Two Remarks on Monge-Ampere Equations (*).

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Summary. – *We consider real Monge-Ampère equations and we present two new properties of these equations. First, we show the existence of the « first eigenvalue of Monge-Ampère equation » i.e. we show the existence of a positive constant possessing all the properties of the first eigenvalue of a 2-nd order elliptic operator (positivity, uniqueness of the eigenfunction, maximum principle, bifurcation ...). The second property concerns variational characterizations of solutions. Both properties are closely related to similar properties of the general class of Hamilton-Jacobi-Bellman equations.*

1. – Introduction.

In this paper, we make two remarks on solutions of real Monge-Ampère equations. Let Ω be a smooth, bounded strongly convex domain in $\boldsymbol{R}^N$ ($N \geqslant 2$) i.e.

$$\exists w \in C^2(\bar{\Omega}), \quad w = 0 \text{ on } \partial\Omega, \quad (D^2 w) > 0 \text{ in } \bar{\Omega}. \tag{1}$$

We consider solutions of

$$\det(D^2 u) = F(x, u, Du) \text{ in } \Omega, \quad u \text{ convex on } \bar{\Omega}, \quad u = \varphi \text{ on } \partial\Omega \tag{2}$$

where F, φ are smooth and $F \geqslant 0$. Existence and regularity results may be found in A. V. POGORELOV [18], [19], [20]; S. Y. CHENG and S. T. YAU [5], [6]; P. L. LIONS [16]; L. CAFFARELLI, L. NIRENBERG and J. SPRUCK [4]; N. V. KRYLOV [11].

Our first observation concerns the existence of a positive constant (depending only on Ω) denoted by λ_1 satisfying the following properties:

i) there exists $\psi_1 \in C^{1,1}(\bar{\Omega}) \cap C^\infty(\Omega)$ such that $\psi_1 < 0$ in Ω and

$$\det(D^2 \psi_1) = (-\lambda_1 \psi_1)^N, \quad \psi_1 \text{ convex on } \bar{\Omega}, \quad \psi_1 = 0 \text{ on } \partial\Omega; \tag{3}$$

ii) if $(\psi, \mu) \in \left(C^{1,1}(\bar{\Omega}) \cap C^\infty(\Omega)\right) \times]0, \infty[$ satisfies (3) with (ψ_1, λ_1) replaced by (ψ, μ) then $\mu = \lambda_1$, $\psi = \theta\psi_1$ for some positive constant θ;

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iii) λ_1 is given by: $\lambda_1 = \mathrm{Inf}\,(\lambda_1^a \colon a \in V)$ where

$$V = \left\{ a = (a_{ij}(x)) = (a_{ij}(x)) \in C(\bar{\Omega}),\ a_{ij} > 0 \text{ in } \bar{\Omega},\ \det a \geqslant \frac{1}{N^N} \right\}$$

and λ_1^a is the first eigenvalue of the linear second-order elliptic operator $-a_{ij}\,\partial_{ij}$ with Dirichlet boundary conditions.

Furthermore, iii) yields easily uniqueness (and existence) results for equations of the following form

$$\det (D^2 u)^{1/N} = H(x, u) \quad \text{in } \Omega\,, \qquad u = \psi \quad \text{on } \partial\Omega\,, \qquad u \text{ convex on } \bar{\Omega} \tag{4}$$

provided $\partial H/\partial t > -\lambda_1$ on $\bar{\Omega} \times \boldsymbol{R}$. Notice that the standard uniqueness results for (4) assume $\partial H/\partial t \geqslant 0$.

Finally, λ_1 acts as a bifurcation point for equations like (4).

All these features of λ_1 suggest the well-known properties of first eigenvalues of linear second-order elliptic operators or more generally of positive operators as given by KREIN-RUTMAN's theorem [10]. This is why, we will call λ_1 the first eigenvalue of the Monge-Ampère operator.

The proof of the above properties is given in section II. In fact, the above could be seen as some application of the general results in P. L. LIONS [13] concerning demi-eigenvalues of Hamilton-Jacobi-Bellman equations (recall that Monge-Ampère equations are very special cases of Hamilton-Jacobi-Bellman equations, cf. [14], [11]). And the proof given in section II follows the corresponding one in [13]. At this stage, it is maybe worth mentionning that these two cases may be included in a single abstract statement concerning an analogue of Krein-Rutman's theorem for nonlinear operators on a cone (positive or negative functions for instance), homogeneous of degree 1: in this abstract setting, the operator with which we deal here is $\det (D^2 u)^{1/N}$. However since we do not know other applications of such an abstract result than the ones in [13] or in this note, we will skip it. Notice also that the existence part may be obtained by applying conveniently some fixed point results due to M. A. KRASNOSELSKII [9] (for instance). However, we will prefer a more constructive proof. Finally, we would like to mention the related work by C. PUCCI [21] on «extremal» eigenvalues of linear elliptic operators.

