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The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory

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Abstract

We give a representation of the parity-even part of the planar two-loop six-gluon MHV amplitude of $\mathcal{N} = 4$ super-Yang-Mills theory, in terms of loop-momentum integrals with simple dual conformal properties. We evaluate the integrals numerically in order to test directly the ABDK/BDS all-loop ansatz for planar MHV amplitudes. We find that the ansatz requires an additive remainder function, in accord with previous indications from strong-coupling and Regge limits. The planar six-gluon amplitude can also be compared with the hexagonal Wilson loop computed by Drummond, Henn, Korchemsky and Sokatchev in arXiv:0803.1466 [hep-th]. After accounting for differing singularities and other constants independent of the kinematics, we find that the Wilson loop and MHV-amplitude remainders are *identical*, to within our numerical precision. This result provides non-trivial confirmation of a proposed n-point equivalence between Wilson loops and planar MHV amplitudes, and suggests that an additional mechanism besides dual conformal symmetry fixes their form at six points and beyond.

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I. INTRODUCTION

Gauge theories play a central role in modern particle physics. How far can we go in understanding their properties quantitatively? Long ago [1], 't Hooft suggested that the theories simplify dramatically in the so-called planar limit, that of a large number of colors N_c. In this limit, he suggested that the theory is expressible as a string theory. This idea was given a concrete realization by Maldacena's [2] conjecture of the anti-de Sitter/conformal field theory (AdS/CFT) duality between weakly-coupled type-IIB string theory on an AdS background and the maximally supersymmetric ($\mathcal{N} = 4$) gauge theory at strong coupling.

The duality requires that the perturbative series for a variety of quantities, including scattering amplitudes, sum up to a simple quantity. In order for this to be possible, it would appear almost essential for the terms in the perturbative series to be related to each other in a simple way. Such a relation is guaranteed for the infrared-singular terms in amplitudes by the requirement that physical quantities are infrared-finite. However, the finite terms are not required to obey such an iterative relation. The $\mathcal{N} = 4$ theory is special, as was shown by Anastasiou and three of the authors (ABDK) [3], in that a simple relation does hold between the finite terms of the planar one- and two-loop four-gluon amplitudes. (Using the Ward identities of $\mathcal{N} = 4$ supersymmetry, this relation extends to four-point amplitudes for arbitrary external states [4, 5].) By demanding that the amplitude have proper factorization as any two external momenta become collinear, this relation was extended to an arbitrary number of external legs, valid for the simplest configuration of gluon helicities, the maximally helicity violating (MHV) one. Smirnov and two of the authors (BDS) formulated [6] an all-orders version of this relation, based on the idea that in the $\mathcal{N} = 4$ theory the finite terms should obey the same (exponential) relations as the divergent terms. This ansatz gave a prediction to all orders in the coupling for planar MHV amplitudes.

In a remarkable paper, Alday and Maldacena [7] suggested a way to compute the dimensionally-regulated planar four-point amplitude at strong coupling, using dual string theory. Their calculation reproduced the strong-coupling limit of the BDS ansatz.

The Alday–Maldacena calculation also provided a link between scattering amplitudes at strong coupling and a special kind of Wilson loop, one composed of light-like segments in a dual coordinate space. Drummond, Korchemsky, and Sokatchev [8] found the surprising result that at weak coupling the lowest-order (one-loop) contribution to a rectangular Wilson loop with light-like edges is equal to the four-point one-loop amplitude (normalized by the tree amplitude, and up to a constant term). Brandhuber, Heslop, and Travaglini [9] showed that the equality extends to one between Wilson n-gons and one-loop MHV n-point amplitudes for all n.

Does this equality extend beyond one loop? The perturbative planar amplitudes exhibit a "dual conformal invariance" [10, 11], distinct from the usual conformal invariance of the $\mathcal{N} = 4$ theory. Roughly speaking, it corresponds to conformal invariance in momentum space. Although the origin of this symmetry is not yet understood, we can use it to guide the calculations. Scattering amplitudes with more than four gluons, when normalized by the tree amplitude, may contain odd powers of the Levi-Civita tensor contracted with the external momenta. These terms flip sign under a parity transformation, which reverses all helicities. We refer to them as odd terms, and to the remaining terms as even. One may write the four-point amplitude through five loops, and the even part of the five-point amplitude at two loops, purely in terms of *pseudo-conformal* integrals. These are dimensionally-regulated integrals whose off-shell continuation is invariant under conformal transformations of dual coordinates whose differences are momenta.

As shown by Drummond, Henn, Korchemsky, and Sokatchev (DHKS), the dual conformal symmetry gives rise [12, 13] to an anomalous Ward identity in dimensional regularization. This Ward identity fixes, to all loop orders, the finite parts of the expectation value of the Wilson loops corresponding to four- and five-point scattering amplitudes (up to a constant term interpretable as an ultraviolet scale adjustment). The same authors also showed that these unique solutions of the anomalous Ward identity coincide with the four- and five-point two-loop Wilson loops and amplitudes as computed in perturbation theory (and again normalized by the tree amplitude). Given as well the one-loop equivalence between Wilson loops and amplitudes [8, 9] and the strong coupling results of Alday and Maldacena [7], we may expect that the same equivalence holds for the four-point amplitude and the even part of the five-point amplitude, to all orders in perturbation theory; indeed, into the strong-coupling regime. The BDS ansatz is in fact a solution to the anomalous Ward identity [12]. Because of the uniqueness imposed by the Ward identity, the BDS ansatz too should be expected to hold to all orders for four- and five-point amplitudes. It has been checked for the four-point amplitude through three loops [3, 6] and for the five-point amplitude through two loops [14, 15].

What about amplitudes with a larger number of legs? Alday and Maldacena have shown [16] that in the limit of a large number of legs, the Wilson loop calculation does not agree with the BDS ansatz. This result might imply that the connection between Wilson loops and the amplitudes breaks down; or it might mean that the BDS ansatz breaks down for more than five external legs. But at how many legs and how many loops might the breakdown occur?

It is possible to examine the ansatz for consistency in different kinematic limits. Recently, the BDS ansatz has been examined in various types of Regge, or high-energy, limits of the scattering. The four- and five-point amplitudes appear to be consistent in all such limits [8, 17–19]. Indeed, higher-order coefficients in the Regge slope parameter and other high-energy quantities can be extracted from such limits. On the other hand, study of a particular multi-Regge limit of $2 \to 4$ scattering, and also of $3 \to 3$ scattering, appears to indicate a difficulty with the ansatz for the six-gluon amplitude starting at two loops [19].

In order to test the BDS ansatz directly, we have computed the parity-even part of the two-loop six-point MHV amplitude in the $\mathcal{N} = 4$ supersymmetric gauge theory. With assistance from the work of Drummond, Henn, Korchemsky, and Sokatchev [20, 21], we can also test the correspondence with the calculation of a hexagonal Wilson loop. Six external legs marks the first appearance of cross ratios invariant under the dual conformal transformations; the finite part of the Wilson loop is no longer fixed by the anomalous Ward identity, but is determined only up to a function of these cross ratios. Six external legs also marks the first appearance of non-MHV amplitudes. The basic Wilson loop is insensitive to the helicities of the external gluons; hence it cannot equal these other six-point helicity amplitudes, even at one loop. The question of an iterative or exponential structure for the non-MHV amplitudes is an interesting one, but we shall not explore it in the present paper.

We perform the calculation using the unitarity-based method, employing a variety of four-dimensional and D-dimensional cuts to express the amplitude in terms of a selected set of six-point two-loop Feynman integrals. The result may be expressed as a sum of pseudo-conformal integrals [10], in close analogy with the four-point amplitude through five loops [6, 11, 22–24] and the parity-even part of the five-point amplitude through two loops [5, 14, 15, 25]. There are some additional integrals in the one- and two-loop six-point amplitudes, whose pseudo-conformal nature is less clear. Their integrands vanish as $D \to 4$, yet their integrals can be nonvanishing in this limit. However, their one- and two-loop contributions conspire to cancel in the logarithm of the amplitude (which is what really is needed to test the BDS ansatz) in the limit $\epsilon \to 0$. We then evaluate the integrals using the packages `AMBRE` [26] and `MB` [27] and compute the amplitude numerically at a variety of kinematic points. The structure of the infrared singularities is known [28, 29], and agrees with the pole terms in our expression. The finite remainders are tested numerically against the BDS ansatz, and against values for the corresponding Wilson loop [21].

The paper is organized as follows. In section II, we review the ABDK and BDS ansätze, the structure of scattering amplitudes at strong coupling, the difficulty that appears as the number of external legs becomes large, and dual conformal invariance. In section III, we give the integrand, and outline its calculation via the unitarity method. In section IV we present our results for the amplitude and compare these to the results of the hexagonal Wilson loop calculation. Section V gives properties that the remainder function must satisfy. We give our conclusions and summarize open problems in section VI. The appendices contain results for the integrals.

II. REVIEW

A. ABDK/BDS Ansatz

In an $SU(N_c)$ gauge theory the leading-color contributions to the L-loop gauge-theory n-point amplitudes can be written as

$$\mathcal{A}_n^{(L)} \;=\; g^{n-2}\, a^L \sum_{\rho} \,\mathrm{Tr}(T^{a_{\rho(1)}} T^{a_{\rho(2)}} \ldots T^{a_{\rho(n)}})\, A_n^{(L)}(\rho(1), \rho(2), \ldots, \rho(n))\,, \tag{2.1}$$

where

$$a \equiv (4\pi e^{-\gamma})^{\epsilon} \frac{\lambda}{8\pi^2}\,. \tag{2.2}$$

Here $\lambda = g^2 N_c$ is the 't Hooft parameter, g is the Yang-Mills coupling, and γ is Euler's constant. The sum is over non-cyclic permutations of the external legs. We have suppressed the momenta and helicities k_i and λ_i, leaving only the index i as a label. This decomposition holds for any amplitude when all particles are in the adjoint representation. We will find it convenient to scale out the tree amplitude, defining

$$M_n^{(L)}(\epsilon) \equiv A_n^{(L)}/A_n^{(0)}\,. \tag{2.3}$$

In planar $\mathcal{N} = 4$ supersymmetric gauge theory, amplitudes computed to date satisfy an iteration relation. At two loops, the iteration conjecture expresses n-point amplitudes entirely in terms of one-loop amplitudes and a set of constants [3]. For two-loop MHV amplitudes the ABDK conjecture reads

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)\,, \tag{2.4}$$

where

$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)\,, \qquad\qquad C^{(2)} = -\zeta_2^2/2\,. \tag{2.5}$$

The form (2.4) was based on explicit computations of both the four-point two-loop amplitude [23, 30] and of the splitting amplitudes [3], which control the behavior of the amplitudes as two momenta become collinear. The collinear splitting amplitude has an iterative property, analogous to eq. (2.4), which guarantees that the ansatz (2.4) for MHV amplitudes has the correct collinear limits for any n. In addition, MHV amplitudes have no poles as a single multi-particle kinematic invariant vanishes, $(k_i + \cdots + k_j)^2 \to 0$ for $j > (i+1)$ mod n; so factorization in such channels is satisfied trivially by eq. (2.4). For the five-point amplitude, eq. (2.4) has also been confirmed by direct calculation [14, 15]. Any violation of eq. (2.4) beyond five external legs must necessarily be expressed by a function which vanishes as pairs of color-adjacent momenta become collinear. We shall see in section V that such a remainder function is, however, detectable in the triple-collinear limit in which three color-adjacent momenta become collinear. This limit can first be achieved for $n = 6$.

The iterative structure in eq. (2.4), together with the exponential nature of infrared divergences [28, 29], suggest that an all-orders resummation should be possible. In ref. [6] the three-loop generalization for $n = 4$ was found by direct calculation, guiding the all-loop order BDS proposal,

$$\ln \mathcal{M}_n = \sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon) \Big)\,, \tag{2.6}$$

where

$$\mathcal{M}_n = \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon) \tag{2.7}$$

is the resummed all-loop amplitude. The quantity $M_n^{(1)}(l\epsilon)$ is the dimensionally-regulated one-loop amplitude, with the tree scaled out according to eq. (2.3), and with $\epsilon \to l\epsilon$. Each

$f^{(l)}(\epsilon)$ is a three-term series in ϵ, beginning at $\mathcal{O}(\epsilon^0)$,

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} \,. \tag{2.8}$$

The constant $f_0^{(l)}$ is the planar cusp anomalous dimension [31], $f_0^{(l)} = \frac{1}{4}\,\hat{\gamma}_K^{(l)}$.

In order to test the ABDK/BDS ansatz, it is convenient to define an l-loop *remainder function* $R_n^{(l)}$ to be the difference between the actual l-loop rescaled amplitude $M_n^{(l)}$ and the ABDK/BDS prediction for it, in the limit $\epsilon \to 0$. This function is finite as $\epsilon \to 0$, because the BDS ansatz has all the correct infrared singularities. It is only defined in the limit $\epsilon \to 0$ because the two-loop ansatz does not hold beyond $\mathcal{O}(\epsilon^0)$, even for $n = 4$. For example, at two loops the remainder function is defined by,

$$R_n^{(2)} \equiv \lim_{\epsilon\to 0} \left[M_n^{(2)}(\epsilon) - \left(\frac{1}{2} (M_n^{(1)}(\epsilon))^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} \right) \right] . \tag{2.9}$$

Notice that the combination $M_n^{(2)} - 1/2(M_n^{(1)})^2$ appearing in eq. (2.9) is the order a^2 term in the logarithm of the amplitude (2.6).

