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Two-Segment Separable Programming — Source link

R. R. Meyer

Institutions: University of Wisconsin-Madison

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ABSTRACT

New iterative separable programming techniques based on two-segment, piecewise-linear approximations are described for the minimization of convex separable functions over convex sets. These techniques have two advantages over traditional separable programming methods. The first is that they do not require the cumbersome "fine grid" approximations employed to achieve high accuracy in the usual separable programming approach. In addition, the new methods yield feasible solutions with objective values guaranteed to be within any specified tolerance of optimality. In computational tests with real-world problems of up to 500 "nonlinear" variables the approach has exhibited rapid convergence and yielded very close bounds on the optimal value.

1. Introduction

In [4,5] techniques are described for the solution of problems of the class

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{s.t.} & x \in C,\ \ell \leq x \leq u, \end{array} \tag{1.1}$$

under the assumptions that $x = (x_1,\ldots,x_n)^T$, that $f(x) = \sum_{i=1}^{n} f_i(x_i)$, that f_i is convex on $[\ell_i,u_i]$ for $i = 1,\ldots,n$, and that $C = \{x \mid Ax=b,\ x \in Z^n\}$, where A is totally unimodular† and Z^n is the set of integer n-vectors (methods that do not require bounds on the x_i are also described there, but in this paper we shall consider only the bounded case). This paper describes appropriate modifications and generalizations of those techniques for the case in which C is a general closed convex set. At each iteration the techniques to be described employ a two-segment, convex, piecewise-linear approximation for each objective function term and perform an optimization over a "corresponding" portion of the feasible set. This approach thus has the advantage of not requiring the traditional [7] but cumbersome "fine grid" separable programming approximation over the full range of each variable. It also provides tight upper and lower bounds on the optimal value by extending error bound concepts of Geoffrion [3], Meyer [4,5], and Thakur [8]. In computational experience with real-world problems of up to 500 "nonlinear" variables the approach has exhibited rapid convergence and yielded very close bounds on the optimal value.

† A matrix is said to be totally unimodular if the determinant of each of its square submatrices has value 0 or ± 1. Coefficient matrices of single commodity networks typically have this property (see Dantzig [2]).

In the next section we consider the relationship between the optimal values of the approximating problems and the given problem (1.1). The most general results obtained depend only on convexity, and thus point the way for extensions of the techniques to the non-separable case, where the local nature of the approximations employed is even more critical in keeping down problem size. Section 3 contains an algorithm for the case in which the $C = \{x|Ax=b\}$, where A is totally unimodular. The results there constitute an extension of the ideas of [4,5] to continuous, "convex" networks. Section 4 deals with the general case in which C is merely assumed to be closed and convex, and develops two finitely convergent algorithms: the first for integer approximations, and the second for arbitrary finite approximations. Computational aspects and numerical results are presented in Section 5, and Conclusions are discussed in Section 6.

2. Approximations and Error Analysis

For notational convenience in the ensuing sections, we denote the set $\{x|\ell\leq x\leq u\}$ by $[\ell,u]$ and the feasible set $C\cap[\ell,u]$ by S. To simplify the discussion we will assume $S \neq \phi$ and $\ell < u$. (If S is empty, then this fact will be determined in the course of attempting to solve the first approximating problem, which is a problem with S as its feasible set.) We will assume that f is finite and convex on $[\ell,u]$. The algorithms do not require that f be continuous on $[\ell,u]$, but we will assume that (1.1) has an optimal solution.

The iterative procedures to be described for (1.1) involve the replacement of the objective function by approximations f^k and the minimization of these approximations over subsets of S of the form $S\cap[\ell^k,u^k]$ where $[\ell^k,u^k]$ is an n-dimensional interval. These subsets are generated in such a way that, if x^* is the optimal solution obtained in the final iteration (the algorithms are finite) and the final problem solved was, say,

$$\begin{aligned} &\min && \tilde{f}(x) \\ &\text{s.t.} && x\in S\cap[\ell^*,u^*], \end{aligned}$$

where $\tilde{f}$ was the final approximation to f, and $\{\ell^*,u^*\}$, the final set of "additional" bounds, then the additional bounds turn out to be non-restrictive in the sense that x^* is also an optimal solution of

$$\begin{aligned} &\min && \tilde{f}(x) \\ &\text{s.t.} && x\in S. \end{aligned} \tag{2.1}$$

(As might be expected, convexity plays an important role in this result.)

More will be said about the nature of these approximations after the relationship of the optimal value of (2.1) to that of (1.1) is discussed. At this stage we assume only that $\tilde{f}$ lies <u>above</u> f in the ("small") set $[\ell^*,u^*]$ and below f outside of $[\ell^*,u^*]$. The former property implies that the optimal value of (2.1) is an <u>upper</u> bound on the optimal value of (1.1). Lemma 2.1 below leads to a <u>lower</u> bound determined by the behavior of $\tilde{f} - f$ in $[\ell^*,u^*]$ alone.

Let $\tilde{g}$ be an <u>approximation</u> to f such that the problem $\min \tilde{g}$ s.t. $x \in S$ has an optimal solution. (The nature of the approximation $\tilde{g}$ is otherwise irrelevant.) The function $e(x)$, assumed to be defined on S, is a bound on the <u>error</u> between f and $\tilde{g}$ in the sense that $f(x) + e(x) \geq \tilde{g}(x)$. For purposes of an error estimate we need only consider a quantity $\hat{e} \geq \sup e(x)$. (Note that in the following Lemma and its Corollary 2.2, convexity and separability of f are <u>not</u> required or assumed. This approach thus provides the basis for extending these ideas to the non-separable case.)

<u>Lemma 2.1</u>: If x^* is an optimal solution of $\min \tilde{g}(x)$ s.t. $x \in S$ and x^{**} is an optimal solution of $\min f(x)$ s.t. $x \in S$, then $f(x^{**}) \geq \tilde{g}(x^*) - \hat{e}$.