Section III is devoted to applications to equations like (4) (uniqueness, bifurcation ...) and to various properties and extensions of λ_1 (stochastic interpretation, stability with respect to Ω, extensions to quasilinear H ...).

In the last section (section IV) we deal with some variational properties of (weak) solutions of (2): assume that F is nondecreasing with respect to u for all $(x, p) \in \bar{\Omega} \times \boldsymbol{R}^N$, then we introduce the set

$$S = \{v \in C(\bar{\Omega}),\ v = \varphi \text{ on } \partial\Omega,\ v \text{ convex on } \bar{\Omega},\ \det (D^2 v) \geqslant F(x, v, Dv) \text{ in } \Omega\}$$

here the inequality is understood in Alexandrov sense (cf. [20] and [5], this only means that the normal measure i.e. the image measure by the subdifferential ∂v

of the Lebesgue measure is more than the L^1_{loc} function $F(x, v, Dv)$). Next if Φ is *any* convex function on $\mathbf{R}^N$, and if J is defined on $W^{1,1}(\Omega)$ by:

$$J(w) = \int_\Omega \Phi(\nabla w)\, dx \leqslant +\infty \tag{5}$$

then the solution u of (2) satisfies:

$$J(u) = \operatorname*{Min}_{v\in S} J(v)\,. \tag{6}$$

This property becomes quite striking if we take $F \equiv 0$ so that u is the convex envelope of φ and thus S becomes the set of continuous, convex functions on $\bar{\Omega}$, equal to φ on $\partial\Omega$. In this case (6) yields that u minimizes *all* convex functionals of ∇w over this set. A similar observation for one particular functional may be found in BEDFORD and TAYLOR [2].

Let us finally mention that using the results of L. CAFFARELLI, J. J. KOHN, L. NIRENBERG and J. SPRUCK [3], all the results of this paper extend (without any changes) to the case of complex Monge-Ampère equations.

2. – The first eigenvalue of Monge-Ampère operators.

Let $a = (a_{ij}(x))_{ij} \in C(\Omega)$ be such that:

$$(a_{ij}) = (a_{ji}) > 0 \quad \text{in } \Omega\,, \qquad \det(a_{ij}(x)) \geqslant \frac{1}{N^N}$$

and let us denote by V the (convex) set of such matrix valued functions. If $a \in V$, the linear operator $-a_{ij}(x)\,\partial_{ij}$ with Dirichlet boundary conditions admits a positive first eigenvalue λ_1^a and the corresponding eigenfunction is positive in Ω and is unique up to multiplicative constants: if $a_{ij} \in C(\bar{\Omega})$ and $(a_{ij}) > 0$ in $\bar{\Omega}$ this result is well-known while the slight extension to a general $a \in V$ is given in the appendix. We then introduce

$$\lambda_1 = \inf_{a\in V} \lambda_1^a\,. \tag{7}$$

THEOREM 1. – i) The infimum in (7) is achieved and $\lambda_1 > 0$.

ii) There exists $\psi_1 \in C^{1,1}(\bar{\Omega}) \cap C^\infty(\Omega)$ solution of:

$$\det(D^2\psi_1) = (-\lambda_1\psi_1)^N \text{ in } \Omega\,, \quad \psi_1 \text{ convex on } \bar{\Omega}\,, \tag{3}$$
$$\psi_1 = 0 \text{ on } \partial\Omega\,, \quad \psi_1 < 0 \text{ in } \Omega\,.$$

iii) Let $(\mu, \psi) \in]0, \infty[\times C^{1,1}(\bar{\Omega})$ satisfy:

$$\det(D^2\psi) = (-\mu\psi)^N \quad \text{a.e. in } \Omega, \quad \psi \text{ convex on } \bar{\Omega}, \quad \psi = 0 \text{ on } \partial\Omega, \quad \psi \not\equiv 0 \tag{8}$$

then $\mu = \lambda_1$ and $\psi = \theta\psi_1$ for some positive constant θ.

REMARKS. – We will see in the next section further properties of λ_1: let us only mention that λ_1 is also given by the infimum of λ_1^a over a satisfying:

$$a_{ij} = a_{ji} \in C(\bar{\Omega}), \quad a_{ij} > 0 \text{ in } \bar{\Omega}, \quad \det(a_{ij}) \geqslant \frac{1}{N^N}. \quad ■$$

PROOF OF THEOREM 1. – We introduce another nonnegative constant (possibly infinite)

$$\mu_1 = \sup\{\lambda \geqslant 0 : \exists u_\lambda \in C^2(\bar{\Omega}) \text{ solution of (9)}\}$$

where equation (9) is given by

$$\det(D^2 u_\lambda) = (1 - \lambda u_\lambda)^N \quad \text{in } \Omega, \quad u_\lambda \text{ convex on } \bar{\Omega}, \quad u_\lambda = 0 \text{ on } \partial\Omega. \tag{9}$$

In view of [4], u_0 exists. We are going to prove that μ_1 is finite, that as $\lambda \to \mu_1$, $u_\lambda \|u_\lambda\|_\infty^{-1}$ converges to a solution of (3) (with λ_1 replaced by μ_1) and then that $\lambda_1 = \mu_1$ and i), iii) hold.