The one-loop MHV amplitudes entering the ABDK/BDS ansatz were computed some time ago [32], with the result

$$M_n^{(1)}(\epsilon) = -\frac{1}{2}\frac{1}{\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) + \mathcal{O}(\epsilon) \,, \tag{2.10}$$

where we use the normalizations of refs. [3, 6, 11]. The "0" argument in the finite part $F_n^{(1)}(0)$ signifies that we have taken $\epsilon \to 0$. These terms have the form,

$$F_n^{(1)}(0) = \frac{1}{2} \sum_{i=1}^{n} g_{n,i} \,, \tag{2.11}$$

where

$$g_{n,i} \;=\; - \sum_{r=2}^{\lfloor n/2 \rfloor -1} \ln\left(\frac{-s_{i\cdots(i+r-1)}}{-s_{i\cdots(i+r)}} \right) \ln\left(\frac{-s_{(i+1)\cdots(i+r)}}{-s_{i\cdots(i+r)}} \right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2 \,, \tag{2.12}$$

in which $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Here $s_{i\cdots j} = (k_i + \cdots + k_j)^2$ are the momentum invariants. (All indices are understood to be mod n.) The form of $D_{n,i}$ and $L_{n,i}$ depends upon whether n is odd or even. For the even case ($n = 2m$) these quantities are given by

$$\begin{aligned}
D_{2m,i} \;&=\; - \sum_{r=2}^{m-2} \mathrm{Li}_2\left(1 - \frac{s_{i\cdots(i+r-1)}\, s_{(i-1)\cdots(i+r)}}{s_{i\cdots(i+r)}\, s_{(i-1)\cdots(i+r-1)}} \right) - \frac{1}{2} \mathrm{Li}_2\left(1 - \frac{s_{i\cdots(i+m-2)}\, s_{(i-1)\cdots(i+m-1)}}{s_{i\cdots(i+m-1)}\, s_{(i-1)\cdots(i+m-2)}} \right) , \\
L_{2m,i} \;&=\; \frac{1}{4} \ln^2\left(\frac{-s_{i\cdots(i+m-1)}}{-s_{(i+1)\cdots(i+m)}} \right) .
\end{aligned} \tag{2.13}$$

In the odd case ($n = 2m+1$), we have,

$$
\begin{aligned}
D_{2m+1,i} &= -\sum_{r=2}^{m-1} \mathrm{Li}_2\left(1 - \frac{s_{i\cdots(i+r-1)}s_{(i-1)\cdots(i+r)}}{s_{i\cdots(i+r)}s_{(i-1)\cdots(i+r-1)}}\right), \\
L_{2m+1,i} &= -\frac{1}{2}\ln\left(\frac{-s_{i\cdots(i+m-1)}}{-s_{i\cdots(i+m)}}\right)\ln\left(\frac{-s_{(i+1)\cdots(i+m)}}{-s_{(i-1)\cdots(i+m-1)}}\right).
\end{aligned}
\tag{2.14}
$$

For $n = 4$ the above formula does not hold; in that case the finite part is simply

$$
F_4^{(1)}(0) = \frac{1}{2}\ln^2\left(\frac{-t}{-s}\right) + 4\zeta_2\,. \tag{2.15}
$$

To make contact with the string theory literature [7, 16, 33] we define,

$$
f(\lambda) = 4\sum_{l=1}^{\infty} a^l f_0^{(l)}\,, \qquad g(\lambda) = 2\sum_{l=2}^{\infty}\frac{a^l}{l} f_1^{(l)}\,, \qquad k(\lambda) = -\frac{1}{2}\sum_{l=2}^{\infty}\frac{a^l}{l^2} f_2^{(l)}\,. \tag{2.16}
$$

In terms of these functions, the BDS ansatz (2.6) may be written as,

$$
\ln\mathcal{M}_n = \mathrm{Div}_n + \frac{f(\lambda)}{4}F^{(1)}(0) + nk(\lambda) + C(\lambda)\,. \tag{2.17}
$$

The infrared-divergent part is

$$
\mathrm{Div}_n = -\sum_{i=1}^{n}\left[\frac{1}{8\epsilon^2}f^{(-2)}\left(\frac{\lambda\mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right) + \frac{1}{4\epsilon}g^{(-1)}\left(\frac{\lambda\mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right)\right], \tag{2.18}
$$

where

$$
\left(\lambda\frac{d}{d\lambda}\right)^2 f^{(-2)}(\lambda) = f(\lambda)\,, \qquad \left(\lambda\frac{d}{d\lambda}\right)g^{(-1)}(\lambda) = g(\lambda)\,, \tag{2.19}
$$

and $\mu_{IR}^2 = 4\pi e^{-\gamma}\mu^2$. The first few orders of both the weak [6, 11, 29, 34, 35] and strong [36–38] coupling expansion for $f(\lambda)$ have been computed, with the result

$$
\begin{aligned}
f(\lambda) &= \frac{\lambda}{2\pi^2}\left(1 - \frac{\lambda}{48} + \frac{11\,\lambda^2}{11520} - \left(\frac{73}{1290240} + \frac{\zeta_3^2}{512\pi^6}\right)\lambda^3 + \cdots\right), \qquad \lambda\to 0\,, & (2.20)\\
f(\lambda) &= \frac{\sqrt{\lambda}}{\pi}\left(1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{\mathrm{K}}{\lambda} + \cdots\right), \qquad \lambda\to\infty\,, & (2.21)
\end{aligned}
$$

where $\mathrm{K} = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2} \simeq 0.9159656\ldots$ is the Catalan constant[1]. The function $g(\lambda)$ has also been computed through four loops [6, 40], but is less well known at strong coupling [7][2],

$$
\begin{aligned}
g(\lambda) &= -\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{2}{3}(6\zeta_5 + 5\zeta_2\zeta_3)\left(\frac{\lambda}{8\pi^2}\right)^3 - (77.56\pm 0.02)\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots\,, \quad \lambda\to 0\,, & (2.22)\\
g(\lambda) &= (1-\ln 2)\frac{\sqrt{\lambda}}{2\pi} + \cdots\,, \qquad \lambda\to\infty\,. & (2.23)
\end{aligned}
$$

[1] The third term in eq. (2.21) was first found in the expansion of an integral equation [33] in ref. [39].

[2] At the next order [38] in the strong-coupling expansion of $g(\lambda)$, one encounters difficulties with the closed-string version of dimensional regularization used in ref. [7].

Beisert, Eden and Staudacher (BES) proposed [33, 41] a striking integral equation giving the cusp anomalous dimension $f(\lambda)$ for all values of the coupling. This integral equation has passed a number of stringent tests at both weak [11, 35] and strong coupling [37, 39, 42], and is therefore very likely the correct expression.

With the cusp anomalous dimension known, the BDS ansatz (2.6) predicts the MHV amplitudes for *all* values of the coupling, up to the undetermined functions $g(\lambda)$, $k(\lambda)$ and $C(\lambda)$, which are independent of the kinematics. The ansatz has been checked through three loops at four points [3, 6]. Integral representations of the four-point amplitude have also been given at four [11] and five loops [24], though these expressions have not yet been integrated at $\mathcal{O}(\epsilon^0)$ to yield explicit functions of the external momenta. If one assumes the dual conformal invariance mentioned in the Introduction, which we shall discuss at greater length below, then the form of the finite parts in the four- and five-point amplitudes are fixed, which provides another way of arriving at the ansatz (2.6) for $n = 4$ or 5. However, beyond five external legs, the assumption of dual conformal invariance does not suffice to fix the functional form of the finite parts of MHV amplitudes.

B. Scattering Amplitudes at Strong Coupling

Alday and Maldacena have proposed a very interesting way to compute color-ordered planar scattering amplitudes at strong coupling, using the AdS/CFT correspondence [7]. They argue that the leading dependence of any amplitude on the coupling has the form $\exp(-\frac{\sqrt{\lambda}}{2\pi}A)$, where A is the regularized area of a special surface in AdS, whose definition and properties are reviewed below. Their result reproduces the BDS ansatz for the four-point amplitude at strong coupling.

Long ago, Gross and Mende [43] showed that in string theory in flat space-time, scattering amplitudes can be computed in the high-energy and fixed-angle regime using a semiclassical approach. Alday and Maldacena noticed that, thanks to the properties of anti-de Sitter space, a semiclassical calculation also suffices to compute the strong-coupling limit of amplitudes in the $\mathcal{N} = 4$ theory, for any energy. Glossing over technical details, scattering amplitudes are given by a saddle-point approximation to the world-sheet partition function with certain vertex operator insertions.

Two-dimensional duality transformations map a vertex operator of momentum k_i^μ to a null (light-like) segment pointing along the direction of the momentum of the corresponding gluon in a dual AdS space. The endpoint coordinates y of each null segment obey

$$\Delta y_i^\mu = 2\pi k_i^\mu \,. \tag{2.24}$$

Momentum conservation then implies that these segments form a closed polygon in these dual coordinates.

Alday and Maldacena do not compute the prefactor of $\exp(-\frac{\sqrt{\lambda}}{2\pi}A)$, which is subleading in the strong-coupling expansion but must contain all the dependence on the polarizations of the external particles. MHV scattering amplitudes in the $\mathcal{N} = 4$ theory are special because supersymmetry Ward identities [4, 5] imply that they are all identical up to simple spinor product factors. Equivalently, MHV amplitudes can be written as a product of the tree amplitude, and an additional factor dependent only on the momentum invariants, and not on the polarizations. In the MHV case, it is natural to identify the tree amplitude with the prefactor. (A similar proposal has also been made for the non-MHV case [44], although it is not clear how it can be consistent with the intricate structure of the one-loop amplitudes [45, 46].) Then the additional factor is given, to leading-order in the strong-coupling expansion, by the saddle-point approximation of a suitably restricted world-sheet partition function. The world sheets must be restricted to unpunctured surfaces whose boundary is the closed polygon of null dual segments. That is, up to some technical details, the factor is just the expectation value of a null Wilson loop in the dual space.

Alday and Maldacena computed the leading term in the strong-coupling expansion of the four-gluon amplitude by building an explicit solution out of cusp solutions written down earlier by Kruczenski [36]. The solution can also be found by solving the sigma-model equations [47] once the Virasoro constraints are imposed [48]. The minimal-surface approach has not yet yielded complete expressions for higher-point amplitudes, because of difficulties in solving the minimal surface conditions with the proper boundary conditions, although there has been some progress [47, 49]. However, Buchbinder [50] has shown that the infrared-divergent terms can be obtained using this approach, and that (as expected) they are consistent with the known exponentiation in gauge theories. Komargodski has argued [51] that the collinear-splitting amplitude obtained from this approach is consistent with the known perturbative splitting amplitudes.

While the original arguments have been formulated at strong coupling, the work of Alday and Maldacena also inspired the discovery of a surprising connection between MHV amplitudes and Wilson loops at weak coupling at one loop [8, 9], for the simplest two-loop amplitudes [12, 13], and perhaps also at higher orders.

C. Trouble at Large n

In the limit of a large number of external legs, Alday and Maldacena [16] argued that the situation simplifies because one may approximate the null Wilson loop by a smooth one. They considered a rectangular Wilson loop with space-like edges of length L and width T as the approximation of a null Wilson loop zig-zagging around the rectangular one. Each edge corresponds to many gluons moving in one direction, alternating with many gluons moving in an opposite direction. The expectation value of the Wilson loop is clearly divergent as the area is scaled to infinity. However, one may focus on the scale-invariant part, which is proportional to T/L. Moreover, one may consider the kinematic configuration corresponding to $T/L \gg 1$. In this case one may ignore the contribution from the sides of length L to leading order. Thus one may further approximate the Wilson loop by two parallel lines of length T at distance L from each other. In other words, the dominant T/L-dependent part of the world-sheet area is essentially T times the heavy-quark potential, computed at strong coupling [52]:

$$\ln\langle W\rangle = \sqrt{\lambda}\,\frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\,\frac{T}{L}\,, \qquad \lambda \gg 1\,. \tag{2.25}$$

Since this expression is finite as T and L are scaled to infinity, one should compare it with the T/L-dependent terms in the logarithm of the finite part of the BDS ansatz.

Alday and Maldacena [16] worked out the behavior of the BDS ansatz in this limit by making use of the all-n one-loop relation between MHV amplitudes and Wilson loops [9], and the known value of the rectangular Wilson loop at one loop. They obtained, at strong coupling and for $T \gg L$,

$$\frac{f(\lambda)}{4}\mathcal{F}_n^{(1)}(0) \;\stackrel{n\to\infty}{\longrightarrow}\; \frac{\sqrt{\lambda}}{4}\,\frac{T}{L}\,. \tag{2.26}$$

This result differs from eq. (2.25), indicating that the BDS ansatz is incomplete in this limit. It is interesting to note that the two formulæ differ by less than 10%, hinting that it may be possible to systematically correct the BDS ansatz.

While this difference appeared in a particular (and somewhat singular) kinematic configuration with a very large number of legs, the fact that it depends on T/L makes clear that it arises from a nontrivial function of momenta. As we discuss below, dual conformal symmetry suggests that the first place such a function can occur is at six points.

D. Pseudo-Conformal Integrals and Dual Conformal Invariance

We now turn to a review of the observed restrictions that dual conformal invariance places on the integrals appearing in the amplitudes [8, 10, 11, 24]. This mysterious symmetry plays an important role in our story. $\mathcal{N} = 4$ super-Yang-Mills theory is a conformal field theory at the quantum level; conformal invariance may be observed in correlation functions of operators of definite (anomalous) dimension. However, the constraints it imposes on on-shell scattering amplitudes are obscured, both by the need for an infrared regulator and by anomalies analogous to the holomorphic anomaly of collinear operators [53].

By inspecting the known results for the one-, two- and three-loop four-gluon amplitudes [6, 22, 23], Drummond, Henn, Sokatchev and Smirnov [10] observed that after continuing the external momenta of the integrals off shell (in a sense we will describe below) and then taking $D = 4$, they exhibit an $SO(2, 4)$ dual conformal symmetry. This symmetry is distinct from the four-dimensional position-space conformal group. Interestingly it holds individually for each contributing integral through five loops [8, 11, 24]. A discussion of the consequences of dual conformal symmetry for complete amplitudes, instead of individual integrals, may be found in refs. [12, 13].

The origin of dual conformal symmetry remains obscure, and its broad validity for amplitudes remains to be proven.[3] Indeed, it is not clear that dual conformal symmetry holds for all contributions to scattering amplitudes. For example, at the five-point level, the dual conformal properties of the parity-odd pieces are not apparent [15]. At least in this case, the parity-odd pieces do not enter the remainder function $R_5^{(2)}$ defined in eq. (2.9). This feature suggests that, as long as we subtract appropriate lower-loop contributions (to be specified further below), we may be able to identify simple conformal properties for the finite remainders $R_n^{(2)}$ of MHV amplitudes to all loop orders. In this vein, an anomalous conformal Ward

[3] Dual conformal symmetry was introduced in the context of multi-loop ladder integrals [54]; it has also cropped up in the two-dimensional theory of Reggeon interactions, again in the planar limit [55].

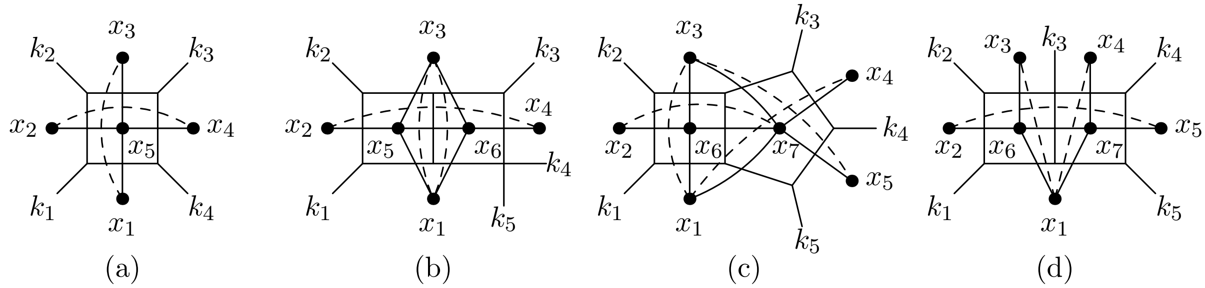

FIG. 1: Examples of pseudo-conformal integrals. Points x_i label the vertices of the dual graph, a solid line connecting two points x_i and x_j corresponds to a factor of $1/x_{ij}^2$, while a dashed line corresponds to a factor of x_{ij}^2. An integral is pseudo-conformal if the difference between the number of solid lines and dashed lines at a vertex equals 4 at the internal vertices and zero at the external vertices. Graphs (*b*), (*c*) and (*d*) show that the integrals appearing in the even part of the five-point two-loop amplitude are pseudo-conformal.

identity has been proven for the finite remainders of Wilson loops [13], and later found at strong coupling [51]. The BDS ansatz was shown to obey the same Ward identity, which was then proposed to hold for MHV amplitudes [12].

Dual conformal symmetry is most transparent in terms of dual variables x_i which are related to the gluon momenta in a way analogous to eq. (2.24),

$$k_i = x_{i+1} - x_i \,, \tag{2.27}$$

and similarly for the loop momenta. Formally, in an integral we may identify the variables x_i as the positions of the vertices of a dual graph. In this construction the momentum conservation constraint is replaced by an invariance under uniform shifts of the dual coordinates x_i. Since the dual variables are unconstrained and the parametrization (2.27) automatically satisfies momentum conservation, one may define an inversion operator

$$I = \sum_i I_i \,, \qquad I_i : \; x_i^\mu \mapsto \frac{x_i^\mu}{x_i^2} \,. \tag{2.28}$$

One may also define conformal boost transformations of the dual variables x; they are generated by

$$K^\mu = \sum_i K_i^\mu \,, \qquad K_i^\mu = 2x_i^\mu x_i \cdot \partial_i - x_i^2 \partial_i^\mu + \cdots \,, \tag{2.29}$$

where the ellipsis stand for terms which are irrelevant when K acts on coordinates. Invariance under I implies invariance under K, because the conformal boosts are generated by two inversions, with an infinitesimal translation of x in between.

