

TWO-SIDED SEARCH GAMES

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Abstract

A target hides himself in one of the boxes $1, 2, \dots, m$ with probability distribution $\mathbf{x} = \langle x_1, \dots, x_m \rangle$ and can move to any of other boxes after each unsuccessful search by a searcher. The searcher is not informed of \mathbf{x} and continues search until the target is found. He is informed of c_i 's, *i.e.*, examination costs of each box, and α_i 's, $i = 1, \dots, m$, where α_i is the probability of overlooking the target if i is searched and the target is in i . A formulation of this two-sided search game in which "non-blind" target and "noisy" searcher are involved is given in a stochastic game framework, and the game is solved in some special cases.

1. Introduction and Summary

In the game which we shall consider in this paper player I (Target) hides himself in one of m boxes, labeled from 1 to m . His opponent, player II (Searcher), has to search for the target by suc-

cessive examinations of the boxes. An examination of the i th box, $i=1, \dots, m$, can be performed at a cost $c_i > 0$ each time and there is a probability α_i , $0 \leq \alpha_i < 1$, of overlooking the target given that the right box is searched. Upon finding the target in the i th box player II receives a reward of $R_i > 0$.

A mixed strategy for player II is a probability m -vector $\mathbf{y} = \langle y_1, \dots, y_m \rangle$ with $y_i \geq 0$, $i=1, \dots, m$, and $\sum_{i=1}^m y_i = 1$ which denotes a probability distribution, by which the box to be examined at each stage is selected. A mixed strategy for player I is also a probability m -vector $\mathbf{x} = \langle x_1, \dots, x_m \rangle$, with $x_i \geq 0$, $i=1, \dots, m$ and $\sum_{i=1}^m x_i = 1$. x_i denotes the probability of hiding in the i th box.

If player I uses a mixed strategy \mathbf{x} and the "memoryless" player II a specified mixed strategy \mathbf{y} , chosen once and for all, the expected return to player I for each stage of the game is

$$(1) \quad M(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^m \{c_i - x_i(1 - \alpha_i)R_i\}y_i,$$

and the probability that the target will be detected during a given search equals

$$(2) \quad S(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^m (1 - \alpha_i)x_i y_i.$$

Hence the discounted expected return to player I for the entire search is given by

$$(3) \quad G(\mathbf{x}, \mathbf{y}) = \frac{M(\mathbf{x}, \mathbf{y})}{1 - \beta(1 - S(\mathbf{x}, \mathbf{y}))}$$

where $0 \leq \beta < 1$ is a discount rate.

In [7], Neuts gives incorrect descriptions of the solutions of this game for the two cases: one in which $0 < \beta < 1$ and the other $\beta = 0$, with the additional restriction that $R_i \equiv R$ in both cases. It is a part of our purpose of this paper to give correct descriptions of the solutions of the game for the case where $\beta = 1$ (Section 2) and that where $\beta = 0$, *i.e.*, two-sided single-stage game (Section 3). The former case

would be a meaningful formulation of the sequential game in which player I is allowed to move after each stage of search by player II. We shall continue studying this class of games in Section 4. A formulation of the sequential search game in which "non-blind" target and "noisy" searcher are involved is given in a stochastic game framework, and the game is solved in some special cases.

2. Solution of the Game with $\beta=1$.

First we consider the case where $\beta=1$. Fortunately this case has every feature of manipulations which will be seen in the general case where $0 < \beta \leq 1$, and, nevertheless, the solution has a simple and remarkable form.