To see that μ_1 is finite, one just remarks that by the algebraic formula (see B. GAVEAU [7])

$$(\det A)^{1/N} = \inf\left\{\operatorname{Tr}(AB)/B = B^T,\ \det B \geqslant \frac{1}{N^N},\ B > 0\right\} \tag{10}$$

we deduce that if u_λ solution of (9) exists then:

$$\frac{1}{N}\Delta u_\lambda \geqslant 1 - \lambda u_\lambda, \quad u_\lambda < 0 \text{ on } \partial\Omega, \quad u_\lambda = 0 \text{ on } \partial\Omega$$

and this implies that λ is less than the first eigenvalue of $(-(1/N)\Delta)$. A similar proof shows that $\mu_1 \leqslant \lambda_1$.

Next, we claim that $\mu_1 > 0$ and that for any $\lambda \in [0, \mu_1[$ there exists a solution u_λ of (9). Indeed, if $\lambda < \|u_0\|_\infty^{-1}$ and if $C = (1 - \lambda\|u_0\|_\infty)^{-1}$ then $\bar{u} = Cu_0$ satisfies: $\bar{u} \in C^2(\bar{\Omega})$ and

$$\det(D^2\bar{u}) = C^N \geqslant (1 - \lambda\bar{u})^N \quad \text{in } \Omega, \quad \bar{u} \text{ convex in } \bar{\Omega}, \quad \bar{u} = 0 \quad \text{on } \partial\Omega.$$

In other words $\bar{u}$ is a subsolution for problem (9) and thus by the results of P. L. LIONS [16], L. CAFFARELLI, L. NIRENBERG and J. SPRUCK [4] there exists a solution u_λ of (9) for $0 < \lambda < \|u_0\|_\infty^{-1}$. Next if $\lambda < \mu_1$ there exists $\lambda' > \lambda$ such that (9) admits a solution $u_{\lambda'}$ for λ': clearly $u_{\lambda'}$ is a subsolution for problem (9) and this yields the existence of u_λ as before. In fact, as we will see later on, u_λ is unique.

The next step consists in showing that $\|u_\lambda\|_\infty$ goes to $+\infty$ as $\lambda \to \mu_1$. Indeed if $\|u_\lambda\|_\infty$ (which is, by the way, increasing in λ) remains bounded, then the a priori estimates in [4] apply and we obtain:

$$\|u_\lambda\|_{C^k(\bar{\Omega})} \leqslant C \ (\text{ind}^t \text{ of } \lambda) \quad \text{for any } k \geqslant 1 .$$

Thus, this would yield the existence of u_{μ_1} solution of (9) for $\lambda = \mu_1$. And again, choosing $\delta < \|u_{\mu_1}\|_\infty$ and $C = (1 - \delta\|u_{\mu_1}\|_\infty)^{-1}$ we find that $\bar{u} = Cu_{\mu_1} \in C^2(\bar{\Omega})$ satisfies:

$$\det(D^2\bar{u}) = C^N(1 - \mu_1 u_{\mu_1})^N \geqslant (1 - (\mu_1 + \delta)\bar{u})^N \quad \text{in } \Omega ,$$
$$\bar{u} \text{ convex on } \bar{\Omega}, \quad \bar{u} = 0 \text{ on } \partial\Omega .$$

This would in turn yield the existence of u_λ solution of (9) for $\mu_1 < \lambda \leqslant \mu_1 + \delta$ and this contradicts the very definition of μ_1. Hence, $\|u_\lambda\|_\infty \to +\infty$ as $\lambda \to \mu_1$.

Next, we consider $v_\lambda = u_\lambda \|u_\lambda\|_\infty^{-1}$ for $\lambda < \mu_1$. Clearly v_λ solves:

$$(11) \qquad \det(D^2 v_\lambda) = (\|u_\lambda\|_\infty^{-1} - \lambda v_\lambda)^N \text{ in } \Omega, \quad v_\lambda \in C^2(\bar{\Omega}),$$
$$v_\lambda \text{ convex on } \bar{\Omega}, \quad v_\lambda = 0 \text{ on } \partial\Omega .$$

since v_λ is bounded, using the defining function w, one shows easily that

$$0 \geqslant v_\lambda \geqslant -Cw \text{ in } \bar{\Omega}, \text{ for some } C \text{ independent of } \lambda \in [0, \mu_1[.$$

