Two Properties of Stochastic KPP Equations: Ergodicity and Pathwise Property

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Abstract

In this paper we study the random approximate travelling wave solutions of the stochastic KPP equations. Two new properties of the stochastic KPP equations are obtained. We prove the ergodicity that for almost all sample paths, behind the wave front $x = \gamma t$, the lower limit of $\frac{1}{t} \int_0^t u(s,x)ds$ as $t \to \infty$ is positive, and ahead of the wave front, the limit is zero. In some cases, behind the wave front, the limit of $\frac{1}{t} \int_0^t u(s,x)ds$ as $t \to \infty$ exists and is positive almost surely. We also prove that behind the wave front, for almost each ω, the solution of some special stochastic KPP equations converges to a stationary trajectory of the corresponding stochastic differential equation. In front of wave front, the solution converges to 0 which is another stationary trajectory of the corresponding SDE. We also study the space derivative of the solution for large time. We show that away from the wave front, for almost all large t the solution is flat in the x-direction for almost all sample paths.

1 Introduction

Approximate random travelling wave solutions for stochastic KPP eqautions were studied by [7], [10] and [25]. These equations arise in the study of the effect of a noise to approximate travelling waves of semi-linear reaction diffusion equations. We say that the noise is mild if $k_m < \sqrt{2c(0)}$, where $k_m = \lim_{t\to\infty} \sqrt{\frac{1}{t} \int_0^t k^2(s)ds}$, assuming the limit exists. It was known that in an environment of mild noise, the solution of the stochastic generalized KPP equation

$$du(t,x) = (\frac{D}{2}\Delta u(t,x) + c(u)u)dt + k(t)udW_t, \tag{1.1}$$

still evolves to an approximate travelling wave solution with a reduced speed $\gamma = \sqrt{D(2c(0) - k_m^2)}$ in the limit as $t \to \infty$. It was proved under some conditions which will be specified in Section 3 that Equation (1.1) has a random travelling wave solution ([10],[7],[25]). The wave front was known as

$$x = \gamma t, \tag{1.2}$$

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and there are constants $d_1, d_2, d_3 > 0$ such that for any $h > 0$

$$\frac{1}{t} \log u(t, x) < -d_1 \text{ if } x > (\gamma + h)t, \text{ for almost all } \omega \tag{1.3}$$

and

$$-d_3 \leq \frac{\log u(t, x)}{\sqrt{2t \log \log t}} \leq d_2 \text{ if } x < (\gamma - h)t, \text{ for almost all } \omega \tag{1.4}$$

for all sufficiently large t.

The travelling wave of the FKPP equation (when $k = 0$ and $c(u) = 1 - u$) was first studied by [18] and [12]. Since then, it has attracted many mathematicians' attentions (e.g. [2],[3],[4],[5],[6],[9], [11],[13],[14],[16],[22],[24],[27], to name but a few.) It was defined as a solution of the type $u(t, x) = U(x - \gamma t)$ and γ is the speed of the travelling wave. Here U satisfies $\lim_{z \to -\infty} U(z) = 1$ and $\lim_{z \to +\infty} U(z) = 0$. Note that $\lim_{t \to \infty} u(t, (\gamma - h)t) = 1$ and $\lim_{t \to \infty} u(t, (\gamma + h)t) = 0$ for any $h > 0$. This was extended to Freidlin's type approximate travelling wave for reaction diffusion equations by [13] so that it is applicable to much more general situations. And a variety of stochastic methods has been developed to study travelling waves ([5],[6],[9],[13],[14],[22]). Approximate travelling waves in a random media were also studied in [13].

For the stochastic KPP equation (1.1), ahead of the wave front, the solution is exponentially small ($\leq e^{-d_1 t}$) almost surely and behaves the same as Freidlin's approximated travelling wave. But behind the wave front, the solution is oscillatory. It was clear that the solution behind the wave front behaves differently from the solution ahead of the wave front as (1.3) and (1.4) have shown. We call the regions behind and ahead of the wave front the crest and the trough of the random approximate travelling wave, respectively. It was remained open to understand the behaviour of the solution on the crest beyond (1.4). It was pointed out in [10] suggested by the numerical works in [15] that in the limit $t \to \infty$, $\frac{1}{t} \int_0^t u(s, x) ds$ should have a simple formula. The integral $\int_0^1 u^\mu(t, x) dt$ was simulated numerically in [15] for rescaled stochastic KPP equations. In this paper, we find a way to study the limit of $\frac{1}{t} \int_0^t u(s, x) ds$ as $t \to \infty$ for Equation (1.1). To explain this ergodicity result, we demonstrate here a special case of Equation (1.1) with $c(u) = c(1 - u)$, the latter c being a constant and $k(t) = k$ being a constant. In this case the equation is

$$du(t, x) = (\frac{D}{2} \Delta u(t, x) + c(1 - u)u)dt + kudW_t, \tag{1.5}$$

with initial condition $u(0, x) = \chi_{x \leq 0}$. Define $\gamma = \sqrt{D(2c - k^2)}$. We have for any $h > 0$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \inf_{x \leq (\gamma - h)t} u(s, x) ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sup_x u(s, x) ds = 1 - \frac{k^2}{2c}, \quad a.s.. \tag{1.6}$$

This result is new in the literature. It is easy to see from (1.3) that on the trough,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sup_{x \geq (\gamma + h)t} u(s, x) ds = 0, \quad a.s.. \tag{1.7}$$

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It is noted that these limits are true for almost all sample paths.

Our method is first to study the corresponding stochastic ordinary differential equation. In Section 2, we find its explicit solution $Y(t)$ to the SODE and thereafter we calculate $\lim_{t\to\infty} \frac{1}{t} \int_0^t Y(s)ds$. In Section 3, we use the Feynman-Kac formula to obtain a comparison result of the solution $u(t,x)$ to Equation (1.1) and $Y(t)$. So we can use the result in Section 2 to obtain $\lim_{t\to\infty} \frac{1}{t} \int_0^t u(s,x)ds$ on the crest.

On the other hand, we study the pathwise property of the solution of a special stochastic KPP equation (1.5). Note the corresponding stochastic differential equation

$$dY(t) = c(1 - Y(t))Y(t)dt + kY(t)dW_t, \tag{1.8}$$

has two stationary trajectories in the sense of Mohammed and Scheutzow. These are 0 and $Z(\omega)$ given by

$$Z(\omega) = (c \int_{-\infty}^0 \exp\{(c - \frac{1}{2}k^2)s + kW_s\}ds)^{-1}. \tag{1.9}$$

We can check 0 is unstable and $Z(\omega)$ is stable by calculating the Lyapunov exponent at each point. The pathwise property for the stochastic KPP equation (1.5) we prove in this paper is that for any $h > 0$, for $x < (\gamma - h)t$ and a.e. ω,

$$|u(t,x,\omega) - Z(\theta_t(\omega))| \to 0 \text{ as } t \to \infty, \tag{1.10}$$

and for $x > (\gamma + h)t$ and a.e. ω,

$$u(t,x,\omega) \to 0 \text{ as } t \to \infty, \tag{1.11}$$

where θ_t is the canonical Brownian shift. Here (1.10) is new while the convergence to 0 result (1.11) was obtained in [7], [10] and [25]. We should point out here that we only obtain the pathwise result for stochastic KPP equation (1.5). But we expect this result is true for a wider class of the stochastic reaction diffusion equations.

In Section 4, we study the convergence of the space derivative of the solution. We prove that $\int_0^t |\nabla u(s,x)|ds$ converges as $t \to \infty$. Therefore for almost all large t, $\nabla u(t,x)$ is small for almost all (t,x) on the crest and the trough. This result is established by considering the logarithmic derivative of the solution $R(t,x) = -\nabla \log u(t,x)$. The space derivatives of the logarithm of solutions for deterministic heat equations were studied by [20], [26]. The space derivatives for nonlinear KPP equations were studied by [19] and [28]. For the nonlinear generalized KPP equations, the fact that $u(t,x) \geq \delta$ on the crest for a constant $\delta > 0$ played a crucial rule in the study of the space derivatives in [19] and [28]. But this is not true for the stochastic generalized KPP Equation (1.1). We find in this paper that $u(t,x) \geq \delta$ can be replaced by (1.6), which is a much weaker requirement.

2 Some preliminary results for certain related stochastic ordinary differential equations

First we consider the following nonlinear stochastic ordinary differential equation,

$$dY(t,y) = \alpha Y(t,y)(\beta - Y(t,y))\, dt + k(t)Y(t,y)\, dW_t, \ Y_0 = y, \tag{2.1}$$

where α, β, and $y > 0$ are constants, W_t is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, $k(t)$ is a continuous function of t. It is easy to check Equation (2.1) has the explicit solution (see e.g. [1])

$$Y(t,y) = \frac{\exp\{\alpha\beta t - \frac{1}{2}\int_0^t k^2(r)dr + \int_0^t k(r)dW_r\}}{\frac{1}{y} + \alpha \int_0^t \exp\{\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr + \int_0^s k(r)dW_r\}\, ds}, \quad t \geq 0. \tag{2.2}$$

For simplicity, we assume that $\lim_{t\to\infty} \frac{1}{t}\int_0^t k^2(s)ds$ exists, and the limit is denoted by k_m^2.

Lemma 2.1 *Let $Y(t)$ be the solution of Equation (2.1). If $\alpha\beta > \frac{1}{2}k_m^2$, then for any $\epsilon > 0$, there exists $t_0 > 0$ such that,*

$$P\{|\frac{1}{t}\int_0^t Y(s)\, ds - (\beta - \frac{k_m^2}{2\alpha})| > \epsilon \text{ for some } t > T\} < \exp\{-\frac{\alpha^2\epsilon^2 T^2}{32\int_0^T k^2(r)dr}\}, \tag{2.3}$$

for all $T > t_0$. In particular, as $t \to \infty$,

$$\frac{1}{t}\int_0^t Y(s)\, ds \to \beta - \frac{k_m^2}{2\alpha}, \tag{2.4}$$

almost surely.