Theorem 1. The single-stage game with the payoff function

$$(4) \quad G(\mathbf{x}, \mathbf{y}) = \frac{M(\mathbf{x}, \mathbf{y})}{S(\mathbf{x}, \mathbf{y})}$$

is solved as follows: Let v_0 be the largest real root of the equation

$$(5) \quad \sum_{i=1}^m \frac{c_i(1-\alpha_i)^{-1}}{R_i+v} = 1.$$

Then v_0 is the value of the game and the optimal strategies are given by

$$(6) \quad \begin{aligned} x_i^0 &= \frac{c_i(1-\alpha_i)^{-1}}{R_i+v_0}, \\ y_i^0 &= \frac{(1-\alpha_i)^{-1}}{R_i+v_0} \bigg/ \sum_{i=1}^m \frac{(1-\alpha_i)^{-1}}{R_i+v_0}, \quad (i=1, \dots, m). \end{aligned}$$

Proof. If we denote by $\mathbf{x}^0 = \langle x_1^0, \dots, x_m^0 \rangle$, $\mathbf{y}^0 = \langle y_1^0, \dots, y_m^0 \rangle$ and v , respectively, a pair of optimal strategies and the value of the game with payoff (4), then we must have

$$G(\mathbf{x}, \mathbf{y}^0) \leq v \leq G(\mathbf{x}^0, \mathbf{y})$$

for all \mathbf{x} and \mathbf{y} . By rearranging and collecting terms this relations can be replaced by the following equivalent systems:

$$(7) \quad c_i - x_i^0(1-\alpha_i)(R_i+v) \geq 0, \quad \text{for all } i,$$

$$(8) \quad \sum_{i=1}^m c_i y_i^0 - (1 - \alpha_i)(R_i + v)y_i^0 \leq 0, \quad \text{for all } i.$$

The inequalities (7) and (8), together with $\sum_{i=1}^m x_i^0 = \sum_{i=1}^m y_i^0 = 1$, can be satisfied as equalities, resulting (6) where $v = v_0$ satisfies the equation (5). This equation has clearly m real roots (including multiplicities if some of R_i 's are equal). Therefore denoting the largest real root of the equation by v_0 , we have

$$\min_i (R_i + v_0) > 0,$$

so that \mathbf{x}^0 and \mathbf{y}^0 given by (6) are mixed strategies. It evidently follows that \mathbf{x}^0 and \mathbf{y}^0 are a pair of optimal strategies and v_0 is the value of the game.

Corollary 2. If $R_1 = \dots = R_m = R$, then we obtain as the solution of the game:

$$(9) \quad \begin{aligned} v_0 &= \sum_{i=1}^m c_i (1 - \alpha_i)^{-1} - R \\ x_i^0 &= c_i (1 - \alpha_i)^{-1} / \sum_{i=1}^m c_i (1 - \alpha_i)^{-1}, \\ y_i^0 &= (1 - \alpha_i)^{-1} / \sum_{i=1}^m (1 - \alpha_i)^{-1}, \quad (i=1, \dots, m). \end{aligned}$$

Each of v_0 , \mathbf{x}^0 and \mathbf{y}^0 in the solution (6) depends on every existing parameters c_i , α_i and R_i 's. But in the identical-reward case, surprising facts happen, *i.e.*, the optimal strategy of each player is independent of the common reward R , and \mathbf{y}^0 is independent of c_i 's too. We also note that

$$(10) \quad v_0 \begin{cases} > \\ = \\ < \end{cases} 0, \quad \text{if } \sum_{i=1}^m \frac{c_i (1 - \alpha_i)^{-1}}{R_i} \begin{cases} > \\ = \\ < \end{cases} 1,$$

i.e., the game is advantageous to the searcher if and only if

$$\sum_i c_i (1 - \alpha_i)^{-1} / R_i < 1.$$

Remark. The same line of arguments as in the proof of Theorem 1 gives the following result for the case of $0 < \beta < 1$. The game with

payoff function (3) with $0 < \beta < 1$ is solved as follows: Assume that $\sum_{i=1}^m c_i(1-\alpha_i)^{-1}/R_i \leq 1$. Let v_0 be the largest real root of the equation

$$\sum_{i=1}^m \frac{c_i - (1-\beta)v}{(1-\alpha_i)(R_i + \beta v)} = 1.$$

Then v_0 is the value of the game and $v_0 \leq 0$. The optimal strategies are given by

$$x_i^0 = \frac{c_i - (1-\beta)v_0}{(1-\alpha_i)(R_i + \beta v_0)},$$

$$y_i^0 = \frac{(1-\alpha_i)^{-1}}{R_i + \beta v_0} \bigg/ \sum_{i=1}^m \frac{(1-\alpha_i)^{-1}}{R_i + \beta v_0}, \quad (i=1, \dots, m).$$

