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TWO-SIDED Γ-α-DERIVATIONS IN PRIME AND SEMIPRIME Γ-NEAR-RINGS

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Abstract. We introduce the notion of two-sided Γ-α-derivation of a Γ-near-ring and give some generalizations of [1, 2].

1. Introduction

The notion of Γ-near-rings, as a generalization of near-rings, was introduced by Satyanarayana in [5]. The concept of Γ-derivations in Γ-near-rings was introduced by Jun, Cho, and Kim [4, 6]. In [1], Argaç defined a two-sided α-derivation d of a near-ring. In a similar way, we introduce the notion of two-sided Γ-α-derivation of a Γ-near-ring and we give some generalizations of [1, 2].

Let M be a prime (resp. a semiprime) Γ-near-ring and U be a right invariant (resp. an invariant) subset of M containing 0, and let d be a two-sided Γ-α-derivation on M which acts as a Γ-homomorphism on U, or an anti-Γ-homomorphism on U under certain conditions on α, then we showed that $d = 0$. Finally, if M is a prime Γ-near-ring, U is a nonzero right invariant of M, and d is a nonzero Γ-$(\alpha, 1)$-derivation of M satisfying $d(x + y - x - y) = 0$ for all $x, y \in U$, then we prove that $(M, +)$ is abelian.

2. Preliminaries

All near-rings considered in this paper are right distributive. A Γ-near-ring is a triple $(M, +, \Gamma)$, where

(i) $(M, +)$ is a group (not necessarily abelian),
(ii) Γ is a non-empty set of binary operations on M such that $(M, +, \gamma)$ is a near-ring for each $\gamma \in \Gamma$,
(iii) $(x\beta y)\gamma z = x\beta(y\gamma z)$ for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$.

A Γ-near-ring M is said to be zero-symmetric Γ-near-ring if $x\gamma 0 = 0$ for all $x \in M$ and $\gamma \in \Gamma$.

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Let M be a Γ-near-ring. A subset U of M is said to be right (resp. left) invariant if $a\gamma x \in U$ (resp. $x\gamma a \in U$) for all $a \in U, \gamma \in \Gamma$ and $x \in M$. We say that U is invariant, if U is both right and left invariant. A Γ-near-ring M is said to be prime Γ-near-ring if $x\Gamma M\Gamma y = \{0\}$ for $x, y \in M$ implies $x = 0$ or $y = 0$, and semiprime Γ-near-ring if $x\Gamma M\Gamma x = \{0\}$ for $x \in M$ implies $x = 0$.

A Γ-derivation on M is defined to be an additive endomorphism d of M satisfying the product rule

$$d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$$

for all $x, y \in M$ and $\gamma \in \Gamma$, or equivalently

$$d(x\gamma y) = x\gamma d(y) + d(x)\gamma y$$

for all $x, y \in M$ and $\gamma \in \Gamma$ (see [6]).

If M and M' are two Γ-near-rings, then a mapping $f : M \to M'$ such that $f(x + y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$ (resp. $f(x\gamma y) = f(y)\gamma f(x)$), for all $x, y \in M$ and $\gamma \in \Gamma$, is called a Γ-near-ring homomorphism (resp. an anti-Γ-near-ring homomorphism) on M. Let S be a nonempty subset of M and let d be a Γ-derivation on M. If $d(x\gamma y) = d(x)\gamma d(y)$(resp. $d(x\gamma y) = d(y)\gamma d(x)$) for all $x, y \in S$ and $\gamma \in \Gamma$, then d is said to act as a Γ-homomorphism (resp. an anti-Γ-homomorphism) on S.

An additive endomorphism $d : M \to M$ of a Γ-near-ring M is called a Γ-(α, β)-derivation on M if there exist two functions $\alpha, \beta : M \to M$ such that the following product rule holds:

$$d(x\gamma y) = d(x)\gamma\alpha(y) + \beta(x)\gamma d(y)$$

for all $x, y \in M$ and $\gamma \in \Gamma$. One can easily show that if d is a Γ-(α, β)-derivation on M such that $\alpha(x + y) = \alpha(x) + \alpha(y)$ and $\beta(x + y) = \beta(x) + \beta(y)$, then

$$d(x\gamma y) = \beta(x)\gamma d(y) + d(x)\gamma\alpha(y).$$

An additive mapping $d : M \to M$ is called a two-sided Γ-α-derivation if d is a Γ-$(\alpha, 1)$-derivation as well as a Γ-$(1, \alpha)$-derivation. We should note that if $\alpha = 1$, then a two-sided Γ-α-derivation is just a Γ-derivation.