Proof. It is easily seen that

$$\int_0^t Y(s)\, ds = \frac{1}{\alpha}\log z(t) - \frac{1}{\alpha}\log z(0), \tag{2.5}$$

where $z(t) = \frac{1}{y_0} + \alpha \int_0^t \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr + \int_0^s k(r)dW_r)\, ds$. Define $z^*(t) = \int_0^t \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\, ds$. Then it is easy to see that for any sufficiently small $\epsilon > 0$, there exists $t_0^* > 0$ such that for $t > t_0^*$,

$$k_m^2 - \frac{1}{8}\alpha\varepsilon \leq \frac{1}{t}\int_0^t k^2(s)ds \leq k_m^2 + \frac{1}{8}\alpha\varepsilon, \tag{2.6}$$

and

$$\int_0^{t_0^*} \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\, ds \leq M \exp(\alpha\beta t - \frac{1}{2}k_m^2 t + \frac{1}{8}\alpha\varepsilon t), \tag{2.7}$$

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for a constant $M > 0$ and

$$\log(\frac{1}{\alpha\beta - \frac{1}{2}k_m^2 + \frac{1}{8}\alpha\varepsilon} + M) \leq \frac{1}{8}\alpha\varepsilon t. \tag{2.8}$$

And there exists $t_0^{**} > t_0^*$ such that for $t > t_0^{**}$,

$$\exp(\alpha\beta t_0^* - \frac{1}{2}k_m^2 t_0^* - \frac{1}{8}\alpha\varepsilon t_0^*) \leq \frac{1}{2}\exp(\alpha\beta t - \frac{1}{2}k_m^2 t - \frac{1}{8}\alpha\varepsilon t), \tag{2.9}$$

and

$$\log(\frac{1}{2(\alpha\beta - \frac{1}{2}k_m^2 - \frac{1}{8}\alpha\varepsilon)}) \geq -\frac{1}{8}\alpha\varepsilon t. \tag{2.10}$$

Therefore for $t > t_0^*$, from (2.6) and (2.7),

$$\begin{aligned}
z^*(t) &= \int_0^t \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\,ds \\
&= \int_0^{t_0^*} \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\,ds + \int_{t_0^*}^t \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\,ds \\
&\leq \int_0^{t_0^*} \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\,ds + \int_{t_0^*}^t \exp(\alpha\beta s - \frac{1}{2}k_m^2 s + \frac{1}{8}\alpha\varepsilon s)\,ds \\
&= \int_0^{t_0^*} \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\,ds \\
&\quad + \frac{1}{\alpha\beta - \frac{1}{2}k_m^2 + \frac{1}{8}\alpha\varepsilon}[\exp(\alpha\beta t - \frac{1}{2}k_m^2 t + \frac{1}{8}\alpha\varepsilon t) - \exp(\alpha\beta t_0^* - \frac{1}{2}k_m^2 t_0^* + \frac{1}{8}\alpha\varepsilon t_0^*)] \\
&\leq (\frac{1}{\alpha\beta - \frac{1}{2}k_m^2 + \frac{1}{8}\alpha\varepsilon} + M)\exp(\alpha\beta t - \frac{1}{2}k_m^2 t + \frac{1}{8}\alpha\varepsilon t). \tag{2.11}
\end{aligned}$$

And similarly, from (2.6) and (2.9), for $t > t_0^{**}$,

$$\begin{aligned}
z^*(t) &= \int_0^t \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\,ds \\
&\geq \int_{t_0^*}^t \exp(\alpha\beta s - \frac{1}{2}\int_0^s k^2(r)dr)\,ds \\
&\geq \int_{t_0^*}^t \exp(\alpha\beta s - \frac{1}{2}k_m^2 s - \frac{1}{8}\alpha\varepsilon s)\,ds \\
&= \frac{1}{\alpha\beta - \frac{1}{2}k_m^2 - \frac{1}{8}\alpha\varepsilon}[\exp(\alpha\beta t - \frac{1}{2}k_m^2 t - \frac{1}{8}\alpha\varepsilon t) - \exp(\alpha\beta t_0^* - \frac{1}{2}k_m^2 t_0^* - \frac{1}{8}\alpha\varepsilon t_0^*)] \\
&\geq \frac{1}{2(\alpha\beta - \frac{1}{2}k_m^2 - \frac{1}{8}\alpha\varepsilon)}\exp(\alpha\beta t - \frac{1}{2}k_m^2 t - \frac{1}{8}\alpha\varepsilon t). \tag{2.12}
\end{aligned}$$

Then taking logarithm to (2.11) and (2.12), it is easy to see from (2.8) and (2.10) that for $t > t_0^{**}$,

$$(\alpha\beta - \frac{k_m^2}{2} - \frac{1}{4}\alpha\epsilon)t \leq \log z^*(t) \leq (\alpha\beta - \frac{k_m^2}{2} + \frac{1}{4}\alpha\epsilon)t.$$

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Recall that $\frac{1}{\sqrt{\int_0^t k^2(r)dr}}\int_0^s k(r)dW_r$ $(0 \leq s \leq t)$ is a time changed Brownian motion $\mathcal{X}(\frac{\int_0^s k^2(r)dr}{\int_0^t k^2(r)dr})$. Here $\mathcal{X}(u)$ is a standard Brownian motion with of time u. Therefore $\mathcal{Y}(s) = \int_0^s k(r)dW_r = \sqrt{\int_0^t k^2(r)dr}\,\mathcal{X}(\frac{\int_0^s k^2(r)dr}{\int_0^t k^2(r)dr})$. Define $C_1 = \log\alpha$ and $C_2 = \log(\frac{1}{y_0} + \alpha)$.

For any $\varepsilon > 0$, take $t_0 \geq t_0^{**}$ such that $\frac{|C_1 - \log y_0|}{\alpha t_0} < \frac{1}{2}\varepsilon$ and $\frac{|C_2 - \log y_0|}{\alpha t_0} < \frac{1}{2}\varepsilon$. For any $T \geq t_0$, define

$$\Omega_T = \{\omega \in \Omega : -\frac{\alpha\varepsilon T}{4\sqrt{\int_0^T k^2(r)dr}} < \mathcal{X}(u) < \frac{\alpha\varepsilon T}{4\sqrt{\int_0^T k^2(r)dr}}, \text{ for all } 0 \leq u \leq 1\}, \quad (2.13)$$

then from the wellknown Doob's inequality (see [21])

$$P\{\Omega_T\} > 1 - \exp\{-\frac{\alpha^2\varepsilon^2}{32\int_0^T k^2(r)dr}T^2\},$$

and for each $\omega \in \Omega_T$, and $t \geq T$, and $s \leq t$,

$$
\begin{aligned}
|\mathcal{Y}(s)| &= |\int_0^s k(r)dW_r| \\
&= \sqrt{\int_0^t k^2(r)dr}\,|\mathcal{X}(\frac{\int_0^s k^2(r)dr}{\int_0^t k^2(r)dr})| \\
&\leq \sqrt{\int_0^t k^2(r)dr}\,\frac{\alpha\varepsilon T}{4\sqrt{\int_0^T k^2(r)dr}} \\
&\leq \frac{\alpha\varepsilon t}{4},
\end{aligned}
\quad (2.14)
$$

for all $0 \leq s \leq t$. Then it follows easily that for $\omega \in \Omega_T$, and $t \geq T$,

$$\alpha z^*(t)e^{-\frac{1}{4}\alpha\varepsilon t} \leq z(t) \leq (\frac{1}{y_0} + \alpha)z^*(t)e^{\frac{1}{4}\alpha\varepsilon t}. \quad (2.15)$$

It turns out that for $\omega \in \Omega_T$, and $t \geq T$,

$$(\alpha\beta - \frac{k_m^2}{2} - \frac{1}{2}\alpha\varepsilon)t + C_1 \leq \log z(t) \leq (\alpha\beta - \frac{k_m^2}{2} + \frac{1}{2}\alpha\varepsilon)t + C_2.$$

So from (2.5), for $\omega \in \Omega_T$, and $t \geq T$,

$$\beta - \frac{k_m^2}{2\alpha} - \frac{\epsilon}{2} + \frac{C_1 - \log y_0}{\alpha t} \leq \frac{1}{t}\int_0^t Y(r)\,ds \leq \beta - \frac{k_m^2}{2\alpha} + \frac{\varepsilon}{2} + \frac{C_2 - \log y_0}{\alpha t}.$$

Then by the definition of t_0, we have for $\omega \in \Omega_T$, and all $t \geq T (\geq t_0)$

$$\beta - \frac{k_m^2}{2\alpha} - \epsilon \leq \frac{1}{t}\int_0^t Y(r)\,ds \leq \beta - \frac{k_m^2}{2\alpha} + \varepsilon.$$

So (2.3) follows. □

We need to generalize the above result to more general nonlinear SDEs. We have the following lemma. The key point of the proof is to use a comparison method.

Lemma 2.2 *Suppose $c \in C(R^+)$, and there exist constants $a \geq b > 0$ such that $by \leq c(0) - c(y) \leq ay$ for all $y \geq 0$, and $c(0) > k_m^2/2$. If $Y(t)$ denotes the solution of*

$$dY(t) = Y(t)c(Y(t))\,dt + k(t)Y(t)\,dW_t, \ Y_0 = y_0 \tag{2.16}$$

then for any $\varepsilon > 0$, there is $t_0 = t_0(\varepsilon) > 0$ such that,

$$P\{\frac{1}{a}\left(c(0) - \frac{k_m^2}{2}\right) - \varepsilon \leq \frac{1}{t}\int_0^t Y(s)\,ds \leq \frac{1}{b}\left(c(0) - \frac{k_m^2}{2}\right) + \varepsilon \text{ for all } t \geq T\}$$

$$> \quad 1 - \exp\{-\frac{a^2\varepsilon^2 T^2}{32\int_0^T k^2(r)dr}\}, \tag{2.17}$$

for all $T > t_0$. In particular, for a.e. ω,

$$\frac{1}{a}\left(c(0) - \frac{k_m^2}{2}\right) \leq \liminf_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds \leq \limsup_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds \leq \frac{1}{b}\left(c(0) - \frac{k_m^2}{2}\right).$$

Proof. If c satisfies the conditions in the lemma, then

$$ay\left(\frac{c(0)}{a} - y\right) \leq yc(y) \leq by\left(\frac{c(0)}{b} - y\right)$$

for all $y \geq 0$. The lemma now follows from the comparison theorem for SDE's (see e.g. [17]) and Lemma 2.1. □

Remark 2.3 *(i) We are only interested in the case $c(0) > \frac{1}{2}k_m^2$. It is easy to see that if $c(0) < \frac{1}{2}k_m^2$, then $Y(t) \to 0$ as $t \to \infty$ a.s.*
(ii) If $\lim_{t\to\infty}\frac{1}{t}\int_0^t k^2(s)ds$ does not exist, we can denote $\underline{k}_m^2 = \liminf_{t\to\infty}\frac{1}{t}\int_0^t k^2(s)ds$ and $\bar{k}_m^2 = \limsup_{t\to\infty}\frac{1}{t}\int_0^t k^2(s)ds$. If $\alpha\beta > \frac{1}{2}\bar{k}_m^2$, then the result is

$$\beta - \frac{\bar{k}_m^2}{2\alpha} = \liminf_{t\to\infty}\frac{1}{t}\int_0^t Y(s)ds \leq \limsup_{t\to\infty}\frac{1}{t}\int_0^t Y(s)ds = \beta - \frac{\underline{k}_m^2}{2\alpha}, \tag{2.18}$$

almost surely. The results in next two sections are still valid with some necessary restatements as (2.18).

In next section, we will also need the following lemma.

