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Two preservation results for countable products of sequential spaces

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We prove two results for the sequential topology on countable products of sequential topological spaces. First we show that a countable product of topological quotients yields a quotient map between the product spaces. Then we show that the reflection from sequential spaces to its subcategory of monotone ω-convergence spaces preserves countable products. These results are motivated by applications to the modelling of computation on non-discrete spaces.

1. Introduction

In the theory of Type Two Effectivity (Weihrauch 2000), computation on non-discrete spaces is performed by type two Turing machines, which compute with infinite words acting as *names* for elements of spaces. The connection between names and their associated elements is specified by a many-to-one relation called a *representation*. Such representations induce a topology on the named elements, namely the quotient topology of the relative product topology on the set of names. For certain *admissible representations*, the naming relations are themselves determined (up to a continuous equivalence) by the topology of the represented space. Moreover, the property of admissibility serves as a well-behavedness criterion for representations. In the case of the real numbers, for example, the property of admissibility exactly captures the distinction between reasonable computable representations (for example, signed digit or Cauchy sequences with specified modulus), which are admissible, and unreasonable ones (for example, ordinary binary/decimal notation), which are not. In general, one can argue that the spaces with admissible quotient representation are exactly the topological spaces that support a good (type two) computability theory, see Weihrauch (2000) and Schröder (2003).

In his Ph.D. thesis (Schröder 2003), the first author characterised those topological spaces that have admissible quotient representations as being exactly the T_0 quotient spaces of countably-based spaces (*qcb spaces*). Qcb spaces are closed under many useful constructions for modelling computation; for example, the category of continuous functions between qcb spaces is cartesian closed and hence models typed lambda-calculus (Schröder 2003; Menni and Simpson 2002). Furthermore, by restricting to qcb spaces

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that are *monotone convergence spaces* in the sense of Gierz *et al.* (2003), one obtains a collection of spaces that have all the desirable closure properties of a category of predomains in domain theory (Simpson 2003; Battenfeld *et al.* 2005). Such *topological predomains* extend the usual scope of domain theory in not being restricted to dcpos with their Scott topology. It is our thesis, supported by the aforementioned work, that qcb spaces and topological predomains are the most general classes of topological spaces that are suitable for the study of computability in topology and domain theory, respectively.

In this paper, we address two technical questions that have arisen in the context of constructing free algebras for equational theories over qcb spaces and topological predomains. The study of such free algebras is interesting for at least two reasons. For us, a main motivation was the identification by Plotkin and Power of free algebras as a general mechanism for modelling *computational effects* (that is, the non-functional aspects of computation), see, for example, Plotkin and Power (2002). More generally, the construction of free algebras is, of course, important in any approach to modelling computable (universal) algebra, *cf.* Stoltenberg-Hansen and Tucker (1995).

One aspect of Plotkin and Power's study of computational effects is that one needs free algebras for equational theories in which the algebraic operations may have infinite arity. Taking account of this, Battenfeld has studied the construction of free algebras for equational theories allowing (parametrised) operations of countable arity (Battenfeld 2006). He has obtained explicit descriptions of such free algebras both in the general case of qcb spaces and in the restricted case of topological predomains. The correctness of his descriptions depends upon two technical results:

(i) Countable products in the category of qcb spaces preserve topological quotients

(ii) The reflection functor from the category of qcb spaces to the category of topological predomains preserves countable products.

The purpose of the present note is to provide proofs of these results.

Since it is the natural level of generality at which the arguments work, we present proofs of (i) and (ii) for arbitrary sequential spaces, rather than restricting to the special case of qcb spaces. In Section 2 we briefly review the necessary technical background on sequential spaces. In Section 3 we develop the technical notion of a function *co-reflecting* convergent sequences, which is used as a tool in the proof of both main results. The proof of the preservation of quotients by countable products is then given in Section 4. Finally, in Section 5, we prove that the reflection from sequential spaces to monotone ω-convergence spaces preserves countable products.

2. Preliminaries

We use $\mathcal{O}(X)$ to denote the topology of a topological space X. A subset U of a topological space X is *sequentially open* if, whenever a sequence (x_n) converges to some $x \in U$ (notation $(x_n) \to x \in U$), all but finitely many x_n are in U. The space X is said to be *sequential* if every sequentially open subset is open (Franklin 1965). This paper is concerned with the category **Seq** of sequential spaces and continuous functions.

Given a sequence of sequential spaces X_i, we use $X_0 \times X_1$, $\prod_{i \leqslant l} X_i$ and $\prod_i X_i$ to denote the binary, finite and countable products, respectively, in this category. The convergence relation on such products is pointwise, that is, $(x_n) \to x$ in $\prod_i X_i$ if and only if $\pi_i(x_n) \to \pi_i(x)$, for all $i \geqslant 0$, where we write $\pi_i(x)$ for the i-th component of x. In general, the sequential topology determined by this convergence relation is finer than the topological product. As notation on infinite sequences, we also write $\pi_{\leqslant l}(x)$ for the prefix $\pi_0(x) \dots \pi_l(x) \in \prod_{i \leqslant l} X_i$, and $\pi_{>l}(x)$ for the suffix $\pi_{l+1}(x)\pi_{l+2}(x) \dots \in \prod_{i>l} X_i$. Moreover, given $y \in \prod_{i \leqslant l} X_i$ and $z \in \prod_{i>l} X_i$, we use $y@z$ to denote the unique $x \in \prod_i X_i$ with $\pi_{\leqslant l}(x) = y$ and $\pi_{>l}(x) = z$. Given functions $q_i : X_i \to Y_i$, we use $\prod_{i \leqslant l} q_i$ and $\prod_{i>l} q_i$ to denote the corresponding product functions from $\prod_{i \leqslant l} X_i$ to $\prod_{i \leqslant l} Y_i$ and from $\prod_{i>l} X_i$ to $\prod_{i>l} Y_i$, respectively.

The category **Seq** is cartesian closed with exponential Y^X given by the set of continuous functions with convergence relation: $(f_n) \to f$ if and only if, for every convergent sequence $(x_n) \to x$ in X, we have $f_n(x_n) \to f(x)$ in Y.

