TWO NOTES ON MATRICES

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I. ON A TYPE OF CIRCULANT MATRIX

1. The properties of the circulant determinant or the circulant matrix are familiar. The circulant matrix C of order 4×4, with elements in the complex field, will serve for illustration.

$$C=\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$$

$$=c_0\begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}+c_1\begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{bmatrix}+c_2\begin{bmatrix} . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \end{bmatrix}+c_3\begin{bmatrix} . & . & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \\ . & . & . & 1 \end{bmatrix}.$$

The four matrix coefficients of c_0, c_1, c_2, c_3 form a reducible matrix representation of the cyclic group $\mathscr{C}_4$, so that C is a group matrix for this. Let ω be a primitive 4th root of 1. Then Ω as below, its columns being normalized latent vectors of C,

$$\Omega=(1/\sqrt{4})\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix},$$

is unitary and symmetric, and reduces C to diagonal form thus,

$$\bar{\Omega}'C\Omega=\begin{bmatrix} \mu_0 & & & \\ & \mu_1 & & \\ & & \mu_2 & \\ & & & \mu_3 \end{bmatrix},$$

where the μ_r, the latent roots of C, are given by

$$\mu_r=c_0+c_1\omega^r+c_2\omega^{2r}+c_3\omega^{3r}\quad (r=0,1,2,3).$$

All of the above extends naturally to the $n\times n$ case.

2. The earliest writers on circulants (see for example Muir, *History of Determinants*, vol. ii, 403, on Catalan, Spottiswoode and others) treated a somewhat different circulant, which we shall denote by $\hat{C}$. To illustrate again by the 4×4 case, it is then

$$\hat{C}=\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & c_0 \\ c_2 & c_3 & c_4 & c_0 \\ c_3 & c_4 & c_0 & c_1 \end{bmatrix}=\begin{bmatrix} 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \end{bmatrix}\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}=JC,\quad \text{say.}$$

In contrast with C it is symmetric, though when the c_r are complex and not all real this does not help to elucidate its properties. It might be called a retrocirculant, since the persymmetry of its elements runs the other way from C. We shall find its latent roots in terms of those of C.

THEOREM. *The latent roots of* $\hat{C}$ *are* μ_0, $\pm\sqrt{(\mu_1\mu_{n-1})}$, $\pm\sqrt{(\mu_2\mu_{n-2})}$, ... *and so on to n roots, except that when* $n = 2m$ *there is a single "positively signed"* μ_m, *namely*

$$\mu_m = c_0 - c_1 + c_2 - \;\ldots\; - c_{n-1}.$$

Proof. It is first noted that

$$\bar{\Omega}'\Omega = I, \quad \Omega' = \Omega, \quad \bar{\Omega}'\bar{\Omega} = \Omega'\Omega = J.$$

From this it follows that

$$\bar{\Omega}'\hat{C}\Omega = \bar{\Omega}'\bar{\Omega}\,\bar{\Omega}'C\Omega$$

$$= J\begin{bmatrix} \mu_0 & & & & & \\ & \mu_1 & & & & \\ & & \mu_2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \mu_{n-1} \end{bmatrix} = \begin{bmatrix} \mu_0 & & & & & \cdot \\ & & & & \cdot & \mu_{n-1} \\ & & & \cdot & \mu_{n-2} & \\ & & \cdot & \cdot & & \\ & \cdot & \mu_2 & & & \\ \cdot & \mu_1 & & & & \end{bmatrix}$$

Hence the characteristic determinant is

$$|\lambda I - \hat{C}| = \begin{vmatrix} \lambda-\mu_0 & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \lambda & \cdot & \ldots & \cdot & -\mu_{n-1} \\ \cdot & \cdot & \lambda & \ldots & -\mu_{n-2} & \cdot \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \cdot & \cdot & -\mu_2 & \ldots & \lambda & \cdot \\ \cdot & -\mu_1 & \cdot & \ldots & \cdot & \lambda \end{vmatrix}$$

$$= (\lambda-\mu_0)\prod_{r=1}^{\frac{1}{2}(n-1)}(\lambda^2-\mu_r\mu_{n-r}) \quad (n \text{ odd}),$$

or

$$= (\lambda-\mu_0)(\lambda-\mu_{\frac{1}{2}n})\prod_{r=1}^{\frac{1}{2}(n-2)}(\lambda^2-\mu_r\mu_{n-r}) \quad (n \text{ even}).$$

The theorem is thus established.