<u>Proof</u>: By applying the definitions above, we have $f(x^{**}) + \hat{e} \geq f(x^{**}) + e(x^{**}) \geq \tilde{g}(x^{**}) \geq \tilde{g}(x^*)$. ■

Note that this result could be improved by replacing $\hat{e}$ by $\sup e(x)$ s.t. $x \in S$, but the latter quantity may be much more difficult to compute. Similar observations hold for the Corollaries below.

The next Corollary follows directly from the Lemma and establishes the validity of obtaining an error bound by considering the difference between $\tilde{g}$ and f only in the (hopefully small) region in which $f(x) \leq \tilde{g}(x)$. The set $\hat{S}$ denotes any set such $S \subseteq \hat{S} \subseteq [\ell,u]$.

Corollary 2.2: Let the hypotheses of Lemma 2.1 hold, and let $L \equiv \{x | x \in \hat{S},\ f(x) \leq \tilde{g}(x)\}$. If $L \neq \phi$ and $\tilde{e}(x)$ is such that $f(x) + \tilde{e}(x) \geq \tilde{g}(x)$ for $x \in L$, then $f(x^{**}) \geq \tilde{g}(x^*) - e^*$, where $e^* \equiv \sup \tilde{e}(x)$ s.t. $x \in L$.

Proof: Define $e(x) \equiv \tilde{e}(x)$ for $x \in L$ and $e(x) \equiv 0$ for $x \in \hat{S} \backslash L$, and note that because $L \neq \phi$ and $\tilde{e}(x)$ is non-negative, $\sup e(x)$ s.t. $x \in \hat{S} = \sup \tilde{e}(x)$ s.t. $x \in L$. ∎

Our final result of this type concerns the case in which $f(x) = \sum_{i=1}^{p} g_i(x)$, $\tilde{g}(x) = \sum_{i=1}^{p} \tilde{g}_i(x)$, and an upper bound is available for each of the differences $\tilde{g}_i(x) - g_i(x)$. Note that this includes the separable case as well as the semi-separable case in which each g_i depends on only a small number of variables.

Corollary 2.3: Let the hypotheses of Lemma 2.1 hold, and assume that there exists a subset L of S such that, for $i = 1,\ldots,p$, $g_i(x) \leq \tilde{g}_i(x)$ if and only if $x \in L$. If $L \neq \phi$ and, for $i = 1,\ldots,p$, $\tilde{e}_i(x)$ is such that $g_i(x) + \tilde{e}_i(x) \geq \tilde{g}_i(x)$ for $x \in L$, then $f(x^{**}) \geq \tilde{g}(x^*) - \sum_{i=1}^{p} e_i^*$, where $e_i^* = \sup \tilde{e}_i(x)$ s.t. $x \in L$.

Proof: By summing the inequalities involving $\tilde{e}_i(x)$, we have $f(x) + \sum_{i=1}^{p} \tilde{e}_i(x) \geq \tilde{g}(x)$ for $x \in L$. The result then follows by letting $e(x) = \sum_{i=1}^{p} \tilde{e}_i(x)$ for $x \in L$, $e(x) = 0$ for $x \in \hat{S} \setminus L$, and noting that $\sum_{i=1}^{p} e_i^* \geq \sup e(x)$. ■

In the separable case, the types of approximations with which we will be dealing in the following sections are sums of those (two-segment or one-segment) convex piecewise-linear approximations to the f_i that coincide with the f_i on sets of the form $\{\bar{\ell}_i, \bar{m}_i, \bar{u}_i\}$, where $\ell_i \leq \bar{\ell}_i \leq \bar{m}_i \leq \bar{u}_i \leq u_i$ and $\bar{\ell}_i < \bar{u}_i$. (See Figure 1). Note that the triple $\{\bar{\ell}_i, \bar{m}_i, \bar{u}_i\}$ uniquely determines the piecewise-linear approximation. The point $\bar{m}_i$ is allowed to coincide with $\bar{\ell}_i$ or $\bar{u}_i$, but for most purposes it is convenient to consider only the case $\bar{\ell}_i < \bar{m}_i < \bar{u}_i$, for which the corresponding approximation $\bar{f}_i$ will generally be a two-segment piecewise-linear function. It is easily shown that $\bar{f}_i \geq f_i$ on $[\bar{\ell}_i, \bar{u}_i]$ and $\bar{f}_i \leq f_i$ "outside" of $[\bar{\ell}_i, \bar{u}_i]$ because f_i is convex. The functions $\bar{f}_i$ are, of course, continuous even if f is not, and the approximating problems in which $\sum_{i=1}^{n} f_i$ is replaced by $\sum_{i=1}^{n} \bar{f}_i = \bar{f}$ thus always have optimal solutions. For ease of description, the sets $\{\bar{\ell}_i, \bar{m}_i, \bar{u}_i\}$ will be said to be the breakpoints of $\bar{f}_i$ and the corresponding vectors $\{\bar{\ell}, \bar{m}, \bar{u}\}$ will be said to be the breakpoints of $\bar{f}$. Conversely, $\bar{f}$ will be said to be the piecewise-linear approximation determined by the breakpoints $\{\bar{\ell}, \bar{m}, \bar{u}\}$.