Example. Let M_1 be a zero-symmetric Γ-near-ring and let M_2 be a Γ-ring (for Γ-rings, see [3]). Let us define $d : M_1 \oplus M_2 \to M_1 \oplus M_2$ by $d\big((m_1, m_2)\big) = \big(0, d_2(m_2)\big)$ and $\alpha : M_1 \oplus M_2 \to M_1 \oplus M_2$ by $\alpha\big((m_1, m_2)\big) = \big(d_1(m_1), 0\big)$, where d_1 is any map on M_1 and d_2 is a Γ-right and left M_2-module map on M_2 which is not a derivation. Then it can be shown that d is a two sided Γ-α-derivation, but not a Γ-derivation.

3. The results

In order to derive our main results we first give the following lemmas.

Lemma 3.1. *Let M be a prime Γ-near-ring and let U be a nonzero invariant of M. If $a + b - a - b = 0$ for all $a, b \in U$, then $(M, +)$ is abelian.*

Proof. Since U is a nonzero invariant of M, we have $x\gamma a, y\gamma a \in U$ for all $a \in U$, $x, y \in M$ and $\gamma \in \Gamma$. Thus, by the hypothesis, we have $x\gamma a+y\gamma a-x\gamma a-y\gamma a = 0$ for all $a \in U$, $x, y \in M$ and $\gamma \in \Gamma$. Then we obtain $(x + y - x - y)\gamma a = 0$ for all $a \in U$, $x, y \in M$ and $\gamma \in \Gamma$. In particularly, $(x + y - x - y)\Gamma U = (x + y - x - y)\Gamma M\Gamma U = 0$. Since M is a prime Γ-near-ring and U is a nonzero invariant, we get $x+y-x-y = 0$ for all $x, y \in M$. Hence $(M, +)$ is abelian. □

Lemma 3.2. *Let M be a prime Γ-near-ring, U be a nonzero invariant of M which contains 0, and d a Γ-$(\alpha, 1)$-derivation on M. If d acts as an anti-Γ-homomorphism on U and $\alpha(0) = 0$, then $x\gamma 0 = 0$ for all $x \in U$ and $\gamma \in \Gamma$.*

Proof. Since $0\gamma x = 0$ for all $x \in U$ and $\gamma \in \Gamma$, and d acts as an anti-Γ-homomorphism on U, it follows that $d(x)\gamma 0 = 0$. If we take $x\gamma 0$ instead of x, we obtain $d(x)\gamma\alpha(0) + x\gamma 0 = 0$ for all $x \in U$ and $\gamma \in \Gamma$. Then we get $x\gamma 0 = 0$ for all $x \in U$ and $\gamma \in \Gamma$. □

Lemma 3.3. *Let M be a Γ-near-ring and U be an invariant of M. If d is a two-sided Γ-α-derivation of M such that $\alpha(x\gamma y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in U$ and $\gamma \in \Gamma$, then*

$$n\mu\big(d(x)\gamma\alpha(y) + x\gamma d(y)\big) = n\mu d(x)\gamma\alpha(y) + n\mu x\gamma d(y)$$

for all $n, x, y \in U$ and $\gamma, \mu \in \Gamma$. Furthermore, if $\alpha(U) = U$, then for all $n, x, y \in U$ and $\gamma, \mu \in \Gamma$

$$n\mu\big(d(x)\gamma y + \alpha(x)\gamma d(y)\big) = n\mu d(x)\gamma y + n\mu\alpha(x)\gamma d(y).$$

Proof. For all $n, x, y \in U$ and $\gamma, \mu \in \Gamma$, we have

$$\begin{aligned} d\big(n\mu(x\gamma y)\big) &= d(n)\mu\alpha(x\gamma y) + n\mu d(x\gamma y) \\ &= d(n)\mu\alpha(x)\gamma\alpha(y) + n\mu\big(d(x)\gamma\alpha(y) + x\gamma d(y)\big). \end{aligned}$$