The properties of the matrix $\hat{C}$ are rather meagre, which probably explains its lack of prominence in the literature. The matrix coefficients of c_r in it, self-reciprocal but not containing the unit matrix among them, do not form a group. Their binary products generate, n times over, the reducible representation types mentioned in § 1.

II. THE LATENT ROOTS OF A SPECIAL MATRIX

1. The question arose from some work of 1959 by Dr M. Marcus, of the University of British Columbia, and was suggested in correspondence. The problem was: to determine the latent roots and latent vectors of the following symmetric matrix of order $n \times n$,

$$A = \begin{bmatrix} & & & & & & 1 \\ & & & & & 1 & 1 \\ & & & & 1 & 1 & 1 \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 1 & \cdots & 1 & 1 & 1 \\ & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

and to obtain reasonable bounds for the sum of the absolute values of the latent roots. One may note first that

$$A^{-1} = \begin{bmatrix} & & & & -1 & 1 \\ & & & -1 & 1 & \\ & & -1 & 1 & & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & \\ -1 & \vdots & & & & \\ 1 & & & & & \end{bmatrix},$$

which is simpler to handle than A.

Let A^{-1} operate on the vector $f = \{f(1) \quad f(2) \quad \ldots \quad f(n)\}$. Then if $f(0) = 0$ be taken as an initial condition we have

$$A^{-1}f = \{\Delta f(n-1) \quad \Delta f(n-2) \quad \ldots \quad \Delta f(1) \quad \Delta f(0)\}.$$

If then f is a latent vector, its elements $f(x)$ must be solutions of the finite difference equation

$$\Delta f(x) = \lambda f(n-x), \quad \text{with} \quad f(0) = 0. \tag{1}$$

This is known in the literature, having been treated by Stirling. In any case its solution can be seen thus: the complementariness of x and $n-x$ in (1), together with the initial condition, suggests a solution in terms of sines; for we have

$$\Delta \sin \frac{\pi x}{2n+1} = \sin \frac{\pi(x+1)}{2n+1} - \sin \frac{\pi x}{2n+1} = 2 \cos \frac{\pi(x+\frac{1}{2})}{2n+1} . \sin \frac{\frac{1}{2}\pi}{2n+1}$$

$$= 2 \sin \frac{\pi(n-x)}{2n+1} . \sin \frac{\pi}{4n+2} .$$

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Hence, taking $x = 1, 2, \ldots, n$, we have n angles in the range $0 < \theta < \frac{1}{2}\pi$ and a latent vector

$$\left\{\sin\frac{\pi}{2n+1} \quad \sin\frac{2\pi}{2n+1} \quad \ldots \quad \sin\frac{n\pi}{2n+1}\right\}, \tag{2}$$

the corresponding latent root being $\mu_1 = 2\sin[\pi/(4n+2)]$.

But complementariness of sine and cosine occurs also with respect to the angles $3\pi/2$, $5\pi/2, \ldots, (2n-1)\pi/2$, the cosines taking here the respective signs $-, +, -, +$ and so on.

In this way we have $n-1$ further distinct latent vectors obtained by writing $3\pi, 5\pi, \ldots,$ $(2n-1)\pi$ instead of π in (2), e.g.

$$\left\{\sin\frac{3\pi}{2n+1} \quad \sin\frac{6\pi}{2n+1} \quad \ldots \quad \sin\frac{3n\pi}{2n+1}\right\}$$

and so on, the associated latent roots being

$$\mu_r = (-)^{r-1}\, 2\sin\frac{(2r-1)\pi}{4n+2} \quad (r = 2, 3, \ldots, n).$$

It may be noted that the μ_r are in ascending order of moduli. Also the set of latent vectors is now complete and necessarily orthogonal, since A^{-1} is real symmetric and the μ_r are distinct. It is easy to show by trigonometrical considerations that the sum of squared elements of each vector is $\frac{1}{4}(2n+1)$, whence the vectors can be orthonormed by multiplying each by $2/\sqrt{(2n+1)}$.