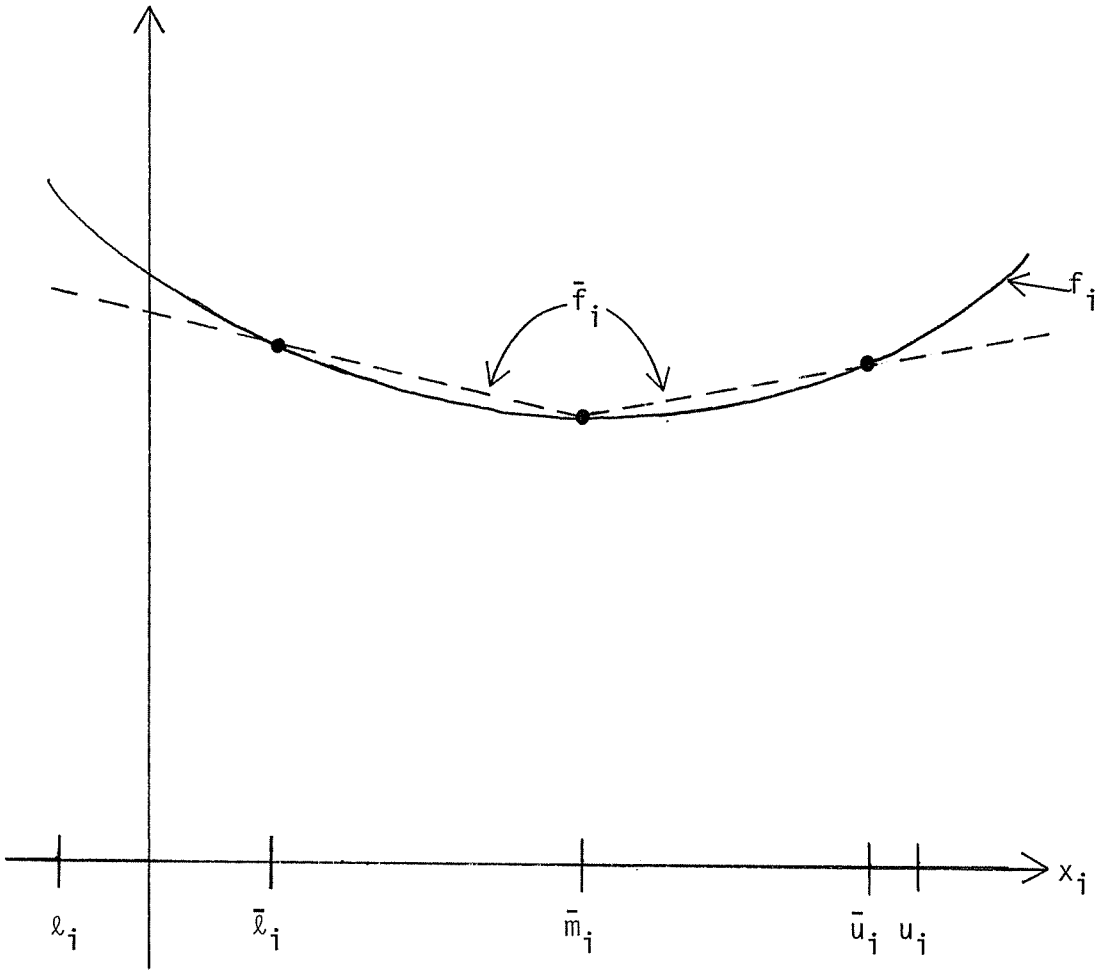

Figure 1: f_i and $\bar{f}_i$.

Note that in this context Corollary 2.3 applies with $g_i(x) = f_i(x_i)$, $\tilde{g}_i(x) = \bar{f}_i(x_i)$, and $L = [\bar{\ell},\bar{u}]$, yielding the following:

Corollary 2.4: If $\bar{x}\in[\bar{\ell},\bar{u}]$ is an optimal solution of

$$\begin{aligned} &\min && \bar{f}(x) \\ &\text{s.t.} && x\in S\cap[\ell,u], \end{aligned}$$

where $\bar{f}(x) = \sum_{i=1}^{n} \bar{f}_i(x_i)$ and, for $i = 1,\ldots,n$, $\bar{f}_i$ is a convex piecewise-linear function determined by f_i and the breakpoint set $[\bar{\ell}_i,\bar{m}_i,\bar{u}_i]$, then the following bounds hold for the optimal value $f(x^{**})$ of (1.1):

$$\bar{f}(\bar{x}) - \sum_{i=1}^{n} \bar{e}_i \leq f(x^{**}) \leq f(\bar{x}) \leq \bar{f}(\bar{x}),$$

where $\bar{e}_i \geq \sup_{\bar{\ell}_i\leq x_i\leq\bar{u}_i} (\bar{f}_i(x_i) - f_i(x_i))$.

3. The Totally Unimodular Case

In this section we will assume that $C = \{x|Ax=b\}$, where A is totally unimodular, and that ℓ and $u \in Z^n$. The algorithm for the case of a general convex feasible set is similar but slightly more complex, and is described in Section 4.

At each iteration: (1) each of the functions f_i is replaced by a convex piecewise-linear approximation of at most two segments, and (2) additional bounds on the variables are added to the original constraints. More precisely, the piecewise-linear approximations will be determined by breakpoint sets $\bar{\ell}$, $\bar{m}$, $\bar{u}$ with the following properties ($\bar{\ell}$ will be the additional lower bounds on x and $\bar{u}$, the additional upper bounds):

(3.1) $\bar{\ell}, \bar{m}, \bar{u} \in Z^n$,

(3.2) $\ell \leq \bar{\ell} \leq \bar{m} \leq \bar{u} \leq u$ and $\bar{\ell} < \bar{u}$,

(3.3) $\bar{\ell}_i < \bar{m}_i$ if $\ell_i < \bar{m}_i$ and $\bar{m}_i < \bar{u}_i$ if $\bar{m}_i < u_i$,

(3.4) $\bar{m}_i = \bar{\ell}_i + 1 = \bar{u}_i - 1$ if $\bar{\ell}_i + 2 = \bar{u}_i$,