On the other hand, for all $n, x, y \in U$ and $\gamma, \mu \in \Gamma$,

$$\begin{aligned} d\big((n\mu x)\gamma y\big) &= d(n\mu x)\gamma\alpha(y) + n\mu x\gamma d(y) \\ &= \big(d(n)\mu\alpha(x) + n\mu d(x)\big)\gamma\alpha(y) + n\mu x\gamma d(y) \\ &= d(n)\mu\alpha(x)\gamma\alpha(y) + n\mu d(x)\gamma\alpha(y) + n\mu x\gamma d(y). \end{aligned}$$

From these two expressions of $d(n\mu x\gamma y)$, we obtain that, for all $n, x, y \in U$ and $\gamma, \mu \in \Gamma$,

$$n\mu\big(d(x)\gamma\alpha(y) + x\gamma d(y)\big) = n\mu d(x)\gamma\alpha(y) + n\mu x\gamma d(y).$$

By a similar way, we obtain that, for all $n, x, y \in U$ and $\gamma, \mu \in \Gamma$,

$$n\mu\big(d(x)\gamma y + \alpha(x)\gamma d(y)\big) = n\mu d(x)\gamma y + n\mu\alpha(x)\gamma d(y).$$

□

Lemma 3.4. *Let M be a prime Γ-near-ring and U be a nonzero invariant of M. Let d be a nonzero Γ-$(\alpha, 1)$-derivation on M such that $\alpha(x\gamma y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in U$ and $\gamma \in \Gamma$. If $x \in M$ and $x\gamma d(U) = \{0\}$, then $x = 0$.*

Proof. Suppose that $x\gamma d(U)=\{0\}$. Then $x\gamma d(u\mu y)=0$ for all $y\in M, u\in U$ and $\mu\in\Gamma$. Thus $0=x\gamma\big(d(u)\mu\alpha(y)+u\mu d(y)\big)=x\gamma u\mu d(y)$ for all $y\in M, u\in U$ and $\mu\in\Gamma$. Then we have $x\Gamma U\Gamma d(y)=\{0\}$ for all $y\in M$. Since d is nonzero, U is a nonzero invariant and M is prime, it follows that $x=0$. □

Lemma 3.5. *Let M be a prime Γ-near-ring and U be a nonzero invariant of M. Let d be a nonzero Γ-$(\alpha,1)$-derivation on M. If $d(x+y-x-y)=0$ for all $x,y\in U$, then $(x+y-x-y)\gamma d(z)=0$ for all $x,y,z\in U$ and $\gamma\in\Gamma$.*

Proof. Assume that $d(x+y-x-y)=0$ for all $x,y\in U$. By taking $y\gamma z$ and $x\gamma z$ instead of y and x, respectively (where $z\in U$ and $\gamma\in\Gamma$) we obtain

$$\begin{aligned}0&=d\big((x+y-x-y)\gamma z\big)\\&=d(x+y-x-y)\gamma\alpha(z)+(x+y-x-y)\gamma d(z)\\&=(x+y-x-y)\gamma d(z)\end{aligned}$$

for all $x,y,z\in U$ and $\gamma\in\Gamma$. □

Lemma 3.6. *Let M be a Γ-near-ring and U be an invariant of M. Let d be a Γ-$(\alpha,1)$-derivation on M such that $\alpha(x\gamma y)=\alpha(x)\gamma\alpha(y)$ for all $x,y\in U$, $\gamma\in\Gamma$ and $\alpha(U)=U$.*

(i) *If d acts as a Γ-homomorphism on U, then*

$$d(y)\mu x\gamma d(y)=y\mu x\gamma d(y)=d(y)\mu x\gamma\alpha(y)$$

for all $x,y\in U$ and $\gamma,\mu\in\Gamma$.

(ii) *If d acts as an anti-Γ-homomorphism on U, then*

$$d(y)\gamma x\gamma d(y)=x\gamma y\gamma d(y)=d(y)\gamma\alpha(y)\gamma x$$

for all $x,y\in U$ and $\gamma\in\Gamma$.

