac{5n-8 \pm \sqrt{9n^2 - 32n + 32}}{2}, 2n - 5^{(n-2)}\right)$
- For  $K_{a,b}$  :  $\left(\frac{5n-8 \pm \sqrt{9(a-b)^2 + 4ab}}{2}, 2n - a - 4^{(b-1)}, 2n - b - 4^{(a-1)}\right)$



## 4. Distance signless Laplacian

### Properties

- For  $Q$ -spectra :

$$q_1(G) \geq q_1(G - e) \geq q_2(G) \geq q_2(G - e) \geq \cdots \geq q_n(G) \geq q_n(G - e)$$

There is no similar result for  $\mathcal{D}^Q$ -spectra

- The  $\mathcal{D}^Q$ -spectra of  $P_6$  and  $C_6$  are  $(25.0838, 12.1755, 11.1743, 8.6727, 7.7418, 5.5118)$  and  $(18, 9, 9, 8, 5, 5)$ , respectively
- If  $e$  is an edge in  $G$  such that  $G - e$  is connected, then  $\partial_i^Q(G - e) \geq \partial_i^Q(G)$ , for  $i = 1, 2, \dots, n$
- $\partial_1^Q(G) \geq \partial_1^Q(K_n) = 2n - 2$  with equality if and only if  $G \cong K_n$
- $\partial_i^Q(G) \geq \partial_i^Q(K_n) = n - 2$ , for  $i = 2, 3, \dots, n$
- $\partial_2^Q(G) \geq n - 2$  with equality if and only if  $G \cong K_n$



## 4. Distance signless Laplacian

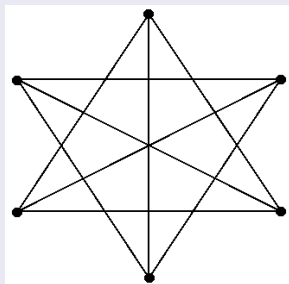
### Transmission regular graphs

- $2Tr_{min} \leq 2\overline{Tr} \leq \partial_1^Q(G) \leq 2Tr_{max}$  with equalities if and only if  $G$  is a transmission regular graph
- If  $G$  is  $k$ -transmission regular with  $\mathcal{D}$ -spectrum  $(\partial_1, \partial_2, \dots, \partial_n)$ , then  $(k + \partial_1, k + \partial_2, \dots, k + \partial_n)$  is the  $\mathcal{D}^Q$ -spectrum of  $G$   
Moreover, the eigenspaces are the same

A 7-transmission regular regular

$\mathcal{D}$ -spectrum :  $(7, 0, 0, -2, -2, -3)$

$\mathcal{D}^L$ -spectrum :  $(14, 7, 7, 5, 5, 4)$



## 4. Distance signless Laplacian

### Bipartite components

For the signless Laplacian :

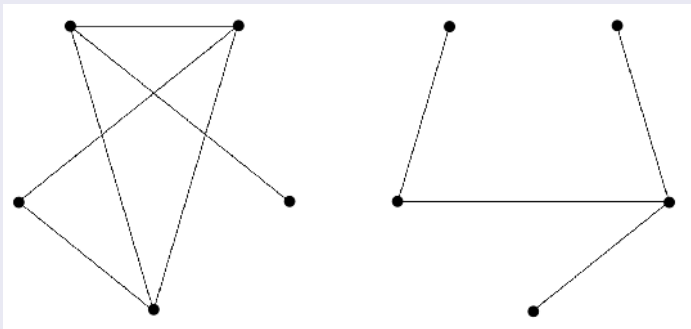
- 0 is a  $Q$ -eigenvalue of  $G$  if and only if  $G$  contains a bipartite component or an isolated vertex
- The multiplicity of 0 is equal to the number of bipartite components and isolated vertices

For the distance signless Laplacian :

- If  $n - 2$  is a  $\mathcal{D}^Q$ -eigenvalue of  $G$  with multiplicity  $m$ , then  $\overline{G}$  contains at least  $m$  components, each of which is bipartite or an isolated vertex
- There exist graphs with a bipartite complement for which  $n - 2$  is not a  $\mathcal{D}^Q$ -eigenvalue

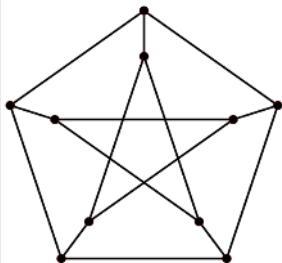
## 4. Distance signless Laplacian

### Bipartite components



$G$  (left) with  $n = 5$ ,  $\partial_5^Q \simeq 3.050286 > 3$  and  $\bar{G}$  (right) bipartite

## The Petersen graph and its spectra



$A$ -spectrum	$3^{(1)}$	$1^{(5)}$	$-2^{(4)}$
$L$ -spectrum	$5^{(4)}$	$2^{(5)}$	$0^{(1)}$
$Q$ -spectrum	$2^{(1)}$	$4^{(5)}$	$1^{(4)}$
$\mathcal{D}$ -spectrum	$15^{(1)}$	$0^{(4)}$	$-3^{(5)}$
$\mathcal{D}^L$ -spectrum	$18^{(5)}$	$15^{(4)}$	$0^{(1)}$
$\mathcal{D}^Q$ -spectrum	$30^{(1)}$	$15^{(4)}$	$12^{(5)}$