Lemma 2.4 *Assume $c(y)$ is decreasing. Let $N(t)$ be a solution of*

$$\frac{d}{dt}N(t) = c(N(t))N(t) - \frac{1}{2}k^2(t)N(t), \quad N(0) = y_0. \tag{2.19}$$

Then there exists $t_0 > 0$ such that for $t > t_0$, the solution to (2.24) satisfies

$$N(t)e^{\inf\limits_{0\leq\sigma\leq t}\int_\sigma^t k(r)dW_r} \leq Y(t) \leq N(t)e^{\sup\limits_{0\leq\sigma\leq t}\int_\sigma^t k(r)dW_r}. \tag{2.20}$$

Proof. Assume $Y(t) \leq N(t)e^{\sup\limits_{0\leq\sigma\leq t}\int_0^t k(r)dW_r}$ is not true for all $t > t_0$ for any t_0. Assume t' is such a large time that $e^{\sup\limits_{0\leq\sigma\leq t'}\int_\sigma^{t'} k(r)dW_r} > 1$ and

$$Y(t') > N(t')e^{\sup\limits_{0\leq\sigma\leq t'}\int_\sigma^{t'} k(r)dW_r}. \tag{2.21}$$

It is evident that $Y(t') > N(t')$. Define τ such that $Y(\tau) = N(\tau)$ and $Y(s) \geq N(s)$ for $\tau < s \leq t'$. First note that

$$N(t') = N(\tau)e^{\int_\tau^{t'} c(N(r))dr - \frac{1}{2}\int_\tau^{t'} k^2(r)dr}. \tag{2.22}$$

Then it is easy to see that

$$
\begin{aligned}
Y(t') &= y_0 e^{\int_0^{t'} c(Y(r))dr - \frac{1}{2}\int_0^{t'} k^2(r)dr + \int_0^{t'} k(r)dW_r} \\
&= Y(\tau)e^{\int_\tau^{t'} c(Y(r))dr - \frac{1}{2}\int_\tau^{t'} k^2(r)dr + \int_\tau^{t'} k(r)dW_r} \\
&\leq N(\tau)e^{\int_\tau^{t'} c(N(r))dr - \frac{1}{2}\int_\tau^{t'} k^2(r)dr} e^{\sup\limits_{0\leq s\leq t'}\int_\sigma^{t'} k(r)dW_r} \\
&\leq N(t')e^{\sup\limits_{0\leq s\leq t'}\int_\sigma^{t'} k(r)dW_r}.
\end{aligned} \tag{2.23}
$$

This contradicts (2.21). Therefore the second inequality in (2.20) is proved. The first part in (2.20) can be proved similarly. $\qquad\square$

Consider equation (2.1) with constant k in the remaining part of this section

$$dY(t,y) = \alpha Y(t,y)(\beta - Y(t,y))\,dt + kY(t,y)\,dW_t, \quad Y_0 = y. \tag{2.24}$$

Here W is a one dimensional Brownian motion on Wiener space $(\Omega, \mathcal{F}, P)$. That is to say that Ω is the space of all continuous paths $\omega : R \to R$ given the topology of uniform convergence, $\mathcal{F}$ is the Borel σ-field, P is the Wiener measure and W is defined by $W(t,\omega) = \omega(t)$. It is easy to see that the solution (2.2) of equation (2.24) defines a stochastic flow $Y(s) : (R^+ \cup \{0\}) \times \Omega \to R^+ \cup \{0\}$. Recall the canonical Brownian shift

$$\theta(t,\omega)(s) = \omega(t+s) - \omega(t).$$

Then by flow property we know that $W(t + s, \omega) = W(s, \theta_t\omega) + W(t, \omega)$. So

$$W(s, \theta_t\omega) = W(t + s, \omega) - W(t, \omega) \tag{2.25}$$

Define

$$Z(\omega) = (\alpha \int_{-\infty}^{0} \exp\{\alpha\beta s - \frac{1}{2}k^2 s + kW(s)\}ds)^{-1}. \tag{2.26}$$

Simple calculation shows

$$
\begin{aligned}
Z(\theta_t\omega) &= (\alpha \int_{-\infty}^{0} \exp\{\alpha\beta s - \frac{1}{2}k^2 s + kW(s+t, \omega) - kW(t, \omega)\}ds)^{-1} \\
&= \frac{\exp\{\alpha\beta t - \frac{1}{2}k^2 t + kW(t, \omega)\}}{\alpha \int_{-\infty}^{0} \exp\{\alpha\beta(s+t) - \frac{1}{2}k^2(s+t) + kW(s+t, \omega)\}ds} \\
&= \frac{\exp\{\alpha\beta t - \frac{1}{2}k^2 t + kW(t, \omega)\}}{\alpha \int_{-\infty}^{t} \exp\{\alpha\beta s - \frac{1}{2}k^2 s + kW(s, \omega)\}ds}.
\end{aligned}
$$

Then it is easy to check that

$$Y(t, Z(\omega), \omega) = Z(\theta_t(\omega)). \tag{2.27}$$

According to Mohammed-Scheutzow's definition of stationary trajectory ([23]), $Z(\omega)$ is a nontrivial stationary trajectory of equation (2.24). One can calculate the Lyapunov exponent of the linearized equation at the stationary trajectory (c.f. [1]). Moreover, for any $y > 0$, consider the derivative flow $V_t = DY(t, y)V$ where V being given and $V_0 = V$. Here D is the derivative with respect to y. First note that V_t satisfies the following linear stochastic differential equation, which can be obtained by differentiating equation (2.24),

$$dV_t = \alpha(\beta - 2Y(t, y))V_t dt + kV_t dW(t), V_0 = V. \tag{2.28}$$

Solving equation (2.28) we obtain

$$V_t = V \exp\{\alpha \int_0^t (\beta - 2Y(s, y))ds - \frac{1}{2}k^2 t + kW(t)\}. \tag{2.29}$$

It turns out that

$$
\begin{aligned}
\lim_{t \to \infty} \frac{1}{t} \log V_t &= \alpha\beta - \lim_{t \to \infty} \frac{2\alpha}{t} \int_0^t Y(s, y)ds - \frac{1}{2}k^2 + \lim_{t \to \infty} \frac{1}{t}kW(t) \\
&= -(\alpha\beta - \frac{k^2}{2}) < 0, \quad \text{P.a.s..} \tag{2.30}
\end{aligned}
$$

Here we used Lemma 2.1 to calculate $\lim_{t \to \infty} \frac{1}{t} \int_0^t Y(s, y)ds$. Similar to [23], there is a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, .)\Omega^* = \Omega^*$ for all $t \in R$ such that for each $\omega \in \Omega^*$, for any $y > 0$, for the discretized flow Y_n,

$$
\begin{aligned}
\lim_{n \to \infty} \frac{1}{n} \log |Y(n, y, \omega) - Z(\theta_n(\omega))| &= \lim_{n \to \infty} \frac{1}{n} \log |Y(n, y, \omega) - Y(n, Z(\omega), \omega)| \\
&= -(\alpha\beta - \frac{k_m^2}{2}) < 0. \tag{2.31}
\end{aligned}
$$

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But according to [1] (P185), $EZ(\omega) < \infty$. Therefore by Jensen's inequality, $E\log^+ Z(\omega) < E\log(1 + Z(\omega)) \le \log(1 + EZ(\omega)) < \infty$. That is to say the integrability condition of Lemma 3.4 in [23] is satisfied. Therefore we can apply the Lemma to pass (2.31) to continuous time to derive the following lemma.

Lemma 2.5 *Let $Y(t, y)$ be the solution of stochastic differential equation (2.24), $Z(\omega)$ be defined by (2.26) and θ_t be the canonical Brownian shift. Then there is a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, .)\Omega^* = \Omega^*$ such that for any $y > 0$, and for each $\omega \in \Omega^*$,*

$$\lim_{t \to \infty} \frac{1}{t} \log |Y(t, y, \omega) - Z(\theta_t(\omega))| = -(\alpha\beta - \frac{k^2}{2}) < 0. \qquad (2.32)$$

3 Ergodic property and pathwise property of the stochastic KPP equations

We consider the following stochastic KPP equation

$$du(t, x) = \left(\frac{D}{2} \Delta u(t, x) + u(t, x)c(u(t, x)) \right) dt + k(t)u(t, x) \, dW_t, \ u|_{t=0} = u_0, \qquad (3.1)$$

where $u_0 = \chi_{(-\infty, 0]}$. Here W_t is a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Recall $k_m^2 = \lim_{t \to \infty} \frac{1}{t} \int_0^t k^2(s)ds$ assuming the limit exists. Assume

Condition (i). Function $c(u)$ is continuous and $c(u) \le c(0)$ and $c(u) < 0$ for $u > 1$.
Condition (ii). Function $c(u)$ is decreasing.
Condition (iii). There exist constants $a \ge b > 0$ such that $b\xi \le c(u) - c(u + \xi) \le a\xi$ for any $\xi \ge 0$ and $u \ge 0$.
Condition (iv). $c(0) > \frac{1}{2}k_m^2$.

Let B_r be a Brownian motion in R^1 on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Note B_r and W_r are independent. Then $u(t, x)$ satisfies the following Feynman-Kac formula:

$$\begin{aligned}
u(t, x) &= \hat{E}u(0, x + \sqrt{D}B_t) \\
&\quad \exp\{\int_0^t c(u(t - r, x + \sqrt{D}B_r))dr - \frac{1}{2}\int_0^t k^2(r) \, dr + \int_0^t k(r) \, dW_r\}. \quad (3.2)
\end{aligned}$$

Here $\hat{E}$ is the expectation with respect to the probability measure $\hat{P}$.

Recall that under Conditions (i), (ii) and (iv) ([10],[7],[25])

$$\begin{aligned}
v(t, x) \exp\left(\inf_{0 \le \sigma \le t} \int_\sigma^t k_s \, dW_s \right) &\le \quad u(t, x, \omega) \qquad (3.3) \\
&\le \quad v(t, x) \exp\left(\sup_{0 \le \sigma \le t} \int_\sigma^t k_s \, dW_s \right),
\end{aligned}$$

where $v(t,x)$ is the solution of the following deterministic reaction diffusion equation

$$\frac{\partial v}{\partial t} = \frac{D}{2}\Delta v + v(c(v) - \frac{1}{2}k^2(t)), \ v(0,\cdot,\omega) = \chi_{(-\infty,0]}; \ (t,x) \in R^+ \times R. \tag{3.4}$$

In this section, we prove some new results about the random travelling waves. First, from (1.3) it is trivial to see

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t u(s, (\gamma + h)s)ds = 0, \tag{3.5}$$

for any $h > 0$. We will investigate the same limit behind the wave front. But generally speaking, we don't know whether or not the limit actually exists except in some special cases (this will be clear later). However we will be able to give the upper limit and lower limit of $\frac{1}{t}\int_0^t u(s,x)ds$ on the crest as $t \to \infty$. The upper limit is straightforward.