A topological space X is a *qcb space* if it can be presented as a topological quotient $q : A \to X$ where A is a countably-based space. It is easy to see that every qcb space is sequential. The category **QCB** of qcb spaces is cartesian closed with countable limits and colimits, and the inclusion of **QCB** in **Seq** preserves this structure (Menni and Simpson 2002; Schröder 2003; Escardó *et al.* 2004).

We write $\mathbb{N}^+$ for the space with underlying set $\mathbb{N} \cup \{\infty\}$, with basic opens: $\{n\}$ and $\{m \mid m \geqslant n\} \cup \{\infty\}$, for every $n \in \mathbb{N}$. This space acts as a generic converging sequence since $(x_n) \to x_\infty$ in X if and only if the function $\alpha \mapsto x_\alpha$ from $\mathbb{N}^+$ to X is continuous.

An *ω-complete partial order* (ωcpo) is a partial order $(X, \sqsubseteq)$ for which every ascending sequence $x_0 \sqsubseteq x_1 \sqsubseteq \dots$ has a least upper bound (lub). A subset U is open in the *ω-Scott topology* on an ωcpo if, for every ascending sequence $x_0 \sqsubseteq x_1 \sqsubseteq \dots$ with lub $x_\infty \in U$, only finitely many (x_n) are outside U. It is readily seen that the ω-Scott topology is sequential.

We write $\sqsubseteq_X$ for the *specialisation order* on a space X, defined by $x \sqsubseteq_X x'$ if x' is contained in every neighbourhood of x. For a general topological space, the specialisation order is a preorder. Recall that a space is T_0 if and only if its specialisation order is a partial order.

We write $\mathbb{S}$ for Sierpinski space $\{\bot, \top\}$ with $\{\top\}$ open but $\{\bot\}$ not open, thus $\bot \sqsubseteq_{\mathbb{S}} \top$ but $\top \not\sqsubseteq_{\mathbb{S}} \bot$. We observe that, for a sequential space X, the exponential $\mathbb{S}^X$ is given by the family of open subsets of X with the ω-Scott topology on the inclusion order. Since this fact is not needed in this note, we do not give a proof. For a proof of an analogous result for compactly generated spaces, see Escardó *et al.* (2004, Corollary 5.16).

3. Co-reflecting convergent sequences

Throughout this paper, $X, Y, \dots$ range over sequential spaces. We begin with two basic lemmas.

Lemma 3.1. Let V be an open set of $X \times Y$, and K be a sequentially compact subset of Y. Then $U := \{x \in X \mid \{x\} \times K \subseteq V\}$ is an open set.

Proof. Assume, in order to show a contradiction, that U is not sequentially open. Then there is a convergent sequence $(x_n)_n \to x_\infty$ in X with $x_\infty \in U$ and $\forall n \in \mathbb{N}\,.\, x_n \notin U$. There exists a sequence $(y_n)_n$ in K with $(x_n, y_n) \notin V$. Sequential compactness yields a subsequence $(y_{\varphi(n)})_n$ converging to some $y_\infty \in K$. As V is open, $(x_{\varphi(n)}, y_{\varphi(n)})$ is eventually in V, which is a contradiction. □

The property stated in the above lemma holds, more generally, for all countably compact subsets K, but this requires a more complex proof. The above is sufficient for the purposes of the present paper.

Lemma 3.2. Let V be an open set of $\prod_i X_i$ and $x \in V$. Then there is some $l \in \mathbb{N}$ such that $\{\pi_{\leqslant l}(x)\} \times \prod_{i>l} X_i \subseteq V$.

Proof. Assume that no such l exists. Then for every $n \in \mathbb{N}$ there exists some $y_n \in \prod_i X_i \setminus V$ with $\pi_{\leqslant n}(y_n) = \pi_{\leqslant n}(x)$. For every $i \in \mathbb{N}$, the sequence $(\pi_i(y_n))_n$ converges in X_i to $\pi_i(x)$ by being eventually constant. Thus $(y_n)_n$ converges in $\prod_i X_i$ to x, which is a contradiction. □

A continuous function $f : X \to Y$ is said to *reflect convergent sequences* if, whenever $(f(x_n)) \to f(x)$ we have $(x_n) \to x$. The following derived notion plays a crucial role in the sequel.

Definition 3.3. We say that a function $q : X \to Y$ *co-reflects convergent sequences* if it is continuous and $\mathbb{S}^q : \mathbb{S}^Y \to \mathbb{S}^X$ reflects convergent sequences.

The next proposition gives a useful characterisation of the property of co-reflecting convergent sequences.

Proposition 3.4. A continuous function $q : X \to Y$ co-reflects convergent sequences if and only if for every convergent sequence $(y_n)_n \to y_\infty$ in Y and every open V containing y_∞ there is a convergent sequence $(x_n)_n \to x_\infty$ in X and a strictly increasing $\psi : \mathbb{N} \to \mathbb{N}$ such that $q(x_\infty) \in V$ and $q(x_n) \sqsubseteq_Y y_{\psi(n)}$ for all $n \in \mathbb{N}$.

Proof. For the only-if direction, let $(y_n)_n \to y_\infty$ be a convergent sequence in Y, and let V be an open neighbourhood of y_∞. For $n \in \mathbb{N}^+$ define $h_n : Y \to \mathbb{S}$ by

$$h_n(y) := \begin{cases} \top & \text{if } (n = \infty \text{ and } y \in V) \text{ or } (n \neq \infty \text{ and } y \not\sqsubseteq_Y y_n) \\ \bot & \text{otherwise.} \end{cases}$$

Then $(h_n)_n$ does not converge to h_∞, because $h_\infty(y_\infty) = \top$ and $h_n(y_n) = \bot$ for all $n \in \mathbb{N}$. Hence $(\mathbb{S}^q(h_n))_n$ does not converge to $\mathbb{S}^q(h_\infty)$. This implies that there are a convergent sequence $(x_n)_n \to x_\infty$ in X and a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ with $\mathbb{S}^q(h_\infty)(x_\infty) = \top$ and $\mathbb{S}^q(h_{\varphi(n)})(x_n) = \bot$, which means $q(x_\infty) \in V$ and $q(x_n) \sqsubseteq_Y y_{\varphi(n)}$.