Proof. (i) Assume that d acts as a Γ-homomorphism on U. Then

$$d(x\gamma y)=d(x)\gamma\alpha(y)+x\gamma d(y)=d(x)\gamma d(y) \tag{1}$$

for all $x,y\in U$ and $\gamma\in\Gamma$. Taking $y\mu x$ instead of x in (1), we get

$$\begin{aligned}d(y\mu x)\gamma\alpha(y)+y\mu x\gamma d(y)&=d(y\mu x)\gamma d(y)\\&=d(y)\mu d(x)\gamma d(y)\\&=d(y)\mu d(x\gamma y)\end{aligned} \tag{2}$$

for all $x,y\in U$ and $\gamma,\mu\in\Gamma$. By Lemma 3.3, we can write

$$\begin{aligned}d(y)\mu d(x\gamma y)&=d(y)\mu d(x)\gamma\alpha(y)+d(y)\mu x\gamma d(y)\\&=d(y\mu x)\gamma\alpha(y)+d(y)\mu x\gamma d(y).\end{aligned}$$

Using this relation in (2), we get

$$y\mu x\gamma d(y)=d(y)\mu x\gamma d(y).$$

Similarly, taking $y\mu x$ instead of y in (1) gives

$$d(x)\gamma\alpha(y\mu x)+x\gamma d(y\mu x)=d(x)\gamma d(y\mu x)=d(x\gamma y)\mu d(x). \tag{3}$$

On the other hand, for all $x, y \in U$ and $\gamma, \mu \in \Gamma$

$$\begin{aligned} d(x\gamma y)\mu d(x) &= \big(d(x)\gamma\alpha(y) + x\gamma d(y)\big)\mu d(x) \\ &= d(x)\gamma\alpha(y)\mu d(x) + x\gamma d(y)\mu d(x) \\ &= d(x)\gamma\alpha(y)\mu d(x) + x\gamma d(y\mu x). \end{aligned}$$

Using this relation in (3), we obtain

$$d(x)\gamma\alpha(y\mu x) = d(x)\gamma\alpha(y)\mu d(x)$$

for all $x, y \in U$ and $\gamma, \mu \in \Gamma$. By hypothesis, we get

$$d(x)\gamma\alpha(y)\mu\alpha(x) = d(x)\gamma\alpha(y)\mu d(x).$$

Since $\alpha(U) = U$, it obvious that

$$d(x)\gamma w\mu\alpha(x) = d(x)\gamma w\mu d(x)$$

for all $x, w \in U$ and $\gamma, \mu \in \Gamma$. That is, for all $x, y \in U$ and $\gamma, \mu \in \Gamma$

$$d(y)\mu x\gamma d(y) = d(y)\mu x\gamma\alpha(y).$$

(ii) Since d acts as an anti-Γ-homomorphism on U, we have

$$d(x\gamma y) = d(x)\gamma\alpha(y) + x\gamma d(y) = d(y)\gamma d(x) \tag{4}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Taking $x\gamma y$ instead of y in (4) gives

$$\begin{aligned} d(x)\gamma\alpha(x\gamma y) + x\gamma d(x\gamma y) &= d(x\gamma y)\gamma d(x) \\ &= \big(d(x)\gamma\alpha(y) + x\gamma d(y)\big)\gamma d(x) \\ &= d(x)\gamma\alpha(y)\gamma d(x) + x\gamma d(y)\gamma d(x) \\ &= d(x)\gamma\alpha(y)\gamma d(x) + x\gamma d(x\gamma y). \end{aligned}$$

From this relation, we get

$$d(x)\gamma\alpha(x\gamma y) = d(x)\gamma\alpha(y)\gamma d(x).$$

Since $\alpha(U) = U$, we have

$$d(x)\gamma\alpha(x)\gamma y = d(x)\gamma y\gamma d(x).$$

Similarly, taking $x\gamma y$ instead of x in (4) gives the relation

$$d(y)\gamma x\gamma d(y) = x\gamma y\gamma d(y).$$

□

Theorem 3.7. *Let M be a semiprime Γ-near-ring and U be an invariant subset of M containing 0. Let d be a two-sided Γ-α-derivation on M such that $\alpha(U) = U$ and $\alpha(x\gamma y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in U$ and $\gamma \in \Gamma$.*

(i) *If d acts as a Γ-homomorphism on U, then $d(U) = \{0\}$.*
(ii) *If d acts as an anti-Γ-homomorphism on U, then $d(U) = \{0\}$.*