The principal conformal-invariance constraints on integrals constructed from the invariants x_{ij}^2 are exposed by performing the inversion I. Since dimensional regularization breaks the dual conformal invariance, for the purposes of exposing the symmetry we adopt a different infrared regularization of the integrals. We take the external legs off shell, letting $k_i^2 \neq 0$, instead of using a dimensional regulator as in the rest of the paper. Under the inversion, the Mandelstam invariants and the integration measure transform as

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}\,, \qquad d^4x_5 \to \frac{d^4x_5}{(x_5^2)^4}\,, \qquad d^4x_6 \to \frac{d^4x_6}{(x_6^2)^4}\,. \tag{2.30}$$

Simple dual diagrams may be used to identify all the integrals that are invariant under I; see refs. [10, 11, 24] for a detailed discussion. We call such integrals "pseudo-conformal" because in the amplitudes we actually use the dimensionally-regulated versions of the integrals, which breaks the conformal invariance but guarantees the gauge-invariance of the amplitudes.

While pseudo-conformal integrals were initially identified in four-point scattering amplitudes, they apparently play a more general role in the planar $\mathcal{N}=4$ theory; they also appear in n-point one-loop MHV amplitudes [32], as well as the even part of the two-loop five-point amplitude [14, 15] (see fig. 1). Their relevance for non-MHV amplitudes, for example at one loop [45], remains to be clarified. In this paper we show that for the two-loop six-point MHV amplitude, they again appear in the even part though some additional integrals also appear. (We shall not compute the odd part in this paper.)

More generally, conformal invariance implies that, as for two-dimensional conformal field theories, conformally invariant quantities can depend only on the cross ratios,

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}\,, \tag{2.31}$$

where i, j, k and l are any four vertices of the dual graph. In scattering amplitudes we must exclude cross ratios that vanish or diverge on account of an x_{ij}^2 vanishing due to the on-shell conditions, $x_{i,i+1}^2 = k_i^2 \to 0$. Consequently, the only cross ratios that may appear in a conformally invariant part of an on-shell amplitude are those with $|i-j| \geq 2 \mod n$ for all $i, j = 1, \ldots, n$; otherwise x_{ij} would correspond to a massless momentum. We remark that there are no conformal cross ratios for $n = 4, 5$, as it is not possible to form a cross ratio without encountering a vanishing x_{ij}^2 [12].

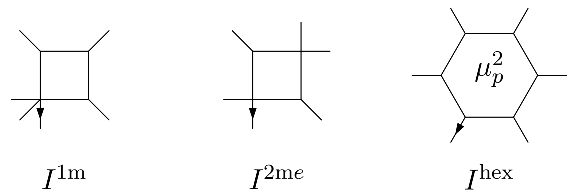

FIG. 2: The three independent integrals which contribute to the six-particle amplitude at one loop. The external momenta are labeled clockwise, beginning with k_1 which is denoted by an arrow. The μ_p^2 in the hexagon integral indicates that a factor of the square of the (-2ϵ)-dimensional components of the loop momentum is to be inserted in the numerator of the integrand.

III. SIX-POINT INTEGRAND

Before turning to our calculation of the two-loop amplitude, we present the one-loop six-point amplitude [5]. We drop the parity-odd terms, which are proportional to the Levi-Civita tensor and are also $\mathcal{O}(\epsilon)$. We split the even part into two pieces,

$$M_6^{(1),D=4-2\epsilon}(\epsilon) = M_6^{(1),D=4}(\epsilon) + M_6^{(1),\mu}(\epsilon)\,, \tag{3.1}$$

where

$$M_6^{(1),D=4}(\epsilon) = -\frac{1}{4}\sum_{6\text{ perms.}}\left[s_{45}s_{56}I^{1m}(\epsilon) + \frac{1}{2}(s_{123}s_{345} - s_{12}s_{45})I^{2me}(\epsilon)\right]\,, \tag{3.2}$$

and

$$M_6^{(1),\mu}(\epsilon) = -\frac{1}{4}\,\mathrm{tr}[123456]I^{hex}(\epsilon)\,. \tag{3.3}$$

The first piece contains scalar box integrals, which are constructible solely from cuts in $D = 4$. The hexagon integral in the second piece contains a numerator factor of μ_p^2, as indicated in fig. 2, which can only be detected by cuts in which the cut loop momentum p has a nonvanishing (-2ϵ)-dimensional component μ_p: $p^2 \equiv p_{[4]}^2 - \mu_p^2$, so that μ_p^2 is positive. (The overall scale μ arising from dimensional regularization, *e.g.* in eq. (2.10), should not be confused with μ_p.)

The three integrals appearing in the one-loop amplitude are defined in fig. 2, using the convention that each loop momentum integral is normalized according to

$$-i\pi^{-D/2}e^{\epsilon\gamma}\int d^D p\,. \tag{3.4}$$

The coefficient of the hexagon integral involves the quantity

$$\mathrm{tr}[123456] \equiv \mathrm{tr}[\not{k}_1\not{k}_2\not{k}_3\not{k}_4\not{k}_5\not{k}_6] = s_{123}s_{234}s_{345} - s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}\,. \tag{3.5}$$

The sum in eq. (3.2) runs over the six cyclic permutations of the external momenta k_i. There is a symmetry factor of 1/2 in front of $I^{2\mathrm{me}}$ to correct for a double count in this case — there are only three separate $I^{2\mathrm{me}}$ integrals.

A. Construction of the Integrand from Unitarity Cuts

We computed the two-loop six-point MHV amplitude using the unitarity method [32, 45, 56]. We know on general grounds that the amplitude can be written in terms of Feynman integrals multiplied by rational coefficients.

In principle, we should consider all two-loop six-point integrals. We can reduce this rather large set of integrals using the no-triangle constraint, which states that one can find a representation for the amplitude in which no integral with a triangle (or bubble) subintegral appears. Using generalized unitarity, we may establish this constraint for two classes of potential triangle contributions: those which may be excised from a multi-loop topology by cutting only gluon lines, and (using supersymmetry Ward identities) those which may be excised by cutting at most five legs [11]. We shall assume that the no-triangle constraint holds as well for the other contributions.

We expect, based on previous calculations of four-point amplitudes at two, three, and four loops (as well as the consistency of the five-loop construction) and of the five-point amplitude at two loops, that essentially only pseudo-conformal integrals will be required for the six-point amplitude. The basic topologies for the integrals that appear in the planar six-point MHV amplitude are shown in figs. 3 and 4. Of these, all are pseudo-conformal, except for (14) and (15), whose integrands vanish as $D \to 4$, yet their integrals are nonvanishing in this limit. The complete set of inequivalent pseudo-conformal integrals (including numerator factors) is presented in fig. 7. The no-triangle constraint would also follow from the stronger assumption of pseudo-conformality, were we to make it. We will not; our calculation can be seen as a test of it instead. The appearance of integrals (14) and (15) shows that additional pieces, unobvious from naive considerations of dual conformal invariance, enter.

Given the no-triangle constraint, iterated two-particle cuts suffice to determine the full integrand. While the four-dimensional cuts do not suffice to determine the integrand completely, they can determine most terms. The four-dimensional double two-particle cuts, shown in fig. 5, are particularly convenient to compute because they can be built out of

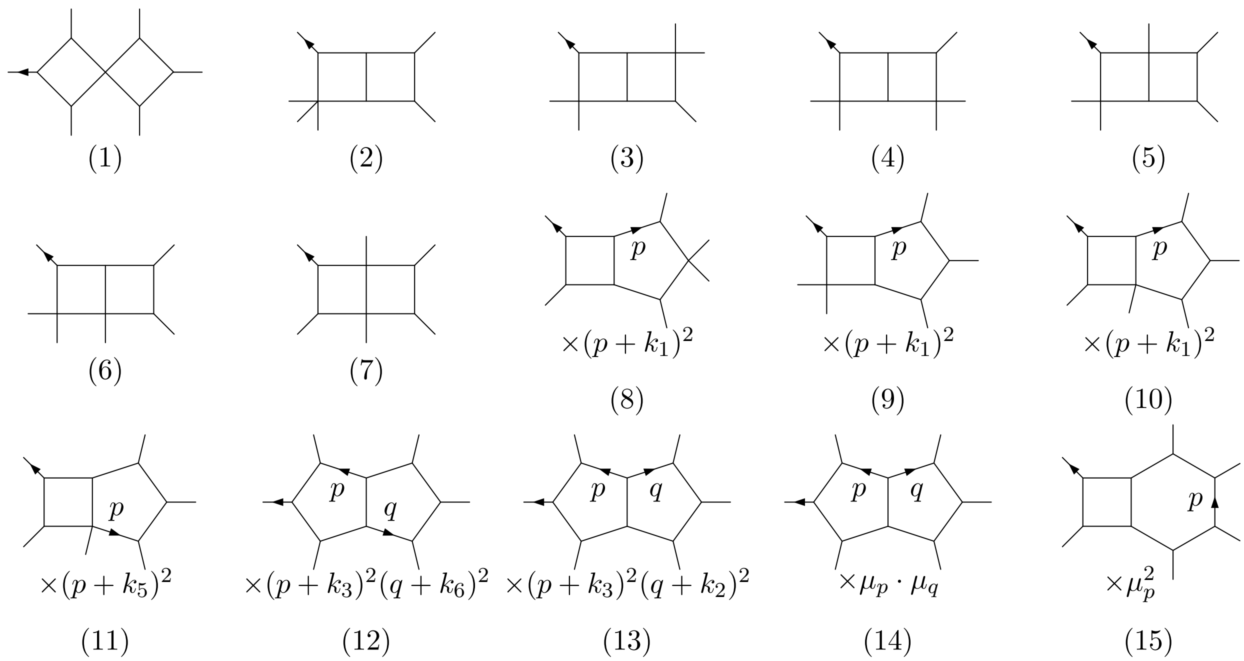

FIG. 3: The 15 independent integrals which contribute to the even part of the six-particle amplitude at two loops. The external momenta are labeled clockwise with k_1 denoted by an arrow. Integrals (8)–(15) are defined to include the indicated numerator factors involving the loop momenta. In the last two integrals, μ_p denotes the (-2ϵ)-dimensional component of the loop momentum p.

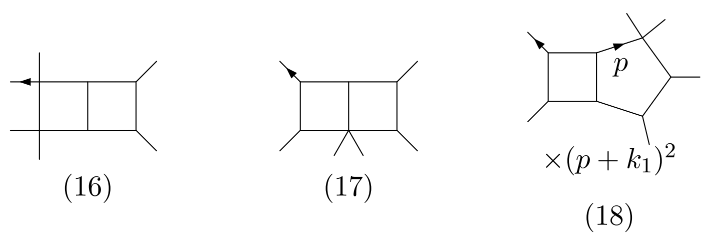

FIG. 4: The three independent two-loop diagrams which can be made pseudo-conformal by including appropriate numerators but which do not contribute to the amplitude (*i.e.*, they enter with zero coefficient).

MHV tree amplitudes.

As example, consider the helicity assignment $(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$, and compute cut a. Because of a Ward identity for $\mathcal{N} = 4$ supersymmetry [4, 5], once we have divided by the tree amplitude, the expression is in fact independent of the placement of the negative helicities. The labeling of the external legs is shown in fig. 6.

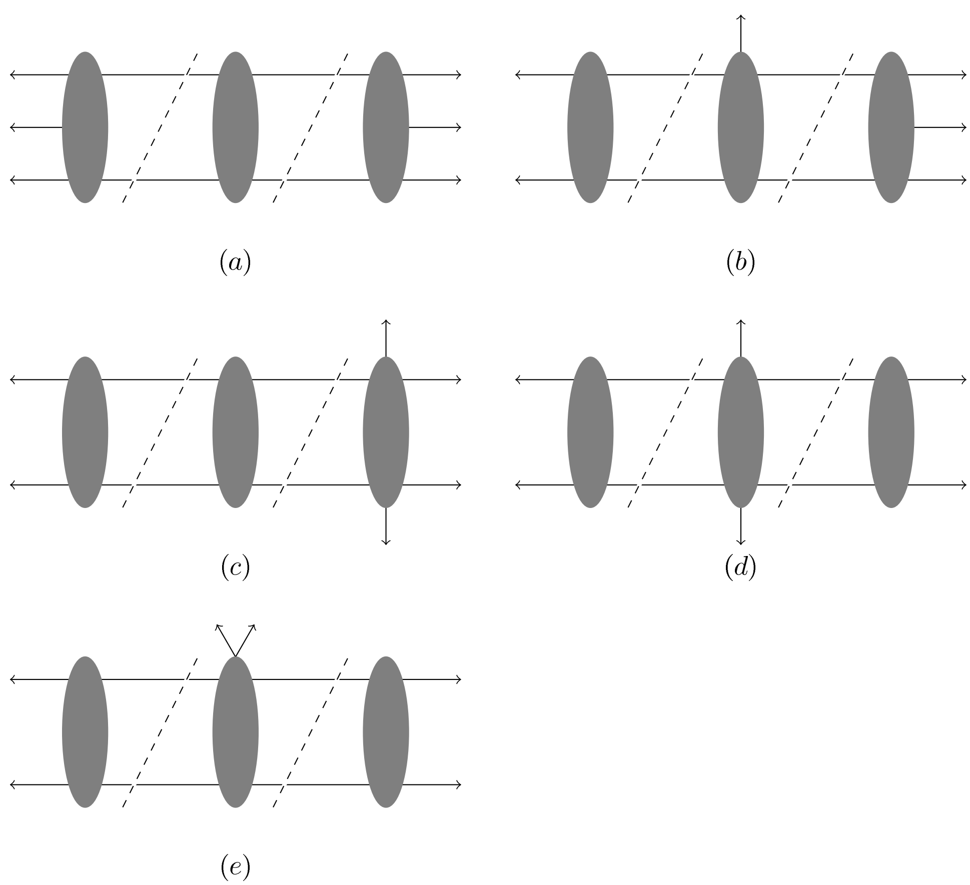

FIG. 5: The double two-particle cuts used to determine the integrand.

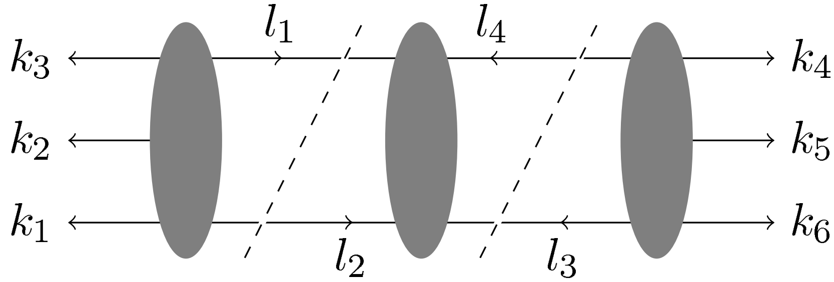

FIG. 6: Labeling of momenta for cut a.

The product of the three tree amplitudes corresponding to cut a is

$$i\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,l_1\rangle\,\langle l_1\,l_2\rangle\,\langle l_2\,1\rangle} \times i\frac{\langle (-l_2)\,(-l_1)\rangle^3}{\langle (-l_1)\,(-l_4)\rangle\,\langle (-l_4)\,(-l_3)\rangle\,\langle (-l_3)\,(-l_2)\rangle} \times i\frac{\langle l_3\,l_4\rangle^3}{\langle l_4\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,l_3\rangle}\,. \tag{3.6}$$

After spinor simplifications, dividing by the tree amplitude, and rationalizing denominators to Lorentz products, we find that the numerator can be written as follows,

$$\text{tr}_+\left[\not{l}_2\,\not{k}_1\,\not{k}_6\,\not{l}_3\,\not{l}_4\,\not{k}_4\,\not{k}_3\,\not{l}_1\right]\ \text{tr}_+\left[\not{l}_2\,\not{l}_3\,\not{l}_4\,\not{l}_1\right]\,, \tag{3.7}$$

where $\mathrm{tr}_+[\cdots] = \frac{1}{2}\,\mathrm{tr}[(1+\gamma_5)\cdots]$.

