TWO NOTES ON RANKIN'S BOOK ON THE MODULAR GROUP

Dedicated to the memory of Hanna Neumann

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We supply proofs that are simple, and possibly partly new, for two theorems that appear in Rankin's book [6].

1.

The first concerns subgroups of the inhomogeneous modular group. Let $\Gamma = SL(2, Z)$, the group of all 2 by 2 matrices with integer coefficients and with determinant 1. For each positive integer n, let $\Gamma(n)$ consist of those T in Γ such that $T \equiv I$ modulo n. Let $\Delta(n)$ be the least normal subgroup of Γ that contains the element $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

THEOREM 1. (a) *If* $1 \leqq n \leqq 5$, *then* $\Gamma(n) = \Delta(n)$.
(b) *For all* n, *the index of* $\Gamma(n)$ *in* Γ *is* $\eta(n) = n^3 \prod_{p/n} (1 - 1/p^2)$.
(c) *For* $n \geqq y$, *the index of* $\Delta(n)$ *in* Γ *is infinite.*

We first prove (b). Note that both $\Gamma(n)$ and $\Delta(n)$ are normal in Γ; thus we may write $\Gamma_n = \Gamma/\Gamma(n)$ and $\Delta_n = \Gamma/\Delta(n)$. Observe that both $|\Gamma_n|$ and $\eta(n)$ are multiplicative functions of n; therefore it suffices to treat the case that n is a power of a prime. The routine solution of a system of congruences shows that, under the natural map, $\Gamma_n \cong SL(2, Z_n)$. A standard argument gives the order of the latter group.

Clearly $\Delta(n) \subseteq \Gamma(n)$, whence $|\Gamma_n| \leqq |\Delta_n|$. Therefore, to establish (a) and (c) it suffices to show that, for $n \leqq 5$, $|\Delta_n| \leqq \eta(n)$ and that for $n \geqq 6$, $|\Delta_n| = \infty$. In the sequel we put aside the trivial case that $n = 1$.

Both Γ and Δ_n have center $Z = \{I, -I\}$. It is well known that $P = \Gamma/Z$ has the presentation

$$P\langle A, B : A^2 = 1, B^3 = 1\rangle,$$

where A and B are given by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Since $(AB)^n$ is

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given by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, the group $G_n = \Delta_n/Z$ obtained from P by imposing the further relation $(AB)^n = 1$ has the presentation

$$G_n = \langle A, B; A^2 = 1, B^3 = 1, (AB)^n = 1 \rangle.$$

These groups are well known, especially from the work of Coxeter (see [1]). It remains to show that G_2 has order $\eta(2)$, that $2\,|G_n| = \eta(n)$ for $n = 3, 4, 5$, and that G_n is infinite for $n \geqq 6$.

For each $n \geqq 2$ we shall construct a graph T_n on the sphere (or better, for $n \geqq 6$, in the plane) such that each region is a triangle and that there are exactly n edges at each vertex. For $n = 2, 3, 4, 5$ we take T_n to be a triangle, the 1-skeleton of a tetrahedron, of an octahedron, of an icosahedron. This reflects the fact that the corresponding Δ_n are the symmetry groups of these figures.

Let $n \geqq 6$. We begin with a triangle ABC and enclose it it a circle K not touching it. Join A and B to a point C' on K, join B and C to a point A' on K, and join C and A to a point B' on K. Now join each of A, B, and C to additional points on K until there are n edges at each of these three vertices. Clearly this can be done so that each bounded region in the resulting figure is a (topological) triangle. We now enclose the figure obtained so far in a larger circle K' not touching it, and repeat the process. It is easy to see that this process can be iterated indefinitely and will yield an infinite graph T_n with the required properties.

We now modify T_n to obtain a new graph S_n. We draw small circles about the vertices of T_n (small enough that no two meet) and delete those parts of T_n interior to these circles. In S_n each vertex lies on exactly three edges: two that are arcs of the small circles and one that is 'straight', that is, the remnant of an edge from T_n.

Let Ω be the set of vertices of S_n. Define two permutations of Ω as follows. If P is any vertex, then $\alpha(P)$ is the other vertex on the straight edge at P, while $\gamma(P)$ is the other vertex on the circular arc proceeding counterclockwise out of P. It is immediate that $\alpha^2 = 1$ and $\gamma^n = 1$. Moreover, inspection shows that, if $\beta = \alpha\gamma$, then $\beta^3 = 1$. If π is the group of permutations of Ω generated by α and γ, it is immediate that setting $\phi(A) = \alpha$ and $\phi(B) = \beta$ define a homomorphism ϕ from G_n onto π. (In fact ϕ is an isomorphism, and S_n is a Cayley diagram for G_n; alternatively, that ϕ is an isomorphism follows by a method known to Poincaré (see Macbeath [3]).

Since π acts regularly on Ω, we have $|\pi| = |\Omega|$, whence $|G_n| \geqq |\Omega|$. This gives the desired inequality for $|\Delta_n|$ in both the finite and infinite cases.

2.

The second theorem is one of Nielsen [4]. We state it in a mildly modified form.