And v_λ being convex, this implies $C^{0,1}(\bar{\Omega})$ bounds on v_λ. Next one observes that $C^{1,1}(\bar{\Omega})$ bounds in [4] do not use the strict positivity of the right-hand side of Monge-Ampère equations (see for example P. L. LIONS [15] for an explicit use of this fact). And we obtain bounds on v_λ in $C^{1,1}(\bar{\Omega})$ independent of $\lambda \in [0, \mu_1[$. Clearly, by the normalization, v_λ (or subsequences) does not converge uniformly on $\bar{\Omega}$ to 0 and thus by the convexity of v_λ, we find that

$$\|u_\lambda\|_\infty^{-1} - \lambda v_\lambda \geqslant \delta \quad \text{on } K \text{ compact} \subset \Omega$$

for some positive constant δ independent of $\lambda \in [0, \mu_1[$. This yields local bounds in $C^k(\Omega)$ for all $k \geqslant 1$ by standard estimates on Monge-Ampère equations. We may pass to the limit as λ goes to μ_1 (extracting subsequences if necessary) and we find that v_λ converges to some $\psi_1 \in C^{1,1}(\bar{\Omega}) \cap C^\infty(\Omega)$ solution of:

$$\det(D^2\psi_1) = (-\mu_1\psi_1)^N \text{ in } \Omega, \quad \psi_1 \text{ convex on } \bar{\Omega}, \quad \psi_1 = 0 \text{ on } \partial\Omega, \quad \psi_1 < 0 \text{ in } \Omega .$$

And if we set $\bar{a} = (1/N)(D^2\psi_1)^{-1}\det(D^2\psi_1)^{1/N}$, $\bar{a} \in V$ and

$$-\bar{a}_{ij}\,\partial_{ij}\psi_1 = +\mu_1\psi_1 \text{ in } \Omega\,, \quad \psi_1 \in C^{1,1}(\bar{\Omega})\,, \quad \psi_1 < 0 \text{ in } \Omega\,, \quad \psi_1 = 0 \text{ on } \partial\Omega\,.$$

therefore, μ_1 is the first eigenvalue of $-\bar{a}_{ij}\partial_{ij}$ i.e.

$$\mu_1 = \lambda_1^{\bar{a}} = \operatorname*{Min}_{a\in V} \lambda_1^a = \lambda_1\,.$$

There just remains to prove part iii) of Theorem 1. Let μ, ψ satisfy (2). Observing first that if μ were strictly less than λ_1, the above argument would yield the existence of some $\tilde{a} \in V$ such that

$$\mu = \lambda_1^{\tilde{a}} < \lambda_1\,,$$

and thus $\mu \geqslant \lambda_1$. Next, ψ_1 being convex (and $\not\equiv 0$) we have:

$$\frac{\partial\psi_1}{\partial n} > 0 \text{ on } \partial\Omega\,, \qquad \text{where } n \text{ is the unit outward normal to } \partial\Omega$$

and thus for $t > 0$ small we have: $0 \leqslant t(-\psi) \leqslant -\psi_1$ in $\bar{\Omega}$. And let $t_0 = \max\big(t > 0,$ $t(-\psi) \leqslant -\psi_1$ in $\bar{\Omega}\big)$. Using formula (10), it is easy to find $a \in V$ such that:

$$-a_{ij}\,\partial_{ij}(t_0\psi - \psi_1) = \mu t_0\psi - \lambda_1\psi_1 \leqslant \lambda_1(t_0\psi - \psi_1) \leqslant \lambda_1^a(t_0\psi - \psi_1) \quad \text{in } \Omega$$

with $t_0\psi - \psi_1 \in C^{1,1}(\bar{\Omega})$, $t_0\psi - \psi_1 \geqslant 0$ in $\bar{\Omega}$, $t_0\psi - \psi_1 = 0$ on $\partial\Omega$.

Thus either $t_0\psi - \psi_1 \equiv 0$ and we conclude, or $t_0\psi - \psi_1 = \theta\varphi_1$ for some $\theta > 0$, where φ_1 is the positive eigenfunction of the operator $(-a_{ij}\partial_{ij})$. In the latter case, using the observations of the appendix,

$$\varphi_1(x) \geqslant \delta \operatorname{dist}(x, \partial\Omega)\,, \qquad \text{for some } \delta > 0$$

and thus $\theta\varphi_1 \geqslant \varepsilon(-\psi)$ on $\bar{\Omega}$ for some $\varepsilon > 0$. But, this yields

$$(-\psi_1) \geqslant (t_0 + \varepsilon)(-\psi) \qquad \text{on } \bar{\Omega}$$

contradicting the definition of t_0. This shows that $\psi_1 \equiv t_0\psi_1$ and thus $\mu = \lambda_1$. ■

REMARK. – The uniqueness argument (as in P. L. LIONS [13]) is an adaptation of the method of T. LAETSCH [12] (see also KRASNOSELSKII [9]). ■

3. – Further properties and variants.