Upon expanding the traces, we find both even and odd terms. The odd terms contain a factor $\epsilon(a,b,c,d) \equiv \epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma$, whose origin lies in the presence of the γ_5 matrix inside the traces. (The product of two epsilon tensors would yield an even term, but only the longer trace here can actually produce an epsilon tensor, as $\epsilon(l_1,l_2,l_3,l_4)$ vanishes because of momentum conservation.) We ignore the odd terms in our calculation.

In order to identify the coefficients of the integrals in fig. 3, we should use momentum conservation to re-express all Lorentz invariants in terms of independent invariants. The required simplifications can be done analytically, but in some cases (for example cut d) it is easier to do them numerically, by matching to a target expression.

Doing so, we obtain for the final result of cut a, in the $(3,4,5,6,1,2)$ permutation with respect to fig. 3,

$$
\begin{aligned}
\frac{1}{4}\Bigg[& \frac{s_{123}^2 s_{34} s_{61} - s_{123}^2 s_{234} s_{345} + s_{123} s_{234} s_{12} s_{45} + s_{123} s_{345} s_{23} s_{56}}{(k_1+l_2)^2 (k_3+l_1)^2 (k_4+l_4)^2 (k_6+l_3)^2} \\
& + \frac{s_{123}^2 s_{345} - s_{123} s_{12} s_{45}}{(k_3+l_1)^2 (l_2+l_3)^2 (k_6+l_3)^2} + \frac{s_{123}^2 s_{234} - s_{123} s_{23} s_{56}}{(k_1+l_2)^2 (l_2+l_3)^2 (k_4+l_4)^2} \\
& + \frac{s_{123}^2 s_{34}}{(k_3+l_1)^2 (l_2+l_3)^2 (k_4+l_4)^2} + \frac{s_{123}^2 s_{61}}{(k_1+l_2)^2 (l_2+l_3)^2 (k_6+l_3)^2} \\
& + \frac{s_{123} s_{12} s_{23} (k_6-l_2)^2}{(k_1+l_2)^2 (k_3+l_1)^2 (l_2+l_3)^2 (k_6+l_3)^2} + \frac{s_{123} s_{12} s_{23} (k_4-l_1)^2}{(k_1+l_2)^2 (k_3+l_1)^2 (l_2+l_3)^2 (k_4+l_4)^2} \\
& + \frac{s_{123} s_{45} s_{56} (k_3-l_4)^2}{(k_3+l_1)^2 (l_2+l_3)^2 (k_4+l_4)^2 (k_6+l_3)^2} + \frac{s_{123} s_{45} s_{56} (k_1-l_3)^2}{(k_1+l_2)^2 (l_2+l_3)^2 (k_4+l_4)^2 (k_6+l_3)^2} \\
& + \frac{1}{(k_1+l_2)^2 (k_3+l_1)^2 (l_2+l_3)^2 (k_4+l_4)^2 (k_6+l_3)^2} \\
& \qquad \times \Big(-s_{123}^2 s_{61} (k_3-l_4)^2 (k_4-l_1)^2 - s_{123}^2 s_{34} (k_1-l_3)^2 (k_6-l_2)^2 \\
& \qquad\quad + s_{123} (s_{123} s_{234} - s_{23} s_{56}) (k_3-l_4)^2 (k_6-l_2)^2 \\
& \qquad\quad + s_{123} (s_{123} s_{345} - s_{12} s_{45}) (k_1-l_3)^2 (k_4-l_1)^2 \Big) \Bigg] .
\end{aligned}
\tag{3.8}
$$

From this expression we can read off the coefficients of every integral in figs. 3 and 4 that is nonvanishing on cut a for four-dimensional values of the cut momenta. These integrals are, in order of their appearance in eq. (3.8), $I^{(1)}$, $I^{(3)}$ (twice), $I^{(4)}$ (twice), $I^{(9)}$ (four times), $I^{(12)}$ and $I^{(13)}$ (each twice). For example, from the second term, up to a symmetry factor, we can simply read off the coefficient of $I^{(3)}$ to be $s_{123}(s_{123}s_{345} - s_{12}s_{45})$, or equivalently $s_{234}(s_{123}s_{234} - s_{23}s_{56})$ in the labeling of fig. 3. The numerator in the second term on the fourth line produces the required loop-momentum dependent factor for a reflected version of $I^{(9)}$, corresponding to the $(4,3,2,1,6,5)$ permutation. The coefficient is $s_{123}s_{12}s_{23}$, or $s_{234}s_{34}s_{23}$ in the figure's labeling. The coefficients of the remaining integrals can be determined by the other cuts in fig. 5.

At one loop in any supersymmetric theory, the improved ultraviolet power-counting ensures that any rational terms in an amplitude are linked to terms with branch cuts. That is, all terms in an amplitude can be determined solely from the standard integral basis, which can be detected in four-dimensional cuts. Beyond one loop, four-dimensional cuts no longer suffice for $\mathcal{N}=1$ supersymmetric theories [57]. In the $\mathcal{N}=4$ theory, four-point amplitudes through five loops are determined solely by their four-dimensional cuts. The same is true for the even terms in the five-point amplitude. (It is no longer true for the odd terms, but in any case we are ignoring the corresponding terms in the six-point amplitude.) However, there is no proof other than explicit computation of this observation. Accordingly, we cannot be certain that four-dimensional cuts will suffice for our calculation.

Indeed, the hexagon integral in fig. 2 will contribute terms of $\mathcal{O}(\epsilon)$ to the one-loop amplitude. In the term in the iteration relation (2.4) in which the one-loop amplitude appears squared, the product of such terms with singular terms in $M_6^{(1)}(\epsilon)$ survives to give $\mathcal{O}(\epsilon^{-1})$ and finite contributions. We will see in section IV A that such contributions are offset by those coming from the last integral (15) from fig. 3, induced by the (-2ϵ)-dimensional components of the loop momentum. These contributions must be computed using D-dimensional cuts, either making use of prior computations [5], or by direct computation. We have computed two cuts, corresponding to fig. 5(a) and (c), using D-dimensional cuts. These cuts determine the coefficients of integrals (14) and (15), respectively, in fig. 3. The calculations were done using the same approach used in refs. [3, 15]. While we can no longer use standard helicity states for the computation, we can take advantage of the equivalence between the $\mathcal{N}=4$ theory and ten-dimensional $\mathcal{N}=1$ super-Yang-Mills theory compactified on a torus. We compute the cuts with the spin algebra performed in the ten-dimensional theory, keeping loop momenta in D dimensions. (External momenta can be taken to be four-dimensional.) The ten-dimensional gluon corresponds to a four-dimensional gluon and six real scalar degrees of freedom, while the ten-dimensional Majorana-Weyl fermions correspond to four flavors of gluinos.

B. Presentation of the Integrand

By analyzing the cuts outlined in the previous section, we find the complete expression for the parity-even part of the two-loop six-particle amplitude to be

$$M_6^{(2),D=4-2\epsilon}(\epsilon) = M_6^{(2),D=4}(\epsilon) + M_6^{(2),\mu}(\epsilon)\,, \tag{3.9}$$

where

$$\begin{aligned} M_6^{(2),D=4}(\epsilon) \;=\; \frac{1}{16}\sum_{12\text{ perms.}} \bigg[& \frac{1}{4}c_1 I^{(1)}(\epsilon) + c_2 I^{(2)}(\epsilon) + \frac{1}{2}c_3 I^{(3)}(\epsilon) + \frac{1}{2}c_4 I^{(4)}(\epsilon) + c_5 I^{(5)}(\epsilon) \\ & + c_6 I^{(6)}(\epsilon) + \frac{1}{4}c_7 I^{(7)}(\epsilon) + \frac{1}{2}c_8 I^{(8)}(\epsilon) + c_9 I^{(9)}(\epsilon) + c_{10} I^{(10)}(\epsilon) \\ & + c_{11} I^{(11)}(\epsilon) + \frac{1}{2}c_{12} I^{(12)}(\epsilon) + \frac{1}{2}c_{13} I^{(13)}(\epsilon) \bigg]\,, \end{aligned} \tag{3.10}$$

and

$$M_6^{(2),\mu}(\epsilon) \;=\; \frac{1}{16}\sum_{12\text{ perms.}} \left[\frac{1}{4}c_{14} I^{(14)}(\epsilon) + \frac{1}{2}c_{15} I^{(15)}(\epsilon)\right], \tag{3.11}$$

involving the 15 independent integrals shown in fig. 3. As with the one-loop amplitude (3.1), we have separated the integrals (1)–(13), which are constructible solely from four-dimensional cuts, from integrals (14) and (15), in which the (-2ϵ)-dimensional components of loop momenta μ_p and μ_q are explicitly present. (The $D = 4$ superscript in $M_6^{(2),D=4}(\epsilon)$ refers to the lack of μ_p and μ_q terms in the numerators of the integrands; the argument ϵ indicates the dependence of the integrals on the dimensional regularization parameter.) Of the 1/16 overall normalization in eqs. (3.10) and (3.11), 1/4 is due to our choice of normalization in eq. (2.1) and the other factor of 1/4 emerges from the calculation of the unitarity cuts.

The sum runs over the 12 cyclic and reflection permutations of external legs,

$$\begin{aligned} &(1,2,3,4,5,6), \quad (2,3,4,5,6,1), \quad (3,4,5,6,1,2), \quad (4,5,6,1,2,3), \\ &(5,6,1,2,3,4), \quad (6,1,2,3,4,5), \quad (6,5,4,3,2,1), \quad (1,6,5,4,3,2), \\ &(2,1,6,5,4,3), \quad (3,2,1,6,5,4), \quad (4,3,2,1,6,5), \quad (5,4,3,2,1,6). \end{aligned} \tag{3.12}$$

The numerical coefficients in each term of eqs. (3.10) and (3.11) are symmetry factors to remove double counts in the permutation sum. The coefficients for the $(1,2,3,4,5,6)$ permutation are,

$$c_1 \;=\; s_{61}s_{34}s_{123}s_{345} + s_{12}s_{45}s_{234}s_{345} + s_{345}^2(s_{23}s_{56} - s_{123}s_{234})\,,$$

$$
\begin{aligned}
c_2 &= 2s_{12}s_{23}^2\,, \\
c_3 &= s_{234}(s_{123}s_{234} - s_{23}s_{56})\,, \\
c_4 &= s_{12}s_{234}^2\,, \\
c_5 &= s_{34}(s_{123}s_{234} - 2s_{23}s_{56})\,, \\
c_6 &= -s_{12}s_{23}s_{234}\,, \\
c_7 &= 2s_{123}s_{234}s_{345} - 4s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}\,, \\
c_8 &= 2s_{61}(s_{234}s_{345} - s_{61}s_{34})\,, \\
c_9 &= s_{23}s_{34}s_{234}\,, \\
c_{10} &= s_{23}(2s_{61}s_{34} - s_{234}s_{345})\,, \\
c_{11} &= s_{12}s_{23}s_{234}\,, \\
c_{12} &= s_{345}(s_{234}s_{345} - s_{61}s_{34})\,, \\
c_{13} &= -s_{345}^2s_{56}\,, \\
c_{14} &= -2s_{126}(s_{123}s_{234}s_{345} - s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})\,, \\
c_{15} &= 2s_{61}(s_{123}s_{234}s_{345} - s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})\,.
\end{aligned} \tag{3.13}
$$

Equation (3.9) omits all odd terms proportional to the Levi-Civita tensor. Because neither the strong-coupling string theory calculations nor the Wilson loop calculations are sensitive to such terms, we may drop them without affecting comparisons to either.

Finally we should note that while the set of integrals in figs. 3 and 4 is convenient for calculation, non-trivial relations between them and other integrals may allow for other equivalent representations of the amplitude (3.9) in which the form of the coefficients (3.13) is substantially altered.

C. Dual Conformal Structure of the Integrand

As mentioned above, we expect that planar amplitudes should manifest dual conformal symmetry. An even stronger statement, that an amplitude can be expressed as a linear combination of integrals, each exhibiting manifest pseudo-conformal invariance, has been observed to hold for the four-particle amplitude through five loops, and for the even part of the five-particle amplitude at two loops.

As with the odd part of $M_5^{(2)}$, the dual conformal properties of the two integrals in $M_6^{(2),\mu}$, $I^{(14)}$ and $I^{(15)}$, are not apparent. It is possible that they can be re-expressed as a linear combination of integrals with manifest properties. However, we will argue in the

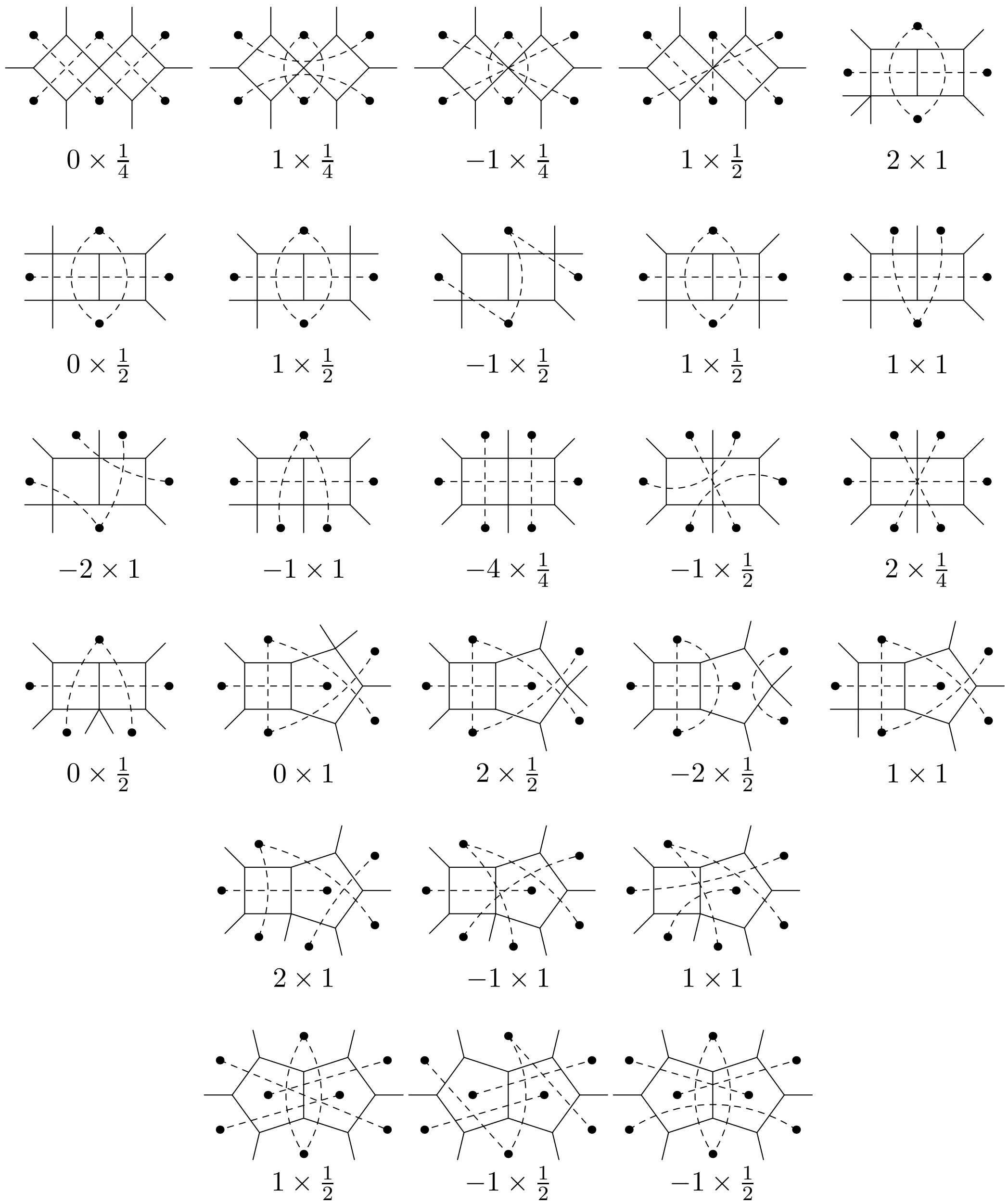

FIG. 7: The 26 different integrals which are allowed, by the hypothesis of dual conformal symmetry, to contribute to the amplitude $M_6^{(2),D=4}$. Beneath each diagram is the coefficient with which the corresponding integral, defined according to the rules reviewed in fig. 1, enters into our result for $M_6^{(2),D=4}$. An overall factor of 1/16 is suppressed and it is understood that one should sum over the 12 cyclic and reflection permutations of the external legs. In each coefficient, the second factor is a symmetry factor that accounts for overcounting in this sum.

next section that they cancel against the analogous one-loop piece $M_6^{(1),\mu}$, in the remainder function (2.9) for the ABDK/BDS ansatz. Hence we focus on the surviving piece $M_6^{(2),D=4}$. We find that this piece can indeed be written as a linear combination of the 26 independent pseudo-conformal integrals, as exhibited in fig. 7.