For the if direction, $\mathbb{S}^q$ is clearly continuous. Let $(h_n)_n$ be a sequence in $\mathbb{S}^Y$ that does not converge to h_∞. Then there are a convergent sequence $(y_n)_n \to y_\infty$ in Y and a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ with $h_\infty(y_\infty) = \top$ and $h_{\varphi(n)}(y_n) = \bot$. By assumption, there exist a convergent sequence $(x_n)_n \to x_\infty$ in X and a strictly increasing function $\psi : \mathbb{N} \to \mathbb{N}$ with $q(x_\infty) \in h_\infty^{-1}\{\top\}$ and $q(x_n) \sqsubseteq_Y y_{\psi(n)}$. Thus we have $\mathbb{S}^q(h_\infty)(x_\infty) = \top$ and $\mathbb{S}^q(h_{\varphi\psi(n)})(x_n) = \bot$, which implies that $(\mathbb{S}^q(h_n))_n$ does not converge to $\mathbb{S}^q(h_\infty)$ in $\mathbb{S}^X$. □

Example 3.5. Every quotient map $q : X \to Y$ between sequential spaces co-reflects convergent sequences.

This fact can be proved abstractly as follows. The quotient maps in **Seq** are exactly the regular epimorphisms. Because the contravariant functor $\mathbb{S}^{(-)} : \mathbf{Seq} \to \mathbf{Seq}^{\mathrm{op}}$ is a left adjoint, it maps regular epimorphisms to regular monos in **Seq**, and the latter are exactly the injective continuous functions that reflect convergent sequences. The following direct argument is included for the benefit of readers who prefer proofs from first principles.

Proof. Let $(h_n)_{n\leqslant\infty}$ be a sequence in $\mathbb{S}^Y$ such that $(\mathbb{S}^q(h_n))_n$ converges to $\mathbb{S}^q(h_\infty)$ in $\mathbb{S}^X$. Let $(y_n)_n \to y_\infty$ be a convergent sequence in Y with $h_\infty(y_\infty) = \top$, and choose $x_\infty \in X$ with $q(x_\infty) = y_\infty$. Since the function $g : X \times \mathbb{N}^+ \to \mathbb{S}$ with $g(x, n) := \mathbb{S}^q(h_n)(x)$ is continuous, there is some $n_0 \in \mathbb{N}$ with $g(x_\infty, n) = \top$ for all $n \geqslant n_0$. Define $V := \{y \in Y \mid \forall n_0 \leqslant n \leqslant \infty\,.\, h_n(y) = \top\}$. Since $q^{-1}[V] = \{x \in X \mid \{x\} \times \{\infty, n \mid n \geqslant n_0\} \subseteq g^{-1}\{\top\}\}$ is open in X by Lemma 3.1, V is open in Y. Thus $(y_n)_n$ is eventually in V, which implies that $h_n(y_n) = \top$ for almost all n. We conclude that $(h_n)_n$ converges to h_∞ in $\mathbb{S}^Y$. □

We show that the property of co-reflecting convergent sequences is preserved by forming countable products of functions.

Proposition 3.6. If the mappings $q_i : X_i \to Y_i$ co-reflect convergent sequences for $i \in \mathbb{N}$, then so does the product mapping $(\prod_i q_i) : (\prod_i X_i) \to (\prod_i Y_i)$.

Proof. With the help of Proposition 3.4 and Lemma 3.1, it is easy to prove that, for any sequential space Z, the product $q_0 \times \mathrm{id}_Z$ co-reflects convergent sequences. From this result one can easily deduce that the finite product $\prod_{i\leqslant n} q_i$ co-reflects convergent sequences for every $n \in \mathbb{N}$.

Let $(y_n)_n \to y_\infty$ be a convergent sequence in $\prod_i Y_i$ and W be an open neighbourhood of y_∞. There is some $l \in \mathbb{N}$ with $\{\pi_{\leqslant l}(y_\infty)\} \times \prod_{i>l} Y_i \subseteq W$.

For $i = l, l+1, \ldots$ we define by recursion strictly increasing functions $\psi_i : \mathbb{N} \to \mathbb{N}$ as follows. We set $\psi_l := \mathrm{id}_{\mathbb{N}}$ and apply for $i > l$ Proposition 3.4 to $(\pi_i(y_{\psi_l\circ\ldots\circ\psi_{i-1}(n)}))_n$ in order to obtain a subsequence $(\pi_i(y_{\psi_l\circ\ldots\circ\psi_i(n)}))_n$ of $(\pi_i(y_n))_n$ and a convergent sequence $(a_{i,n})_n \to a_{i,\infty}$ in X_i with $q_i(a_{i,n}) \sqsubseteq_{Y_i} \pi_i(y_{\psi_l\circ\ldots\circ\psi_i(n)})$. Then we define the strictly increasing function ψ by $\psi(n) := \psi_l \circ \ldots \circ \psi_{l+n}(n)$. Using $(a_{i,n})_{i,n}$, we construct a convergent sequence $(z_n)_n \to z_\infty$ in $\prod_{i>l} X_i$ with $(\prod_{i>l} q_i)(z_n) \sqsubseteq_{\prod_{i>l} Y_i} \pi_{>l}(y_{\psi(n)})$.

As $V := \{y \in \prod_{i\leqslant l} Y_i \mid y@(\prod_{i>l} q_i)(z_\infty) \in W\}$ is a neighbourhood of $\pi_{\leqslant l}(y_\infty)$ by Lemma 3.1 and $\prod_{i\leqslant l} q_i$ co-reflects convergent sequences, there is a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ and a convergent sequence $(x_n)_n \to x_\infty$ in $\prod_{i\leqslant l} X_i$ with $(\prod_{i\leqslant l} q_i)(x_n) \sqsubseteq_{\prod_{i\leqslant l} Y_i} \pi_{\leqslant l}(y_{\psi\varphi(n)})$ and $(\prod_{i\leqslant l} q_i)(x_\infty) \in V$. It is obvious that

$$\left(\prod_i q_i\right)\left(x_n @ z_{\varphi(n)}\right) \sqsubseteq_{\prod_i Y_i} y_{\psi\varphi(n)}$$

and $(\prod_i q_i)(x_\infty @ z_\infty) \in W$. Hence $\prod_i q_i$ co-reflects convergent sequences by Proposition 3.4. □

Lemma 3.7. Suppose $q_i : X_i \to Y_i$ co-reflects convergent sequences for $i \in \mathbb{N}$. Let W be a subset of $\prod_i Y_i$ satisfying:

(a) The preimage $(\prod_i q_i)^{-1}[W]$ is open.
(b) For every $y \in W$ there is some $l \in \mathbb{N}$ with $\{\pi_{\leqslant l}(y)\} \times \prod_{i>l} Y_i \subseteq W$.
(c) The set $\{y \in \prod_{i\leqslant l} Y_i \mid y@(\prod_{i>l} q_i)(x) \in W\}$ is open in $\prod_{i\leqslant l} Y_i$ for every $l \in \mathbb{N}$ and every $x \in \prod_{i>l} X_i$.
(d) $(\prod_i q_i)(x) \in W$ and $(\prod_i q_i)(x) \sqsubseteq_{\prod Y_i} y$ imply $y \in W$ for all $x \in \prod_i X_i$ and $y \in \prod_i Y_i$.

