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TWO LOCAL PROPERTIES OF GRAPHS

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Summary. For a graph G and its vertex v the symbol $N_G(v)$ denotes the subgraph of G induced by the set of all vertices which are adjacent to v in G. We say that a graph G has locally the property P, if $N_G(v)$ has the property P for each v. Two local properties are studied: the local disconnectedness ($N_G(v)$ is disconnected for each v) and the local cyclicity ($N_G(v)$ is a circuit for each v).

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Lately, various local properties of graphs have been studied by various authors. The first impulse was the problem of A. A. Zykov at the symposium on the graph theory in Smolenice in 1963 [1]. A survey on this topic is given in the paper [2] by J. Sedláček.

Let G be an undirected graph, let v be its vertex. The symbol $N_G(v)$ denotes the subgraph of G induced by the set of all vertices which are adjacent to v. L. Szamkołowicz [3] suggested to study graphs G in which $N_G(v)$ for each vertex v belongs to a given class K of graphs. In particular, he mentioned the case when K is the class of all circuits. This case will be studied in the second part of this paper; the first concerns the case when K is the class of all disconnected graphs.

1. LOCAL DISCONNECTEDNESS

We say that a graph G is locally disconnected, if for each vertex v of G the graph $N_G(v)$ is disconnected.

Theorem 1. *Let G be a finite planar locally disconnected graph, let n be its number of vertices, $n \geqq 4$. Then the number of edges of G is at most $\frac{5}{2}n - 6$.*

Remark. For $n \leqq 3$ locally disconnected graphs with n vertices do not exist; the proof of this assertion is left to the reader.

Proof. Consider a representation of G in the plane; then the faces of this representation will be called faces of G. The symbols n, m, f, t will denote the numbers of vertices, edges, faces and triangular faces, respectively. Let v be a vertex of G. If all faces of G incident with v except at most one were triangular, then $N_G(v)$ would have a Hamiltonian path (the edges of this path are the edges of those triangulars faces which are opposite to v) and therefore it would be connected. Hence, as G is locally disconnected, each vertex v is incident with at least two non-triangular faces. If $\delta(v)$ is the degree of v in G, then the number of all faces incident with v is $\delta(v)$ and the number of the triangular ones is at most $\delta(v) - 2$. As each triangular face is incident with three vertices, the total number of triangular faces of G is at most $\frac{1}{3} \sum_{v \in V(G)} (\delta(v) - 2)$. We have

$$\frac{1}{3} \sum_{v \in V(G)} (\delta(v) - 2) = \frac{1}{3} \sum_{v \in V(G)} \delta(v) - \frac{2}{3}n .$$

As $\frac{1}{2} \sum_{v \in V(G)} \delta(v) = m$, we have

$$t \leqq \tfrac{2}{3}(m - n) .$$

From Euler's Formula we have

$$f = 2 - n + m .$$

The number of non-triangular faces of G is $f - t$ and we have

$$f - t \geqq 2 - n + m - \tfrac{2}{3}(m - n) = 2 - \tfrac{1}{3}n + \tfrac{1}{3}m . \tag{1}$$

At each non-triangular face we may choose two non-adjacent vertices which are both incident with this face, and join them by an edge (a diagonal of the face); the resulting graph is also planar. Hence we may add $f - t$ edges to G without violating its planarity. The resulting graph has n vertices and $m + f - t$ edges. The upper bound for the number of edges of a planar graph with n vertices is $3n - 6$, hence

$$m + f - t \leqq 3n - 6 .$$

On the other hand, (1) yields

$$m + f - t \geqq 2 - \tfrac{1}{3}n + \tfrac{4}{3}m .$$

These two inequalities imply

$$2 - \tfrac{1}{3}n + \tfrac{4}{3}m \leqq 3n - 6 ,$$

which yields

$$m \leqq \tfrac{5}{2}n - 6 ,$$

which was to be proved. □

The following theorem will show that this upper bound cannot be improved.