The integrals appearing in the four-point amplitude have a number of striking features, partly explained by heuristics such as the rung rule [23] and box substitution rule [24]. Three interesting features are observed in the four-point amplitudes through five loops:

- All pseudo-conformal integrals appear with relative weights of ± 1 or 0 [3, 11, 23, 24].
- Moreover, an integral appears with coefficient zero if and only if the integral is ill-defined (unregulated) after taking its external legs off shell and taking $\epsilon \to 0$ [8].
- Finally, it has been proposed that the signs ± 1 of the contributing integrals can be understood by the requirement of cancelling unphysical singularities [58].

It is clear from fig. 7 that the first of these does not hold for our representation of the six-point amplitude; in particular, some relative weights are ± 2, and there is one weight of -4. An examination of the integrals in fig. 4, shows that the second observation also requires some modification. This should not be too surprising since at six points we expect that some of the well-defined integrals appear in non-MHV amplitudes but not in the MHV ones. It would certainly be very interesting to determine whether any of these considerations could be generalized or modified to explain the pattern of coefficients appearing in fig. 7.

IV. RESULTS

Because the two-loop iteration formula (2.4) incorporates the known infrared singularities, it must hold for infrared-singular terms. We have evaluated the integrals in figs. 3 and 4 through $\mathcal{O}(\epsilon^{-2})$ analytically in terms of ordinary polylogarithms. In appendix A we have collected their values though $\mathcal{O}(\epsilon^{-3})$; we refrain from presenting the much lengthier $\mathcal{O}(\epsilon^{-2})$ contributions. By inserting the values of the two-loop integrals into the assembly equation (3.9) we find that eq. (2.4) holds analytically through $\mathcal{O}(\epsilon^{-2})$. This provides a non-trivial check on our cut construction, evaluation of integrals, and assembly of contributions. The reader may check agreement through $\mathcal{O}(\epsilon^{-3})$ using the values of the two-loop integrals in appendix A and the one-loop amplitude (2.10) with $n = 6$.

Beyond $\mathcal{O}(\epsilon^{-2})$, we resort to numerical integration. We first constructed Mellin-Barnes representations, in order to make use of the package `MB` [27]. The package `AMBRE` [26] provides a simple means for obtaining Mellin-Barnes representations that can be integrated using `MB`. One must treat the most complicated double-pentagon integrals, $I^{(12)}$ and $I^{(13)}$, with some care to produce a numerically suitable representation, so we give Mellin-Barnes representations for these two integrals in appendix C.

In four dimensions, there are at most four linearly independent momenta. For the six-point amplitude, therefore, the Gram determinant of any five external momenta must vanish,

$$\det(k_i \cdot k_j) = 0\,, \qquad i,j = 1,2,3,4,5. \tag{4.1}$$

This constraint turns out not to be relevant for any of our checks, but it is important to choose at least a few kinematic points satisfying this constraint, in order to ensure that any deviation from the BDS ansatz does not arise from choosing momentum invariants that cannot be realized in four dimensions.

A. The μ-Dependent Terms

In section III we split both the one- and two-loop amplitudes into a $D = 4$ part and a part containing explicit dependence on μ, the (-2ϵ)-dependent part of the loop momentum, according to eqs. (3.1) and (3.9). The μ-dependent part of the one-loop amplitude, $M_6^{(1),\mu}(\epsilon)$, vanishes as $\epsilon \to 0$. It only contributes to the remainder function $R_6^{(2)}$, defined in eq. (2.9), because it appears in $(M_n^{(1)}(\epsilon))^2$ multiplied by the singular terms in the one-loop amplitude, which are given in eq. (2.10). Thus the contribution of the one- and two-loop μ-dependent terms to $R_6^{(2)}$ is

$$R_6^{(2),\mu} \;=\; \lim_{\epsilon\to 0}\left[M_6^{(2),\mu}(\epsilon) - \left(-\frac{1}{2}\frac{1}{\epsilon^2}\sum_{i=1}^{6}(-s_{i,i+1})^{-\epsilon}M_6^{(1),\mu}(\epsilon)\right)\right], \tag{4.2}$$

where we set the dimensional regularization scale $\mu \to 1$ here to avoid confusion with μ_p.

Integrals containing numerator factors of μ_p and μ_q can be computed by differentiating integrands for scalar integrals with respect to Schwinger parameters [57]. This result holds because the dependence of the integrals on the (-2ϵ) components of the loop momenta is very simple. At two loops, it is given by,

$$\int d^{-2\epsilon}\mu_p\; d^{-2\epsilon}\mu_q \exp\Big[-\mu_p^2\, T_p - \mu_q^2\, T_q - \mu_{p+q}^2\, T_{pq}\Big] \;\propto\; \Delta^{\epsilon}\,, \tag{4.3}$$

where T_p, T_q and T_{pq} are the sums of Schwinger parameters for propagators carrying loop momenta p, q and $p+q$, respectively, and $\Delta = T_pT_q + T_pT_{pq} + T_qT_{pq}$. Differentiation leads to the parameter insertions (see eq. (4.26) of ref. [57]),

$$\mu_p^2 \;\to\; -\epsilon\frac{T_q+T_{pq}}{\Delta}\,, \tag{4.4}$$

$$\mu_p\cdot\mu_q \;\to\; \frac{-\epsilon}{2}\frac{(T_p+T_q)-(T_q+T_{pq})-(T_p+T_{pq})}{\Delta} = \epsilon\frac{T_{pq}}{\Delta}\,. \tag{4.5}$$

At one loop, $\mu_p^2 \to -\epsilon/T$, where T is the sum of all the Schwinger parameters.

At one loop, the insertion of $1/T$ shifts the dimension of the integral from $D = 4-2\epsilon$ to $D = 6-2\epsilon$, which makes the integral infrared finite (and it remains ultraviolet finite). Thus $I^{\rm hex}(\epsilon)$, and hence $M_6^{(1),\mu}(\epsilon)$, vanish as $\epsilon \to 0$. At two loops, the factor of $1/\Delta$ in eqs. (4.4) and (4.5) also shifts the dimension to $D = 6-2\epsilon$. However, the Schwinger parameters in the numerator lead to doubled propagators (see, for example, the discussion in ref. [57]), which can cause infrared divergences, even near $D = 6$.

In the case of the double pentagon integral $I^{(14)}$, eq. (4.5) shows that the doubled propagator is the central one. Because this propagator does not touch any on-shell external legs, doubling it is "safe", and we expect that $I^{(14)} = \mathcal{O}(\epsilon)$. We have confirmed this expectation, both numerically, and by checking analytically the analogous planar double box integral for massless four-point kinematics.

In the case of the "hexabox" integral, $I^{(15)}$, eq. (4.4) leads to doubled propagators on the box loop, which create infrared divergences. However, the hexagon loop remains infrared safe. Reasoning by analogy to the factorization of soft and collinear singularities at the amplitude level [28], the μ_p^2-hexagon inside the hexabox can be thought of as a hard process. Thus one can shrink it to zero size in space-time, and decorate it by a one-loop scalar triangle integral representing the infrared-divergent contributions. Thus we expect,

$$I^{(15)}(\epsilon) = -\frac{1}{\epsilon^2}(-s_{61})^{-1-\epsilon}\, I^{\rm hex}(\epsilon) \;+\; \mathcal{O}(\epsilon)\,. \tag{4.6}$$

The reason the equation is valid to $\mathcal{O}(\epsilon)$ rather than $\mathcal{O}(\epsilon^0)$ is simply because the "hard" part $I^{\rm hex}$ is itself $\mathcal{O}(\epsilon)$. Once again, we have checked eq. (4.6) numerically. We have also checked analytically that the same relation holds for the analogous planar double box integral. Inserting eq. (4.6) into the explicit expressions for the one- and two-loop μ-dependent contributions $M_6^{(1),\mu}$ and $M_6^{(2),\mu}$ in eq. (4.2), we see that $R_6^{(2),\mu}$ vanishes.

B. Evaluation of Remainder Function

Explicit computations [3, 14, 15] have demonstrated that the remainder function $R_n^{(2)}$ defined in eq. (2.9) vanishes for $n = 4, 5$. In this section we shall evaluate $R_6^{(2)}$ numerically at a few kinematic points, and find that it is nonzero and nonconstant.

We choose Euclidean kinematics for all points, as this simplifies the numerical evaluation. A particularly convenient kinematic point is

$$K^{(0)} : s_{i,i+1} = -1\,, \quad s_{i,i+1,i+2} = -2\,, \tag{4.7}$$

which we take to be our standard reference point. This point has several advantages. Firstly, because it is symmetric under cyclic relabeling $i \to i+1$ and under the reflection $i \to 6-i+1$, we do not need to evaluate any relabelings of the integrals to obtain the amplitude. We have also exploited $s_{i,i+1} = -1$ to simplify the Mellin-Barnes representations. Moreover, this kinematic point satisfies the Gram determinant constraint (4.1).

The numerical values of all the integrals in fig. 3 at the standard kinematic point are given in appendix B. (For completeness we also give the values of the non-contributing integrals in fig. 4.) Inserting these values into the amplitude (3.10), we obtain,

$$M_6^{(2),D=4} = \frac{9}{2\epsilon^4} - \frac{12.2457}{\epsilon^2} - \frac{21.99}{\epsilon} - 20.8534 \pm 0.0057 + \mathcal{O}(\epsilon)\,, \tag{4.8}$$

at the standard kinematic point $K^{(0)}$. The values of the μ-dependent contributions (3.11) are,

$$M_6^{(2),\mu} = \frac{2.3510}{\epsilon} + 8.6024 \pm 0.0010 + \mathcal{O}(\epsilon)\,. \tag{4.9}$$

We have included estimated errors from the numerical integration reported by `CUBA` [59], added in quadrature. In many cases the errors appear to be overestimated. However, there can be correlations between different subintegrals in which the amplitude is expanded, because the same random seed is used. The reported errors do appear to give us a reliable measure of how many digits are trustworthy.

The values (4.8) and (4.9) may be compared to those for the ABDK/BDS ansatz (2.4). Separating it out, in analogy to (3.9), as

$$M_6^{\rm BDS} = M_6^{{\rm BDS},\, D=4} + M_6^{{\rm BDS},\mu}\,, \tag{4.10}$$

where the second term arises from $I^{\rm hex}$ in $M_6^{(1),\mu}$, we find that

$$M_6^{{\rm BDS},\, D=4} = \frac{9}{2\epsilon^4} - \frac{12.2457}{\epsilon^2} - \frac{21.995}{\epsilon} - 21.9471 + \mathcal{O}(\epsilon)\,, \tag{4.11}$$

and that the μ pieces are given by

$$M_6^{\mathrm{BDS},\mu} = \frac{2.3510}{\epsilon} + 8.6017 + \mathcal{O}(\epsilon)\,. \tag{4.12}$$

Since these formulæ involve computing only one-loop integrals, the numerical integration errors are much smaller and do not affect the answer to the quoted precision.

By comparing eqs. (4.9) and (4.12), we see that the μ terms agree. This result is in accord with the general vanishing of $R_6^{(2),\mu}$ described in section IV A. However, there is a difference in the $D = 4$ terms, between our explicit calculation of the amplitude and the BDS ansatz. Defining it as

$$R_A \equiv R_6^{(2)} = M_6^{(2)} - M_6^{\mathrm{BDS}}\,, \tag{4.13}$$

we find that at our standard kinematic point (4.7) it equals

$$R_A^0 \equiv R_A(K^{(0)}) = 1.0937 \pm 0.0057\,. \tag{4.14}$$

Although the remainder is only 5 percent of the finite term in eq. (4.8), it is nonzero at very high confidence level, demonstrating that the ABDK/BDS ansatz needs to be modified.

Besides our standard kinematic point, we also evaluated R_A at various other kinematic points,

$$\begin{aligned}
K^{(1)}\ :\ & s_{12} = -0.7236200, \quad s_{23} = -0.9213500, \quad s_{34} = -0.2723200, \quad s_{45} = -0.3582300, \\
& s_{56} = -0.4235500, \quad s_{61} = -0.3218573, \quad s_{123} = -2.1486192, \quad s_{234} = -0.7264904, \\
& s_{345} = -0.4825841, \\
K^{(2)}\ :\ & s_{12} = -0.3223100, \quad s_{23} = -0.2323220, \quad s_{34} = -0.5238300, \quad s_{45} = -0.8237640, \\
& s_{56} = -0.5323200, \quad s_{61} = -0.9237600, \quad s_{123} = -0.7322000, \quad s_{234} = -0.8286700, \\
& s_{345} = -0.6626116, \\
K^{(3)}\ :\ & s_{i,i+1} = -1, \quad s_{123} = -1/2, \quad s_{234} = -5/8, \quad s_{345} = -17/14, \\
K^{(4)}\ :\ & s_{i,i+1} = -1, \quad s_{i,i+1,i+2} = -3, \\
K^{(5)}\ :\ & s_{i,i+1} = -1, \quad s_{i,i+1,i+2} = -9/2.
\end{aligned} \tag{4.15}$$

With six-point kinematics we have sufficient freedom to construct three nontrivial conformal cross ratios:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}\,,$$

TABLE I: The numerical remainder compared with the ABDK ansatz (2.4) for various kinematic points. The second column gives the conformal cross ratios defined in eq. (4.16).

kinematic point	(u_1, u_2, u_3)	R_A
$K^{(0)}$	$(1/4, 1/4, 1/4)$	1.0937 ± 0.0057
$K^{(1)}$	$(1/4, 1/4, 1/4)$	1.076 ± 0.022
$K^{(2)}$	$(0.547253,\ 0.203822,\ 0.881270)$	-1.659 ± 0.014
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-3.6508 ± 0.0032
$K^{(4)}$	$(1/9, 1/9, 1/9)$	5.21 ± 0.10
$K^{(5)}$	$(4/81, 4/81, 4/81)$	11.09 ± 0.50

$$\begin{aligned} u_2 &= \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} = \frac{s_{23} s_{56}}{s_{234} s_{123}}\,, \\ u_3 &= \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2} = \frac{s_{34} s_{61}}{s_{345} s_{234}}\,. \end{aligned} \tag{4.16}$$

The three conformal cross ratios (4.16) for these kinematic points are given in the second column of Table I. Here $K^{(1)}$, $K^{(2)}$ and $K^{(3)}$ satisfy the Gram determinant constraint (4.1) while $K^{(4)}$ and $K^{(5)}$ do not.