The first result we mention concerns equations of the form (4). We will consider a smooth function $H(x, t)$ satisfying:

(12) $$H(x, t) > 0 \text{ for } (x, t) \in \Omega \times \boldsymbol{R}; \qquad \frac{\partial H}{\partial t} \geqslant -\lambda_0 > -\lambda_1 \text{ on } \bar{\Omega} \times \boldsymbol{R};$$

and we wish to solve:

$$\det(D^2u)^{1/N} = H(x,u) \text{ in } \Omega\,, \quad u \text{ convex on } \bar{\Omega}\,, \quad u = 0 \text{ on } \partial\Omega\,. \tag{13}$$

COROLLARY 2. – We assume (12). Then there exists a unique solution of (13) in $C^{1,1}(\bar{\Omega}) \cap C^2(\Omega)$.

REMARKS. – i) The above result is only an example of applications of Theorem 1. Much more general results hold for weak solutions (in Alexandrov's sense), for general nonnegative H and for arbitrary Dirichlet boundary conditions.

ii) An analogous result holds for equations like:

$$\begin{cases} \det\big(D^2u + g_{ij}(x)\big) = H(x,u)^N \text{ in } \Omega\,, & (\partial_{ij}u + g_{ij}) \geqslant 0 \quad \text{in } \Omega \\ u = \varphi \quad \text{on } \partial\Omega \end{cases} \tag{14}$$

where $g_{ij} = g_{ji}$ (see [4] for a treatment of these equations). ■

PROOF OF COROLLARY 2. – To show the existence part, we just need to build a subsolution. But, choosing $\lambda \in]\lambda_0, \lambda_1[$, we have for $C > 0$

$$\det(D^2Cu_\lambda)^{1/N} = C(1 - \lambda u_\lambda) \geqslant \|H(x,0)\|_\infty - C\lambda_0 u_\lambda \geqslant H(x, Cu_\lambda) \quad \text{in } \Omega$$

provided $C \geqslant \|H(x,0)\|_\infty$.

The uniqueness is also straightforward since if u, v are two solutions one may find $a \in V$ such that

$$-a_{ij}\partial_{ij}(u - v) + H(x,u) - H(x,v) = 0 \text{ in } \Omega\,, \quad u - v \in C^{1,1}(\bar{\Omega})\,,$$

$$u - v = 0 \text{ on } \partial\Omega\,.$$

Since

$$H(x,u) - H(x,v) = \int_0^1 \frac{\partial H}{\partial t}\big(x, v + s(u - v)\big)\, ds (u - v) \quad \text{and} \quad \frac{\partial H}{\partial t} \geqslant -\lambda_0\,,$$

we easily reach a contradiction by comparing eigenvalues. ■

At this stage, let us mention without proof a few properties of λ_1, φ_1. First of all, the uniqueness characterizations imply easy stability results of λ_1, φ_1 (normalizing φ_1 if necessary) with respect to variations of the domain (preserving convexity of course). Next, we approximate Monge-Ampère equations by Hamilton-Jacobi-Bellman equations as in [16] that is we consider the operators

$$\mathcal{A}^\varepsilon\varphi = \inf\left\{-a_{ij}\,\partial_{ij}\varphi / (a_{ij}) = (a_{ji}) > 0,\ \det(a_{ij}) \geqslant \frac{1}{N^N},\ \operatorname{Tr} a \leqslant \frac{1}{\varepsilon}\right\}.$$

The results of P. L. LIONS [13] show the existence of two demi-eigenvalues of $\mathcal{A}^\varepsilon$ namely $\underline{\lambda}_1^\varepsilon$, $\bar{\lambda}_1^\varepsilon$ corresponding, respectively, to negative and positive eigenfunctions φ_1^ε, Φ_1^ε. Then, one can prove that as ε goes to 0, $\underline{\lambda}_1^\varepsilon$ goes to λ_1, $\bar{\lambda}_1^\varepsilon$ goes to $+\infty$ and φ_1^ε (normalized so that $\|\varphi_1^\varepsilon\|_\infty = 1$) converges to φ_1 the solution of (3) with $\|\varphi_1\|_\infty = 1$.

Finally, λ_1 has an interesting stochastic interpretation: let B_t be a standard Brownian motion on some given probability space and let

$$\mathcal{K} = \left\{\sigma_t \text{ bounded progressively measurable, } \sigma_t \text{ is a } N\times N \text{ matrix, } |\det \sigma_t| \geqslant \left(\frac{2}{N}\right)^{N/2} \text{a.s.}\right\}.$$

For $\sigma_t \in \mathcal{K}$, we denote by $X_t = x + \int_0^t \sigma_s \, dB_s$ $(x \in \bar{\Omega})$.

Then we have for any fixed x_0

$$\begin{aligned}\lambda_1 &= \sup\left\{\lambda > 0 \colon \sup_{x\in\bar{\Omega}} \inf_{\sigma_t} E[\exp[\lambda \tau_x]] < +\infty\right\} \\ &= \sup\left\{\lambda > 0 \colon \inf_{\sigma_t} E[\exp[\lambda \tau_{x_0}]] < +\infty\right\}\end{aligned}$$

where $\tau_x = \inf(t \geqslant 0, X_t \notin \bar{\Omega})$.