Theorem 2. *Let n be an even positive integer, $n \geqq 8$. Then there exists a planar locally disconnected graph with n vertices and $\frac{5}{2}n - 6$ edges.*

Proof. If $n \geqq 8$ and $n \equiv 2 \pmod 6$, there exists a positive integer k such that $n = 6k + 2$. Let $V(G) = \{v(i, j) \mid 1 \leqq i \leqq k, 1 \leqq j \leqq 6\} \cup \{x, y\}$. If for two vertices

$v(i_1, j_1)$, $v(i_2, j_2)$ we have either $i_1 = i_2$, $|j_1 - j_2| \equiv 1 \pmod 6$, or $|i_1 - i_2| = 1$, $j_1 = j_2$, then these two vertices are adjacent in G. Further, if $i - j$ is even and $i < k$, then $v(i, j)$ is adjacent to $v(i + 1, j + 1)$, where the sum $j + 1$ is taken modulo 6. Finally, the vertex x is adjacent to the vertices $v(1, 2)$, $v(1, 3)$, $v(1, 5)$, $v(1, 6)$ and the vertex y is adjacent to the vertices $v(k, 2)$, $v(k, 3)$, $v(k, 5)$, $v(k, 6)$. This graph G has the required properties.

If $n \equiv 4 \pmod 6$, we construct a graph G' with $n - 2$ vertices which has the required properties, and choose a face of G' with four vertices. Into that face we add two vertices and five edges as is shown in Fig. 1. The graph G thus obtained has the required properties. If $n \equiv 0 \pmod 6$, we construct a graph G' with $n - 4$ vertices and with the required properties, and choose again a face of G' with four vertices. Into that face we add four vertices and ten edges as is shown in Fig. 2. The graph G thus obtained has again the required properties. □

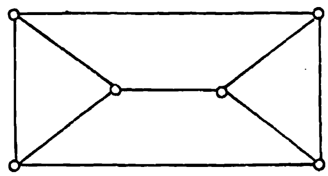

Fig. 1

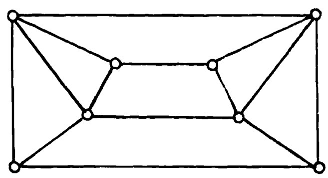

Fig. 2

For n odd the number $\frac{5}{2}n - 6$ is not an integer, therefore there exists no graph with this number of edges. Nonetheless, there exists a graph with n vertices and $[\frac{5}{2}n] - 6$ edges. It suffices to construct a graph with $n - 1$ vertices and $\frac{5}{2}(n - 1) - 6$ edges according to the proof of Theorem 2 and to add one vertex and two edges into a face with four vertices as is shown in Fig. 3.

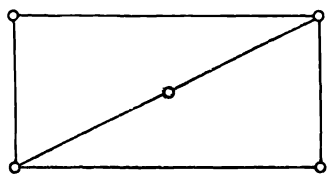

Fig. 3

In Fig. 4 we see an example of a graph from Theorem 2 for $n = 26$. Instead of $v(i, j)$ we write shortly ij.

We will study properties of locally cyclic graphs.

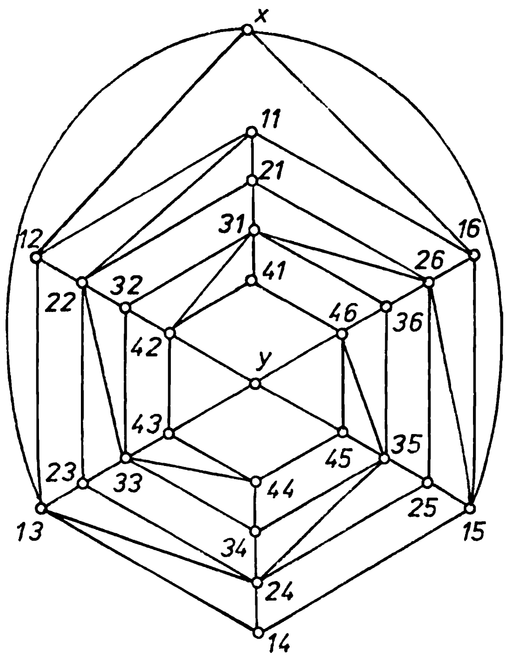

Fig. 4

2. LOCAL CYCLICITY

If $N_G(v)$ is a circuit for each vertex v of a graph G, the graph G is called locally cyclic.

