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TWO SHORTER PROOFS IN SPECTRAL GRAPH THEORY [1]

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We give shorter proofs of two inequalities already known in spectral graph theory.

1. INTRODUCTION

Let G be a simple graph with V (E) as its vertex (respectively, edge) set. The spectrum of G is the spectrum of its adjacency matrix. For all other definitions (or notation not given here), especially those related to graph spectra, see [**1**].

The purpose of this note is to provide shorter proofs of two inequalities already known in spectral graph theory. The first bounds vertex eccentricities of a connected graph in terms of some spectral quantities, while the second bounds the spectral radius of a graph in terms of vertex degrees. The first inequality is a slight generalization of a bound due to C.D. GODSIL, while the second, also known as the RUNGE-HOFMEISTER conjecture, was first proved by A.J. HOFFMAN et al. (The proofs given here were obtained by the first and second author, respectively.)

2. A NEW PROOF OF THE BOUNDS ON ECCENTRICITIES

Let G be a simple graph, with adjacency matrix A. Assume that $\mu_1, \mu_2, \ldots, \mu_m$ are the distinct eigenvalues of A (and therefore of G), and let $P_1, P_2, \ldots, P_m$ be the matrices of the orthogonal projections of the whole space onto the eigenspaces corresponding to these eigenvalues. Then, by the spectral decomposition theorem,

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we can write $A = \sum_{i=1}^{m} \mu_i P_i$ (see, for example, [**2** p.3]). It follows that

$$A^k = \sum_{i=1}^{m} \mu_i^k P_i \quad (k = 0, 1, 2, \ldots).$$

Let $N_k(s,t)$ be the number of walks of length k between vertices s and t of G. Then we have

$$N_k(s,t) = \sum_{i=1}^{m} \mu_i^k (P_i)_{st}.$$

Denote by $d(s,t)$ the distance between vertices s and t in G. From here onwards, we will assume that the graph G in question is connected.

Let m_{st} be the number of non-zero members in the list

$$(P_1)_{st}, \ldots, (P_m)_{st}.$$

Then the following lemma holds:

Lemma 1. *If s and t are two vertices of a connected graph G then $d(s,t) \leq m_{st} - 1$.*

Proof. For brevity, put $d = d(s,t)$, and assume, to the contrary, that $d \geq m_{st}$. Then

$$\sum_{i=1}^{m} \mu_i^k (P_i)_{st} = 0 \quad (k = 0, 1, \ldots, d-1), \tag{1}$$

while

$$\sum_{i=1}^{m} \mu_i^d (P_i)_{st} \neq 0. \tag{2}$$

If we consider (1) as a system of linear equations in m_{st} unknowns (the non-zero members from the list $(P_1)_{st}, \ldots, (P_m)_{st}$), then, since $d \geq m_{st}$, we get that $(P_i)_{st} = 0$ for all $i = 1, \ldots, m$. But this contradicts (2). □

We will next find some bounds on m_{st}. Observe first that $(P_i)_{st}$ is equal to 0 if (i) $||P_i e_s|| = 0$, or (ii) $||P_i e_t|| = 0$, or (iii) the vectors $P_i e_s$, $P_i e_t$ are orthogonal. Consider now conditions (i) and (ii). Recall also that $||P_i e_j|| = \alpha_{ij}$ (here α_{ij} is the (i,j)-th entry of the angle matrix $\mathcal{A} = (\alpha_{ij})$, where index i corresponds to the i-th eigenspace, and index j to the vertex j of the graph in question; note also that the angle matrix is usually ordered so that it represents a graph invariant, cf. [**2**, p.75]). Now, if $\alpha_{is} = 0$ or $\alpha_{it} = 0$, then $(P_i)_{st} = 0$. Guided by this fact, we define $\hat{m}_{st} = |\{i \mid \alpha_{is} \neq 0 \ \wedge \ \alpha_{it} \neq 0\}|$, and arrive at the following result:

Lemma 2. *If s and t are two vertices of a graph G then $m_{st} \leq \hat{m}_{st}$.*

Remark. In general, $m_{st} \neq \hat{m}_{st}$, in view of (iii).

We now switch to the bounds on vertex eccentricities. Recall, $ecc(s) = \max\{d(s,t) \mid t \in V\}$. By Lemma 1, we first get that $ecc(s) \leq \max\{m_{st} \mid t \neq s\} - 1$; we next get, by Lemma 2, that $ecc(s) \leq \max\{\hat{m}_{st} \mid t \neq s\} - 1$. Let $m_s = \max\{\hat{m}_{st} \mid t \neq s\}$. We therefore obtain:

Theorem 3. *If s is any vertex of a (connected) graph G then $ecc(s) \leq m_s - 1$.*

Let m'_s be the number of non-zero entries in the s-th column of the angle matrix of G. Clearly, $m_s \leq m'_s$ (since only condition (i) from above has been used to define m'_s); note also that $m'_s = m_{ss}$. Accordingly we get the following result of C. D. Godsil (see [**3**]) as a corollary:

Corollary 4. *If s is any vertex of G, then $ecc(s) \leq m'_s - 1$.*

Remark. It is well known that $diam(G) \leq m-1$ for any connected graph G (see [**1**, p.88]). Based on the above observations, we have $diam(G) \leq \max\{m_s \mid s \in V\} - 1$. In many situations this bound can be better than the previous one. Note also that the bound is completely extracted from the angle matrix of a graph, in contrast to (lower) bounds where the eigenvalues are involved (see, for example, [**2**, p.84]).