(3.5) $S \cap [\bar{\ell}, \bar{u}] \neq \phi$

(Hypotheses (3.1), (3.2), and (3.5) merely serve to rule out trivial or uninteresting cases; hypotheses (3.3) and (3.4) are needed in order to allow certain conclusions about optimal solutions.) A triple $\{\bar{\ell}, \bar{m}, \bar{u}\}$ satisfying (3.1) - (3.5) will be said to be admissible. The admissible breakpoint triple employed in the k^{th} iteration will be denoted by $\{\ell^k, m^k, u^k\}$. It determines the

k^{th} approximating problem (denoted by P^k) in the sense that $f_i^k(x_i)$ is the piecewise-linear approximation determined by $\{\ell_i^k, m_i^k, u_i^k\}$, $f^k(x) \equiv \sum_{i=1}^{n} f_i^k(x_i)$, and the feasible set for P^k is taken to be $S \cap [\ell^k, u^k]$, so that

$$P^k \equiv \begin{array}{ll} \min\limits_{x} & f^k(x) \\ \text{s.t.} & x \in S^k \end{array}$$

Note that since $S^k \neq \phi$ by (3.5), P^k will have an optimal solution in Z^n that may be determined by solving a single LP. Note also for future reference that, given any problem of the form (1.1), there are only a finite number of distinct problems of the form P^k, where $\{\ell^k, m^k, u^k\}$ is admissible.

In order to establish a result relating solutions of problems with restricted ranges to (1.1), we will introduce some additional terminology. Given an $\hat{x} \in [\bar{\ell}, \bar{u}]$, where $\ell \leq \bar{\ell} \leq \bar{u} \leq u$, the bounds $\bar{\ell}$, $\bar{u}$ will be said to be non-restrictive (relative to $[\ell, u]$) at $\hat{x}$ if, for any $\tilde{x} \in [\ell, u]$, $\lambda\tilde{x} + (1-\lambda)\hat{x} \in [\bar{\ell}, \bar{u}]$ for sufficiently small positive λ. Note that if $\{\bar{\ell}, \bar{m}, \bar{u}\}$ is an admissible breakpoint triple, $\bar{\ell}$, $\bar{u}$ are non-restrictive at $\bar{m}$ because of (3.2) and (3.3). On the other hand, if, for some j, either $\ell_j < \bar{\ell}_j = \hat{x}_j$ or $\hat{x}_j = \bar{u}_j < u_j$, $\hat{x}$ will be said to be artifically bounded by the pair $\{\bar{\ell}, \bar{u}\}$. If a function g_1 is convex on an interval $[\ell', u'] \subseteq [\ell, u]$, g_2 will be said to be a convex extension of g_1 to $[\ell, u]$ if g_2 is convex on $[\ell, u]$ and $g_1(x) = g_2(x)$ for $x \in [\ell', u']$.

Lemma 3.1: If x' is an optimal solution of the problem

$$\begin{aligned} &\min_{x} && g_1(x) \\ &\text{s.t.} && x \in C \cap [\ell', u'], \end{aligned} \tag{3.6}$$

where g_1 is convex on the convex set C, and if ℓ', u' are non-restrictive at x', then x' is also an optimal solution of

$$\begin{aligned} &\min_{x} && g_2(x) \\ &\text{s.t.} && x \in C \cap [\ell, u], \end{aligned} \tag{3.7}$$

where g_2 is any convex extension of g_1 to $[\ell, u]$.

Proof: Suppose that x' is not optimal for (3.7). Let $\tilde{x}$ be any element of $C \cap [\ell, u]$ such that $g_2(\tilde{x}) < g_2(x') = g_1(x')$. Since x' is not artificially bounded, $\lambda\tilde{x} + (1-\lambda)x' \in C \cap [\ell', u']$ for sufficiently small positive λ. However, for such λ, $g_1(\lambda\tilde{x} + (1-\lambda)x') = g_2(\lambda\tilde{x} + (1-\lambda)x') < g_2(x') = g_1(x')$, a contradiction. ∎

Thus, identifying g with the piecewise-linear approximation determined by an admissible breakpoint triple $\{\ell', m', u'\}$, it follows that if m' solves the corresponding approximating problem, then m' is also optimal for any convex extension of g to $[\ell, u]$. We will see that the algorithm to be described terminates with such a breakpoint triple.

We will now describe a basic algorithm and then prove its finite convergence to an optimal solution of an integer approximation of

(1.1). Specific computational techniques that satisfy the assumptions of the basic algorithm are described in Section 5.

Algorithm 1 (A assumed totally unimodular)

Step 0: (Initialization): Set $\ell^0 = \ell$, $u^0 = u$, and $m^0 = [\frac{1}{2}(\ell+u)]$ (this is the vector whose i^{th} component is the greatest integer equal to or less than $\frac{1}{2}(\ell_i+u_i)$. Let x^0 be an integer optimal solution of P^0. Set $k = 0$.

Step 1: If $k = 0$ or if $k \geq 1$ and $x^k \neq x^{k-1}$, set $m^{k+1} = x^k$ and choose ℓ^{k+1} and u^{k+1} as vectors satisfying (3.1)-(3.4), and go to Step 3.

Step 2: Set $m^{k+1} = x^k$ and derive ℓ^{k+1} and u^{k+1} from ℓ^k and u^k by increasing at least one lower bound and/or decreasing at least one upper bound while maintaining (3.1)-(3.4).

Step 3: If x^k solves P^{k+1} and $\ell^{k+1} + 2e \geq u^{k+1}$, where $e = (1,1,\ldots,1)^T$ terminate. If x^k solves P^{k+1} and the latter inequality is not satisfied, set $x^{k+1} = x^k$, increase k by 1, and go to Step 2.

Step 4: If x^k does not solve P^{k+1}, let x^{k+1} be any integer optimum of P^{k+1}. Increase k by 1 and go to Step 1.