The idea to study such graphs comes from [3].

Theorem 3. *Let G be a finite undirected graph. Then the following two assertions are equivalent:*

(i) *G is locally cyclic.*

(ii) *Each edge of G is contained in exactly two triangles and there are no two distinct wheels in G with a common centre.*

Proof. (i) $\Rightarrow$ (ii). Let e be an edge of G, let u, v be its end vertices. Thus v is a vertex of $N_G(u)$. As $N_G(u)$ is a circuit, there exist exactly two vertices x and y of $N_G(u)$ which are adjacent to v. The sets $\{u, v, x\}$, $\{u, v, y\}$ induce two triangles in G which contain e.

There exists no other triangle with this property; otherwise v would have a degree greater than 2 in $N_G(u)$ and $N_G(u)$ would not be a circuit. If there were two distinct wheels in G with a common centre u, then $N_G(u)$ would contain two distinct circuits and G would not be locally cyclic.

(ii) $\Rightarrow$ (i). Let u be a vertex of G. If v is a vertex of $N_G(u)$, then uv is an edge of G. This edge belongs to exactly two triangles in G, therefore the degre of v in $N_G(u)$ is 2. As v was chosen arbitrarily, the graph $N_G(u)$ is regular of degree 2. If $N_G(u)$ contained two distinct circuits, then G would contain two distinct wheels with the common centre u. Therefore $N_G(u)$ is a circuit. As u was chosen arbitrarily, G is locally cyclic. $\square$

This assertion enables us to generalize in a certain sense the concept of the dual graph of a planar graph.

Let G be a finite connected locally cyclic graph. We assign a certain graph $D(G)$ to it. The vertex set of $D(G)$ is the set of all triangles in G. Two vertices of $D(G)$ are adjacent if and only if they have an edge in common (as triangles). The graph $D(G)$ will be called the dual graph to G.

Let $\mathfrak{D}$ be the class of finite connected graphs with the following properties:

(a) Each graph from $\mathfrak{D}$ is regular of degree 3.

(b) In each graph H from $\mathfrak{D}$ there exists a system $\mathscr{C}$ of circuits in H such that any two of them have at most one edge in common and each edge of H is contained in exactly two circuits of $\mathscr{C}$.

(c) If three circuits of $\mathscr{C}$ have the property that any two of them have a common edge, then all three have a common vertex.

Theorem 4. *Let G be a finite connected locally cyclic graph. Then its dual graph $D(G) \in \mathfrak{D}$.*

Proof. Each triangle in G has three edges and thus it has common edges with exactly three other triangles and (a) holds. Now for each vertex u of G let $C(u)$ be the subgraph of $D(G)$ induced by the set of all vertices corresponding to triangles of G which contain u; it is evidently a circuit. By C let us denote the set of all $C(u)$ for vertices u of G. If u, v are two adjacent vertices of G, then there exist exactly two triangles in G containing both u and v (and obviously also the edge uv). These triangles form a pair of adjacent vertices in $D(G)$. The edge joining them is the unique common edge of $C(u)$ and $C(v)$ in $D(G)$. If u and v are not adjacent in G, then there is no triangle in G containing both u and v and the circuits $C(u)$ and $C(v)$ are edge-disjoint. Now let e be an edge of $D(G)$, let its end vertices be x and y. These two vertices are triangles in G with a common edge f. Let a and b be the end vertices of f in G. Then e is a common edge of $C(a)$ and $C(b)$, and (b) holds. Now let C_1, C_2, C_3 be three circuits of C and let any two of them have a common edge. Let c_1, c_2, c_3 be vertices of G such that $C_1 = C(c_1)$, $C_2 = C(c_2)$, $C_3 = C(c_3)$. Then any two of the vertices c_1, c_2, c_3 are adjacent and thus the set $\{c_1, c_2, c_3\}$ induces a triangle in G. Let x be the vertex of $D(G)$ corresponding to this triangle; then x is the common vertex of C_1, C_2 and C_3 and (c) holds.