The last property of λ_1 that we wish to mention concerns bifurcation properties: to this end, we will just give a very particular example of nonlinearities. Let $f \in C^\infty(\boldsymbol{R})$ satisfy:

$$(15)\qquad \begin{cases} f(0) = 0, \ f'(0) = 1, \ f(t) > 0 \text{ for } t > 0, \quad 0 < \dfrac{f(s)}{s} < \dfrac{f(t)}{t} \text{ if } 0 \leqslant t < s \\ \text{and} \\ \lim\limits_{t\to+\infty} f(t)/t = 0 . \end{cases}$$

We will consider solutions of:

$$(16)\quad \det (D^2 u)^{1/N} = \lambda f(-u), \quad u \text{ convex}, \quad u \in C^{1,1}(\bar{\Omega}) \cap C^\infty(\Omega), \quad u|_{\partial\Omega} = 0 .$$

We have the

PROPOSITION 3. – If $\lambda \leqslant \lambda_1$, the unique solution of (16) is $u \equiv 0$. For $\lambda > \lambda_1$ there exists a unique solution $\underline{u}_\lambda \not\equiv 0$. In addition $\underline{u}_\lambda$ is continuous, decreasing with respect to λ and $\underline{u}_\lambda$ converges to 0 as λ goes to λ_1. ■

PROOF. – The case $\lambda \leqslant \lambda_1$ may be deduced from Theorem 1 and Corollary 2. The properties on $\underline{u}_\lambda$ are easy consequences of the uniqueness; and uniqueness may be proved by another application of Laetsch argument. Therefore, we will only prove the existence of $\underline{u}_\lambda$ and we do so by building a nontrivial supersolution of (16). Because of (15), we have for ε small enough

$$\det\left(D^2(\varepsilon\psi_1)\right)^{1/N} = -\varepsilon\lambda_1\psi_1 \leqslant \lambda f(-\varepsilon\psi_1) \quad \text{in } \Omega$$

if $\lambda > \lambda_1$. On the other hand we have for some $C_0 \geqslant 0$

$$\lambda f(t) \leqslant \frac{\lambda_1}{2} t + C_0 \quad \text{for all } t \geqslant 0$$

and thus for $C \geqslant C_0$, C large enough the solution of

$$\det(D^2 \underline{u}) = C - \frac{\lambda_1}{2}\underline{u} \quad \text{in } \Omega\,, \quad \underline{u} \in C^2(\bar{\Omega})\,, \quad \underline{u} \text{ convex in } \bar{\Omega}\,, \quad \underline{u} = 0 \text{ on } \partial\Omega$$

satisfies $\underline{u} \leqslant \varepsilon\psi_1$ in $\bar{\Omega}$ and

$$\det(D^2 \underline{u})^{1/N} \geqslant \lambda f(-\underline{u}) \quad \text{in } \Omega\,.$$

This yields the existence of $\underline{u}_\lambda$ solution of (16) satisfying: $\underline{u} \leqslant \underline{u}_\lambda \leqslant \varepsilon\psi_1$ in $\bar{\Omega}$ ∎

We conclude this section by a variant of Theorem 1: it concerns equations of the following form:

$$(17) \qquad \left(\det(D^2\psi_1)\right)^{1/N} = H(x, D\psi_1) - \lambda_1 K(x)\psi_1\,, \quad \psi_1 \text{ convex}\,, \quad \psi_1 < 0 \quad \text{on } \Omega\,, \quad \psi_1 = 0 \quad \text{on } \partial\Omega\,.$$

We will assume that K is smooth, $K > 0$ in $\bar{\Omega}$, $H(x, p)$ is smooth on $\bar{\Omega} \times (\mathbf{R}^N - \{0\})$, H is smooth in x (two derivatives bounded for example) uniformly for p bounded in $\mathbf{R}^N$, $\partial H/\partial x$ is Lipschitz in p uniformly in x and that

$$(18) \qquad \begin{cases} H(x, \mu p) = \mu H(x, p) > 0,\ \forall \mu > 0,\ \forall x \in \bar{\Omega},\ \forall p \in \mathbf{R}^N,\ |p| = 1; \\ H \text{ is convex in } p \text{ for } x \in \bar{\Omega}; \quad \left(\dfrac{\partial H}{\partial p}(x, n(x)), n(x)\right) > 0 \quad \text{on } \partial\Omega\,. \end{cases}$$

A typical example is: $K \equiv 1$, $H(x, p) = H(x)|p|$ with $H > 0$ in $\bar{\Omega}$.