Proof. (i) Suppose that d acts as a Γ-homomorphism on U. Then Lemma 3.6 gives

$$d(y)\mu x\gamma d(y) = d(y)\mu x\gamma\alpha(y) \tag{5}$$

for all $x, y \in U$ and $\gamma, \mu \in \Gamma$. Right multiplying (5) by $d(z)$, and using the hypothesis that d acts as a Γ-homomorphism on U together with Lemma 3.3, we get $d(y)\mu x\gamma d(y)\mu\alpha(z) = 0$ for all $x, y, z \in U$ and $\gamma, \mu \in \Gamma$. Since $\alpha(U) = U$, $d(y)\mu x\gamma d(y)\mu z = 0$ for all $x, y, z \in U$ and $\gamma, \mu \in \Gamma$. Taking $z\eta m$ instead of x, where $m \in M, z \in U$, and $\eta \in \Gamma$, we obtain $d(y)\mu z\eta m\gamma d(y)\mu z = 0$ for all $x, y, z \in U, m \in M$ and $\mu, \eta, \gamma \in \Gamma$. In particular, $d(y)\mu z\Gamma M\Gamma d(y)\mu z = 0$. By the semiprimness of M we obtain that $d(y)\mu z = 0$. Since $\alpha(U) = U$, it is clear that

$$d(y)\mu\alpha(z) = 0 \tag{6}$$

for all $y, z \in U$ and $\mu \in \Gamma$. Substituting $y\eta n$ for y in (6) and left multiplying (6) by $d(z)$, we get $d(z)\beta d(y)\eta n\mu\alpha(z) + d(z)\beta y\eta d(n)\mu\alpha(z) = 0$, where $z \in U, \beta \in \Gamma$. Since the second summand is zero by (6), we get

$$\begin{aligned} 0 &= d(z)\beta d(y)\eta n\mu\alpha(z) = d(z\beta y)\eta n\mu\alpha(z) \\ &= d(z)\beta\alpha(y)\eta n\mu\alpha(z) + z\beta d(y)\eta n\mu\alpha(z) \\ &= z\beta d(y)\eta n\mu\alpha(z) \end{aligned}$$

for all $n \in M, x, y, z \in U$ and $\gamma, \mu, \beta \in \Gamma$. Since $\alpha(U) = U$, $z\beta d(y)\eta n\mu w = 0$, where $w \in U$. Taking $z\beta d(y)$ instead of w, we obtain $z\beta d(y)\eta n\mu z\beta d(y) = 0$. Since M is semiprime, we have

$$z\beta d(y) = 0 \tag{7}$$

for all $y, z \in U$ and $\beta \in \Gamma$. Combining (6) and (7) gives that $d(y)\beta\alpha(z) + y\beta d(z) = d(y\beta z) = 0$ for all $y, z \in U$ and $\beta \in \Gamma$. In particular, $d(z\gamma m\beta z) = 0$ for all $m \in M, z \in U$ and $\gamma, \beta \in \Gamma$. Since d acts as a Γ-homomorphism on U, we have

$$0 = d(z\gamma m)\beta d(z) = d(z)\gamma\alpha(m)\beta d(z) + z\gamma d(m)\beta d(z).$$

The second summand is zero by (7). Thus, since $\alpha(U) = U$ and by the semiprimness of M we conclude that $d(z) = 0$ for all $z \in U$.

(ii) Assume that d acts as an anti-Γ-homomorphism on U. First, we note that $a\gamma 0 = 0$ for all $a \in U$ and $\gamma \in \Gamma$ by Lemma 3.2. By Lemma 3.6 we have

$$x\gamma y\gamma d(y) = d(y)\gamma x\gamma d(y) \tag{8}$$

for all $x, y \in U$ and $\gamma \in \Gamma$,

$$d(y)\gamma\alpha(y)\gamma x = d(y)\gamma x\gamma d(y) \tag{9}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Replacing x by $x\gamma d(y)$ in (8) and using Lemma 3.6, we get

$$\begin{aligned} x\gamma d(y)\gamma y\gamma d(y) &= d(y)\gamma x\gamma d(y\gamma y) = d(y)\gamma x\gamma(d(y\gamma\alpha(y) + y\gamma d(y)) \\ &= d(y)\gamma x\gamma d(y)\gamma\alpha(y) + d(y)\gamma x\gamma y\gamma d(y). \end{aligned} \tag{10}$$