Theorem 3.1: Let $f_i^Z (i=1,\ldots,n)$ be the piecewise-linear approximation to f_i determined by the integers in $[\ell_i,u_i]$ and let $f^Z \equiv \sum_{i=1}^{n} f_i^Z$. Algorithm 1 will terminate in a finite number of iterations with an optimal solution of the problem:

$$(3.8)\qquad \begin{array}{ll} \min\limits_{x} & f^Z(x) \\ \text{s.t.} & Ax = b,\ \ell \leq x \leq u. \end{array}$$

Proof: If the algorithm terminates, then the optimality conditions of the preceding Lemma are satisfied by the last iterate because of (3.1)-(3.4), so it is optimal for the convex extension f^Z. Finite termination will be established by verifying that no approximating problem is ever repeated by Algorithm 1. Note that, because of convexity, $x \in [\ell^k, u^k]$ implies $f^k(x) \geq f(x)$, but, since, $m^k = x^{k-1}$ for $k \geq 1$, $f^k(x^{k-1}) = f(x^k)$. If $x^k \neq x^{k-1}$ and $k \geq 1$, then $v^k \equiv f^k(x^k) < f^k(x^{k-1}) = f^k(m^k) = f(m^k) \leq f^{k-1}(x^{k-1}) = v^{k-1}$. If $x^k = x^{k-1}$, an analogous argument shows $v^k \leq v^{k-1}$, so that an approximating problem may be repeated only if it is repeated for two indices in a set of iteration indices $\{r, r+1, \ldots, r+s\}$ such that $x^r = x^{r+1} = \ldots = x^{r+s}$. However, Step 2 of the algorithm guarantees that the corresponding approximating problems are all distinct. ∎

The error analysis of Section 2 now allows us to derive bounds on the optimal value of the original problem (1.1).

Theorem 3.2: If x^* is an optimal solution of (3.8) obtained via Algorithm 1, then $f(x^*) - \sum_{i=1}^{n} e_i^Z \leq f(x^{**}) \leq f(x^*)$, where x^{**} is an optimal solution of (1.1), $e_i^Z \geq \sup_{\ell_i^* \leq x_i \leq u_i^*} (f_i^Z(x_i) - f_i(x_i))$, and ℓ^*, u^* are the bounds employed in the last iteration.

(Note that, with an appropriate scaling of the variables, e_i^Z can be made arbitrarily small, since $u_i^* - \ell_i^* \leq 2$.)

4. The General Case

In this section we consider the case in which the feasible set is only assumed to be of the form $C \cap [\ell,u]$, where C is a convex set. The initial algorithm to be considered below, Algorithm 2, is similar to Algorithm 1 in that it employs approximations determined by integer breakpoints. Algorithm 3 then follows as a straightforward extension that allows breakpoints to be selected from arbitrary finite sets. As in the totally unimodular case, at each iteration each f_i is approximated by a piecewise-linear convex function of at most two segments and the feasible set is correspondingly restricted. The notation used to define approximating problems is the same as that of the preceding section.

Algorithm 2 (Integer breakpoints)

Step 0 Set $\ell^0 = \ell$, $u^0 = u$, and $m^0 = [\frac{1}{2}(\ell+u)]$. Let x^0 be an optimal solution of P^0. Set $k = 0$.

Step 1: If $k = 0$ or if $k \geq 1$ and $x^k \neq x^{k-1}$, then set $m^{k+1} = [x^k]$ and choose ℓ^{k+1} and u^{k+1} as integer vectors satisfying $x^k \leq u^{k+1}$ and (3.1)-(3.4), and go to Step 3.

Step 2: Set $m^{k+1} = [x^k]$ and derive ℓ^{k+1} and u^{k+1} from ℓ^k and u^k by increasing at least one lower bound and/or decreasing at least one upper bound while maintaining (3.1)-(3.4) and $x^k \leq u^{k+1}$.

Step 3: If x^k solves P^{k+1} and $\ell^{k+1} + 2e \geq u^{k+1}$, terminate.

Step 4: If x^k solves P^{k+1} and $x^k \in Z^n$, set $x^{k+1} = x^k$ and $v^{k+1} = f(x^k)$, increase k by 1 and go to Step 2. Otherwise, let y^* be any optimal solution of P^{k+1} and let v^* be the optimal value of P^{k+1}. If $v^* < v^k$, set $x^{k+1} = y^*$ and $v^{k+1} = v^*$, increase k by 1, and return to Step 1.

Step 5: If $v^* \geq v^k$, then modify ℓ^{k+1} and/or u^{k+1} by increasing at least one lower bound and/or decreasing at least one upper bound while maintaining $x^k \leq u^{k+1}$ and (3.1)-(3.4) (note that k is unchanged). Return to Step 3.

Since Algorithm 2 deals with problems determined by integer breakpoints, only a finite number of distinct problems of that form exist. As with Algorithm 1, we will show that Algorithm 2 cannot consider the same problem twice. In this regard we will first demonstrate that the algorithm cannot cycle infinitely often between Steps 4 and 5. Note that in Step 5 only the lower and upper bounds of P^{k+1} are modified, since m^{k+1} is maintained at $[x^k]$. Clearly, because of the contraction of the range in Step 5, after a finite number of applications of Step 5 additional contraction of the range will not be possible and $\ell^{k+1} + 2e \geq u^{k+1}$ will hold. When this is indeed the case, we will show that the complete cycle cannot again be repeated. Thus, we will assume that none of the conditions for the alternatives to repeating Step 5 are satisfied, and show that this leads to a contradiction. That is, suppose that no further contraction of the ranges is possible, that x^k does not solve P^{k+1}, and that $v^* \geq v^k$. Because of the

impossibility of contracting the ranges, it must be the case that $f_i^{k+1}(x_i^k) \leq f_i^k(x_i^k)$ for all i. (See Appendix). However, this means that $v^* = f^{k+1}(y^*) < f^{k+1}(x^k) \leq f^k(x^k) = v^k$, a contradiction.