Theorem 5. *Let H be a graph contained in $\mathfrak{D}$. Then there exists a finite connected locally cyclic graph G such that $D(G) \cong H$.*

Proof. Let $\mathscr{C}$ be a system of circuits in H satisfying (b) and (c). We shall construct the graph G. The vertex set of G is $\mathscr{C}$; two vertices are adjacent in G if and only if they have a common edge (as circuits in H). The graph H is regular of degree 3 and therefore for each vertex u of H there exist exactly three circuits of $\mathscr{C}$ containing u. Any two of them are adjacent and thus they form a triangle $T(u)$ in G. Hence to each vertex u of H a certain triangle $T(u)$ in G is assigned. Two triangles $T(u)$, $T(v)$ have a common edge if and only if u and v are adjacent in H. Now let T be a triangle in G, let u, v, w be its vertices. These three vertices are circuits of H with the property that any two of them have a common edge. According to (c) there exists a vertex u of G contained in all three circuits; evidently $T(u) = T$. Hence we have a one-to-one correspondence between the vertices of H and the triangles of G with the property that two vertices of H are adjacent if and only if the corresponding triangles of G have a common edge; this implies $D(G) \cong H$.

Now we recall the concept of an independent system of circuits in a graph. Let G be a graph, let $\mathscr{S}$ be a certain set of subgraphs of G. Then the composition of graphs from $\mathscr{S}$ is a subgraph of G whose edge set consists of all edges e with the property that the number of graphs from $\mathscr{S}$ containing e is odd, and whose vertex set is the set of the end vertices of these edges. If $\mathscr{S}$ is a system of circuits in G with the property that no circuit $C \in \mathscr{S}$ is the composition of circuits from a subset of $\mathscr{S} - \{C\}$, the system $\mathscr{S}$ is called independent. The maximum number of independent circuits in a graph G is called the cyclomatic number of G and denoted by $c(G)$. The equality $c(G) = m - n + p$ holds, where m, n, p are respectively the numbers of edges, vertices and connected components of G.

Theorem 6. *Let $H \in \mathfrak{D}$, let $\mathscr{C}$ be a system of circuits from the definition of $\mathfrak{D}$. Let C_0 be an arbitrary circuit of $\mathscr{C}$. Then $\mathscr{C} - \{C_0\}$ is an independent system of circuits in H.*

Proof. Let $C \in \mathscr{C} - \{C_0\}$. Suppose there exists a subset $\mathscr{C}' \subseteq \mathscr{C} - \{C_0\}$ such that the composition of all circuits of $\mathscr{C}'$ is C. Then the composition of $\mathscr{C}' \cup \{C\}$ is the empty graph and thus any edge of H is in an even number of circuits of $\mathscr{C}' \cup \{C\}$. As $\mathscr{C}' \cup \{C\} \subseteq \mathscr{C}$, this number is 0 or 2. If $\mathscr{C}' \cup \{C\} = \mathscr{C} - \{C_0\}$, then we have a contradiction, because any edge of C_0 lies exactly in one circuit of this set. If $\mathscr{C}' \cup \{C\}$ is a proper subset of $\mathscr{C} - \{C_0\}$, then there exists a circuit C_1 belonging to $\mathscr{C} - \{C_0\}$ and not belonging to $\mathscr{C}' \cup \{C\}$, and having a common edge with a circuit of $\mathscr{C}' \cup \{C\}$. Then this common edge is contained in exactly one circuit of $\mathscr{C}' \cup \{C\}$, which is again a contradiction. Hence the system $\mathscr{C} - \{C_0\}$ is independent.

Theorem 7. *Let G be a finite connected locally cyclic graph with n vertices and m edges. Let $D(G)$ have n' vertices and m' edges. Then*

$$n' = \tfrac{2}{3}m\,,$$
$$m' = m\,,$$
$$m \geqq 3n - 6\,.$$

Proof. The number n' of vertices of $D(G)$ is equal to the number of triangles in G. If u is a vertex of G and $d(u)$ its degree, it follows from Theorem 1 that the number of triangles containing u is equal to $d(u)$. As each triangle has three vertices, we have