The point $K^{(1)}$ is chosen so that the conformal cross ratios $u_i = 1/4$ are identical to the cross ratios for our reference kinematic point $K^{(0)}$. The agreement, within the errors, between the remainder functions for these two kinematic points, suggests that R_A is a function of only the cross ratios, *i.e.* is invariant under dual conformal transformations.

C. Comparison with Wilson Loop

Drummond, Henn, Korchemsky and Sokatchev have already shown [20] that the Wilson loop expectation value $\langle W_6^{(2)} \rangle$ corresponding to the two-loop six-point MHV amplitude, is not equal to that suggested by the ABDK/BDS ansatz for amplitudes. That is, they found a nonvanishing remainder function,

$$R_W \equiv \langle W_6^{(2)} \rangle - W_6^{\rm BDS}\,. \tag{4.17}$$

Here W_6^{BDS} is the Wilson loop analog of M_6^{BDS} defined in eq. (2.6). It involves the same function $M_6^{(1)}(\epsilon)$, but different constants appear [20, 21] than the ones for amplitudes (which

TABLE II: The comparison between the remainder functions R_A and R_W for the MHV amplitude and the Wilson loop. To account for various constants of the kinematics, we subtract from the remainders their values at the standard kinematic point $K^{(0)}$, denoted by R_A^0 and R_W^0. The third column contains the difference of remainders for the amplitude, while the fourth column has the corresponding difference for the Wilson loop. The numerical agreement between the third and fourth columns provides strong evidence that the finite remainder for the Wilson loop is identical to that for the MHV amplitude.

kinematic point	(u_1, u_2, u_3)	$R_A - R_A^0$	$R_W - R_W^0$
$K^{(1)}$	$(1/4, 1/4, 1/4)$	-0.018 ± 0.023	$< 10^{-5}$
$K^{(2)}$	$(0.547253,\ 0.203822,\ 0.881270)$	-2.753 ± 0.015	-2.7553
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.7445 ± 0.0075	-4.7446
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.12 ± 0.10	4.0914
$K^{(5)}$	$(4/81, 4/81, 4/81)$	10.00 ± 0.50	9.7255

are captured by $f^{(l)}(\epsilon)$ and $C^{(l)}$). We are motivated by the correspondence between MHV amplitudes and Wilson loops at one loop [8, 9], to ask how these two remainder functions, R_A and R_W, compare.

As explained in ref. [20], in the collinear limits, corresponding to $u_1 = 0, u_3 = 1 - u_2$, the Wilson loop remainder function R_W becomes a constant, corresponding to eq. (17) that paper. On the other hand, as explained in section V, the MHV amplitude remainder function must vanish in the collinear limits in order to be consistent with collinear factorization. This suggests a simple relation between the two remainders,

$$R_A = R_W - c_W \,, \tag{4.18}$$

which we test numerically. From DHKS [21], the constant c_W takes on the value,

$$c_W = 12.1756 \,, \tag{4.19}$$

with a precision of $\sim 10^{-3}$.

The numerical determination of the constant c_W from the collinear limits of the Wilson loop leads to some loss of precision, so instead in Table II we compare the Wilson loop and MHV-amplitude remainder functions by considering a “difference of differences”. That is, for both the Wilson loop and MHV amplitude the remainders R_A and R_W are found by subtracting the value of the appropriate ABDK/BDS formula at that point. From R_A and R_W we subtract the corresponding values R_A^0 and R_W^0 at the standard kinematic point $K^{(0)}$. From DHKS the remainder at the standard point is [21] $R_W^0 = 13.26530$. This subtraction eliminates any dependence on c_W. The Wilson loop results are obtained from ref. [21]. In general, the Wilson loop results have much smaller errors than those of the amplitude. This is due to the much simpler integral representations appearing in the Wilson loop computation [21].

Using the value of c_W from eq. (4.19), we can also compare R_A and R_W directly, albeit at lower precision. We find that within errors eq. (4.18) is satisfied for all six kinematic points in eqs. (4.7) and (4.15). As mentioned above, we tested dual conformal invariance directly at one point. However, this invariance was tested much more extensively (also numerically) for the Wilson loop [20, 21]. Thus our numerical agreement with the Wilson loop remainder, displayed in Table II, obviously provides considerable additional evidence that R_A possesses dual conformal invariance.

V. THE REMAINDER FUNCTION

In the previous section we found a numerical difference between the ABDK/BDS ansatz for the two-loop six point amplitude and the explicit calculation. In this section we constrain the analytic form of the remainder and point out that its functional form can be determined from triple-collinear limits.

A. Constraints on the Remainder Function

The form of the ABDK/BDS ansatz is tightly constrained by factorization properties and also exhibits dual conformal invariance. As discussed in the previous section, numerical evidence confirms that, while it departs from the ansatz, the even part of the two-loop six-point amplitude is invariant under dual conformal transformations. We can therefore discuss further constraints imposed by this symmetry on the remainder function. We will discuss the case of n-particle amplitudes and specialize to $n = 6$ at the end.

DHKS [13] argue that MHV amplitudes (like Wilson loops), should obey anomalous dual conformal Ward identities, the anomaly due to infrared (ultraviolet) divergences. The fact that the BDS ansatz accounts for all infrared divergences of MHV amplitudes to all loop orders implies that R_n is finite and thus independent of the regulator. It also means that R_n should satisfy *non*-anomalous Ward identities; that is, it must actually be invariant under dual conformal transformations and thus depend only on conformally-invariant cross ratios:

$$R_n = R_n(\{u_{ijkl}\})\,. \tag{5.1}$$

Here $(ijkl)$ denote the allowed quartets of the external legs leading to well-defined cross ratios (2.31). The symmetry properties of the remainder function R_n under the permutation of its arguments follow from the reflection and cyclic and identities obeyed by the rescaled MHV scattering amplitudes $M_n^{(L)}$.

Further restrictions on R_n come from the fact that the BDS ansatz correctly captures the two-particle collinear factorization of MHV amplitudes. In general, the L-loop rescaled planar amplitudes $M_n^{(L)}(1,2,\ldots,n)$ satisfy simple relations as the momenta of two color-adjacent legs k_i, k_{i+1} become collinear, [32, 60–62],

$$M_n^{(L)}(\ldots,i^{\lambda_i},(i+1)^{\lambda_{i+1}},\ldots) \longrightarrow \sum_{l=0}^{L}\sum_{\lambda=\pm} r_{-\lambda}^{(l)}(z;i^{\lambda_i},(i+1)^{\lambda_{i+1}})M_{n-1}^{(L-l)}(\ldots,P^{\lambda},\ldots)\,. \tag{5.2}$$

The index l sums over the different loop orders of the rescaled splitting amplitudes,

$$r_{-\lambda}^{(l)}(z;i^{\lambda_i},(i+1)^{\lambda_{i+1}}) \equiv \frac{\mathrm{Split}_{-\lambda}^{(l)}(z;i^{\lambda_i},(i+1)^{\lambda_{i+1}})}{\mathrm{Split}_{-\lambda}^{(0)}(z;i^{\lambda_i},(i+1)^{\lambda_{i+1}})}\,, \tag{5.3}$$

while λ sums over the helicities of the intermediate leg $k_P = (k_i + k_{i+1})$, and z is the longitudinal momentum fraction of k_i, $k_i \approx z k_P$.

The relevant two-loop splitting amplitudes were calculated in refs. [3, 63]. If we assume that dual conformal symmetry holds to all orders, then the five-point amplitudes are fully determined by this symmetry. By taking the collinear limit, the all-loop splitting amplitude,

$$r^{\mathrm{full}} \equiv 1 + \sum_{L=1}^{\infty} a^L r^{(L)}\,, \tag{5.4}$$

must have the form

$$\ln r^{\mathrm{full}} = \sum_{l=1}^{\infty} a^l\, f^{(l)}(\epsilon)\, r^{(1)}(l\epsilon) + \mathcal{O}(\epsilon)\,. \tag{5.5}$$

As discussed in ref. [6], this iterative structure yields the correct collinear behavior to all loop orders.

Since the BDS ansatz accounts for collinear factorization, to all orders in perturbation theory, the remainder functions must have a trivial behavior under collinear factorization. The n- and $(n-1)$-point remainder functions must be related by

$$\lim_{x_{i,i+2}^2 \to 0} R_n(\{u_{i_1,i_2,i_3,i_4}\}) = R_{n-1}(\{u_{i_1,i_2,i_3,i_4}\}') \,, \tag{5.6}$$

for any two-particle Mandelstam invariant $x_{i,i+2}^2 = s_{i,i+1}$. The arguments of R_{n-1} are the subset of the n-point conformal cross ratios that are non-vanishing and well-defined in the collinear limit $s_{i,i+1} \to 0$.

For five-point amplitudes, no conformal cross ratio with the required properties may be constructed. Thus no remainder function can exist consistent with collinear factorization and the requirement of dual conformal invariance. A constant remainder is ruled out by collinear factorization.

Nontrivial conformal cross ratios can be first constructed with six-particle kinematics, u_1, u_2 and u_3 in eq. (4.16). The cyclic and reflection symmetries imply that $R_6(u_1, u_2, u_3)$ is a totally symmetric function of its arguments. Because the remainder function $R_5^{(2)}$ for the five-point amplitude vanishes [14, 15], $R_6^{(2)}(u_1, u_2, u_3)$ must vanish in all collinear limits. If dual conformal symmetry is valid to all loop orders then R_5 vanishes exactly and therefore R_6 must vanish in all collinear limits to all loop orders. DHKS [20] reached a similar conclusion in their analysis of the two-loop six-sided Wilson loop, namely that there is a remainder $\hat{f}$ above and beyond the ABDK/BDS ansatz with the properties described above.

B. Remainder Function from Triple-Collinear Limits

Planar color-ordered scattering amplitudes exhibit singularities when several adjacent momenta become collinear. General all-order factorization properties of scattering amplitudes have been discussed in ref. [62]. The most familiar of these limits are when just two particles become collinear. However, the limits when more particles become collinear simultaneously can provide additional constraints[4]. (Multi-collinear configurations should not be confused with multi-particle factorization limits, in which amplitudes factorize into products of lower-point, non-degenerate scattering amplitudes. The latter limits are trivial for MHV amplitudes in supersymmetric theories.)

[4] We thank Gregory Korchemsky and Emery Sokatchev for discussion on the triple-collinear limits.

As discussed previously, the ABDK/BDS ansatz incorporates the correct two-particle collinear factorization of MHV amplitudes. It also makes definite predictions, which remain to be tested, for the multi-collinear factorization of MHV amplitudes in $\mathcal{N}=4$ super-Yang-Mills theory. These tests amount to constraints on the remainder functions R_n. For the six-gluon amplitude we only have the triple-collinear limit. As we will see, it is possible to completely determine the remainder function R_6 from this limit.

Let us consider three adjacent momenta $k_{a,b,c}$ in the limit that they become collinear and introduce the three momentum fractions

$$k_a = z_1 P\,, \qquad k_b = z_2 P\,, \qquad k_c = z_3 P\,, \qquad z_1+z_2+z_3=1\,, \qquad 0 \le z_i \le 1\,, \qquad P^2 \to 0\,. \tag{5.7}$$

An n-point amplitude at l loops factorizes as follows:

$$A_n^{(l)}(k_1,\ldots,k_{n-2},k_{n-1},k_n) \mapsto \sum_{\lambda=\pm}\sum_{s=0}^{l} A_n^{(l-s)}(k_1,\ldots,P^\lambda)\ \mathrm{Split}^{(s)}_{-\lambda}(k_{n-2}k_{n-1}k_n;P)\,. \tag{5.8}$$

Taking into account parity and reflection symmetries, there are six independent triple-collinear splitting amplitudes:

$$\mathrm{Split}_{+}(k_a^+ k_b^+ k_c^+;P), \tag{5.9}$$

$$\mathrm{Split}_{-\lambda_P}(k_a^{\lambda_a} k_b^{\lambda_b} k_c^{\lambda_c};P), \qquad \lambda_a+\lambda_b+\lambda_c-\lambda_P = 2\,, \tag{5.10}$$

$$\mathrm{Split}_{-\lambda_P}(k_a^{\lambda_a} k_b^{\lambda_b} k_c^{\lambda_c};P), \qquad \lambda_a+\lambda_b+\lambda_c-\lambda_P = 0\,. \tag{5.11}$$

The first one (5.9) vanishes in any supersymmetric theory. The three triple-collinear splitting amplitudes of the second type (5.10), an example of which is $\lambda_a=\lambda_b=\lambda_c=\lambda_P=1$, appear in limits of MHV amplitudes. The $\mathcal{N}=4$ supersymmetry Ward identities for MHV amplitudes imply that their rescaled forms[5] are all equal,

$$\frac{\mathrm{Split}^{(l)}_{\mp}(k_a^{\pm}k_b^+k_c^+;P)}{\mathrm{Split}^{(0)}_{\mp}(k_a^{\pm}k_b^+k_c^+;P^{\mp})} = \frac{\mathrm{Split}^{(l)}_{\mp}(k_a^{+}k_b^{\pm}k_c^+;P)}{\mathrm{Split}^{(0)}_{\mp}(k_a^{+}k_b^{\pm}k_c^+;P)} = r_S^{(l)}\big(\tfrac{s_{ab}}{s_{abc}},\tfrac{s_{bc}}{s_{abc}},z_1,z_3\big)\,. \tag{5.12}$$

The two splitting amplitudes of the third kind (5.11) arise only in limits of NMHV amplitudes and do not have a simple factorized form similar to (5.12).[6]

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[5] We omit a trivial dimensional dependence on s_{abc} from the argument list of $r_S^{(l)}$.

[6] The spin-averaged absolute values squared of tree-level triple-collinear splitting amplitudes have been computed in ref. [64]; without spin-averaging they have been computed in refs. [65]. The tree-level triple (and higher) collinear splitting amplitudes themselves have been computed in ref. [66] using the MHV rules [67]. The one-loop correction to the $q \to q\bar{Q}Q$ triple-collinear splitting amplitude in QCD was computed in ref. [68].

On general grounds, the six-gluon amplitude exhibits a nontrivial triple-collinear limit. In the limit (5.7), for $a = 4$, $b = 5$, $c = 6$, the three conformal cross ratios (4.16) are all nonvanishing and arbitrary:

$$\bar{u}_1 = \frac{s_{45}}{s_{456}} \frac{1}{1-z_3}\,, \qquad \bar{u}_2 = \frac{s_{56}}{s_{456}} \frac{1}{1-z_1}\,, \qquad \bar{u}_3 = \frac{z_1 z_3}{(1-z_1)(1-z_3)}\,. \tag{5.13}$$

The remainder function therefore survives the triple-collinear limit, and is evaluated at $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$. Thus, assuming dual conformal invariance, finding the remainder function in this limit is equivalent to finding it for generic six-particle kinematics.