The same proof as in Theorem 1 gives the:

COROLLARY 4. – Assume (18). Then there exists $\lambda_1 > 0$ such that there exists $\psi_1 \in C^{1,1}(\bar{\Omega}) \cap C^2(\Omega)$ solution of (17). In addition, if $(\mu, \psi) \in]0, \infty[\times C^{1,1}(\bar{\Omega})$ solves

$$\det(D^2\psi) = H(x, D\psi) - \mu K(x)\psi \quad \text{in } \Omega\,, \quad \psi \text{ convex}\,, \quad \psi = 0 \text{ on } \partial\Omega\,, \quad \psi \not\equiv 0$$

then $\mu = \lambda_1$ and $\psi = \theta\psi_1$ for some positive constant θ. Finally, if $K \equiv 1$ and if we denote by $C_x = \{q \in \mathbf{R}^N, \sup_{p\in\mathbf{R}^N} \{q\cdot p - H(x, p)\} < \infty\}$, then we have

$$\lambda_1 = \text{Inf}\,\{\lambda_1(a, b): a \in V,\ b \in W,\ a \in C(\bar{\Omega}),\ a > 0 \text{ in } \bar{\Omega}\}$$

where

$$W = \{b \in L^\infty(\Omega; \mathbf{R}^N),\ b(x) \in C_x \text{ a.e. in } \Omega\}$$

and $\lambda_1(a, b)$ is the first eigenvalue of the operator $(-a_{ij}\partial_{ij} + b_i\partial_i)$. ■

REMARKS. – i) If $K \equiv 1$ and $H(x, p) = H(x)|p|$ then $C_x = \{q \in \mathbf{R}^N: |q| \leqslant H(x)\}$ and $W = \{b \in L^\infty(\Omega; \mathbf{R}^N),\ |b(x)| \leqslant H(x) \text{ a.e.}\}$.

ii) Of course, the analogue of Corollary 2 holds in the above setting. This, in particular, yields uniqueness results for solutions of

$$\det\,(D^2u)^{1/N} = F(x, u, Du) \quad \text{in } \Omega\,, \quad u \text{ convex}\,, \quad u = \psi \text{ on } \partial\Omega$$

provided

$$\left|\frac{\partial F}{\partial p}\right| \leqslant H(x)\,, \qquad \frac{\partial F}{\partial t} > -\lambda_1 K(x)$$

where λ_1 corresponds to the choice $H(x, p) = H(x)|p|$. ■

4. – Variational properties of solutions of Monge-Ampère equations.

Assume $u \in C^{0,1}(\bar{\Omega})$ is a weak solution of (2) where $F(x, t, p)$ is continuous, nonnegative, nondecreasing in t, convex in (t, p) and $\varphi \in C^{0,1}(\partial\Omega)$. We then denote by $\mathcal{S}$ the set of subsolutions of (2) i.e.

$$\mathcal{S} = \{v \in C(\bar{\Omega}),\ v = \varphi \text{ on } \partial\Omega,\ \det\,(D^2v) \geqslant F(x, v, Dv) \text{ in } \Omega,\ v \text{ convex on } \Omega\}$$

then (cf. [1], [5], [16]) u is the maximum element of the convex set $\mathcal{S}$. We claim that *for any convex function Φ on $\mathbf{R}^N$, nonnegative, u is the minimum over $\mathcal{S}$ of*

$$J_\Phi(v) = \int_\Omega \Phi(\nabla v)\,dx\,.$$

Indeed, by some easy density argument, we may assume without loss of generality that Φ satisfies: $\Phi \in C^2(\mathbf{R}^N)$;

$$|\Phi'(p)| \leqslant C + C|p|; \quad \exists \nu > 0\,, \quad (\Phi'(p) - \Phi'(q), p - q) \geqslant \nu|p - q|^2\,, \quad \forall p, q\,.$$

Then by standard arguments J_Φ admits a minimum over the closed convex set $\mathcal{S}$ or equivalently over $\mathcal{S} \cap H^1(\Omega)$. Let us denote by u_0 the minimum point: u_0 is the unique solution in $\mathcal{S} \cap H^1(\Omega)$ of the following variational inequality

$$\int_\Omega (\Phi'(\nabla u_0), \nabla v - \nabla u_0)\, dx \geqslant 0 , \quad \forall v \in \mathcal{S} \cap H^1(\Omega) . \tag{19}$$

Therefore we just need to show that u is the solution of (19). Recalling that u is the maximum element of $\mathcal{S}$, we just have to prove that if v is convex, belongs to (say) $C^{0,1}(\bar{\Omega})$ and if $\varphi \in H_0^1(\Omega)$, $\varphi \geqslant 0$

$$\int_\Omega (\Phi'(\nabla v), \nabla \varphi)\, dx \leqslant 0 .$$

Formally this is obtained by integrating by parts and observing that

$$\operatorname{div}[\Phi'(\nabla v)] = \frac{\partial^2 \Phi}{\partial p_i\, \partial p_j} \partial_{ij} v \geqslant 0 .$$

This is easily justified by choosing $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geqslant 0$ and by regularizing u: set $u_\varepsilon = u * \varrho_\varepsilon u_\varepsilon = u * \varrho_\varepsilon$ on $\{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ where $\varrho_\varepsilon = (1/\varepsilon^N)\, \varrho(\cdot/\varepsilon)$, $\varrho \in \mathcal{D}_+(\boldsymbol{R}^N)$, $\operatorname{Supp} \varrho \subset B_1$, $\int \varrho\, dx = 1$.