Substituting $x\gamma y$ for x in (8), we have

$$x\gamma y\gamma y\gamma d(y) = d(y)\gamma x\gamma y\gamma d(y) \tag{11}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Right-multiplying (8) by $\alpha(y)$, we obtain

$$x\gamma y\gamma d(y)\gamma\alpha(y) = d(y)\gamma x\gamma d(y)\gamma\alpha(y) \tag{12}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Replacing x by y in (8) we get $y\gamma y\gamma d(y) = d(y)\gamma y\gamma d(y)$. Now left-multiplying this relation by x gives

$$x\gamma y\gamma y\gamma d(y) = x\gamma d(y)\gamma y\gamma d(y) \tag{13}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Putting (11), (12), and (13) in (10) gives

$$x\gamma y\gamma d(y)\gamma\alpha(y) = 0.$$

In particular, $y\gamma n\gamma y\gamma d(y)\gamma\alpha(y) = 0$, where $n \in M$. Hence

$$y\gamma d(y)\gamma\alpha(y)\gamma M\gamma y\gamma d(y)\gamma\alpha(y) = 0.$$

By the semiprimeness of M

$$y\gamma d(y)\gamma\alpha(y) = 0 \tag{14}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. According to (12) we get $d(y)\gamma x\gamma d(y)\gamma\alpha(y) = 0$. Using this relation in (9), we have

$$d(y)\gamma\alpha(y)\gamma x\gamma\alpha(y) = 0 \tag{15}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Replacing x by $x\gamma n\gamma d(y)$ in (15), we have

$$d(y)\gamma\alpha(y)\gamma x\gamma\alpha(y) = d(y)\gamma\alpha(y)\gamma x\gamma n\gamma d(y)\gamma\alpha(y)\gamma x = 0$$

for all $x, y \in U, n \in M$ and $\gamma \in \Gamma$. Hence

$$d(y)\gamma\alpha(y)\gamma x = 0 \tag{16}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Using (16) in (9), we obtain $d(y)\gamma x\gamma d(y) = 0$, and so we get

$$d(y)\gamma x\gamma n\gamma d(y)\gamma x = 0$$

for all $x, y \in U, n \in M$ and $\gamma \in \Gamma$. Hence

$$d(y)\gamma x = 0 \tag{17}$$

for all $x, y \in U$ and $\gamma \in \Gamma$. Therefore $x\gamma d(z)\gamma d(y\gamma n)\gamma x = 0$ for all $x, y, z \in U, n \in M$ and $\gamma \in \Gamma$. Thus

$$0 = x\gamma d(z)\gamma(d(y)\gamma n + \alpha(y)\gamma d(n))\gamma x = x\gamma d(z)\gamma d(y)\gamma\alpha(y)\gamma d(n)\gamma x$$

for all $x, y, z \in U, n \in M$ and $\gamma \in \Gamma$. Since $\alpha(U) = U$ the second summand is zero by (17). Hence $x\gamma d(z)\gamma d(y)\gamma N\gamma x = \{0\}$, and then

$$x\gamma d(z)\gamma d(y)\gamma N\gamma x\gamma d(z)\gamma d(y) = \{0\}.$$

Because M is semiprime, we get

$$0 = x\gamma d(z)\gamma d(y) = x\gamma d(y\gamma z).$$

Therefore

$$0 = x\gamma d(y)\gamma z + x\gamma\alpha(y)\gamma d(z) = x\gamma\alpha(y)\gamma d(z).$$

In particular

$$0 = \alpha(y)\gamma d(z)\gamma n\gamma\alpha(y)\gamma d(z).$$

Hence $0 = \alpha(y)\gamma d(z)$. By (17), we obtain $0 = d(x\gamma y)$ for all $x, y \in U$. Thus $d(x\gamma x\gamma n) = 0$ for all $x \in U, n \in M$ and $\gamma \in \Gamma$. Thus

$$\begin{aligned} 0 &= d(x\gamma n)\gamma d(x) \\ &= \big(d(x)\gamma n + \alpha(x)\gamma d(n)\big)\gamma d(x) \\ &= d(x)\gamma n\gamma d(x) + \alpha(x)\gamma d(n)\gamma d(x) \\ &= d(x)\gamma n\gamma d(x) + \alpha(x)\gamma d(x\gamma n). \end{aligned}$$