Theorem 3.1 *Suppose c satisfies Conditions (i), (ii) and (iii) and let u be the solution of (3.1). Then for almost all ω,*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \sup_{x\in R} u(s,x)\,ds \le \frac{1}{b}\left(c(0) - \frac{k_m^2}{2}\right).$$

Proof. Let $Y(t)$ be the solution of (2.24) with $Y(0) = 1$. First note that $Y(t,x) = Y(t)$ is the solution of (3.1) with the initial condition $Y(0,x) = 1$. Therefore for any $\tau \le t$, it is easy to see

$$Y(t) = Y(t-\tau)e^{\int_0^\tau c(Y_{t-r})dr - \frac{1}{2}\int_0^\tau k^2(t-r)dr - \int_0^\tau k(t-r)dW_{t-r}}. \tag{3.6}$$

We will prove $0 \le u(t,x) \le Y(t,x) = Y(t)$ for all $x \in R$. Assume it is wrong, then $u(t,x) > Y(t)$ for certain t and x. Define a stopping time $\tau = \inf\{r : u(t-r, x+\sqrt{D}B_r) \le Y(t-r)\}$. Then by Markov property of Brownian motion, as c is decreasing,

$$
\begin{aligned}
u(t,x) &= \hat{E}u(t-\tau, x+\sqrt{D}B_\tau) \\
&\quad \exp\{\int_0^\tau c(u(t-r, x+\sqrt{D}B_r))dr - \frac{1}{2}\int_0^\tau k^2(t-r)\,dr - \int_0^\tau k(t-r)\,dW_{t-r}\} \\
&\le Y(t-\tau)\exp\{\int_0^\tau c(Y(t-r))dr - \frac{1}{2}\int_0^\tau k^2(t-r)\,dr - \int_0^\tau k(t-r)\,dW_{t-r}\} \\
&= Y(t).
\end{aligned} \tag{3.7}
$$

Therefore we have desored inequality. The result now follows from Lemma 2.2. □

The proof of a lower bound is more complicated. The following lemma is needed.

Lemma 3.2 *Assume Conditions (i) and (ii), then a.s.*

$$\lim_{t\to\infty} \frac{1}{t}\log u(t, t\sqrt{(2c(0) - k_m^2)D}) = 0. \tag{3.8}$$

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Proof. Let v be the solution of (3.4). Recall (e.g. [9]) that for any $\epsilon > 0$, there exists $t_0 > 0$ such that

$$-\frac{\varepsilon t}{2} \leq \log v(t, t\sqrt{(2c(0) - k_m^2)D}) \leq \frac{\varepsilon t}{2}, \ t \geq t_0.$$

Similar to the proof of Lemma 2.1, for any $T \geq t_0$, there exists $\Omega_T \subset \Omega$ with $P\{\Omega_T\} > 1 - \exp\{-\frac{\varepsilon^2}{8\int_0^T k^2(r)dr}T^2\}$, such that for each $\omega \in \Omega_T$, and $t \geq T$,

$$|\int_0^s k(r)dW_r| \leq \frac{\varepsilon t}{2}, \tag{3.9}$$

for all $0 \leq s \leq t$. It follows from (3.3) that for all $t \geq T$ and $\omega \in \Omega_T$,

$$-\varepsilon < \frac{1}{t} \log u(t, t\sqrt{(2c(0) - k_m^2)D}) < \varepsilon,$$

which completes the proof. □

Lemma 3.3 *Suppose c satisfies Conditions (i), (ii), (iii) and (iv) and let $u(t,x)$ be the solution of (3.1) and $Y(t)$ be the solution of (2.16) with $Y(0) = 1$. Let $h > 0$ be arbitrary. Then for any λ, there exist $t_0 > 0$ and $\delta > 0$ such that*

$$P\{\inf_{x < t(\gamma b/a - h)} \frac{u(t,x)}{Y(t)} \geq 1 - \lambda, \ \text{for all } t \geq T\} > 1 - \exp\{-\delta T\}, \tag{3.10}$$

for any $T \geq t_0$.

Proof. Let $h \in (0, \gamma\frac{b}{2a})$. For any $\lambda > 0$, let $\epsilon > 0$ be small enough such that

$$\lambda h(c(0) - \frac{k_m^2}{2})/(4a) - b\lambda\epsilon - 2\epsilon \geq 0. \tag{3.11}$$

From Lemma 2.2, we know there exists $t_0^* > 0$ such that for any $T \geq t_0^*$, there exists $\Omega_T^{1,\epsilon} \subset \Omega$ with $P(\Omega_T^{1,\varepsilon}) > 1 - e^{-\frac{a^2\varepsilon^2(\frac{b}{2a})^2 T^2}{32\int_0^{\frac{b}{2a}T} k^2(r)dr}}$ such that for any $\omega \in \Omega_T^{1,\varepsilon}$, and for all $s \geq \frac{b}{2a}T$

$$\frac{1}{a}(c(0) - \frac{k_m^2}{2}) - \varepsilon \leq \frac{1}{s}\int_0^s Y(r)dr \leq \frac{1}{b}(c(0) - \frac{k_m^2}{2}) + \varepsilon \tag{3.12}$$

From Lemma 3.2 and Lemma 2.4, we know there exist $t_1 = t_1(\epsilon(\lambda))$ and $\Omega_T^{2,\epsilon} \subset \Omega$ for all $T \geq t_1$, with $P(\Omega_T^{2,\epsilon}) > 1 - e^{-\delta_2 T}$ for a constant $\delta_2 > 0$, such that if $\omega \in \Omega_T^{2,\epsilon}$,

$$u(s, \gamma s) \geq e^{-\epsilon s}, \ \text{for all } s \geq T, \tag{3.13}$$

and

$$e^{-\epsilon s} \leq Y(s)(\omega) \leq e^{\epsilon s}, \ \text{for all } s \geq T. \tag{3.14}$$

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It is trivial to see that there is a $t_2 > 0$ such that for all $s \geq t_2$,

$$\mathrm{e}^{-\frac{h^2 s}{8D(1-\frac{b}{a}+\frac{h}{2\gamma})}} + \mathrm{e}^{-\frac{(h-\frac{b}{2a}\gamma)^2}{2D}s} \leq \frac{\lambda}{4}. \tag{3.15}$$

Define $t_0 = \max\{\frac{2a}{b}t_0^*, t_1, t_2\}$. Let $T \geq t_0$ be arbitrary. Define $\Omega_T^{0,\epsilon} = \Omega_T^{1,\epsilon} \cap \Omega_T^{2,\epsilon}$. First, it is evident that there is a $\delta > 0$ such that $P(\Omega_T^{0,\epsilon}) > 1 - \mathrm{e}^{-\delta T}$ and for any $\omega \in \Omega_T^{0,\epsilon}$ and $s \geq T$, we have (3.12), (3.13-3.15). Now we shall show that for all $\omega \in \Omega_T^{0,\varepsilon}$, and all $t \geq T$,

$$\inf_{x < t(\gamma b/a - h)} \frac{u(t,x)}{Y(t)} \geq 1 - \lambda. \tag{3.16}$$

Suppose (3.16) is false, i.e., there exist $\lambda > 0$, $\omega \in \Omega_T^{0,\epsilon}$ and $t \geq T$, $x^* = x^*(t) < t(\gamma b/a - h)$ such that $u(t, x^*(t), \omega) \leq (1-\lambda)Y(t,\omega)$. Here T and $\Omega_T^{0,\epsilon}$ are defined as above. Since $u(t, \cdot, \omega)$ is decreasing we have that

$$0 \leq u(t, (\frac{\gamma b}{a} - h)t, \omega) \leq u(t, x^*(t), \omega) \leq (1 - \lambda)Y(t, \omega). \tag{3.17}$$

For such a t and ω, define a stopping time

$$\hat{\tau}^t = \inf\{r \geq 0; \ \hat{X}_r^t \notin F\},$$

where

$$\hat{X}_r^t = (t - r, (\gamma b/a - h)t + \sqrt{D}B_r).$$

Here

$$F = F_t = \{(s, x); \ s > 0, \ x < \gamma s, \ u(s, x) \leq (1 - \frac{\lambda}{2})Y(s)\}.$$

Note $\tau < t$ as $u(0, x) = 1 = Y_0$ for $x < 0$ therefore line $s = 0$ lies out side of F.

The boundary of F consists of two parts. We denote the part along $x = \gamma s$ by ∂F_1 and the remaining part by ∂F_2. Define

$$\hat{\Omega}_1 = \{\hat{\omega} \in \hat{\Omega}; \ \hat{X}_{\hat{\tau}^t}^t \in \partial F_1, \ \hat{\tau}^t \notin [(1 - b/a + h/2a)t, (1 - b/2a)t]\}$$
$$\hat{\Omega}_2 = \{\hat{\omega} \in \hat{\Omega}; \ \hat{X}_{\hat{\tau}^t}^t \in \partial F_1, \ \hat{\tau}^t \in [(1 - b/a + h/2a)t, (1 - b/2a)t]\},$$
$$\hat{\Omega}_3 = \{\hat{\omega} \in \hat{\Omega}; \ \hat{X}_{\hat{\tau}^t}^t \in \partial F_2\}.$$

From the Feynman-Kac formula for $u(t, x)$ and the strong Markov property of Brownian motions, we see that

$$u(t, (\gamma b/a - h)t, \omega)$$
$$= \hat{E}\left[u(\hat{X}_{\hat{\tau}^t}^t)\exp\left(\int_0^{\hat{\tau}^t} c(u(\hat{X}_r^t, \omega))\, dr + \int_{t-\hat{\tau}^t}^t k(r)\, dW_r(\omega) - \frac{1}{2}\int_{t-\hat{\tau}^t}^t k^2(r)\, dr\right)\right]$$
$$= \sum_{i=1}^3 u_i(t, (\gamma b/a - h)t, \omega),$$

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where

$$
\begin{aligned}
& u_i(t, (\gamma b/a - h)t, \omega) \\
= \ & \hat{E}\chi_{\hat{\Omega}_i} u(\hat{X}^t_{\hat{\tau}t}, \omega) \\
& \times \exp\left(\int_0^{\hat{\tau}^t} c(u(\hat{X}^t_r, \omega))\, dr + \int_{t-\hat{\tau}^t}^t k(r)\, dW_r(\omega) - \frac{1}{2}\int_{t-\hat{\tau}^t}^t k^2(r)\, dr \right),
\end{aligned} \tag{3.18}
$$

for $i = 1, 2, 3$. We shall obtain lower bound on each of the u_i's. It is trivial to see that

$$
u_1(t, x) \geq 0, \quad \text{for any } (t, x). \tag{3.19}
$$

To obtain a lower bound on $u_2(t, (\gamma b/a - h)t)$ we start by considering the first integral in the exponential. Since $u(r, x) \leq (1 - \lambda/2)Y(r)$ for $(r, x) \in F$, from Conditions (ii) and (iii), it is easy to see

$$
\int_0^{\hat{\tau}^t} c(u(\hat{X}^t_r))\, dr \geq \int_0^{\hat{\tau}^t} c((1 - \lambda/2)Y_{t-r})\, dr \geq \int_0^{\hat{\tau}^t} c(Y_{t-r})\, dr + \frac{b\lambda}{2}\int_0^{\hat{\tau}^t} Y_{t-r}\, dr. \tag{3.20}
$$