We can expose the two-loop remainder function by comparing eq. (5.8) for the rescaled MHV six-point amplitude,

$$\lim_{4||5||6} M_6^{(2)} = M_4^{(2)} + M_4^{(1)} r_S^{(1)}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon) + r_S^{(2)}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon)\,, \tag{5.14}$$

with the triple-collinear limit of eq. (2.9). The second term in eq. (5.14) is already determined by one-loop calculations, and it is incorporated in the ABDK/BDS ansatz. Therefore $R_6^{(2)}$ enters only in the two-loop splitting amplitude $r_S^{(2)}$, as the deviation from the ABDK/BDS prediction $r_S^{(2)\,\mathrm{BDS}}$:

$$r_S^{(2)}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon) = r_S^{(2)\,\mathrm{BDS}}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon) + R_6^{(2)}(\bar{u}_1, \bar{u}_2, \bar{u}_3)\,, \tag{5.15}$$

with

$$\begin{aligned} r_S^{(2)\,\mathrm{BDS}}&(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon) = \\ &\frac{1}{2}\Big(r_S^{(1)}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon)\Big)^2 + f^{(2)}(\epsilon)\, r_S^{(1)}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon)\,. \end{aligned} \tag{5.16}$$

Thus, the two-loop remainder function is *completely determined* by the two-loop triple-collinear splitting amplitude, *e.g* for the helicity configuration (5.10). While *a priori* it may depend on all four arguments of the splitting amplitude, dual conformal invariance requires that it depend only on the three cross ratios.

The triple-collinear splitting amplitudes may be computed using the unitarity method following the strategy in refs. [62, 63]. It is important to understand whether they satisfy an iteration relation generalizing that of the double-collinear splitting amplitude [3]. If such an iteration relation exists, then it should be straightforward to construct an all-order iteration relation for the six-point gluon amplitude. This would allow us to add in a correction term to the BDS ansatz, at least for the six-point case.

The remainder function beyond two loops can also be extracted from the triple-collinear splitting amplitude, though they are no longer equal. Instead, they are related iteratively via,

$$R_6^{(l)}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = \sum_{s=2}^{l} M_4^{(l-s)}(\epsilon) \left[r_S^{(s)}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon) - r_S^{(s)\,\mathrm{BDS}}(\tfrac{s_{45}}{s_{456}}, \tfrac{s_{56}}{s_{456}}, z_1, z_3, \epsilon) \right]. \quad (5.17)$$

For amplitudes with additional external legs, it is unclear whether the triple-collinear limits suffice to constrain the remainder functions completely. If these limits do not suffice, we can formulate additional constraints along the lines above. In particular, it is easy to see that the remainder function of the two-loop n-point MHV amplitude is completely determined by the difference between the two-loop $(n-3)$-point splitting amplitude and the iteration of the one-loop $(n-3)$-point splitting amplitude. Also, the consideration of the m-particle collinear limit with $m \leq n-4$ leads to consistency conditions analogous to eq. (5.6).

VI. SUMMARY AND CONCLUSIONS

The $\mathcal{N} = 4$ supersymmetric gauge theory has proven an important laboratory and testing ground for inquiry into the properties of gauge theories, both at weak and strong coupling. The BDS ansatz for planar MHV scattering amplitudes [6] in this theory, along with the BES integral equation for the cusp anomalous dimension [33] and the strong-coupling calculation of Alday and Maldacena [7], point to the possibility of computing planar amplitudes for *any* value of the coupling.

In this paper we have checked the BDS ansatz directly by computing the parity-even parts of the leading-color part of the planar two-loop six-point amplitude. Using the unitarity method, we have obtained an integral representation for it. Numerical evaluation of this representation shows that there is a remainder beyond the ABDK/BDS prediction for this amplitude. Strikingly, the remainder agrees with the corresponding remainder for the hexagonal Wilson loop [20, 21].

This remainder must vanish in any limit where two color-adjacent momenta become collinear, because the ABDK/BDS construction accounts for all terms with collinear singularities. As we showed, it should be possible to fully reconstruct the remainder function for the six-point amplitude by evaluating triple-collinear splitting amplitudes.

There are a number of interesting open issues which remain to be clarified. The origin of the dual conformal symmetry remains mysterious. In the context of the AdS/CFT correspondence it has been suggested [16] that it is related to symmetries of the space defined by the coordinates y^μ introduced in eq. (2.24). Based on this, one might wonder whether dual conformal symmetry can be found in the planar amplitudes of all four-dimensional CFTs with a string-theory dual.

We have found that the integrals appearing in the amplitude either vanish as the loop momenta are taken to be four dimensional or are pseudo-conformal (with the external momenta taken off shell to make them infrared finite). Contributions containing the (-2ϵ)-dimensional components of loop momenta in the numerator factors satisfy the BDS ansatz and drop out of the remainder function. It seems reasonable to expect that this pattern continues to all loop orders. It would therefore be very useful to have a set of rules for writing the coefficients of all pseudo-conformal integrals directly, without resorting to evaluations of the cuts.

So far dual conformal transformations have only been discussed in the context of their action on Wilson loops or planar MHV amplitudes after dividing out the tree amplitude prefactor. Can we extend this to non-MHV amplitudes? At least at one loop the integrals appearing in non-MHV amplitudes are pseudo-conformal scalar box integrals [32], hinting that dual conformal invariance might be a general property of the planar limit of the theory. However, in this case the tree amplitude does not factor out [45, 46], leaving the question of how the dual conformal symmetry might act on the spinor products that enter the relative factors of different pseudo-conformal integrals. Related to this is the question of whether the dual conformal symmetry can be extended to the Lagrangian of $\mathcal{N} = 4$ super-Yang-Mills theory. It is not clear how it should act on the Lagrangian, given that it is only understood at present for planar MHV amplitudes.

What other properties can we employ to constrain the scattering amplitudes? Integrability [33, 41, 69] of the dilatation operator in planar $\mathcal{N} = 4$ super-Yang-Mills theory has not yet been used. Similarly, we expect the amplitudes to have simple structures in twistor space [70]. At one loop the coefficients in front of each integral have been shown to lie on simple curves in twistor space, though the precise structure of complete loop amplitudes has not been determined [46, 53, 71]. It would be very interesting to explore the structure at higher loops.

In order to shed light on the remainder function it is very important to find its analytic form for the two-loop six-point amplitude. This form would be extremely useful for understanding the missing terms in the BDS ansatz at higher loops. It would also be useful for analytic continuation into the physical high energy or Regge limits of $2 \to 4$ and $3 \to 3$ scattering that have been discussed recently [18, 19]. In section V we proposed a possible means for constructing it at six points from triple-collinear limits. The equality of the six-point remainder function and the corresponding Wilson loop quantity [21] is rather surprising. It would obviously be very desirable to understand whether this equality holds to all loop orders, as well as at strong coupling.

In summary, our computation demonstrates that the BDS ansatz requires modification for amplitudes with six or more external legs. The surprising equality of the Wilson loop and MHV amplitude remainders, however, points to an additional structure in the theory which constrains its form. This in turn provides hope of determining the remainder function analytically, first at two loops, and eventually to all loop orders.

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on FeynArts [75]).

APPENDIX A: ANALYTIC VALUES OF INTEGRALS

1. One-loop integrals

In the Euclidean region with all $s_{i,i+1}$ and $s_{i,i+1,i+2}$ negative, the one-mass box, $I^{1\text{m}}$ in fig. 2, is given through $\mathcal{O}(\epsilon^0)$ by

$$\begin{aligned} I^{1\text{m}}(s_{45}, s_{56}, s_{123}) \;=\; & \frac{2}{s_{45}s_{56}}\left[\frac{1}{\epsilon^2}\Big((-s_{45})^{-\epsilon} + (-s_{56})^{-\epsilon} - (-s_{123})^{-\epsilon}\Big) - \frac{1}{2}\ln^2\left(\frac{s_{45}}{s_{56}}\right)\right. \\ & \left. - \operatorname{Li}_2\left(1 - \frac{s_{123}}{s_{45}}\right) - \operatorname{Li}_2\left(1 - \frac{s_{123}}{s_{56}}\right) - \frac{\pi^2}{4}\right] + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{A1}$$

Similarly, the two-mass "easy" box, $I^{2\text{me}}$, is

$$\begin{aligned} I^{2\text{me}}(s_{123}, s_{345}, s_{12}, s_{45}) \;=\; & \frac{2}{(s_{12}s_{45} - s_{123}s_{345})}\left[\frac{1}{\epsilon^2}\Big((-s_{12})^{-\epsilon} + (-s_{45})^{-\epsilon}\right. \\ & - (-s_{123})^{-\epsilon} - (-s_{345})^{-\epsilon}\Big) + \frac{1}{2}\ln^2\left(\frac{s_{123}}{s_{345}}\right) \\ & + \operatorname{Li}_2\left(1 - \frac{s_{12}}{s_{123}}\right) + \operatorname{Li}_2\left(1 - \frac{s_{45}}{s_{123}}\right) + \operatorname{Li}_2\left(1 - \frac{s_{12}}{s_{345}}\right) \\ & \left. + \operatorname{Li}_2\left(1 - \frac{s_{45}}{s_{345}}\right) - \operatorname{Li}_2\left(1 - \frac{s_{12}s_{45}}{s_{123}s_{345}}\right)\right] + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{A2}$$

2. Two-loop integrals

We have computed analytic expressions for the two-loop integrals appearing in fig. 3 through $\mathcal{O}(\epsilon^{-2})$. The $\mathcal{O}(\epsilon^{-2})$ expressions are rather cumbersome, so we display here only the results through $\mathcal{O}(\epsilon^{-3})$, omitting $I^{(1)}$ as it is given by a product of two one-loop integrals of type $I^{1\text{m}}$:

$$\begin{aligned} I^{(2)} \;&=\; -\frac{1}{(-s_{12})^{1+2\epsilon}\, s_{23}^2}\left[\frac{1}{\epsilon^4} + \frac{2}{\epsilon^3}\ln\left(\frac{s_{123}}{s_{23}}\right)\right] + \mathcal{O}(\epsilon^{-2})\,, \\ I^{(3)} \;&=\; \mathcal{O}(\epsilon^{-2})\,, \\ I^{(4)} \;&=\; -\frac{1}{(-s_{12})^{1+2\epsilon}\, s_{234}^2}\left[\frac{1}{4\epsilon^4} + \frac{1}{2\epsilon^3}\ln\left(\frac{s_{34}s_{56}}{s_{234}^2}\right)\right] + \mathcal{O}(\epsilon^{-2})\,, \\ I^{(5)} \;&=\; \frac{3}{2\epsilon^3}\,\frac{1}{s_{34}(s_{23}s_{56} - s_{123}s_{234})}\,\ln\left(\frac{s_{123}s_{234}}{s_{23}s_{56}}\right) + \mathcal{O}(\epsilon^{-2})\,, \\ I^{(6)} \;&=\; -\frac{1}{(-s_{12})^{1+2\epsilon}\, s_{23}s_{234}}\left[\frac{3}{4\epsilon^4} + \frac{1}{2\epsilon^3}\ln\left(\frac{s_{34}s_{56}^3}{s_{234}^4}\right)\right] + \mathcal{O}(\epsilon^{-2})\,, \end{aligned}$$

$$I^{(7)} = -\frac{1}{s_{61}s_{34}(-s_{123})^{1+2\epsilon}}\left[\frac{1}{\epsilon^4}+\frac{1}{\epsilon^3}\ln\!\left(\frac{s_{12}s_{23}s_{45}s_{56}}{s_{61}s_{34}s_{123}^2}\right)\right]+\mathcal{O}(\epsilon^{-2})\,,$$

$$I^{(8)} = \frac{1}{s_{61}(s_{61}s_{34}-s_{234}s_{345})}\frac{3}{\epsilon^3}\ln\!\left(\frac{s_{234}s_{345}}{s_{61}s_{34}}\right)+\mathcal{O}(\epsilon^{-2})\,,$$

$$\begin{aligned}I^{(9)} = -\frac{1}{(-s_{23})^{1+2\epsilon}\,s_{34}s_{234}}\Bigg[&\frac{1}{2\epsilon^4}+\frac{1}{\epsilon^3}\ln\!\left(\frac{s_{23}s_{56}}{s_{12}s_{234}}\right)\\&+\frac{3}{2\epsilon^3}\frac{s_{123}s_{234}}{s_{123}s_{234}-s_{23}s_{56}}\ln\!\left(\frac{s_{23}s_{56}}{s_{123}s_{234}}\right)\Bigg]+\mathcal{O}(\epsilon^{-2})\,,\end{aligned}$$

$$I^{(10)} = -\frac{1}{(-s_{61})^{1+2\epsilon}s_{23}s_{34}}\left[\frac{5}{2\epsilon^4}+\frac{1}{2\epsilon^3}\ln\!\left(\frac{s_{234}^5s_{45}^2s_{345}^3s_{61}^5}{s_{12}^4s_{23}^2s_{34}^5s_{123}^4}\right)\right]+\mathcal{O}(\epsilon^{-2})\,,$$

$$\begin{aligned}I^{(11)} = &-\frac{1}{\epsilon^4 s_{23}}\left[\frac{s_{45}}{s_{61}s_{34}(-s_{123})^{1+2\epsilon}}+\frac{3}{4s_{12}(-s_{234})^{1+2\epsilon}}+\frac{3s_{345}}{2(-s_{12})^{1+2\epsilon}\,s_{61}s_{34}}\right]\\&+\frac{1}{2\epsilon^3 s_{23}}\Bigg[\frac{2s_{45}}{s_{61}s_{34}s_{123}}\ln\!\left(\frac{s_{12}s_{23}s_{45}s_{56}}{s_{61}s_{34}s_{123}^2}\right)+\frac{1}{s_{12}s_{234}}\ln\!\left(\frac{s_{34}^3s_{123}}{s_{12}^3s_{23}}\right)\\&\qquad+\frac{s_{61}s_{23}s_{34}s_{56}+2s_{12}s_{45}s_{234}^2}{s_{12}s_{61}s_{34}s_{234}(s_{23}s_{56}-s_{123}s_{234})}\ln\!\left(\frac{s_{23}s_{56}}{s_{123}s_{234}}\right)\\&\qquad+\frac{s_{345}}{s_{12}s_{61}s_{34}}\ln\!\left(\frac{s_{123}^2s_{234}^3s_{345}^3}{s_{61}^3s_{23}^2s_{34}^3}\right)\Bigg]+\mathcal{O}(\epsilon^{-2})\,,\end{aligned}$$

$$\begin{aligned}I^{(12)} = &-\frac{1}{\epsilon^4}\left[\frac{3s_{123}}{(-s_{12})^{1+2\epsilon}\,s_{61}s_{34}s_{45}}+\frac{s_{23}s_{56}}{s_{12}s_{61}s_{34}s_{45}(-s_{234})^{1+2\epsilon}}+\frac{1}{s_{61}s_{34}(-s_{345})^{1+2\epsilon}}\right]\\&+\frac{1}{\epsilon^3}\Bigg[\frac{s_{123}}{s_{12}s_{61}s_{34}s_{45}}\ln\!\left(\frac{s_{234}^2s_{345}^6}{s_{23}s_{34}^3s_{45}^3s_{56}}\right)+\frac{s_{23}s_{56}}{s_{12}s_{61}s_{34}s_{45}s_{234}}\ln\!\left(\frac{s_{23}s_{56}s_{345}^2}{s_{12}^2s_{34}s_{45}}\right)\\&\quad+\frac{1}{s_{61}s_{34}s_{345}}\ln\!\left(\frac{s_{45}s_{234}s_{345}}{s_{23}s_{34}s_{56}}\right)\\&\quad+\frac{1}{s_{61}s_{34}-s_{234}s_{345}}\,\frac{s_{45}s_{234}s_{12}+2s_{345}s_{23}s_{56}}{s_{45}s_{234}s_{12}s_{345}}\,\ln\!\left(\frac{s_{61}s_{34}}{s_{234}s_{345}}\right)\\&\quad+\frac{s_{12}s_{45}s_{234}+(s_{23}s_{56}+3s_{123}s_{234})s_{345}}{s_{12}s_{61}s_{34}s_{45}s_{234}s_{345}}\ln\!\left(\frac{s_{12}}{s_{61}}\right)\Bigg]+\mathcal{O}(\epsilon^{-2})\,,\end{aligned}$$