Then W is open in $\prod_i Y_i$.

Proof. Assume that W is not open. Then there is a convergent sequence $(y_n)_n \to y_\infty$ with $y_\infty \in W$ and $\forall n \in \mathbb{N}\,.\, y_n \notin W$. By (b), there is some $l \in \mathbb{N}$ such that $\{\pi_{\leqslant l}(y_\infty)\} \times \prod_{i>l} Y_i \subseteq W$. By Propositions 3.4 and 3.6 there exist a convergent sequence $(z_n)_n \to z_\infty$ in $\prod_{i>l} X_i$ and some strictly increasing $\varphi : \mathbb{N} \to \mathbb{N}$ with $(\prod_{i>l} q_i)(z_n) \sqsubseteq_{\prod_{i>l} Y_i} \pi_{>l}(y_{\varphi(n)})$. By (c), the set $V = \{y \in \prod_{i\leqslant l} Y_i \mid y@(\prod_{i>l} q_i)(z_\infty) \in W\}$ is an open neighbourhood of $\pi_{\leqslant l}(y_\infty)$. Again by Proposition 3.6, there is a convergent sequence $(x_n)_n \to x_\infty$ in $\prod_{i\leqslant l} X_i$ and a strictly increasing $\psi : \mathbb{N} \to \mathbb{N}$ such that $(\prod_{i\leqslant l} q_i)(x_n) \sqsubseteq_{\prod_{i\leqslant l} Y_i} \pi_{\leqslant l}(y_{\varphi\psi(n)})$ and $(\prod_{i\leqslant l} q_i)(x_\infty) \in V$. Clearly, $(x_n@z_{\psi(n)})_n$ converges to $x_\infty@z_\infty$ in $\prod_i X_i$. Since $(\prod_i q_i)^{-1}[W]$ is open, there is some $n_0 \in \mathbb{N}$ with $(\prod_i q_i)(x_{n_0}@z_{\psi(n_0)}) \in W$. This implies $y_{\varphi\psi(n_0)} \in W$ by (d), which is a contradiction. $\square$

4. Preservation of quotients

Let $q_i : X_i \to Y_i$ be quotient maps between sequential spaces X_i and Y_i. We define $q_\infty : \prod_{i\in\mathbb{N}} X_i \to \prod_{i\in\mathbb{N}} Y_i$ by $q_\infty(x_0, x_1, \dots) := \big(q_0(x_0), q_1(x_1), \dots\big)$ and use Y_∞ to denote the sequential space having $\prod_{i\in\mathbb{N}} Y_i$ as its underlying set and as its topology $\mathcal{O}(Y_\infty)$ the final topology induced by q_∞. The aim of this section is to prove that $Y_\infty = \prod_{i\in\mathbb{N}} Y_i$, and thus that q_∞ exhibits $\prod_{i\in\mathbb{N}} Y_i$ as a quotient of $\prod_{i\in\mathbb{N}} X_i$. In other words, we show that countable products in **Seq** preserve quotient maps.

First, we recall the standard fact that finite products in **Seq** preserve quotients.

Lemma 4.1. If $q : X \to Y$ is a quotient map and Z is a sequential space, then $(q \times \mathrm{id}_Z) : X \times Z \to Y \times Z$ is a quotient map.

For an abstract proof of the lemma, since **Seq** is cartesian closed, the functor $(-) \times Z : \mathbf{Seq} \to \mathbf{Seq}$ preserves regular epis and hence topological quotients. Again we give a self-contained proof for readers who prefer such arguments.

Proof. Clearly, $r := q \times \mathrm{id}_Z$ is continuous. Now let $W \subseteq Y \times Z$ be a set such that $r^{-1}[W]$ is open in $X \times Z$. Let $(y_n, z_n)_n$ be a sequence that converges in $Y \times Z$ to some $(y_\infty, z_\infty) \in W$. Choose some $x_\infty \in q^{-1}\{y_\infty\}$. There is some $n_1 \in \mathbb{N}$ with $\forall n \geqslant n_1\,.\,(x_\infty, z_n) \in r^{-1}[W]$. Define $K := \{z_\infty, z_n \mid n \geqslant n_1\}$ and $V := \{y \in Y \mid \{y\} \times K \subseteq W\}$. Since K is sequentially compact, $q^{-1}[V] = \{x \in X \mid \{x\} \times K \subseteq r^{-1}[W]\}$ of V is open. Hence V is an open neighbourhood of y_∞. Thus $(y_n)_n$ is eventually in V, hence $(y_n, z_n)_n$ is eventually in W. We conclude that W is open in $Y \times Z$. $\square$

Since quotient maps are closed under composition, it follows easily from Lemma 4.1 that the product $q \times q'$ in **Seq** of two quotient maps is again a quotient map. Thus finite products in **Seq** do indeed preserve quotients.

Theorem 4.2. Let $q_i : X_i \to Y_i$ be quotient maps for all $i \in \mathbb{N}$. Then the countable product $(\prod_{i\in\mathbb{N}} q_i) : \prod_{i\in\mathbb{N}} X_i \to \prod_{i\in\mathbb{N}} Y_i$ is a quotient map.

Proof. Let Y_∞ be as defined at the start of the section. We show $\mathcal{O}(Y_\infty) = \mathcal{O}(\prod_i Y_i)$. The '$\supseteq$' inclusion follows easily from the fact that the projection functions $\pi_k : Y_\infty \to Y_k$ are continuous. It remains to show that $\mathcal{O}(Y_\infty) \subseteq \mathcal{O}(\prod_i Y_i)$.