Since the second summand is zero, we get $d(x)\gamma n\gamma d(x) = 0$. Therefore $d(x) = 0$ for all $x \in U$. □

Corollary 3.8. *Let M be a semiprime Γ-near-ring and d be a two-sided Γ-α-derivation of M such that $\alpha(x\gamma y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$.*

(i) *If d acts as a Γ-homomorphism on M, then $d = 0$.*
(ii) *If d acts as an anti-Γ-homomorphism on M and $\alpha(0) = 0$, then $d = 0$.*

Corollary 3.9. *Let M be a prime Γ-near-ring and let U be a nonzero invariant subset of M such that $0 \in U$. Let d be a two-sided Γ-α-derivation of M such that $\alpha(U) = U$ and $\alpha(x\gamma y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$.*

(i) *If d acts as a Γ-homomorphism on M, then $d = 0$.*
(ii) *If d acts as an anti-Γ-homomorphism on M and $\alpha(0) = 0$, then $d = 0$.*

Proof. By Theorem 3.7, we have $d(x) = 0$ for all $x \in U$. Taking $x\gamma a$ instead of x, where $x \in U, a \in M$ and $\gamma \in \Gamma$, then we have $0 = d(x\gamma a) = d(x)\gamma\alpha(a) + x\gamma d(a) = x\gamma d(a)$. Substituting $x\mu b$ for x in the last expression, where $x \in U, b \in M$ and $\gamma \in \Gamma$, we get $x\mu b\gamma d(a) = 0$. In particular, $x\Gamma M\Gamma d(a) = \{0\}$. By the primness of M, since U is a nonzero invariant subset of M, we have $d(a) = 0$ for all $a \in M$. □

Theorem 3.10. *Let M be a prime Γ-near-ring, U be a nonzero invariant of M and d be a nonzero Γ-$(\alpha, 1)$-derivation of M such that $\alpha(x\gamma y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in U$ and $\gamma \in \Gamma$. If $d(x + y - x - y) = 0$ for all $x, y \in U$, then $(M, +)$ is abelian.*

Proof. Suppose that $d(x + y - x - y) = 0$ for all $x, y \in U$. By Lemma 3.5, we have $(x + y - x - y)\gamma d(z) = 0$ for all $x, y, z \in U$ and $\gamma \in \Gamma$. Since $d \neq 0$, it follows that $x + y - x - y = 0$ for all $x, y \in U$ by Lemma 3.4. Hence $(M, +)$ is abelian by Lemma 3.1. □

Corollary 3.11. *Let M be a prime Γ-near-ring and U be a nonzero invariant of M and d be a nonzero Γ-$(\alpha, 1)$-derivation of M such that $\alpha(x\gamma y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in U$ and $\gamma \in \Gamma$. If $d + d$ is additive on U, then $(M, +)$ is abelian.*

Example. Let $M = M_1 \oplus M_2$, where M_1 and M_2 are prime Γ-near-rings. Let us define $d : M \to M$ by $d\big((m_1, m_2)\big) = (0, m_2)$ and $\alpha : M \to M$ by $\alpha\big((m_1, m_2)\big) = (m_1, 0)$ for all $(m_1, m_2) \in M$. Then d is a two-sided Γ-α-derivation on M. On the other hand, it can be shown that d acts as a Γ-homomorphism on M and

$$\alpha\big((m_1, m_2)\gamma(m_1', m_2')\big) = \alpha\big((m_1, m_2)\big)\gamma\alpha\big((m_1', m_2')\big)$$

for all $(m_1, m_2), (m_1', m_2') \in M$ and $\gamma \in \Gamma$. One can also show that if M_2 is commutative, then d acts as an anti-homomorphism on M. Now, if M_2 is abelian, then $d(m+m'-m-m') = 0$ for all $m = (m_1, m_2), m' = (m_1', m_2') \in M$. But $d \neq 0$ and $(M, +)$ is not abelian. Therefore primeness condition on M in Corollary 3.9 and Theorem 3.10 cannot be omitted.

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