Indeed for ε small enough (so that $\operatorname{Supp} \varphi \subset \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$)

$$\int_\Omega (\Phi'(\nabla u_\varepsilon), \nabla \varphi)\, dx \leqslant 0$$

since u_ε is also convex, and we conclude passing to the limit. ∎

Observing that if $F \equiv 0$, $\mathcal{S} = \{v \in C(\bar{\Omega}),\ v \text{ convex on } \Omega,\ v = \varphi \text{ on } \partial\Omega\}$ and u is the convex enveloppe of φ on $\bar{\Omega}$, we deduce that u is the minimum over $\mathcal{S}$ of all functionals J_Φ for any convex, nonnegative function Φ on $\boldsymbol{R}^N$. A similar result holds for the convex enveloppe of a given function φ degined on $\bar{\Omega}$ where $\mathcal{S}$ is now given by

$$\mathcal{S} = \{v \in C(\bar{\Omega}),\ v \text{ convex on } \bar{\Omega},\ v \leqslant \varphi \text{ on } \bar{\Omega}\} .$$

Appendix. Hopf boundary lemma and first eigenvalues.

Let $a \in V$ and denote by $A = -a_{ij}\partial_{ij}$. By standard arguments one proves that for any $f \in L^N(\Omega)$, there exists a unique solution $u \in W_{\mathrm{loc}}^{2,N}(\Omega) \cap C(\bar{\Omega})$ of

$$Au = f \quad \text{in } \Omega , \qquad u = 0 \quad \text{on } \partial\Omega . \tag{A.1}$$

Furthermore using the defining function w of Ω, it is quite clear that if $f \in L^\infty(\Omega)$ then u satisfies:

$$|u(x)| \leqslant C \operatorname{dist}(x, \partial\Omega) \quad \text{on } \bar{\Omega}. \tag{A.2}$$

Furthermore if $f \geqslant 0$, then $u \geqslant 0$ and either $u \equiv f \equiv 0$, or $u > 0$ in Ω. Finally, we have the:

PROPOSITION A.1. – With the above notations, if $f \not\equiv 0$, $f \geqslant 0$, then u satisfies for some $\delta > 0$:

$$u(x) \geqslant \delta \operatorname{dist}(x, \partial\Omega). \tag{A.3}$$

This is of course a formulation of the well-known Hopf boundary lemma which is proved by the standard method (see [8], [17]). Using this observation, one may prove the

PROPOSITION A.2. – There exists a constant $\lambda_1 > 0$ such that:

i) If $\lambda \in]0, \lambda_1[$, there exists a unique solution of

$$Au - \lambda u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad u \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$$

where $f \in L^N(\Omega)$. In addition if $f \geqslant 0$, $u \geqslant 0$.

ii) If $u \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ satisfies: $u \geqslant 0$, $u \not\equiv 0$ and

$$Au - \lambda u \geqslant 0 \quad (\text{resp. } \leqslant 0) \text{ a.e. in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

then $\lambda \leqslant \lambda_1$ (resp. $\lambda \geqslant \lambda_1$). And if $\lambda = \lambda_1$, $Au \equiv \lambda u$ in Ω.

iii) There exists $\psi_1 \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$ $(\forall p < \infty)$ solution of:

$$A\psi_1 = \lambda_1 \psi_1 \text{ in } \Omega, \qquad \psi_1 > 0 \text{ in } \Omega, \qquad \psi_1 = 0 \text{ on } \partial\Omega.$$

iv) If $\tilde{\psi} \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\bar{\Omega})$, $\lambda \in \mathbf{R}$ satisfy:

$$A\tilde{\psi} = \lambda\tilde{\psi} \quad \text{in } \Omega, \qquad \tilde{\psi} \geqslant 0 \quad \text{in } \Omega, \qquad \tilde{\psi} \not\equiv 0, \ \tilde{\psi} = 0 \quad \text{on } \partial\Omega$$

then $\lambda = \lambda_1$ and $\tilde{\psi} = \theta\psi_1$ for some positive constant $\theta > 0$. ∎

Similar results hold if we replace A by

$$B = -a_{ij}\partial_{ij} + b_i\partial_i$$

where $a \in V$, $b \in L^\infty(\Omega; \mathbf{R}^N)$, b is continuous near $\partial\Omega$ and

$$\big(b(x), n(x)\big) > 0 \quad \text{on } \partial\Omega. \tag{A.4}$$

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