By Example 3.5, it suffices to show that every set $W \in \mathcal{O}(Y_\infty)$ has the properties of Lemma 3.7.

(a) By the definition of Y_∞, it follows that $(\prod_i q_i)^{-1}[W]$ is open.
(b) Let $y \in W$. Assume, to show a contradiction, that for every $n \in \mathbb{N}$ there is some $z_n \in \prod_i Y_i \setminus W$ with $\pi_{\leqslant n}(z_n) = \pi_{\leqslant n}(y)$. Choose x_∞ with $(\prod_i q_i)(x_\infty) = y$. For every $n \in \mathbb{N}$ there is some $x_n \in \prod_i X_i$ with $(\prod_i q_i)(x_n) = z_n$ and $\pi_{\leqslant n}(x_n) = \pi_{\leqslant n}(x_\infty)$. Obviously, $(x_n)_n$ converges to x_∞ in $\prod_i X_i$. By the continuity of $\prod_i q_i$, we have $(z_n)_n$ converges to y in the quotient space Y_∞, which is a contradiction.
(c) Let $l \in \mathbb{N}$ and $x \in \prod_{i>l} X_i$. Then $V := \{y \in \prod_{i\leqslant j} Y_i \,|\, y@(\prod_{i>l} q_i)(x) \in W\}$ is open in the quotient topology induced by $\prod_{i\leqslant l} q_i$, because its preimage $(\prod_{i\leqslant l} q_i)^{-1}[V] = \{z \in \prod_{i\leqslant l} X_i \,|\, z@x \in (\prod_i q_i)^{-1}[W]\}$ is open in $\prod_{i\leqslant l} X_i$. Since quotient maps are preserved by finite products in **Seq**, V is open in the product $\prod_{i\leqslant l} Y_i$ as well.
(d) Let $x \in (\prod_i q_i)^{-1}[W]$ and $y \in \prod_i Y_i$ with $(\prod_i q_i)(x) \sqsubseteq_{\prod Y_i} y$. By (b) there is some $l \in \mathbb{N}$ such that $\pi_{\leqslant l}((\prod_i q_i)(x)) \times \prod_{i>l} Y_i \subseteq W$. As there is some $a \in \prod_{i>l} X_i$ with $(\prod_{i>l} q_i)(a) = \pi_{i>l}(y)$, by (c) the set $V = \{z \in \prod_{i\leqslant l} Y_i \,|\, z@\pi_{i>l}(y) \in W\}$ is open in $\prod_{i\leqslant l} Y_i$. Since $(\prod_{i\leqslant l} q_i)(\pi_{\leqslant l}(x)) \in V$ and $(\prod_{i\leqslant l} q_i)(\pi_{\leqslant l}(x)) \sqsubseteq_{\prod_{i\leqslant l} Y_i} \pi_{\leqslant l}(y)$, we have $\pi_{\leqslant l}(y) \in V$ and thus $y \in W$. □

5. Monotone ω-convergence spaces

A topological space X is a *monotone convergence space* if its specialisation order is a directed-complete partial order and every open subset of X is Scott-open under the specialisation order. (This notion was introduced by Wyler under the name *d-space* (Wyler 1981).) Analogously, we say that X is a *monotone ω-convergence space* if the specialisation order is an ωcpo and every open set of X is ω-Scott open. Note that monotone (ω-)convergence spaces are automatically T_0. A qcb space is a monotone convergence space if and only if it is a monotone ω-convergence space, see Battenfeld *et al.* (2005, Proposition 4.7).

In this section we show that countable products in **Seq** are preserved by the reflection functor into the subcategory of monotone ω-convergence sequential spaces. It will follow that the reflection functor from qcb spaces to the subcategory of monotone (ω-)convergence qcb spaces also preserves countable products.

We use ω**MC** to denote the category of monotone ω-convergence spaces and of continuous functions and ω**MCSeq** for its full subcategory of sequential monotone ω-convergence spaces.

A closed set A is said to be *irreducible* if $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$ implies $A \cap U \cap V \neq \emptyset$ for all $U, V \in \mathcal{O}(X)$. Recall that the *sobrification* $\mathfrak{S}(X)$ of a topological space X has the set of irreducible closed sets of X as its underlying set (equivalently, one can use the set of completely prime filters). The topology of $\mathfrak{S}(X)$ is defined by the family of sets of the form $\{A \in \mathfrak{S}(X) \,|\, A \cap U \neq \emptyset\}$, where $U \in \mathcal{O}(X)$. Clearly, the specialisation order $\sqsubseteq_{\mathfrak{S}(X)}$ is given by set-inclusion. This implies that $(\mathfrak{S}(X), \sqsubseteq_{\mathfrak{S}(X)})$ forms a directed complete partial order, where the least upper bound, $\mathrm{lub}_X(\beta)$, of a directed family $\beta \subseteq \mathfrak{S}(X)$ is given by the closure of the union over β, which is, indeed, an irreducible closed set.

We define a functor $\mathfrak{M}_\omega$ from the category **Top** of topological spaces to ω**MC** as follows. For $X \in$ **Top**, let $\mathfrak{M}_\omega(X)$ be the topological subspace of $\mathfrak{S}(X)$ whose underlying set is the smallest subset $\mathscr{D}$ of $\mathfrak{S}(X)$ that contains, for all $x \in X$, the irreducible closed set $\eta_X(x) := \mathrm{Cls}(x)$ and is closed under the formation of lubs of increasing sequences in $\mathscr{D}$. For an open set $U \in \mathcal{O}(X)$ we use U^+ to denote the open set $\{A \in \mathfrak{M}_\omega(X) \,|\, A \cap U \neq \emptyset\}$ in $\mathfrak{M}_\omega(X)$. Given a morphism $f : X \to Y$ in **Top**, we define the function $\mathfrak{M}_\omega(f) : \mathfrak{M}_\omega(X) \to \mathfrak{M}_\omega(Y)$ by $\mathfrak{M}_\omega(f)(A) := \mathrm{Cls}(f[A])$.