Theorem 4.1: Let $f_i^Z (i=1,\ldots,n)$ be the piecewise-linear approximation to f_i determined by the integers in $[\ell_i, u_i]$, and let $f^Z = \sum_{i=1}^{n} f_i^Z$. Algorithm 2 will terminate in a finite number of iterations with an optimal solution of

$$\begin{aligned} &\min_{x} && f^Z(x) \\ &\text{s.t.} && x \in C \cap [\ell, u]. \end{aligned} \tag{4.1}$$

Proof: If termination occurs, then by Lemma 3.1 an optimal solution of (4.1) has been obtained, so we need only show that termination occurs in a finite number of iterations. Note that, in Step 4, if $v^* < v^k$, then $v^{k+1} < v^k$, and if $x^{k+1} = x^k \in Z^n$, then $v^{k+1} = f(x^k) \leq f^k(x^k) = v^k$, so we need only show that $v^{k+1} = v^k$ can only occur for a finite number of successive iterations. However, this follows by an argument analogous to that used in the TU case, since repetition of the optimal value in this way forces a contraction of the ranges in Step 2, and this contraction can occur only finitely often. ∎

Letting x^* be the optimal solution of (4.1) determined by Algorithm 2, an error bound analogous to that of Theorem 3.2 can now be stated:

$$f^Z(x^*) - \sum_{i=1}^{n} e_i^Z \leq f(x^{**}) \leq f(x^*).$$

(The lower bound can be improved in this case - see Reference [6] for a disscussion of this point.)

The techniques in Algorithm 2 are easily extended to the case in which the breakpoints for x_i are chosen from an arbitrary finite set T_i that contains ℓ_i and u_i. (From a computational viewpoint the most interesting case is perhaps the one in which each T_i is chosen as the set of numbers representable on a given (finite precision) computer.) If $T \in \mathbb{R}^n$ is the product of such sets $T_1, \ldots, T_n$, the hypotheses (3.1)-(3.5) are replaced by

(4.2) $\bar{\ell}, \bar{m}, \bar{u} \in T$

(4.3) $\ell \leq \bar{\ell} \leq \bar{m} \leq \bar{u} \leq u$ and $\bar{\ell} < \bar{u}$

(4.4) $\ell_i \leq \bar{m}_i$ if $\ell_i < m_i$ and $\bar{u}_i > \bar{m}_i$ if $u_i > \bar{m}_i$

(4.5) if $(\bar{\ell}_i, \bar{u}_i) \cap T_i$ consists of exactly one point α, then $\bar{m}_i = \alpha$

(4.6) $S \cap [\bar{\ell}, \bar{u}] \neq \phi$

For notational convenience we will define for any $\omega \geq \ell_i$ the function

$$[\omega]_{T_i} \equiv \max_{\alpha} \ \alpha$$

$$\text{s.t.} \quad \alpha \leq \omega, \ \alpha \in T_i,$$

and, for an n-vector $v \geq \ell$, $[v]_T$ is defined componentwise.

Algorithm 3 (General breakpoints):

Step 0: Set $\ell_0 = \ell$, $u_0 = u$, and $m^0 = [\frac{1}{2}(\ell+u)]_T$. Let x^0 be an optimal solution of P^0. Set $k = 0$ and $v^0 = f^0(x^0)$.

Step 1: If $k = 0$ or if $k \geq 1$ and $x^k \neq x^{k-1}$, set $m^{k+1} = [x^k]_T$, and then choose ℓ^{k+1} and u^{k+1} so as to satisfy (4.2)-(4.5) and go to Step 3.

Step 2: Set $m^{k+1} = [x^k]_T$ and derive ℓ^{k+1} and u^{k+1} from ℓ^k and u^k by increasing at least one lower bound and/or decreasing at least one upper bound while maintaining (4.2)-(4.5) and $x^k \leq u^{k+1}$.

Step 3: If x^k solves P^{k+1} and the range contraction procedure cannot be performed, terminate.

Step 4: If x^k solves P^{k+1} and $x^k \in T$, set $x^{k+1} = x^k$ and $v^{k+1} = f(x^k)$, increase k by 1 and go to Step 2. Otherwise, let y^* be any optimal solution of P^{k+1} and let v^* be the optimal value of P^{k+1}. If $v^* < v^k$, set $x^{k+1} = y^*$ and $v^{k+1} = v^*$, increase k by 1, and return to Step 1.

Step 5: If $v^* \geq v^k$, then modify ℓ^{k+1} and/or u^{k+1} by increasing at least one lower bound and/or decreasing at least one upper bound while maintaining (4.2)-(4.5) and $x^k \leq u^{k+1}$ (note that k is unchanged). Return to Step 3.

Note that if each T_i is chosen as the set of machine-representable numbers, $[\omega]_{T_i} = \omega$ if ω is machine-representable. In practice, then, Algorithm 3 may be simplified by taking this into account, noting that Step 5 is not needed when $m^k = x^k$, since $v^* \leq f^{k+1}(m^k) = f(m^k) = f(x^k) \leq f^k(x^k)$. (Alternatively, in [6] a two-segment algorithm is discussed in which $m^{k+1} = x^k$, and ℓ^{k+1} and u^{k+1} are determined by using the error bound obtained from

P^k. For this algorithm, convergence to the optimal value of the original problem (1.1) may be proved.)