Note if $s \in [(1 - \frac{b}{a} + \frac{h}{2a})t, (1 - \frac{b}{2a})t]$, then $\frac{b}{2a}t \leq t - s \leq (\frac{b}{a} - \frac{h}{2a})t$. It is evident that as $\omega \in \Omega_T^{1,\epsilon}$, $t \geq T$

$$
\begin{aligned}
\frac{b\lambda}{2}\int_0^s Y(t-r)\, dr \ = \ & \frac{b\lambda}{2}\left(\int_0^t Y(r)\, dr - \int_0^{t-s} Y(r)\, dr \right) \\
\geq \ & \frac{b\lambda}{2}[(\frac{1}{a}(c(0) - \frac{k_m^2}{2}) - \epsilon)t - ((\frac{1}{b}(c(0) - \frac{k_m^2}{2}) + \epsilon)(\frac{b}{a} - \frac{h}{2a})t] \\
= \ & \frac{b\lambda}{2}[\frac{h}{2ab}(c(0) - \frac{k_m^2}{2})t - \epsilon t - \epsilon(\frac{b}{a} - \frac{h}{2a})t] \\
\geq \ & \frac{\lambda h}{4a}\left(c(0) - \frac{k_m^2}{2} \right)t - b\lambda\epsilon t.
\end{aligned} \tag{3.21}
$$

Since $Y(t)$ satisfies (3.1) with $u_0 \equiv 1$, there is an implicit Feynman-Kac formula for $Y(t)$. Using the strong Markov property as above we find that

$$
\exp\left(\int_0^{\hat{\tau}^t} c(Y_{t-r})\, dr + \int_{t-\hat{\tau}^t}^t k(r)\, dW_r - \frac{1}{2}\int_{t-\hat{\tau}^t}^t k^2(r)\, dr \right) = Y(t)(Y(t - \hat{\tau}^t))^{-1}. \tag{3.22}
$$

Recall (3.13) and (3.14). As $t \geq 2aT/b$, so

$$
u(t - \hat{\tau}^t, \gamma(t - \hat{\tau}^t)) \geq e^{-\epsilon(t-\hat{\tau}^t)} > e^{-\epsilon t} \text{ for all } \hat{\omega} \in \hat{\Omega}_2, \tag{3.23}
$$

and

$$
e^{-\epsilon(t-\hat{\tau}^t)} \leq Y(t - \hat{\tau}^t)(\omega) \leq e^{\epsilon(t-\hat{\tau}^t)}. \tag{3.24}
$$

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Hence for $\omega \in \Omega_T^{0,\varepsilon}$, and $t \geq T$, from (3.18),(3.20-3.22),

$$
\begin{aligned}
u_2(t, (\gamma b/a - h)t) &\geq \hat{E}\chi_{\hat{\Omega}_2} Y(t)(Y(t - \hat{\tau}^t))^{-1} e^{\lambda h t(c(0) - k_m^2/2)/(4a) - b\lambda \epsilon t - \epsilon t} \\
&\geq \hat{P}(\hat{\Omega}_2) Y(t) e^{(\lambda h(c(0) - k_m^2/2)/(4a) - b\lambda \epsilon - 2\epsilon)t} \\
&\geq \hat{P}(\hat{\Omega}_2) Y(t).
\end{aligned} \tag{3.25}
$$

We also have that

$$
\begin{aligned}
u_3(t, (\gamma b/a - h)t) &\geq \hat{E}\chi_{\hat{\Omega}_3}(1 - \frac{\lambda}{2})Y(t - \hat{\tau}^t) e^{\int_0^{\hat{\tau}^t} c(Y_{t-r})\,dr + \int_{t-\hat{\tau}t}^t k(r)\,dW_r - \int_{t-\hat{\tau}t}^t k(r)^2\,dr/2} \\
&\qquad e^{b\lambda \int_0^{\hat{\tau}^t} Y_{t-r}\,dr/2} \\
&\geq \hat{P}(\hat{\Omega}_3)(1 - \frac{\lambda}{2})Y_t,
\end{aligned} \tag{3.26}
$$

since $Y(t, \omega)$ satisfies an implicit Feynman-Kac formula (3.22) and $\int_0^s Y_{t-r}\,dr \geq 0$ for all $s \in [0, t]$.

Note if $\hat{\omega} \in \hat{\Omega}_1$ and $\tau < (1 - \frac{b}{a} + \frac{h}{2\gamma})t$, then X_s has to meet $x = \gamma s$ at a time $\tau \leq (1 - \frac{b}{a} + \frac{h}{2\gamma})t$. Therefore $(\gamma \frac{b}{a} - h)t + \sqrt{D}B_\tau \geq \gamma(\frac{b}{a} - \frac{h}{2\gamma})t$. This is followed by $\sqrt{D}B_\tau > \frac{h}{2}t$. It turns out from Doob's inequality (see [21]) that

$$
\begin{aligned}
&\hat{P}\{\hat{\omega} \in \hat{\Omega}_1 \text{ and } \tau < (1 - \frac{b}{a} + \frac{h}{2\gamma})t\} \\
&\leq \hat{P}\{\sqrt{D}B_\tau > \frac{h}{2}t, \text{ and } \tau < (1 - \frac{b}{a} + \frac{h}{2\gamma})t\} \\
&\leq \hat{P}\{\sup_{0 \leq s \leq (1 - \frac{b}{a} + \frac{h}{2\gamma})t} \sqrt{D}B_s > \frac{h}{2}t\} \leq \exp\{-\frac{(\frac{h}{2}t)^2}{2D(1 - \frac{b}{a} + \frac{h}{2\gamma})t}\} \\
&= \exp\{-\frac{h^2 t}{8D(1 - \frac{b}{a} + \frac{h}{2\gamma})}\}.
\end{aligned} \tag{3.27}
$$

Similarly, if $\hat{\omega} \in \hat{\Omega}_1$ and $\tau > (1 - \frac{b}{2a})t$, then X_s has to meet $x = \gamma s$ at a time $\tau > (1 - \frac{b}{2a})t$. Therefore $(\gamma \frac{b}{a} - h)t + \sqrt{D}B_\tau \leq \gamma(\frac{b}{2a})t$. This is followed by $\sqrt{D}B_\tau \leq (h - \frac{b}{2a})t$. It turns out that

$$
\begin{aligned}
&\hat{P}\{\hat{\omega} \in \hat{\Omega}_1 \text{ and } \tau > (1 - \frac{b}{2a})t\} \\
&\leq \hat{P}\{\sqrt{D}B_\tau \leq (h - \frac{b}{2a})t \text{ and } (1 - \frac{b}{2a})t \leq \tau \leq t\} \\
&\leq \hat{P}\{\inf_{0 \leq s \leq t} \sqrt{D}B_s < (h - \frac{b}{2a}\gamma)t\} \\
&\leq \exp\{-\frac{(h - \frac{b}{2a}\gamma)^2}{2D}t\}.
\end{aligned} \tag{3.28}
$$

Recall (3.15). It is obvious that (3.27) and (3.28) imply $\hat{P}(\hat{\Omega}_1) \leq \frac{\lambda}{4}$ as $t \geq T$. It follows from (3.18), (3.19), (3.25) and (3.26) that for $\omega \in \Omega_T^{0,\varepsilon}$, and $t \geq T$

$$
\begin{aligned}
u(t, (\gamma b/a - h)t) &= \sum_{i=1}^{3} u_i(t, (\gamma b/a - h)t) \\
&\geq (1 - \frac{\lambda}{2})Y(t)(\hat{P}(\hat{\Omega}_2) + \hat{P}(\hat{\Omega}_3)) \\
&= (1 - \frac{\lambda}{2})Y(t) - (1 - \frac{\lambda}{2})Y(t)\hat{P}(\hat{\Omega}_1) \\
&= (1 - \frac{\lambda}{2})Y(t) - \frac{\lambda}{4}Y(t) \\
&\geq (1 - \frac{3\lambda}{4})Y(t),
\end{aligned}
\tag{3.29}
$$

if T is sufficiently large. This contradicts (3.17). Therefore we have proved that for any $\lambda > 0$, and sufficiently large T, if $\omega \in \Omega_T^{0,\varepsilon(\lambda)}$ and and $t \geq T$,

$$
\inf_{x < t(\gamma b/a - h)} u(t, x) \geq (1 - \lambda)Y(t).
\tag{3.30}
$$

$\square$

Theorem 3.4 *Suppose c satisfies Conditions (i), (ii), (iii) and (iv) and u solves the SPDE (3.1). Let $h > 0$ be arbitrary. For any $\varepsilon > 0$, there exist $t_0 = t_0(\varepsilon)$ and $\delta(\varepsilon) > 0$ such that,*

$$
P\{\frac{1}{t} \int_0^t \inf_{x < s(\gamma b/a - h)} u(s, x) \, ds > \frac{1}{a}\left(c(0) - \frac{k_m^2}{2}\right) - \varepsilon \text{ for all } t \geq T\} > 1 - \exp\{-\delta T\},
\tag{3.31}
$$

for all $T \geq t_0$

Proof. According to Lemma 3.3 and Lemma 2.2, there exist t_1, $\delta_1 > 0$ and $\Omega_{1,T}^{\varepsilon} \subset \Omega$ with $P(\Omega_{1,T}^{\varepsilon}) > 1 - \exp\{-\delta_1 T\}$ for any $T \geq t_1$ such that for all $s \geq T$ and $\omega \in \Omega_{1,T}^{\varepsilon}$,

$$
\inf_{x < s(\gamma b/a - h)} u(s, x) > (1 - \frac{a\varepsilon}{3(c(0) - \frac{k_m^2}{2})})Y(s),
\tag{3.32}
$$

and

$$
\frac{1}{t} \int_0^t Y(s)ds \geq \frac{1}{a}(c(0) - \frac{k_m^2}{2}) - \frac{\varepsilon}{3}.
\tag{3.33}
$$

Let $t_0 > t_1$ satisfy

$$
\frac{1}{t_0}\ln 2 + \frac{c(0)t_1 + t_0^{\frac{1}{2}}}{t_0} < \frac{\varepsilon}{3}.
\tag{3.34}
$$

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Then for any $T \geq t_0$, define

$$\Omega_{2,T} = \{\omega \in \Omega : \sup_{0 \leq s \leq t_1} \int_0^s k(r) dW(r) \leq T^{\frac{1}{2}}\}. \tag{3.35}$$

It is easy to see that $P(\Omega_{2,T}) > 1 - \exp\{-\delta_2 T\}$ for a constant $\delta_2 > 0$. Moreover, for all $\omega \in \Omega_{2,T}$ with $T \geq t_0$,

$$\frac{1}{t} \int_0^{t_1} Y(s) ds < \frac{\varepsilon}{3}. \tag{3.36}$$

for all $t \geq T$.