We show that $\mathfrak{M}_\omega$ exhibits ω**MC** as a full reflective subcategory of **Top**. This is analogous to the reflection of the category of monotone convergence spaces in **Top** established in Wyler (1981), see also Battenfeld *et al.* (2005).

Proposition 5.1. The functor $\mathfrak{M}_\omega$ constitutes a reflection functor from **Top** to ω**MC**.

Proof. Let $X \in$ **Top**. From the fact that the specialisation order $\sqsubseteq_{\mathfrak{M}_\omega(X)}$ of $\mathfrak{M}_\omega(X)$ is given by set-inclusion, it follows that $\big(\mathfrak{M}_\omega(X), \sqsubseteq_{\mathfrak{M}_\omega(X)}\big)$ forms an ωcpo. Every open set W of $\mathfrak{M}_\omega(X)$ is ω-Scott-open by being of the form $W = U^+$ for some $U \in \mathcal{O}(X)$. Hence $\mathfrak{M}_\omega(X)$ is indeed a monotone ω-convergence space.

Let Y be a topological space and $f : X \to Y$ be continuous. It is easy to verify that $\mathfrak{M}_\omega(f)(A) \in \mathfrak{S}(Y)$ for every $A \in \mathfrak{M}_\omega(X)$ and $\mathfrak{M}_\omega(f) \circ \eta_X = \eta_Y \circ f$. Thus $\mathscr{D} := \{A \in \mathfrak{M}_\omega(X) \,|\, \mathfrak{M}_\omega(f)(A) \in \mathfrak{M}_\omega(Y)\}$ contains $\{\mathrm{Cls}(x) \,|\, x \in X\}$. Moreover, if $(A_i)_i$ is an increasing sequence of elements in $\mathscr{D}$, then $(\mathrm{Cls}(f[A_i]))_i$ is an increasing sequence in $\mathfrak{M}_\omega(Y)$, hence $\mathrm{lub}(\mathrm{Cls}(f[A_i]))_i \in \mathfrak{M}_\omega(Y)$. Since $\mathfrak{M}_\omega(f)(\mathrm{lub}(A_i)_i) = \mathrm{lub}(\mathrm{Cls}(f[A_i]))_i$, it follows that $\mathrm{lub}(A_i)_i \in \mathscr{D}$. Therefore, $\mathscr{D}$ is closed under countable lubs, so $\mathfrak{M}_\omega(X) \subseteq \mathscr{D}$. This shows that $\mathfrak{M}_\omega(f)$ indeed maps $\mathfrak{M}_\omega(X)$ into $\mathfrak{M}_\omega(Y)$. Furthermore, $\mathfrak{M}_\omega(f)$ is continuous, because $(\mathfrak{M}_\omega(f))^{-1}[V^+] = (f^{-1}[V])^+$ holds for all $V \in \mathcal{O}(Y)$. Clearly, $\mathfrak{M}_\omega$ preserves composition. Thus $\mathfrak{M}_\omega$ is a functor.

Let Z be a monotone ω-convergence space. Using the fact that every open of Z is ω-Scott-open in the ωcpo $(Z, \sqsubseteq_Z)$, one can show that $\mathfrak{M}_\omega(Z) = \{\eta_Z(z) \,|\, z \in Z\}$. Hence $\varepsilon_Z : \mathfrak{M}_\omega(Z) \to Z$ can be defined by $\varepsilon_Z\big(\eta_Z(z)\big) := z$. Clearly, ε_Z is continuous and $\eta_Z \circ \varepsilon_Z = \mathrm{id}_{\mathfrak{M}_\omega(Z)}$, hence $\mathfrak{M}_\omega(Z)$ and Z are isomorphic. Thus for any morphism $h : X \to Z$, the function $h' := \varepsilon_Z \circ \mathfrak{M}_\omega(h)$ satisfies $h = h' \circ \eta_X$. It is unique, because for any continuous function $g : \mathfrak{M}_\omega(X) \to Z$ with $g \circ \eta_X = h$ and any open $V \in \tau_Z$, we have $g^{-1}[V] = (f^{-1}[V])^+$. This implies $g = h'$, because Z is a T_0-space.

Therefore, $\mathfrak{M}_\omega$ is a left adjoint to the embedding **MC** $\hookrightarrow$ **Top**, with η being the unit and ε being the counit of the adjunction. □

Proposition 5.2. The reflection functor $\mathfrak{M}_\omega$ preserves sequential spaces and qcb-spaces.

Proof. Let X be sequential. In order to show that $\mathfrak{M}_\omega(X)$ is sequential, we first define for every countable ordinal α the family $X^{(\alpha)}$ by transfinite induction as follows:

$$X^{(0)} := \{\eta_X(x) \mid x \in X\} \tag{1}$$

$$X^{(\alpha)} := \left\{ \mathrm{lub}(A_n)_n \mid \forall n \in \mathbb{N} \,.\, A_n \subseteq A_{n+1} \wedge A_n \in \bigcup_{\beta<\alpha} X^{(\beta)} \right\} \quad (\alpha > 0).$$

Clearly, $\bigcup_{\alpha<\omega_1} X^{(\alpha)}$ is the underlying set of $\mathfrak{M}_\omega(X)$.

Let V be a sequentially open subset of $\mathfrak{M}_\omega(X)$. Since η_X is continuous, the set $U := (\eta_X)^{-1}[V]$ is sequentially open, and thus open in the sequential space X. By transfinite induction, we now show $V \cap X^{(\alpha)} = U^+ \cap X^{(\alpha)}$ for $\alpha < \omega_1$:

1 For $\alpha = 0$, $V \cap X^{(\alpha)} = U^+ \cap X^{(\alpha)}$ follows from the definition of U.

2 Let $\alpha > 0$ and $A \in X^{(\alpha)}$. Then there is an increasing sequence $(A_n)_n$ in $\bigcup_{\beta<\alpha} X^{(\beta)}$ with $A = \mathrm{Cls}(\bigcup_{n\in\mathbb{N}} A_n)$. If $A \in U^+$, then there is some $m \in \mathbb{N}$ with $A_m \in U^+$, and thus $A_m \in V$ by the induction hypothesis. Since the constant sequence $(A)_j$ converges to A_m in $\mathfrak{M}_\omega(X)$, it follows that $A \in V$. On the other hand, if $A \in V$, there is some $n_0 \in \mathbb{N}$ with $A_{n_0} \in V$, because $(A_n)_n$ converges to A in $\mathfrak{M}_\omega(X)$. By the induction hypothesis, we have $A_{n_0} \in U^+$, which implies that $A \in U^+$.