Theorem 4.2: Let f_i^T be the piecewise-linear approximation of f_i determined by the breakpoints $T_i (i=1,\ldots,n)$, and let $f^T \equiv \sum_{i=1}^{n} f_i^T$. Algorithm 3 will terminate in a finite number of iterations with an optimal solution $\hat{x}$ of the problem

$$\begin{aligned} &\min_{x} && f^T(x) \\ &\text{s.t.} && x \in C \cap [\ell, u], \end{aligned} \tag{4.7}$$

and the optimal value $f(x^{**})$ of (1.1) has the bounds $f^T(\hat{x}) - \sum_{i=1}^{n} e_i^T \leq f(x^{**}) \leq f(\hat{x})$, where $e_i^T \equiv \sup_{\hat{\ell}_i \leq x_i \leq \hat{u}_i} (f_i^T(x_i) - f_i(x_i))$ and where $\hat{\ell}$, $\hat{u}$ are the final sets of bounds used in the Algorithm.

In the case that the f_i are continuous and Algorithm 3 is applied to a sequence of approximating problems in which the distance between the points in the sets T_i tends to 0, convergence of the corresponding sequence of optimal values for the approximating problems to the optimal value of the given problem is guaranteed by Theorem 4.2. This is easily seen since the terms e_i^T tend to 0, while each $f^T(\hat{x})$ will be an upper bound on $f(x^{**})$.

5. Computational Aspects

The algorithms described in the preceding two sections have the property that they are finitely convergent for arbitrary choices of admissible breakpoint sets, however, the actual rate of convergence is significantly affected by the specific rules used for breakpoint selection. A specific strategy that has been found to be computationally effective is described below. Moreover, for large problems it is computationally more realistic to terminate when the upper and lower bounds differ by less than a specified tolerance rather to iterate until an optimal solution is obtained for the approximating problem. It turns out to be the case that lower bounds can be obtained even in the case in which the optimal solution of P^k is artificially bounded by ℓ^k, u^k. In that case, the dual variable values corresponding to the active artificial bounds may be used to construct two-segment approximations for which x^k will be optimal over S. These dual variable values are also used to determine the breakpoint sets for the next iteration. Details of this method will be given in [6]. Here we will only summarize results obtained for the two problems at the extreme ranges of the test problems solved. The fundamental elements of the strategy for breakpoint determination that proved most efficient in the computational tests with a variety of problems are as follows: (1) if the optimal value of i^{th} variable in P^{k+1} coincides with its value in P^k, then the length of the range of x_i for P^{k+2} is set approximately 0.2 times its length

in P^{k+1} (an exact factor of 0.2 may not be allowable due to admissibility considerations), (2) if the optimal value of the i^{th} variable in P^{k+1} is near the limit of its range (i.e., near ℓ_i^{k+1} or u_i^{k+1}), then the length of the range of x_i is increased by approximately a factor of 1.2 for P^{k+2}, (3) the scale factor used for range modification varies linearly between 0.2 and 1.2 for optimal values of x_i in $(\ell_i^{k+1}, u_i^{k+1})$, and (4) if $x_i^{k+1} = \ell_i^{k+1}$ or u_i^{k+1} (and these values do not coincide with ℓ_i or u_i), then the value of the dual variable corresponding to the active bound is taken into account in determining the range of x_i for P^{k+2} as well as in the error analysis cited above (for details, see [6]).

Test Problem A:

An unscaled version of this problem was described in [5], where results were given for procedures requiring more than three breakpoints per variable. The algebraic statement of the problem is:

$$\min_x \quad \sum_{i=1}^{15} w_i (1-q_i)^{x_i/1000}$$

$$\text{s.t.} \quad \sum_{i=1}^{10} x_i = 75{,}000 \qquad \sum_{i=6}^{15} x_i = 67{,}000$$

$$0 \le x_i \le u_i \qquad (i=1,\ldots,15),$$

where the constants w_i, q_i, and u_i are given in the following table.

i	w_i	q_i	u_i
1	9.2	0.31	16,000
2	1.0	0.45	16,000
3	7.6	0.23	19,000
4	0.6	0.09	10,000
5	8.8	0.15	10,000
6	4.2	0.21	11,000
7	3.2	0.15	17,000
8	3.4	0.01	20,000
9	8.8	0.79	16,000
10	6.6	0.41	15,000
11	1.2	0.71	17,000
12	4.6	0.77	12,000
13	0.8	0.79	13,000
14	3.0	0.21	20,000
15	1.2	0.07	20,000

Table 1. Data for Test Problem A

An optimal solution to integer-breakpoint approximation to this problem was obtained via Algorithm 1 in 17 iterations (with optimality verified on the 18^{th} iteration). The optimal solution is

$x^* = (11602, 4287, 14400, 2176, 10000, 10638, 11469, 0, 3292, 7136, 3038, 3590, 2298, 12805, 12734)^T$

with objective value ≈ 7.738141. The lower bound from Theorem 3.2 on the optimal value for the given continuous problem is 7.738140. This problem was also solved by the gradient projection code GPM, and yielded a nearby solution with an objective function value of 7.738495 after 23 iterations.

Test Problem B:

This is a model of the Dallas water supply network with data provided to us by Jeff Kennington of SMU. A detailed description of the formulation of this problem as an optimization problem is given in [1]. The version that we employed has 452 nodes, 551 arcs, and an objective function involving 14 cubic functions and 516 terms of the form $c_i|x_i|^{2.85}$. In 23 iterations a feasible solution with objective value -29212×10^6 was obtained along with a lower bound on the optimal value of -29238×10^6. This lower bound was obtained via an estimation technique that used only three available values of each f_i, and could be improved through the use of first and second derivative information (see [6] and [8]) on the f_i. Thus the actual relative error in the objective value of the approximate solution is likely to be closer to 0.01% than to the 0.1% figure available from the computed lower bound.