3. A NEW PROOF OF THE RUNGE-HOFMEISTER CONJECTURE

Let G be a simple graph with n vertices and m edges. Recall that the vertex set and the edge set of G are denoted by V and E, respectively. The spectral radius ρ of G is the largest eigenvalue of A, the adjacency matrix of G. The degree of a vertex $i \in V$ is denoted by d_i.

The graph G is called (r_1, r_2)-semiregular if it is bipartite with bipartition (V_1, V_2) such that all vertices in V_i have the same degree r_i for $i = 1, 2$. The graph G is called almost regular if there is a non-negative real number r such that every component of G is either r-regular or (r_1, r_2)-semiregular with $r_1 r_2 = r^2$. F. Runge [**6**] (see also [**1**, p.49]) showed that if G is a regular or semiregular graph, then

$$m = \rho^2 \sum_{(i,j)\in E} \frac{1}{d_i d_j}, \tag{3}$$

and conjectured that the condition (3) is sufficient for a graph to be regular or semiregular. Since F. Runge had not explicitly required the graph to be connected, M. Hofmeister [**5**] weakened this to the conjecture that the condition (3) is sufficient for a graph to be almost regular. This conjecture was proved in [**4**], and here we give a new and shorter proof.

Theorem 5. *For any graph G we have*

$$\rho^2 \geq \frac{m}{\sum_{(i,j)\in E} \frac{1}{d_i d_j}}. \tag{4}$$

The case of equality holds if and only if G is almost regular.

Proof. It is known that

$$(5) \qquad \begin{aligned} \rho &= \sup \{ \, x^{\mathrm{T}} A x : \, x \in \mathbf{R}^n, \|x\| = 1 \, \} \\ &= \sup \{ \, 2 \textstyle\sum_{(i,j)\in E} x_i x_j \, : \, x \in \mathbf{R}^n, \|x\| = 1 \, \}, \end{aligned}$$

with the supremum attained if and only if x is an eigenvector of A corresponding to ρ. Setting $x_i = \sqrt{\frac{d_i}{2m}}$ for $i \in V$ (giving $\|x\| = 1$, since $\sum_{i\in V} d_i = 2m$), we get from (5) that

$$\rho \geq 2 \sum_{(i,j)\in E} \sqrt{\frac{d_i}{2m}} \sqrt{\frac{d_j}{2m}} = \frac{1}{m} \sum_{(i,j)\in E} \sqrt{d_i d_j}.$$

It follows from the inequality between arithmetic and geometric means that

$$\rho \geq \left(\prod_{(i,j)\in E} \sqrt{d_i d_j} \right)^{1/m}.$$

Squaring both sides of this inequality, we get

$$\rho^2 \geq \left(\prod_{(i,j)\in E} d_i d_j \right)^{1/m}.$$

Finally, from the inequality between geometric and harmonic means, we obtain

$$(6) \qquad \rho^2 \geq \frac{m}{\sum_{(i,j)\in E} \frac{1}{d_i d_j}}.$$

Equality holds in (6) if and only if $x = (x_i)_{i\in V}$ is an eigenvector of A corresponding to ρ and

$$(7) \qquad d_i d_j = \rho^2 \quad \text{for each } (i,j) \in E.$$

Let u be any vertex of G, and let C be the component of G containing u. From (7) we conclude that $d_v = \rho^2/d_u$ holds for all vertices v of C at an odd distance from u, while $d_v = d_u$ for all those at an even distance from u. Thus, if $d_u = \rho$ then C is ρ-regular graph, while if $d_u \neq \rho$ then C is a $(d_u, \rho^2/d_u)$-semiregular graph with bipartition $(\{v \in C \,|\, d_v = d_u\}, \{v \in C \,|\, d_v = \rho^2/d_u\})$. □

REFERENCES

1. D. Cvetković, M. Doob, H. Sachs: *Spectra of Graphs—Theory and Application.* Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.
2. D. Cvetković, P. Rowlinson, S. Simić: *Eigenspaces of Graphs.* Cambridge University Press, Cambridge, 1997.

3. C.D. Godsil: *Matching and walks in graphs.* J. Graph Theory, **5** (1981), 285–297.
4. A.J. Hoffman, P. Wolfe, M. Hofmeister: *A Note on Almost Regular Matrices.* Linear Algebra Appl. 226–228 (1995), 105–108.
5. M. Hofmeister: *A Note on Almost Regular Graphs.* Math. Nachr. **166** (1994), 259–262.
6. F. Runge: *Beiträge zur Theorie der Spektren von Graphen und Hypergraphen.* Dissertation, TH Ilmenau, 1976.

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