So for $T \geq t_0$, define $\Omega_T = \Omega_{1,T} \cap \Omega_{2,T}$. It is easy to see $P(\Omega_T) > 1 - \exp\{-\delta T\}$ for a costant $\delta > 0$ and $\omega \in \Omega_T$,

$$
\begin{aligned}
\frac{1}{t} \int_0^t \inf_{x < s(\gamma b/a - h)} u(s, x) ds &\geq \frac{1}{t} \int_{t_1}^t \inf_{x < s(\gamma b/a - h)} u(s, x) ds \\
&\geq (1 - \frac{a\varepsilon}{3(c(0) - \frac{k_m^2}{2})}) \frac{1}{t} \int_{t_1}^t Y(s) ds \\
&= (1 - \frac{a\varepsilon}{3(c(0) - \frac{k_m^2}{2})}) \frac{1}{t} (\int_0^t Y(s) ds - \int_0^{t_1} Y(s) ds) \\
&\geq (1 - \frac{a\varepsilon}{3(c(0) - \frac{k_m^2}{2})}) \{\frac{1}{a}(c(0) - \frac{k_m^2}{2}) - \frac{\varepsilon}{3} - \frac{\varepsilon}{3}\} \\
&> \frac{1}{a}(c(0) - \frac{k_m^2}{2}) - \varepsilon, \tag{3.37}
\end{aligned}
$$

for all $t \geq T$. Here we have used (3.32), (3.33) and (3.36). The proof is completed. $\square$

Remark 3.5 *(i). Although the solutions of the stochastic generalized KPP equations may not converge as time and space tend to ∞, Theorem 3.4 tells us that the time average grows linearly with time for large time. In some special case, $\frac{1}{t} \int_0^t u(s, x) ds$ converges on the crest. For this see Remark 3.5 (ii). It is trivial to see $\frac{1}{t} \int_0^t u(s, x) ds$ converges to zero on the trough.*
(ii). Note that if $a = b$, i.e. if c is linear, we find that for any $h > 0$ almost all $\omega \in \Omega$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \inf_{x < (\gamma - h)s} u(s, x) \, ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sup_{x \in R} u(s, x) \, ds = c(0) - \frac{k_m^2}{2}. \tag{3.38}$$

Now we consider the following stochastic KPP equation (3.1) when $c(u) = c(1 - u)$, and k is a constant

$$du(t, x) = \left(\frac{D}{2} \Delta u(t, x) + cu(t, x)(1 - u(t, x))\right) dt + ku(t, x) dW_t, \ u|_{t=0} = u_0, \tag{3.39}$$

where $u_0 = \chi_{(-\infty, 0]}$. Here W_t is a Brownian motion on the Wiener space $(\Omega, \mathcal{F}, P)$, and c is a constant.

Theorem 3.6 *Let $u(t,x)$ be the solution of (3.39) and θ_t be the canonical Brownian shift. Then*

$$\lim_{t\to\infty} \sup_{x<(\gamma-h)t} |u(t,x,\omega) - Z(\theta_t\omega)| = 0, \quad P.a.s \qquad (3.40)$$

for any $h > 0$. Here $\gamma = \sqrt{2D(c - \frac{k_m^2}{2})}$ and $Z(\omega)$ is given by

$$Z(\omega) = (c\int_{-\infty}^{0} \exp\{(c - \frac{1}{2}k^2)s + kW(s)\}ds)^{-1}. \qquad (3.41)$$

Proof. Let $Y(t) = Y(t,1)$. The proof follows from Lemma 2.5, Lemma 3.3 and the inquality $u(t,x) \le Y(t)$ immediately. $\qquad\square$

4 Behaviour of the derivatives

It is easy to see that the solution $u(t,x)$ is C^2 in x for all $t > 0$ and almost all $\omega \in \Omega$ (see [25]). Let $R(t,x) = -D\nabla \log u(t,x)$ for $t > 0$, then it is easy to verify that $R(t,x)$ satisfies the following Burgers' equation for $t > 0$ and a.e. $\omega \in \Omega$:

$$\frac{\partial}{\partial t}R(t,x) + R(t,x)\nabla R(t,x) = \frac{D}{2}\Delta R(t,x) + c'(u(t,x))u(t,x)R(t,x). \qquad (4.1)$$

Let y_r be the solution of the following stochastic differential equation up to explosion time

$$dy_r = \sqrt{D}dB_r - R(t-r, y_r)dr, \quad y_0 = x. \qquad (4.2)$$

Here B_r is a Brownian motion on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_r, \hat{P}\}$. See [8] for explosion time.

We first prove that y_r is nonexplosive.

Lemma 4.1 *Assume Conditions (i) and (ii). Then for any $t > 0$, the solution y_r to the Equation (4.2) is nonexplosive up to any time $t_0 < t$ for almost all $\hat{\omega}$ and ω.*

Proof. We fix $t > 0$. Let

$$\mathcal{E}_t = e^{-\frac{1}{2}\int_0^t k^2(r)dr + \int_0^t k(r)dW_r} \qquad (4.3)$$

and

$$\mathcal{V}(t,x) = u(t,x)\mathcal{E}_t^{-1}. \qquad (4.4)$$

Then $\mathcal{V}$ satisfies the following reaction diffusion equation (see [25]),

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{V}(t,x) &= \frac{D}{2}\Delta\mathcal{V} + c(\mathcal{E}_t\mathcal{V})\mathcal{V}, \\
\mathcal{V}(0,x) &= u_0(x) = \chi_{x\le 0}.
\end{aligned} \qquad (4.5)$$

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Applying Feynman-Kac formula, we have for any x,

$$
\begin{aligned}
\mathcal{V}(t,x) &= \hat{E}u_0(x_t)\mathrm{e}^{\int_0^t c(\mathcal{E}_{t-r}\mathcal{V}(t-r,x_r))dr} \\
&\leq \mathrm{e}^{c(0)t}\hat{E}u_0(x_t) \\
&\leq M(t).
\end{aligned}
\tag{4.6}
$$

Here $M(t) = \mathrm{e}^{c(0)t}$, $x_t = x + \sqrt{D}B(t)$, $B(t)$ is a Brownian motion on the probability space $(\hat{\Omega}, \hat{F}, \hat{P})$ and $\hat{E}$ is the expectation with respect to $\hat{P}$. As $c(u)$ is decreasing, so

$$
c(\mathcal{E}_{t-r}\mathcal{V}(t-r, x_r)) \geq c(\mathcal{E}_{t-r}M(t-r)).
\tag{4.7}
$$

Therefore

$$
\begin{aligned}
\mathcal{V}(t,x) &\geq \hat{E}u_0(x_t)\mathrm{e}^{\int_0^t c(\mathcal{E}_{t-r}M(t-r))dr} \\
&= \mathrm{e}^{\int_0^t c(\mathcal{E}_{t-r}M(t-r))dr}H(t,x).
\end{aligned}
\tag{4.8}
$$

Here $H(t,x)$ is the solution of the heat equation $\frac{\partial}{\partial t}H(t,x) = \frac{D}{2}\Delta H(t,x)$ with $H(0,x) = \chi_{x\leq 0}$. It is easy to see that $H(t,x)$ is decreasing with respect to x, so for all $x \leq 0$,

$$
H(t,x) \geq H(t,0) = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi Dt}}\mathrm{e}^{-\frac{y^2}{2Dt}}dy = \frac{1}{2}.
\tag{4.9}
$$

It follows from (4.8) and (4.9) that for all $x \leq 0$,

$$
\mathcal{V}(t,x) \geq \tfrac{1}{2}\mathrm{e}^{\int_0^t c(\mathcal{E}_{t-r}M(t-r))dr}.
\tag{4.10}
$$

On the other hand, $\mathcal{V}(t,x)$ satisfies

$$
\mathcal{V}(t,x) = \int_R p_t(y,x)u_0(y)dy + \int_0^t \int_R p_{t-r}(y,x)c(\mathcal{E}_r\mathcal{V}(r,y))\mathcal{V}(r,y)dydr,
\tag{4.11}
$$

where

$$
p_t(y,x) = \frac{1}{\sqrt{2\pi Dt}}\mathrm{e}^{-\frac{(y-x)^2}{2Dt}}.
$$

Differentiating (4.11) with respect to x, we have

$$
\nabla_x \mathcal{V}(t,x) = \int_R (\nabla_x p_t(y,x))u_0(y)dy + \int_0^t \int_R (\nabla_x p_{t-r}(y,x))c(\mathcal{E}_r\mathcal{V}(r,y))\mathcal{V}(r,y)dydr. \tag{4.12}
$$

It follows that there exists a constant $C > 0$ such that

$$
|\nabla v(t,x)| \leq \int_R \frac{1}{\sqrt{Dt}}\sqrt{\frac{(y-x)^2}{Dt}}p_t(y,x)u_0(y)dy
$$

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$$+ \int_0^t \int_R \frac{1}{\sqrt{D(t-r)}} \sqrt{\frac{(y-x)^2}{D(t-r)}} p_{t-r}(y,x) c(\mathcal{E}_r \mathcal{V}(r,y)) \mathcal{V}(r,y) dy dr$$

$$\leq \frac{C}{\sqrt{t}} \int_R \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(y-x)^2}{4Dt}} u_0(y) dy$$

$$+ \int_0^t \frac{C}{\sqrt{t-r}} \int_R \frac{1}{\sqrt{4\pi D(t-r)}} e^{-\frac{(y-x)^2}{4D(t-r)}} dy \max\{c(0), |c(\mathcal{E}_r M(r))|\} M(r) dr$$

$$\leq \frac{C}{\sqrt{t}} + C \int_0^t \frac{\max\{c(0), |c(\mathcal{E}_r M(r))|\} M(r)}{\sqrt{t-r}} dr. \qquad (4.13)$$

It then follows from (4.10) and (4.13) that for any $x \leq 0$,

$$\begin{aligned} |R(t,x)| &= |-\frac{\nabla u(t,x)}{u(t,x)}| = |-\frac{\nabla \mathcal{V}(t,x)}{\mathcal{V}(t,x)}| \\ &\leq \frac{\frac{C}{\sqrt{t}} + C \int_0^t \frac{\max\{c(0), |c(\mathcal{E}_r M(r))|\} M(r)}{\sqrt{t-r}} dr}{\frac{1}{2} e^{\int_0^t c(\mathcal{E}_{t-r} M(t-r)) dr}} \\ &\leq (\frac{2C}{\sqrt{t}} + 2CE(t,\omega)) e^{-\int_0^t c(\mathcal{E}_{t-r} M(t-r)) dr}. \qquad (4.14) \end{aligned}$$

Here $E(t,\omega)$ is bounded for almost all ω and so $R(t,x)$ is bounded for almost all $\omega \in \Omega$ for any $x \leq 0$.