$$n' = \tfrac{1}{3} \sum_{u \in V(G)} d(u)\,.$$

On the other hand,

$$m = \tfrac{1}{2} \sum_{u \in V(G)} d(u)\,;$$

this implies

$$n' = \tfrac{2}{3}m\,.$$

Now we define a mapping φ of the edge set $E(G)$ of G onto the edge set $E(D(G))$ of $D(G)$. Let $e \in E(G)$. Then e is contained in exactly two triangles; these triangles are adjacent vertices of $D(G)$. The edge of $D(G)$ joining these vertices will be $\varphi(e)$. It is easy to prove that the mapping φ is a one-to-one mapping of $E(G)$ onto $E(D(G))$ and thus

$$m' = m\,.$$

The number n of vertices of G is equal to the number of circuits in $\mathscr{C}$. Theorem 4 implies

$$n - 1 \leqq c(D(G)) = m' - n' + 1\,.$$

If we substitute $n' = \tfrac{2}{3}m$, $m' = m$, we obtain

$$n - 1 \leqq m - \tfrac{2}{3}m + 1\,,$$

which implies

$$m \geqq 3n - 6\,.$$

Theorem 8. *Let $H \in \mathfrak{D}$, let $\mathscr{C}$ be the system of circuits from the definition of $\mathfrak{D}$. Let $C \in \mathscr{C}$. Then the edge set of C is not an edge cut of H.*

Proof. Suppose that the vertex set of C is an edge cut of H. The graph H' obtained from H by deleting the edges of C is disconnected. There exist two adjacent vertices u_1, u_2 of C which lie in distinct connected components of H'. Let e_1 (or e_2) be the edge of H not belonging to C and incident to u_1 (or u_2, respectively). Let e be the edge joining u_1 and u_2. Let C_0 be the (uniquely determined) circuit of $\mathscr{C}$ which has the common edge e with C. As C_0 cannot have more than one common edge with C, it must contain e_1 and e_2. Let P be the path obtained from C_0 by deleting e. Then P is a path in H connecting the vertices u_1, u_2 from distinct connected components of H'. Thus it must contain an edge e_0 of C distinct from e. The edges e, e_0 are distinct common edges of C and C_0, which is a contradiction.

The results which were presented here enable us to construct locally cyclic graphs, outgoing from their dual graphs. If we find a graph $H \in \mathfrak{D}$, we can construct the locally cyclic graph whose dual graph is H.

In the end we prove a theorem showing a recurrent method of constructing graphs from $\mathfrak{D}$.

Theorem 9. *Let $H_0 \in \mathfrak{D}$. Choose a circuit $C \in \mathscr{C}$ and two edges e_1, e_2 of C. Replace the edge e_1 (or e_2) by a path of the length 2 with the inner vertex u_1 (or u_2, respectively). Join u_1 and u_2 by an edge. The graph thus constructed belongs to $\mathfrak{D}$.*

Proof. By this transformation the circuit C is replaced by two new circuits C_1, C_2 with the common edge u_1u_2. It is easy to prove that the new graph is again regular of degree 3 and that the set $(\mathscr{C} - \{C\}) \cup \{C_1, C_2\}$ satisfies (b) and (c).

In Fig. 5 we see a graph which satisfies (a) and (b), but not (c). Any two of the circuits C_1, C_2, C_3 have a common edge, but there is no vertex belonging to all three.

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Souhrn

DVĚ LOKÁLNÍ VLASTNOSTI GRAFŮ

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Symbolem $N_G(v)$ se označuje podgraf grafu G indukovaný množinou uzlů spojených hranami s uzlem v. Je-li $N_G(v)$ nesouvislý graf pro každý uzel v, graf G se nazývá lokálně nesouvislý. Je-li $N_G(v)$ kružnice pro každý uzel v, graf G se nazývá lokálně cyklický. Zkoumají se lokálně nesouvislé grafy a lokálně cyklické grafy.

Резюме

ДВА ЛОКАЛЬНЫХ СВОЙСТВА ГРАФОВ

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Пусть $N_G(v)$ обозначает подграф графа G, порождённый множеством вершин смежных с вершиной v. Граф G называется локально несвязным, если граф $N_G(v)$ несвязен для каждой его вершины v, и локально циклическим, если $N_G(v)$ является контуром для каждой его вершины v. В статье изучаются локально несвязные и локально циклические графы.

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