6. Conclusions

The two-segment approach to separable programming has been shown to yield a finite method for solving approximating problems of any specified accuracy, and, in computational tests, has exhibited rapid convergence in test problems ranging from 15 to more than 500 variables. Taking into account the additional features that the method does not require initial values for either the variables or for parameters such as penalty multipliers, and that it also provides tight bounds on the optimal value, it offers decided advantages over more general nonlinear programming methods, especially for "large" problems. The strictly local nature of the approximations needed at each iteration also holds out promise for efficient extensions of this approach to the non-separable case. Some convergence results for generalizations of these algorithms to the non-separable case are given in [6].

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Appendix. The Monotonicity Condition of Algorithm 2

We will consider only the case $x_i^k \notin Z$, since the inequality holds trivially otherwise. In this case, since $m_i^{k+1} = [x_i^k]$, it follows that the distance from x_i^k to the closest breakpoint of f_i^k that is less than x_i^k is at least $|x_i^k - m_i^{k+1}|$ and that the distance from x_i^k to the closest breakpoint of f_i^k that is greater than x_i^k is at least $|x_i^k - u_i^{k+1}|$. Because of the convexity of f_i, this quarantees that the required inequality by the following Lemma:

Lemma: Let g and h be piecewise-linear convex functions derived from a convex function ω by interpolation at breakpoints $g^- \leq g^+$ and $h^- \leq h^+$, respectively, where $g^- \leq h^- \leq h^+ \leq g^+$. If $y \in [h^-, h^+]$, then $h(y) \leq g(y)$.

Proof: Let $y = \lambda h^- + (1-\lambda)h^+$, where $\lambda \in [0,1]$, let $h^- = \mu^- g^- + (1-\mu^-)g^+$, where $\mu^- \in [0,1]$, and let $h^+ = \mu^+ g^- + (1-\mu^+)g^+$, where $\mu^+ \in [0,1]$. Then $h(y) = \lambda h(h^-) + (1-\lambda)\, h\, (h^+) = \lambda\omega(h^-) + (1-\lambda)\, \omega\, (h^+) \leq \lambda(\mu^- g(g^-)$ $+ (1-\mu^-)\, g\, (g^+)) + (1-\lambda)(\mu^+ g(g^-) + (1-\mu^+)\, g\, (g^+)) = (\lambda\mu^- + (1-\lambda)\mu^+)\, g\, (g^-)$ $+ (\lambda(1-\mu^-) + (1-\lambda)(1-\mu^+))\, g\, (g^+) = g(y)$, since $y = (\lambda\mu^- + (1-\lambda)\mu^+)g^-$ $+ (\lambda(1-\mu^-) + (1-\lambda)(1-\mu^+))g^+$.

Appendix A. Properties of the Piecewise-Linear Approximations

In this section we will verify properties of the piecewise-linear approximations used in Section 2. For simplicity in notation, we assume that ω is a convex function on $[\alpha,\beta]$ and that h is the piecewise-linear that agrees with ω at the points $y^- < y^0 < y^+$, where $\alpha \leq y^- < y^+ \leq \beta$, i.e.,

$$h(y) \equiv \begin{cases} \lambda\omega(y^-) + (1-\lambda)\,\omega\,(y^0), \\ \text{where} \quad \lambda y^- + (1-\lambda)y^0 = y, \text{ for} \quad y \leq y^0; \\ (1-\lambda)\,\omega\,(y^0) + \lambda\omega(y^+), \\ \text{where} \quad (1-\lambda)y^0 + \lambda y^+ = y, \text{ for} \quad y \geq y^0 \end{cases}$$

(In the case of one-segment approximations, the required properties follow as a special case of the results below.) An alternative representation of h that is useful for our purposes is

$$h(y) \equiv \begin{cases} h^L(y) \quad \text{for} \quad y \leq y^0, \\ h^R(y) \quad \text{for} \quad y \geq y^0, \end{cases}$$

where $h^L(y) = \omega(y^0) + \dfrac{(\omega(y^0)-\omega(y^-))}{y^0-y^-}\,(y-y^0)$

and $h^R(y) = \omega(y^0) + \dfrac{(\omega(y^+)-\omega(y^0))}{y^+-y^0}\,(y-y^0)$.

Note that h is <u>convex</u> because of the slope inequality

$$(\omega(y^0)-\omega(y^-))/(y^0-y^-) \leq (\omega(y^+)-\omega(y^0))/(y^+-y^0),$$

which follows from $\omega(y^0) \leq \left(\dfrac{1}{y^0-y^-} + \dfrac{1}{y^+-y^0}\right)^{-1}\left(\dfrac{\omega(y^-)}{y^0-y^-} + \dfrac{\omega(y^+)}{y^+-y^0}\right)$,

which in turn is implied by the convexity of ω. Also because of the convexity of ω, $\omega(y) \leq \lambda\omega(y^-) + (1-\lambda)\,\omega\,(y^0) = h^L(y)$, where $y \in [y^-, y^0]$ and $\lambda y^- + (1-\lambda)y^0 = y$. Similarly, $\omega(y) \leq h^R(y)$ for $y \in [y^0, y^+]$. For a $\hat{y} < y^-$, suppose that $h^L(\hat{y}) > \omega(\hat{y})$. Then if $\hat{\lambda}$ is chosen such that $\hat{\lambda}\hat{y} + (1-\hat{\lambda})y^0 = y^-$, we have $h^L(y^-) = \hat{\lambda}h^L(\hat{y}) + (1-\hat{\lambda})\,h\,(y^0) > \hat{\lambda}\omega(\hat{y}) + (1-\hat{\lambda})\,h\,(y^0) = \hat{\lambda}\omega(\hat{y}) + (1-\hat{\lambda})\,\omega\,(y^0) \geq \omega(y^-) = h^L(y^-)$, a contradiction. Thus for $y \leq y^-$, $h^L(y) \leq \omega(y)$, and similarly we can verify that $h^R(y) \leq \omega(y)$ for $y \geq y^+$.

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