We conclude $U^+ = V$, and hence that $\mathfrak{M}_\omega(X)$ is sequential.

Since **QCB** is cartesian closed, **QCB** contains the function space $\mathbb{S}^{\mathbb{S}^X}$. The function $e : \mathfrak{M}_\omega(X) \to \mathbb{S}^{\mathbb{S}^X}$ defined by $e(A)(h) = \top :\Longleftrightarrow A \in (h^{-1}[\top])^+$ has the property that a sequence $(A_n)_n$ converges to A_∞ in $\mathfrak{M}_\omega(X)$ if and only if $(e(A_n))_n$ converges to $e(A_\infty)$ in $\mathbb{S}^{\mathbb{S}^X}$. Together with the fact that $\mathfrak{M}_\omega(X)$ is sequential, this implies that $\mathfrak{M}_\omega(X)$ is homeomorphic to the space $\{e(A) \mid A \in \mathfrak{M}_\omega(X)\}$ equipped with the subspace topology induced by $\mathbb{S}^{\mathbb{S}^X}$. Any sequential subspace of a qcb space is a qcb space as well: this follows from the characterisation of qcb spaces as those spaces that have a countable pseudobase (*cf.* Schröder (2003) and Escardó *et al.* (2004)) and from the fact that subspaces inherit the existence of a countable pseudobase (*cf.* Schröder (2002)). Hence, $\mathfrak{M}_\omega(X)$ is a qcb space. □

The goal of this section is to show that the functor $\mathfrak{M}_\omega$ from **Seq** to ω**MCSeq** preserves countable products. We show first that there is a bijection between the underlying sets of $\prod_i \mathfrak{M}_\omega(X_i)$ and $\mathfrak{M}_\omega(\prod_i X_i)$.

Lemma 5.3. The function $\iota : \prod_i \mathfrak{M}_\omega(X_i) \to \mathfrak{M}_\omega(\prod_i X_i)$ defined by $\iota((A_i)_i) := \prod_i A_i$ is bijective. Its inverse ι^{-1} is continuous.

Proof. In order to prove that ι maps $\prod_i \mathfrak{M}_\omega(X_i)$ into $\mathfrak{M}_\omega(\prod_i X_i)$, we show

$$\mathrm{Cls}\left(\bigcup_{n\in\mathbb{N}} \prod_{i\in\mathbb{N}} A_{i,n} \right) = \prod_{i\in\mathbb{N}} \mathrm{Cls}\left(\bigcup_{n\in\mathbb{N}} A_{i,n} \right) \tag{2}$$

for every $i \in \mathbb{N}$ and every increasing sequence $(A_{i,n})_n$ of closed sets in X_i:

'$\subseteq$' This follows since $\bigcup_n \prod_i A_{i,n} \subseteq \prod_i (\bigcup_n A_{i,n})$ and $\prod_i \mathrm{Cls}(\bigcup_n A_{i,n})$ is closed in $\prod_i X_i$.

'$\supseteq$' Let $x \in \prod_i(\bigcup_n A_{i,n})$ and V be an open neighbourhood of x in $\prod_i X_i$. By Lemma 3.2, there is some $l \in \mathbb{N}$ such that $\{\pi_{\leqslant l}(x)\} \times \prod_{i>l} X_i \subseteq V$. Choose some $y \in \prod_{i>l} A_{i,0}$. Since $\{a \in X_l \mid \pi_{<l}(x)@a@y \in V\}$ is a neighbourhood of $\pi_l(x)$, there is some $a_l \in \bigcup_n A_{l,n}$ with $\pi_{<l}(x)@a_l@y \in V$. By repeated application of this argument, we can find elements $a_{l-1} \in \bigcup_n A_{l-1,n}, \dots, a_0 \in \bigcup_n A_{0,n}$ with $z := (a_0@\dots@a_l@y) \in V$. Since there is some $n_0 \in \mathbb{N}$ with $z \in \prod_i A_{i,n_0}$, we conclude $x \in \mathrm{Cls}(\bigcup_n \prod_i A_{i,n})$.

For every ordinal $\alpha < \omega_1$ and every $k \in \mathbb{N}$ we define $X_k^{(\alpha)} \subseteq \mathfrak{S}(X_k)$ as in Equation (1) and show $\iota[\prod_i X_i^{(\alpha)}] \subseteq \mathfrak{M}_\omega(\prod_i X_i)$ by transfinite induction:

— Case $\alpha = 0$: For every $x \in \prod_i X_i$, we have $\prod_i \eta_{X_i}(\pi_i(x)) = \eta_{(\prod_i X_i)}(x) \in \mathfrak{M}_\omega(\prod_i X_i)$.

— Case $\alpha > 0$: Let $(A_i)_i \in \prod_i X_i^{(\alpha)}$. For every $i, n \in \mathbb{N}$ there is an ordinal $\beta_{i,n} < \alpha$ and a set $A_{i,n} \in X_i^{(\beta_{i,n})}$ such that $(A_{i,n})_n$ is increasing and $\mathrm{lub}(A_{i,n})_n = A_i$. Choose $a \in \prod_{i \in \mathbb{N}} A_{i,0}$ and define

$$B_{i,n} := \begin{cases} A_{i,n} & \text{if } i \leqslant n \\ \mathrm{Cls}(\{\pi_i(a)\}) & \text{otherwise.} \end{cases}$$

Then $(B_{i,n})_i \in \prod_i X_i^{(\max\{\beta_{j,n} \mid j \leqslant n\})}$ for every $n \in \mathbb{N}$. The induction hypothesis implies $\prod_i B_{i,n} \in \mathfrak{M}_\omega(\prod_i X_i)$. By Equation (2), we obtain $\prod_i A_i = \prod_i \mathrm{lub}(B_{i,n})_n = \mathrm{lub}(\prod_i B_{i,n})_n \in \mathfrak{M}_\omega(\prod_i X_i)$, which concludes the induction.