All of the above refers to A^{-1}. For A, the sole change is to take reciprocals of the μ_r; whence finally the latent roots of A are

$$\lambda_r = (-)^{r-1}\tfrac{1}{2}\operatorname{cosec}\frac{(2r-1)\pi}{4n+2} \quad (r = 1, 2, \ldots, n).$$

Example. $n = 4$. The latent roots are

$$2{\cdot}879385, \qquad -1{\cdot}00000, \qquad 0{\cdot}65270, \qquad -0{\cdot}53209.$$

The four latent orthonormal vectors, brought together to make the columns of an orthogonal matrix, are

$$\tfrac{2}{3}\begin{bmatrix} 0{\cdot}34202 & 0{\cdot}86603 & 0{\cdot}98481 & 0{\cdot}64279 \\ 0{\cdot}64279 & 0{\cdot}86603 & -0{\cdot}34202 & -0{\cdot}98481 \\ 0{\cdot}86603 & 0{\cdot}00000 & -0{\cdot}86603 & 0{\cdot}86603 \\ 0{\cdot}98481 & -0{\cdot}86603 & 0{\cdot}64279 & -0{\cdot}34202 \end{bmatrix}.$$

2. The next problem was to set effective bounds to $\sum|\lambda_r|$.

Clearly, since over $0 < \theta < \frac{1}{2}\pi$ we have $1 < \theta/(\sin\theta) < \frac{1}{2}\pi$, a lower bound is

$$\frac{2n+1}{\pi}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right).$$

The bracketed expression can be approximated asymptotically from below by

$$\tfrac{1}{2}\log_e n + \log_e 2 + \tfrac{1}{2}\gamma,$$

where γ is Euler's constant.

For an upper bound, we may easily show that $\theta/\sin\theta$ is convex in $0 < \theta < \frac{1}{2}\pi$, whence

$$\sum|\lambda_r| < \frac{2n+1}{\pi}\left[1+\frac{\frac{1}{2}\pi-1}{2n+1}+\frac{1}{3}\left\{1+\frac{3(\frac{1}{2}\pi-1)}{2n+1}\right\}+\ \dots\ +\frac{1}{2n-1}\left\{1+\frac{(2n-1)(\frac{1}{2}\pi-1)}{2n+1}\right\}\right],$$

i.e.
$$< \frac{2n+1}{\pi}\left\{1+\frac{1}{3}+\frac{1}{5}+\ \dots\ +\frac{1}{2n-1}+\frac{n}{2n+1}\left(\frac{\pi}{2}-1\right)\right\},$$

i.e.
$$< \frac{2n+1}{\pi}\left(1+\frac{1}{3}+\frac{1}{5}+\ \dots\ +\frac{1}{2n-1}\right)+n\left(\frac{1}{2}-\frac{1}{\pi}\right).$$

The expression in the first bracket can be asymptotically approximated from above by

$$\log_e(4n+1)-\tfrac{1}{2}\log_e(2n+1)+\tfrac{1}{2}\gamma-\tfrac{1}{2}\log_e 2,$$

so that, replacing the bracket by this, we now have $\sum|\lambda_r|$ enclosed between tolerable bounds. It would be possible to refine on either bound, but it was unnecessary for the purpose in view. Two numerical examples may serve to exhibit these bounds.

Examples. (i) $n = 4$. Lower bound 4·8019, upper bound 5·5287, actual value

$$\sum|\lambda_r| = 5{\cdot}0642.$$

(ii) $n = 22$. Lower bound 36·201, upper bound 40·198, actual value

$$\sum|\lambda_r| = 37{\cdot}842.$$

For large values of n the bounds are very satisfactory.

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