Now it is easy to see $\nabla \mathcal{V}(t,x) \leq 0$ (this is because $u(0,x)$ is nonincreasing and can be approximated by a regular and nonincreasing function. Then directly differentiating Equation (4.5) with the regular initial condition and using the Feynman-Kac formula, we can easily see the space derivative is nonpositive. The claim follows then from taking the limit.) Therefore $R(t-r, y_r) \geq 0$. Integrating (4.2), we have

$$y_s = \sqrt{D} B_s - \int_0^s R(t-r, y_r) dr \leq \sqrt{D} B_s. \qquad (4.15)$$

That is to say y_s cannot be explosive to $+\infty$ at any finite time $\hat{\omega}$ almost surely and for all ω. But according to (4.14), the solution cannot be explosive to $-\infty$ at any time $s \leq t_0 < t$ $\hat{\omega}$ almost surely and for almost all $\omega \in \Omega$ as the drift $-R(t-r, y_r)$ is bounded for almost all ω, $y_r < 0$ and $r \leq t_0 < t$. Here t_0 is any time $t_0 < t$. $\qquad \square$

Before we prove the main theorem of this section, we present some priori estimate for $v(t,x)$ which will be used in the proof. Here $v(t,x)$ was defined by eqauation (3.4). We don't strive to get the best possible estimates here. By comparison, under Conditions (i) and (ii), it is easy to see

$$v(t,x) \leq 1. \qquad (4.16)$$

Therefore $c(v) \geq 0$. It follows that

$$v(t,x) = \hat{E} u_0(x_t) e^{\int_0^t c(v(t-s,x_s)) ds - \frac{1}{2} \int_0^t k^2(s) ds}$$

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$$
\begin{aligned}
&\geq \quad \mathrm{e}^{-\frac{1}{2}\int_0^t k^2(s)ds}\hat{E}u_0(x_t)\\
&= \quad \mathrm{e}^{-\frac{1}{2}\int_0^t k^2(s)ds}\int_{-\infty}^0 \frac{1}{\sqrt{2\pi Dt}}\mathrm{e}^{-\frac{(y-x)^2}{2Dt}}dy\\
&= \quad \mathrm{e}^{-\frac{1}{2}\int_0^t k^2(s)ds}\int_{-\infty}^{-\frac{x}{\sqrt{Dt}}} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}dy.
\end{aligned}
$$

It is easy to see if $x \leq 1$,

$$
\begin{aligned}
v(t,x) &\geq \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\int_0^t k^2(s)ds}\int_{-\infty}^{-\frac{1}{\sqrt{Dt}}}\mathrm{e}^{-\frac{y^2}{2}}dy\\
&\geq \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\int_0^t k^2(s)ds}\frac{\sqrt{Dt}}{Dt+1}\mathrm{e}^{-\frac{1}{2t}}.
\end{aligned}
$$

Here we used a wellknown inequality $\int_a^\infty \mathrm{e}^{-\frac{y^2}{2}}dy \geq \frac{1}{a+\frac{1}{a}}\mathrm{e}^{-\frac{a^2}{2}}$ (see e.g. [21]). Therefore if $x \leq 1$,

$$
\begin{aligned}
\log v(t,x) &\geq \log\frac{1}{\sqrt{2\pi}} - \frac{1}{2}\int_0^t k^2(s)ds + \frac{1}{2}\log Dt - \log(Dt+1) - \frac{1}{2Dt}\\
&\geq \log\frac{1}{\sqrt{2\pi}} - \log 2 - \frac{1}{2}\log Dt - \frac{1}{2}\int_0^t k^2(s)ds - \frac{1}{2Dt}. \quad (4.17)
\end{aligned}
$$

And if $x > 1$,

$$
\begin{aligned}
v(t,x) &\geq \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\int_0^t k^2(s)ds}\frac{1}{\frac{x}{\sqrt{Dt}}+\frac{1}{\frac{x}{\sqrt{Dt}}}}\mathrm{e}^{-\frac{x^2}{2Dt}}\\
&= \frac{1}{\sqrt{2\pi}}\frac{x\sqrt{Dt}}{x^2+Dt}\mathrm{e}^{-\frac{1}{2}\int_0^t k^2(s)ds-\frac{x^2}{2Dt}}.
\end{aligned}
$$

It turns out that for $1 \leq x \leq \gamma t$,

$$
\begin{aligned}
&\log v(t,x)\\
&\geq \log\frac{1}{\sqrt{2\pi}} + \log x + \frac{1}{2}\log Dt - \log(x^2+Dt) - \frac{1}{2}\int_0^t k^2(s)ds - \frac{x^2}{2Dt}\\
&\geq \log\frac{1}{\sqrt{2\pi}} - \log(\gamma^2+D) + \frac{1}{2}\log D - \frac{1}{2}\log t - \frac{1}{2}\int_0^t k^2(s)ds - \frac{\gamma^2}{2D}t. \quad (4.18)
\end{aligned}
$$

We are going to prove the main theorem of this section. In addition to Conditions (i)-(iv), set

Condition (v). For any $u > 0$, $c'(u) \leq -d_0$ for a constant $d_0 > 0$.

Theorem 4.2 *Assume Conditions (i)-(v). There exist $t_0 > 0$ and $C_1 > 0, C_2 > 0$ and $\delta > 0$, such that,*

$$
P\{\omega \in \Omega : \sup_{x \leq (\frac{b}{a}\gamma-h)t} |\frac{\nabla u(t,x)}{u(t,x)}| \leq C_1\exp\{-C_2\sqrt{t} \text{ for all } t \geq T\} > 1 - \exp\{-\delta T\}, \quad (4.19)
$$

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and

$$P\{\omega \in \Omega : \int_t^\infty \sup_{x \leq (\frac{b}{a}\gamma - h)t} |\nabla u(s,x)| ds \leq C_1 \exp\{-C_2\sqrt{t}\} \text{ for all } t \geq T\}$$

$$> \quad 1 - \exp\{-\delta T\}, \tag{4.20}$$

for any $T > t_0$, and $h > 0$.

Proof. Recall $R = -D\nabla \log u$, y_s is defined by (4.2) and R satisfies (4.1). Similar to [28], applying Ito's formula, we can prove that $R(t-s, y_s) \exp\{\int_0^s c'(u(t-r,y_r))u(t-r,y_r)dr\}$ is a martingale with respect to $\hat{\mathcal{F}}_s$ on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_s, \hat{P})$ for $0 \leq s < t$. Then by martingale property we have for a.e. $\omega \in \Omega$,

$$R(t,x) = \hat{E} R\left(\frac{t}{2}, y_{\frac{t}{2}}\right) \exp\left\{\int_0^{\frac{t}{2}} c'(u(t-r,y_r))u(t-r,y_r)dr\right\}.$$

Note by integration by parts formula, Ito's formula and (4.1),

$$\int_0^{\frac{t}{2}} R(t-s, y_s) \exp\left\{\int_0^s c'(u(t-r,y_r))u(t-r,y_r)dr\right\} ds$$

$$= \quad \frac{t}{2} R\left(\frac{t}{2}, y_{\frac{t}{2}}\right) \exp\left\{\int_0^{\frac{t}{2}} c'(u(t-r,y_r))u(t-r,y_r)dr\right\}$$

$$- \int_0^{\frac{t}{2}} (s) \exp\left\{\int_0^s c'(u(t-r,y_r))u(t-r,y_r)\, dr\right\}$$

$$\times \left[-\frac{\partial}{\partial s} R(t-s, r_s) + \sqrt{D}\nabla R(t-s, y_s)dB_s - \nabla R(t-s, y_s)R(r-s, y_s)ds \right.$$

$$\left. + \frac{D}{2}\Delta R(t-s, y_s)ds + c'(u(t-s, y_s))u(t-s, y_s)ds \right]$$

$$= \quad \frac{t}{2} R\left(\frac{t}{2}, y_{\frac{t}{2}}\right) \exp\left\{\int_0^{\frac{t}{2}} c'(u(t-r,y_r))u(t-r,y_r)dr\right\}$$

$$- \sqrt{D} \int_0^{\frac{t}{2}} (s) \exp\left\{\int_0^s c'(u(t-r,y_r))u(t-r,y_r)\, dr\right\} \nabla R(t-s, y_s)dB_s.$$

It turns out that

$$R(t,x) \quad = \quad \frac{2}{t}\hat{E} \int_0^{\frac{t}{2}} R(t-s, y_s) \exp\left\{\int_0^s c'(u(t-r,y_r))u(t-r,y_r)dr\right\} ds$$

$$+ \frac{2\sqrt{D}}{t}\hat{E} \int_0^{\frac{t}{2}} (s)\nabla R(t-s, y_s) \exp\left\{\int_0^s c'(u(t-r,y_r))u(t-r,y_r)\, dr\right\} dB_s$$

$$= \quad \frac{2}{t}\hat{E} \int_0^{\frac{t}{2}} R(t-s, y_s) \exp\left\{\int_0^s c'(u(t-r,y_r))u(t-r,y_r)\, dr\right\} ds.$$

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Applying Cauchy-Schwarz inequality we have

$$|R(t,x)| \leq \frac{\sqrt{2}}{\sqrt{t}} \left(\hat{E} \int_0^{\frac{t}{2}} <R(t-s,y_s), R(t-s,y_s)> \right.$$

$$\left. \exp \left\{ 2 \int_0^s c'(u(t-r,y_r))u(t-r,y_r)\, dr \right\} ds \right)^{\frac{1}{2}}. \qquad (4.21)$$

Now we establish some preliminary estimate for $R(t,x)$. Let $J(t,x) = -D\log u(t,x)$, then we can check that J satisfies the following nonlinear stochastic Hamilton Jacobi Bellman equation,

$$dJ(t,x) + \frac{1}{2}|\nabla J(t,x)|^2 dt = \frac{D}{2}\Delta J(t,x)dt + c(u)dt - \frac{1}{2}k^2(t)dt + k(t)dW_t. \qquad (4.22)$$

Applying Itô's formula to $J(t-r,y_r)$ for $0 \leq r < t$ we have

$$
\begin{aligned}
dJ(t-r,y_r) = \quad & -\frac{\partial}{\partial r}J(t-r,y_r)\, dr + \nabla J(t-r,y_r)(\sqrt{D}dB_r - R(t-r,y_r)\, dr) \\
& +\frac{D}{2}\Delta J(t-r,y_r)\, dr \\
= \quad & \sqrt{D}\nabla J(t-r,y_r)\, dB_r - \frac{1}{2}\langle \nabla J(t-r,y_r), \nabla J(t-r,y_r)\rangle\, dr \\
& -c(u(t-r,y_r))\, dr + \frac{1}{2}k^2(t-r)dr - k(t-r)dW_{t-r}.
\end{aligned}
$$

For any integer N, let $\tau_N = \inf\{s \in [0,t] : |\nabla J(t-r,y_r)| \geq N\}$ and $\tau_N = \infty$ if $|\nabla J(t-r,y_r)| < N$ for any $0 \leq s \leq t$. Therefore

$$
\begin{aligned}
& J\left(t - \left(\frac{t}{2} \wedge \tau_N\right), y_{\frac{t}{2}\wedge\tau_N}\right) - J(t,x) \\
= \quad & \sqrt{D}\int_0^{\frac{t}{2}\wedge\tau_N} \nabla J(t-r,y_r)\, dB_r - \frac{1}{2}\int_0^{\frac{t}{2}\wedge\tau_N} \langle \nabla J(t-r,y_r), \nabla J(t-r,y_r)\rangle\, dr \\
& -\int_0^{\frac{t}{2}\wedge\tau_N} c(u(t-r,y_r))\, dr + \frac{1}{2}\int_0^{\frac{t}{2}\wedge\tau_N} k^2(t-r)dr - \int_0^{\frac{t}{2}\wedge\tau_N} k(t-r)dW_{t-r}.
\end{aligned}
$$

Taking expectations and noticing that $\int_0^{\frac{t}{2}\wedge\tau_N} \nabla J(t-r,y_r)\, dB_r$ is a martingale we have

$$
\begin{aligned}
& \frac{1}{2}\hat{E}\int_0^{\frac{t}{2}\wedge\tau_N} \langle \nabla J(t-r,y_r), \nabla J(t-r,y_r)\rangle\, dr \\
= \quad & J(t,x) - \hat{E}J\left(t - \left(\frac{t}{2} \wedge \tau_N\right), y_{\frac{t}{2}\wedge\tau_N}\right) \\
& -\hat{E}\int_0^{\frac{t}{2}\wedge\tau_N} c(u(t-r,y_r))\, dr + \frac{1}{2}\int_0^{\frac{t}{2}\wedge\tau_N} k^2(t-r)dr - \int_0^{\frac{t}{2}\wedge\tau_N} k(t-r)dW_{t-r}.
\end{aligned}
$$