Thus ι maps $\prod_i \mathfrak{M}_\omega(X_i)$ into $\mathfrak{M}_\omega(\prod_i X_i)$. It is clear that ι is injective. Moreover, $\iota[\prod_i \mathfrak{M}_\omega(X_i)]$ contains $\eta_{\prod_i X_i}(x)$ for all $x \in \prod_i X_i$ and is closed under lub's by Equation 2. Therefore, $\mathfrak{M}_\omega(\prod_i X_i) \subseteq \iota[\prod_i \mathfrak{M}_\omega(X_i)]$. We conclude that ι is a bijection between $\prod_i \mathfrak{M}_\omega(X_i)$ and $\mathfrak{M}_\omega(\prod_i X_i)$.

Since $\iota^{-1}(Q) = (\mathfrak{M}_\omega(\pi_0)(Q), \mathfrak{M}_\omega(\pi_1)(Q), \dots)$, we have ι^{-1} is continuous. □

Lemma 5.4. The unit $\eta_X : X \to \mathfrak{M}_\omega(X)$ co-reflects convergent sequences.

Proof. Since $\phi : \mathbb{S}^X \to \mathbb{S}^{\mathfrak{M}_\omega(X)}$ defined by $\phi(h)(A) = \top :\Longleftrightarrow A \in (h^{-1}\{\top\})^+$ is a continuous inverse of $\mathbb{S}^{\eta_X}$, we have that η_X co-reflects convergent sequences. □

Proposition 5.5. The functor $\mathfrak{M}_\omega : \mathbf{Seq} \to \omega\mathbf{MCSeq}$ preserves finite products.

Proof. Let X, Y be sequential spaces. By Lemma 5.3, the function $\iota : \mathfrak{M}_\omega(X) \times \mathfrak{M}_\omega(Y) \to \mathfrak{M}_\omega(X \times Y)$ with $\iota(A, B) := A \times B$ is bijective and its inverse is continuous. In order to show that ι is continuous, let $(A_n, B_n)_n$ converge to (A_∞, B_∞) in $\mathfrak{M}_\omega(X) \times \mathfrak{M}_\omega(Y)$. Let $W \in \mathcal{O}(X \times Y)$ with $A_\infty \times B_\infty \in W^+$. Assume, to show a contradiction, that there is a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ with $A_{\varphi(n)} \times B_{\varphi(n)} \notin W^+$. Choose $(a, b) \in W \cap (A_\infty, B_\infty)$. Since $U := \{x \in X \mid (x, b) \in W\}$ is open with $A_\infty \in U^+$ and η_X co-reflects convergent sequences, there is some strictly increasing function $\psi : \mathbb{N} \to \mathbb{N}$ and a converging sequence $(x_n)_n \to x_\infty$ in X with $x_\infty \in U$ and $\forall n \in \mathbb{N} \,.\, x_n \in A_{\varphi\psi(n)} \cap U$. By Lemma 3.1, the set $V := \{y \in Y \mid \{x_\infty, x_n \mid n \in \mathbb{N}\} \times \{y\} \subseteq W\}$ is open in Y. As $B_\infty \in V^+$, there is some $n_1 \in \mathbb{N}$ with $\forall n \geqslant n_1 \,.\, B_{\varphi\psi(n)} \in V^+$. Hence $(A_{\varphi\psi(n_1)} \times B_{\varphi\psi(n_1)}) \cap W \neq \emptyset$, which is a contradiction. □

Theorem 5.6. The functor $\mathfrak{M}_\omega : \mathbf{Seq} \to \omega\mathbf{MCSeq}$ preserves countable products.

Proof. Let X_i be a sequential space for $i \in \mathbb{N}$. We only need to show that the function $\imath$ defined in Lemma 5.3 is continuous. It suffices to prove that for every open O in $\prod_i X_i$ the preimage $W := \imath^{-1}\{O^+\}$ satisfies the requirements of Lemma 3.7.

(a) Since $\imath \circ \prod_i \eta_{X_i} = \eta_{\prod_i X_i}$, we have $(\prod_i \eta_{X_i})^{-1}[W] = O$.
(b) Let $(A_i)_i \in W$. Then there is some $x \in O \cap \prod_i A_i$. Lemma 3.2 yields some $l \in \mathbb{N}$ with $\{\pi_{\leqslant l}(x)\} \times \prod_{i>l} X_i \subseteq O$. Hence $\{A_0\} \times \ldots \times \{A_l\} \times \prod_{i>l} \mathfrak{M}_\omega(X_i) \subseteq W$.
(c) Let $l \in \mathbb{N}$ and $x \in \prod_{i>l} X_i$. By Lemma 3.1, $U := \{z \in \prod_{i\leqslant l} X_i \,|\, z@x \in O\}$ is open. Since U^+ is open in $\mathfrak{M}_\omega(\prod_{i\leqslant l} X_i)$ and $\mathfrak{M}_\omega$ preserves finite products, the set

$$\Big\{(A_0,\ldots,A_l) \in \prod_{i\leqslant l} \mathfrak{M}_\omega(X_i) \,|\, A_0@\ldots@A_l@(\prod_{i>l}\eta_{X_i})(x) \in W\Big\}$$
$$= \Big\{(A_0,\ldots,A_l) \in \prod_{i\leqslant l} \mathfrak{M}_\omega(X_i) \,|\, \exists z \in \prod_{i\leqslant l} A_i \,.\, z@x \in O\Big\}$$
$$= \Big\{(A_0,\ldots,A_l) \in \prod_{i\leqslant l} \mathfrak{M}_\omega(X_i) \,|\, \prod_{i\leqslant l} A_i \in U^+\Big\}$$

is open in $\prod_{i\leqslant l} \mathfrak{M}_\omega(X_i)$.
(d) Let $x \in \prod_i X_i$ and $(A_i)_i \in \prod_i \mathfrak{M}_\omega(X_i)$ with $(\prod_i \eta_{X_i})(x) \in W$ and

$$\Big(\prod_i \eta_{X_i}\Big)(x) \sqsubseteq_{\prod_i \mathfrak{M}_\omega(X_i)} (A_i)_i\,.$$

Then $x \in \prod_i A_i \cap O$, which implies $(A_i)_i \in W$. □

It follows that the restriction of $\mathfrak{M}_\omega$ to a reflection from **QCB** to its subcategory of monotone convergence spaces, ω**MCQCB**, also preserves countable products.

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