$$\begin{aligned}I^{(13)} = &-\frac{1}{\epsilon^4}\Bigg[\frac{3s_{23}}{s_{12}s_{61}s_{34}(-s_{45})^{1+2\epsilon}}+\frac{s_{123}s_{234}}{s_{12}s_{34}s_{45}(-s_{56})^{1+2\epsilon}\,s_{61}}+\frac{1}{4(-s_{56})^{1+2\epsilon}\,s_{345}^2}\\&\qquad\qquad+\frac{s_{123}}{2s_{12}s_{45}s_{56}(-s_{345})^{1+2\epsilon}}+\frac{s_{234}}{2s_{61}s_{34}(-s_{56})^{1+2\epsilon}s_{345}}\Bigg]\\&+\frac{1}{\epsilon^3}\Bigg[\frac{s_{23}}{s_{12}s_{61}s_{34}s_{45}}\ln\!\left(\frac{s_{345}^6s_{56}^2s_{45}^3}{s_{12}^3s_{61}^3s_{34}^3s_{123}s_{234}}\right)+\frac{s_{123}s_{234}}{s_{12}s_{34}s_{45}s_{56}s_{61}}\ln\!\left(\frac{s_{123}s_{234}s_{345}^2}{s_{12}s_{61}s_{34}s_{45}}\right)\\&+\frac{1}{2}\frac{1}{s_{56}s_{345}^2}\ln\!\left(\frac{s_{12}s_{34}}{s_{345}^2}\right)+\frac{s_{23}}{s_{61}s_{34}s_{123}s_{345}}\ln\!\left(\frac{s_{12}s_{45}}{s_{123}s_{345}}\right)\\&+\frac{s_{123}}{s_{12}s_{45}s_{56}s_{345}}\ln\!\left(\frac{s_{34}}{s_{56}}\right)+\frac{1}{2}\frac{s_{234}}{s_{61}s_{34}s_{56}s_{345}}\ln\!\left(\frac{s_{12}^2s_{234}}{s_{61}s_{34}s_{345}}\right)\\&+2\frac{s_{23}}{s_{12}s_{61}s_{34}s_{45}-s_{61}s_{34}s_{123}s_{345}}\ln\!\left(\frac{s_{12}s_{45}}{s_{123}s_{345}}\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{s_{123}^2}{-s_{12}^2 s_{45}^2 s_{56}+s_{12}s_{45}s_{56}s_{123}s_{345}}\ln\Big(\frac{s_{123}s_{345}}{s_{12}s_{45}}\Big)\\
&+\frac{s_{23}}{s_{12}s_{61}s_{34}s_{45}-s_{12}s_{45}s_{234}s_{345}}\ln\Big(\frac{s_{61}s_{34}}{s_{234}s_{345}}\Big)\\
&-\frac{s_{12}s_{23}s_{45}}{s_{12}s_{61}s_{34}s_{45}s_{123}s_{345}-s_{61}s_{34}s_{123}^2 s_{345}^2}\ln\Big(\frac{s_{12}s_{45}}{s_{123}s_{345}}\Big)\\
&+\frac{1}{2}\frac{s_{234}}{s_{61}s_{34}s_{56}s_{345}-s_{56}s_{234}s_{345}^2}\ln\Big(\frac{s_{61}s_{34}}{s_{234}s_{345}}\Big)\Bigg]\\
&+\mathcal{O}(\epsilon^{-2})\,,
\end{aligned}
$$

$$
\begin{aligned}
I^{(14)} &= \mathcal{O}(\epsilon)\,,\\
I^{(15)} &= \mathcal{O}(\epsilon^{-1})\,,\\
I^{(16)} &= -\frac{1}{s_{34}^2(-s_{123})^{1+2\epsilon}}\left[\frac{1}{4\epsilon^4}+\frac{1}{2\epsilon^3}\ln\Big(\frac{s_{12}s_{56}}{s_{34}s_{123}}\Big)\right]+\mathcal{O}(\epsilon^{-2})\,,\\
I^{(17)} &= -\frac{1}{(-s_{12})^{1+2\epsilon}\,s_{61}s_{23}}\left[\frac{9}{4\epsilon^4}+\frac{3}{2\epsilon^3}\ln\Big(\frac{s_{123}s_{345}}{s_{16}s_{23}}\Big)\right]+\mathcal{O}(\epsilon^{-2})\,,\\
I^{(18)} &= -\frac{1}{s_{61}(-s_{45})^{1+2\epsilon}\,s_{234}}\left[\frac{3}{2\epsilon^4}+\frac{1}{\epsilon^3}\ln\Big(\frac{s_{23}s_{45}^2}{s_{56}^2 s_{234}}\Big)\right]+\mathcal{O}(\epsilon^{-2})\,. \qquad \text{(A3)}
\end{aligned}
$$

APPENDIX B: NUMERICAL VALUE OF INTEGRALS

In this appendix we give numerical values of the one- and two-loop integrals at the symmetric kinematic point $K^{(0)}$.

The numerical values of the one-loop integrals (A1) and (A2) are needed through $\mathcal{O}(\epsilon^2)$,

$$
\begin{aligned}
I^{1\mathrm{m}}(\epsilon) &= \frac{2}{\epsilon^2}+\frac{2\ln 2}{\epsilon}-2.125387-6.638772\epsilon-9.871006\epsilon^2+\mathcal{O}(\epsilon^3)\,,\\
I^{2\mathrm{me}}(\epsilon) &= -\frac{4\ln 2}{3\epsilon}-0.580026+1.033726\epsilon+3.0089373\epsilon^2+\mathcal{O}(\epsilon^3)\,. \qquad \text{(B1)}
\end{aligned}
$$

The one-loop hexagon in fig. 2 is

$$
I^{\mathrm{hex}}(\epsilon) = -1.56735\epsilon-(5.73447\pm 0.00067)\epsilon^2\,. \qquad \text{(B2)}
$$

The numerical values of the two-loop integrals at $K^{(0)}$ are, through $\mathcal{O}(\epsilon^0)$,

$$
\begin{aligned}
I^{(1)}(\epsilon) &= \frac{4}{\epsilon^4}+\frac{8\ln 2}{\epsilon^3}-\frac{6.57974}{\epsilon^2}-\frac{32.4479}{\epsilon}-53.373339+\mathcal{O}(\epsilon)\,,\\
I^{(2)}(\epsilon) &= -\frac{1}{\epsilon^4}-\frac{2\ln 2}{\epsilon^3}-\frac{0.822467}{\epsilon^2}-\frac{0.121225}{\epsilon}+3.655417+\mathcal{O}(\epsilon)\,,\\
I^{(3)}(\epsilon) &= -\frac{0.64646}{\epsilon^2}-\frac{0.55042}{\epsilon}+1.767276+\mathcal{O}(\epsilon)\,,\\
I^{(4)}(\epsilon) &= -\frac{1}{16\epsilon^4}+\frac{\ln 2}{4\epsilon^3}+\frac{0.27382}{\epsilon^2}+\frac{0.57483}{\epsilon}+0.511132\pm 0.000013+\mathcal{O}(\epsilon)\,,
\end{aligned}
$$

$$
\begin{aligned}
I^{(5)}(\epsilon) &= \frac{\ln 2}{\epsilon^3} - \frac{0.53467}{\epsilon^2} - \frac{4.06579}{\epsilon} - 2.47430 \pm 0.00070 + \mathcal{O}(\epsilon)\,, \\
I^{(6)}(\epsilon) &= -\frac{3}{8\epsilon^4} + \frac{\ln 2}{\epsilon^3} + \frac{3.11575}{\epsilon^2} + \frac{5.60293}{\epsilon} + 6.7248 + \mathcal{O}(\epsilon)\,, \\
I^{(7)}(\epsilon) &= -\frac{1}{2\epsilon^4} + \frac{2\ln 2}{\epsilon^3} + \frac{2.48496}{\epsilon^2} - \frac{5.9524}{\epsilon} - 28.8375 + \mathcal{O}(\epsilon)\,, \\
I^{(8)}(\epsilon) &= \frac{2\ln 2}{\epsilon^3} + \frac{3.45587}{\epsilon^2} + \frac{2.62502}{\epsilon} - 12.847078 + \mathcal{O}(\epsilon)\,, \\
I^{(9)}(\epsilon) &= -\frac{1}{4\epsilon^4} + \frac{5\ln 2}{2\epsilon^3} + \frac{2.46847}{\epsilon^2} - \frac{3.82945}{\epsilon} - 7.799036 + \mathcal{O}(\epsilon)\,, \\
I^{(10)}(\epsilon) &= -\frac{5}{2\epsilon^4} - \frac{2\ln 2}{\epsilon^3} + \frac{7.89144}{\epsilon^2} + \frac{17.0922}{\epsilon} + 8.592639 + \mathcal{O}(\epsilon)\,, \\
I^{(11)}(\epsilon) &= -\frac{31}{8\epsilon^4} - \frac{7\ln 2}{\epsilon^3} + \frac{10.37440}{\epsilon^2} + \frac{49.447}{\epsilon} + 107.1558 \pm 0.0027 + \mathcal{O}(\epsilon)\,, \\
I^{(12)}(\epsilon) &= -\frac{7}{\epsilon^4} - \frac{17\ln 2}{\epsilon^3} + \frac{9.99506}{\epsilon^2} + \frac{70.50}{\epsilon} + 148.0118 \pm 0.0013 + \mathcal{O}(\epsilon)\,, \\
I^{(13)}(\epsilon) &= -\frac{129}{16\epsilon^4} - \frac{87\ln 2}{4\epsilon^3} + \frac{9.74788}{\epsilon^2} + \frac{94.31}{\epsilon} + 236.1222 \pm 0.0016 + \mathcal{O}(\epsilon)\,, \\
I^{(14)}(\epsilon) &= \mathcal{O}(\epsilon)\,, \\
I^{(15)}(\epsilon) &= \frac{1.56735}{\epsilon} + 5.73494 \pm 0.00069 + \mathcal{O}(\epsilon)\,, \\
I^{(16)}(\epsilon) &= -\frac{1}{8\epsilon^4} + \frac{\ln 2}{2\epsilon^3} - \frac{0.68607}{\epsilon^2} - \frac{2.83047}{\epsilon} + 2.218047 + \mathcal{O}(\epsilon)\,, \\
I^{(17)}(\epsilon) &= -\frac{9}{4\epsilon^4} - \frac{3\ln 2}{\epsilon^3} + \frac{8.61834}{\epsilon^2} + \frac{36.07160}{\epsilon} + 78.922647 + \mathcal{O}(\epsilon)\,, \\
I^{(18)}(\epsilon) &= -\frac{3}{4\epsilon^4} + \frac{\ln 2}{2\epsilon^3} + \frac{3.76700}{\epsilon^2} + \frac{7.50556}{\epsilon} + 7.57613 + \mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{B3}
$$

If no errors are quoted, the integration errors are smaller than the quoted precision.

APPENDIX C: MELLIN-BARNES REPRESENTATIONS

In this appendix we present Mellin-Barnes representations of the most complicated integrals of fig. 3, $I^{(12)}$ and $I^{(13)}$. In both cases it is necessary to introduce an auxiliary parameter η in order to render the integral well-defined. The value of the integral in the desired limit $\eta \to 0$ is obtained by analytic continuation. Using the notation $z_{i,j,\dots} = z_i + z_j + \cdots$ we have:

$$
\begin{aligned}
I^{(12)} = {} & \frac{(-1)^{1+2\eta} e^{2\epsilon\gamma}}{\Gamma(-1-2\epsilon-\eta)\Gamma(\eta)} \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \prod_{j=1}^{18} \frac{dz_j}{2\pi i} \Gamma(-z_j) \frac{\Gamma(3+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})}{\Gamma(4+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})} \\
& \times (-s_{12})^{z_{8,13}} (-s_{23})^{z_{14}} (-s_{34})^{z_{1,18}} (-s_{45})^{z_{3,15}} (-s_{61})^{z_{11}} (-s_{123})^{z_{9,16}} (-s_{234})^{z_{17}} (-s_{345})^{z_{2,12}} \\
& \times (-s_{56})^{-5-2\epsilon-2\eta-z_{1,2,3,8,9,11,12,13,14,15,16,17,18}} \\
& \times \frac{\Gamma(-3-\epsilon-z_{1,2,3,4,5,6,7})\Gamma(5+2\epsilon+2\eta+z_{1,2,3,8,9,10,11,12,13,14,15,16,17,18})}{\Gamma(1-z_4)\Gamma(\eta-z_5)\Gamma(-z_6)\Gamma(1-z_7)\Gamma(-3-3\epsilon-2\eta-z_{1,2,3,8,9,10})}
\end{aligned}
$$

$$\times\Gamma(-5-2\epsilon-2\eta-z_{1,2,3,6,8,9,10,11,12,13,14,15,16})\Gamma(-1-\epsilon-\eta+z_{4,5,6,7}-z_{11,12,14,15,17,18})$$
$$\times\Gamma(-2-\epsilon-\eta-z_{1,2,3,8,9,10})\Gamma(\eta-z_5+z_{14,15,16})\Gamma(1-z_4+z_{12,13,18})\Gamma(1+z_{1,2,4,8})$$
$$\times\Gamma(1-z_7+z_{11,12,15})\Gamma(1+z_{1,6,10})\Gamma(1+z_{2,3,7})\Gamma(1+z_{3,5,9})\Gamma(1+z_{11,14,17})\,, \qquad \text{(C1)}$$

$$I^{(13)} = \frac{(-1)^{1+2\eta}e^{2\epsilon\gamma}}{\Gamma(-1-2\epsilon-\eta)\Gamma(\eta)}\int_{-i\infty}^{+i\infty}\cdots\int_{-i\infty}^{+i\infty}\prod_{j=1}^{18}\frac{dz_j}{2\pi i}\Gamma(-z_j)\frac{\Gamma(3+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})}{\Gamma(4+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})}$$
$$\times(-s_{12})^{z_{3,17}}(-s_{23})^{z_{8,14}}(-s_{34})^{z_{11}}(-s_{45})^{z_{9,13}}(-s_{61})^{z_{1,16}}(-s_{123})^{z_{10,18}}(-s_{234})^{z_{15}}(-s_{345})^{z_{2,12}}$$
$$\times(-s_{56})^{-5-2\epsilon-2\eta-z_{1,2,3,8,9,10,11,12,13,14,15,16,17,18}}\Gamma(\eta-z_4+z_{14,15,16})$$
$$\times\frac{\Gamma(-3-\epsilon-z_{1,2,3,4,5,6,7})\Gamma(5+2\epsilon+2\eta+z_{1,2,3,8,9,10,11,12,13,14,15,16,17,18})\Gamma(1+z_{1,2,6,9})}{\Gamma(\eta-z_4)\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(-z_7)\Gamma(-3-3\epsilon-2\eta-z_{1,2,3,8,9,10})}$$
$$\times\Gamma(-5-2\epsilon-2\eta-z_{1,2,3,7,8,9,10,11,12,13,14,15,16})\Gamma(4+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10,13,14,18})$$
$$\times\Gamma(-4-2\epsilon-2\eta-z_{1,2,3,8,9,10,12,13,14,16,17,18})\Gamma(-2\epsilon-\eta-z_{1,2,3,8,9,10})\Gamma(1+z_{2,3,5})$$
$$\times\Gamma(1-z_5+z_{11,12,17})\Gamma(1-z_6+z_{12,13,16})\Gamma(1+z_{1,4,8})\Gamma(1+z_{3,7,10})\,. \qquad \text{(C2)}$$

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