Let $N \to \infty$ we obtain

$$\frac{1}{2}\hat{E} \int_0^{\frac{t}{2}} \langle \nabla J(t-r, y_r), \nabla J(t-r, y_r) \rangle \, dr$$

$$= \quad J(t,x) - \hat{E}J\left(\frac{t}{2}, y_{\frac{t}{2}}\right) - \hat{E}\int_0^{\frac{t}{2}} c(u(t-r, y_r)) \, dr$$

$$+ \frac{1}{2}\int_0^{\frac{t}{2}} k^2(t-r)dr - \int_0^{\frac{t}{2}} k(t-r)dW_{t-r}. \tag{4.23}$$

From Condition (iii), it is easy to see $-c(u) \le a(u-1) < au$. Recall (4.17) and (4.18). It is easy to see that there exist constants $\delta > 0$, $M > 0$ and $\Omega_T^1 \subset \Omega$ with $P(\Omega_T^1) > 1 - \exp\{-\delta T\}$ such that for $\omega \in \Omega_T^1$, and $t \ge T$, and $x \le \gamma t$,

$$\inf_{s \le s \le t} \int_0^s k(r)dW_r \quad \ge \quad -\frac{M}{8}t, \tag{4.24}$$

$$-\log v(t,x) \quad \le \quad \frac{M}{8}t, \tag{4.25}$$

$$\sup_{s \le s \le t} \int_0^s k(r)dW_r \quad \le \quad \frac{M}{4}t, \tag{4.26}$$

$$-\hat{E}\int_0^{\frac{t}{2}} c(u(t-r, y_r)) \, dr \quad \le \quad a\hat{E}\int_0^{\frac{t}{2}} u(t-r, y_r)dr$$

$$\le \quad a\int_0^{\frac{t}{2}} \sup_y u(t-r, y)dr$$

$$\le \quad \frac{M}{4}t, \tag{4.27}$$

$$\left| \int_0^{\frac{t}{2}} k(t-r)dW_{t-r} \right| \quad \le \quad \frac{M}{4}t. \tag{4.28}$$

It follows from (4.24-4.26), (4.16) and (3.3) that if $\omega \in \Omega_T^1$, and $t \ge T$, and $x \le \gamma t$,

$$J(t,x) \quad \le \quad \frac{M}{4}t, \tag{4.29}$$

$$-\hat{E}J(\frac{t}{2}, y_{\frac{t}{2}}) \quad \le \quad \frac{M}{4}t. \tag{4.30}$$

It turns out from (4.23), (4.27-4.30) that if $\omega \in \Omega_T^1$, and $t \ge T$, and $x \le \gamma t$, then

$$\frac{1}{2}\hat{E} \int_0^{\frac{t}{2}} \langle \nabla J(t-r, y_r), \nabla J(t-r, y_r) \rangle \, dr \le Mt.$$

Therefore by (4.21) we have if $\omega \in \Omega_T^1$, and $t \ge T$, and $x \le \gamma t$,

$$|R(t,x)| \le 2\sqrt{M}. \tag{4.31}$$

Now we are in a position to estimate $R(t,x)$ on the crest. Denote y_r the solution of (4.2) with $y_0 = x = (\frac{b}{a}\gamma - h)t$. Denote $\mathcal{K}_1 = \{(s,y) : y \le (\frac{b}{a}\gamma - \frac{1}{2}h)s\}$. Define for each $\omega \in \Omega_T^1$,

$$\tau = \inf\{s : (t-r, y_r) \notin \mathcal{K}_1\}.$$

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Then if $\tau \leq \sqrt{t}$, the process $(t - r, y_r)$ meets the line $y = (\frac{b}{a}\gamma - \frac{1}{2}h)s$ with $y_\tau = (\frac{b}{a}\gamma - \frac{1}{2}h)(t - \tau) \geq (\frac{b}{a}\gamma - \frac{1}{2}h)(t - \sqrt{t}) \geq (\frac{b}{a}\gamma - \frac{1}{4}h)t$ for $t > t_0$ where t_0 is a constant such that $(\frac{b}{a}\gamma - \frac{1}{2}h)\sqrt{t} < \frac{1}{4}ht$ for all $t > t_0$. Notice also (4.31), we have for $t > t_0$ (we require t_0 large enough such that $2\sqrt{Mt} \leq \frac{1}{8}ht$ for all $T > t_0$) and all $\omega \in \Omega_T^1$, and $t \geq T$,

$$
\begin{aligned}
& \hat{P}\{\tau \leq \sqrt{t}\} \\
\leq\ & \hat{P}\{\max_{0 \leq r \leq \sqrt{t}} y_r \geq \frac{1}{4}ht\} \\
=\ & \hat{P}\{\max_{0 \leq r \leq \sqrt{t}}\{\sqrt{D}B_r - \int_0^r R(t - s, y_s)ds\} \geq \frac{1}{4}ht\} \\
\leq\ & \hat{P}\{\max_{0 \leq r \leq \sqrt{t}} \sqrt{D}B_r \geq \frac{1}{8}ht\} \\
\leq\ & \exp\{-\frac{h^2}{128D}t^{\frac{3}{2}}\}.
\end{aligned}
$$

Therefore we have for $T > t_0$ and each $\omega \in \Omega_T^1$, $t \geq T$,

$$
\begin{aligned}
R(t, x) =\ & \hat{E}R(t - \tau \wedge \frac{t}{2}, y_{\tau \wedge \frac{t}{2}}) \exp\{\int_0^{\tau \wedge \frac{t}{2}} c'(u(t - r, y_r))u(t - r, y_r)dr\} \\
=\ & \hat{E}\chi_{\tau \leq \sqrt{t}} R(t - \tau, y_\tau) \exp\{\int_0^\tau c'(u(t - r, y_r))u(t - r, y_r)dr\} \\
& + \hat{E}\chi_{\sqrt{t} < \tau \leq \frac{t}{2}} R(t - \tau, y_\tau) \exp\{\int_0^\tau c'(u(t - r, y_r))u(t - r, y_r)dr\} \\
& + \hat{E}\chi_{\tau > \frac{t}{2}} R(\frac{t}{2}, y_{\frac{t}{2}}) \exp\{\int_0^{\frac{t}{2}} c'(u(t - r, y_r))u(t - r, y_r)dr\}.
\end{aligned}
$$

It turns out using (4.31) that for $T > t_0$ and each $\omega \in \Omega_T^1$, $t \geq T$,

$$
\begin{aligned}
|R(t, x)| \leq\ & \sqrt{2M}\hat{P}(\tau \leq \sqrt{t})) \\
& + \sqrt{2M}\hat{E}\chi_{\sqrt{t} < \tau \leq \frac{t}{2}} \exp\{-d_0\sqrt{t}(\frac{1}{\sqrt{t}} \int_0^{\sqrt{t}} u(t - r, y_r)dr)\} \\
& + \sqrt{2M}\hat{E}\chi_{\tau > \frac{t}{2}} \exp\{-\frac{d_0 t}{2}(\frac{1}{\frac{t}{2}} \int_0^{\frac{t}{2}} u(t - r, y_r)dr)\}.
\end{aligned}
$$

Recall Theorem 3.4, then for $T > t_0$ and each $\omega \in \Omega_T^1$, $t \geq T$,

$$
\begin{aligned}
|R(t, x)| \leq\ & \sqrt{2M}\exp\{-\frac{h^2}{128D}t^{\frac{3}{2}}\} \\
& + \sqrt{2M}\hat{E}\chi_{\sqrt{t} < \tau \leq \frac{t}{2}} \exp\{-d_0\sqrt{t}(\frac{1}{\sqrt{t}} \int_0^{\sqrt{t}} \inf_{y \leq (\frac{b}{a}\gamma - \frac{1}{2}h)(t - r)} u(t - r, y)dr)\} \\
& + \sqrt{2M}\hat{E}\chi_{\tau > \frac{t}{2}} \exp\{-d_0\frac{t}{2}(\frac{1}{\frac{t}{2}} \int_0^{\frac{t}{2}} \inf_{y \leq (\frac{b}{a}\gamma - \frac{1}{2}h)(t - r)} u(t - r, y)dr)\}
\end{aligned}
$$

$$\leq \quad \sqrt{2M}\exp\{-\frac{\delta^2}{128}t^{\frac{3}{2}}\}$$
$$+\sqrt{2M}\exp\{-d_0\frac{1}{2a}(c(0)-\frac{k^2}{2})\sqrt{t}\}$$
$$+\sqrt{2M}\exp\{-d_0\frac{1}{2a}(c(0)-\frac{k^2}{2})\frac{t}{2}\}$$
$$\leq \quad C_1\exp\{-C_2\sqrt{t}\},$$

for constants $C_1 > 0$ and $C_2 > 0$. That is to say for each $\omega \in \Omega_T^1$ and $T^* > t \geq T > 0$,

$$\int_t^{T^*} \sup_{x \leq (\frac{b}{a}\gamma-h)s} |\nabla u(s,x)|ds$$

$$\leq \quad \int_t^{T^*} C_1\exp\{-C_2\sqrt{s}\} \sup_{x \leq (\frac{b}{a}\gamma-h)s} u(s,x)ds$$

$$\leq \quad C_1\exp\{-\frac{1}{2}C_2\sqrt{t}\}\int_t^{T^*}\exp\{-\frac{1}{2}C_2\sqrt{s}\}d(\int_t^s \sup_{x \leq (\frac{b}{a}\gamma-h)r} u(r,x)dr)$$

$$\leq \quad C_1\exp\{-\frac{1}{2}C_2\sqrt{t}\}\exp\{-\frac{1}{2}C_2\sqrt{T^*}\}\int_t^T \sup_{x \leq (\frac{b}{a}\gamma-h)r} u(r,x)dr$$

$$+\frac{1}{2}C_1C_2\exp\{-\frac{1}{2}C_2\sqrt{t}\}\int_t^{T^*}\frac{1}{\sqrt{s}}\exp\{-\frac{1}{2}C_2\sqrt{s}\}\int_t^s \sup_{x \leq (\frac{b}{a}\gamma-h)r} u(r,x)drds$$

$$\leq \quad C_3\exp\{-\frac{1}{2}C_2\sqrt{t}\}T^*\exp\{-\frac{1}{2}C_2\sqrt{T^*}\}+C_3\exp\{-C_2\sqrt{T^*}\},$$

for a constant $C_3 > 0$. Therefore taking the limit $T^* \to +\infty$,

$$\int_t^{\infty} \sup_{x \leq (\frac{b}{a}\gamma-h)s} |\nabla u(s,x)|ds \leq 2C_3\exp\{-C_2\sqrt{t}\}.$$

$\square$

Remark 4.3 *One can improve* $\exp\{-C_2\sqrt{t}\}$ *in (4.20) to* $\exp\{-C_2t^r\}$ *for any* $0 < r < 1$.

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