THEOREM 2. *Let $G_1, \cdots, G_n$ be arbitrary groups, and let N be the kernel*

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of the natural map from the free product $G = G_1 * \cdots * G_n$ *onto the direct product* $\bar{G} = G_1 \times \cdots \times G_n$. *Then* N *is free group with a basis* X *consisting of all non trivial elements of the form*

$$x = (a_1 \cdots a_{i-1} a_{i+1} \cdots a_n a_i)(a_1 \cdots a_n)^{-1}$$

where $a_1 \in G_1, \cdots, a_n \in G_n$.

We begin the proof by showing that X generates N. Let H be the subgroup generated by X. Clearly $H \subseteq N$. It will suffice to show that $G = HG_1 \cdots G_n$. For this it suffices to show that, for all $i = 1.2, \cdots, n$ one has $HG_1 \cdots G_n G_i = HG_1 \cdots G_n$. Now $Hx = H$ for all $x \in X$ implies that $HG_1 \cdots G_{i-1} G_{i+1} \cdots G_n G_i = HG_1 \cdots G_n$. Using this relation we find that $HG_1 \cdots G_n G_i = HG_1 \cdots G_{i-1} G_{i+1} \cdots G_n G_i G_i$ $HG_1 \cdots G_{i-1} G_{i+1} \cdots G_n G_i = HG_1 \cdots G_n$.

It remains to show that X is a basis for N. Note that an element x as above is not trivial just in case $a_i \neq 1$ and that $a_j \neq 1$ for some $j > i$. We write $x = UV^{-1}$ where $U = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n a_i$ and $V = a_1 \cdots a_n$. Then U and V have the same length $|U| = |V| = m \leqq n$ and $|x| = 2m$. (Here m is the number of non-trivial factors a_i.) Let $x' = U'V'^{-1}$ denote analogously another element of X.

We make several observations.

(1) If x and x' have the initial segment U in common, then $x = x'$. This follows from the fact that a_i, as the first (non trivial) syllable of x with decreasing subscript, must match the first such syllable of x'. This implies that $U = U'$ whence also $V = V'$.

(2) If x^{-1} and x'^{-1} have an initial segment longer than V in common, then $x = x'$. Suppose they had such an initial segment in common, and hence the initial segment Va_i^{-1}. Since a_i^{-1} is the first syllable of x^{-1} with decreasing subscript, it must match the first such syllable of x'^{-1}, whence $Va_i^{-1} = V'a_{i'}^{-1}$. From $V = V'$ it follows that $a_1 = a_1', \cdots, a_n = a_n'$, and we have also that $a_i^{-1} = a_{i'}^{-1}$, whence $i = i'$. Thus $U = U'$ and $x = x'$.

(3) If x^{-1} and x' have the initial segment V in common, then they have no longer initial segment in common, and the segment V is less than half of x'. This follows from the facts that V, with increasing subscripts, must be a proper initial segment of U', and that Va_i^{-1}, containing two syllables a_i and a_i^{-1} from G_i, cannot be a segment of U'.

(4) In a product $xx'^e \neq 1$, with $e = \pm 1$, at most the right half V^{-1} of x cancels. If $e = +1$ this follows from (3), and if $e = -1$ from (2).

(5) In a product $x'^e x \neq 1$, with $e = \pm 1$, not all of the left half U of x cancels. If $e = +1$ this follows from (3), and if $e = -1$ from (1).

Now a classical argument of Nielsen [5] shows that in a product $w = x_1^{e_1} \cdots x_k^{e_k}$ where $k \geqq 1$ and no $x_i^{e_i} x_{i+1}^{e_{i+1}} = 1$, some part of each factor remains after cancellation, whence $w \neq 1$. This proves that X is a basis. (It is curious that Nielsen himself used at this point a different argument.)

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We conclude with two remarks. The first is a minor point, that Nielsen used a slightly different basis consisting of elements of the form

$$x' = (a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1})(a_i a_j a_i^{-1} a_j^{-1})(a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1})^{-1}.$$

It is easy to pass from one basis to the other by Nielsen transformations. The second point is that Nielsen treated only the case that the G_i are all finite cyclic groups, and in this case gave a formula for the rank $r(N) = |X|$ of N. For the slightly more general case that all the groups G_i are arbitrary finite groups, and hence that $\bar{G}$ is finite and N again of finite rank, a formula for $r(N)$ can be recovered easily by counting how many n-tuples $a_1, \cdots, a_n$ yield non trivial elements $x \in X$. The result can be stated as a generalization of Schreier's index theorem. Assuming all the G_i finite, define the 'free rank' of G_i to be $r(G_i) = 1 - 1/|G_i|$ and $r(G) = r(G_1) + \cdots + r(G_n)$. Then one has

$$[G : N] = \frac{r(N) - 1}{r(G) - 1}.$$

In Nielsen's case this is indeed a case of a classical formula for surface groups, for which a combinatorial proof is given in [2]. One may conjecture that such a formula holds for a subgroup N, not neccesarily normal, in a free product G of some more extensive class of groups G_i for which a reasonable definition of $r(G_i)$ can be provided.

Postscript (August, 1972). Mr. I. Chiswell has established that for G and the function r as defined above, the formula stated above holds for any subgroup N of finite index in G, without the assumption that N is normal.

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