If $R_1 = \dots = R_m = R$, then these reduce to

$$v_0 = \left\{ \sum_{i=1}^m c_i(1-\alpha_i)^{-1} - R \right\} \bigg/ \left\{ \beta + (1-\beta) \sum_{i=1}^m (1-\alpha_i)^{-1} \right\},$$

$$x_i^0 = \frac{c_i - (1-\beta)v_0}{(1-\alpha_i)(R + \beta v_0)},$$

$$y_i^0 = (1-\alpha_i)^{-1} \bigg/ \sum_{i=1}^m (1-\alpha_i)^{-1}, \quad (i=1, \dots, m).$$

We shall state another corollary which will be used later, and a numerical example will be attached to it.

Let

$$(11) \quad M = \begin{bmatrix} c_1 - (1-\alpha_1)R_1 & c_2 & \dots & c_m \\ c_1 & c_2 - (1-\alpha_2)R_2 & \dots & c_m \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_m - (1-\alpha_m)R_m \end{bmatrix}.$$

and

$$(12) \quad S = \begin{bmatrix} 1-\alpha_1 & & 0 \\ & \ddots & \\ 0 & & 1-\alpha_m \end{bmatrix}.$$

Corollary 3. Let v_0 be the largest real root of the equation (5). Then v_0 is a root of the equation¹⁾

¹⁾ Given a matrix game B , let $\text{val}(B)$ denote its max-min value to player I.

$$(13) \quad \text{val}(\mathbf{M}-v\mathbf{S})=0,$$

and \mathbf{x}^0 and \mathbf{y}^0 defined by (6) are a pair of the optimal strategies for the $m \times m$ matrix game $\mathbf{M}-v_0\mathbf{S}$. Moreover both of \mathbf{x}^0 and \mathbf{y}^0 are completely mixed.

Example 1. Let $c_1=1, c_2=2; \alpha_1=1/2, \alpha_2=1/3; R_1=4, R_2=9$. Then

$$\frac{c_1(1-\alpha_1)^{-1}}{R_1} = \frac{1}{2}, \quad \frac{c_2(1-\alpha_2)^{-1}}{R_2} = \frac{1}{3},$$

and the equation (5) becomes

$$\frac{2}{4+v} + \frac{3}{9+v} = 1, \quad \text{i.e., } v^2+8v+6=0$$

which has two real roots $-4 \pm \sqrt{10}$. Hence the value of the game is $v_0 = -4 + \sqrt{10} \doteq -0.8377$, being consistent with (10). The optimal strategies are from (6)

$$\mathbf{x}^0 = \left\langle \frac{\sqrt{10}}{5}, 1 - \frac{\sqrt{10}}{5} \right\rangle \doteq \langle 0.6325, 0.3675 \rangle,$$

$$\mathbf{y}^0 = \left\langle \frac{2}{3}(\sqrt{10}-2), \frac{1}{3}(7-2\sqrt{10}) \right\rangle \doteq \langle 0.7749, 0.2251 \rangle.$$

Now since

$$\mathbf{M} = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2/3 \end{bmatrix},$$

we obtain

$$f(v) \equiv \text{val}(\mathbf{M}-v\mathbf{S}) = \text{val} \begin{bmatrix} -1 - \frac{1}{2}v & 2 \\ 1 & -4 - \frac{2}{3}v \end{bmatrix} \\ = \begin{cases} \frac{-(v^2+8v+6)}{24+(7/2)v}, & \text{if } v < -9 \text{ or } v > -4 \\ 2, & \text{if } -9 < v < -6 \\ -1 - \frac{1}{2}v, & \text{if } -6 < v < -4. \end{cases}$$

We find (see, Fig. 1) that the equation (13) has only one root $v=v_0 = -4 + \sqrt{10}$.

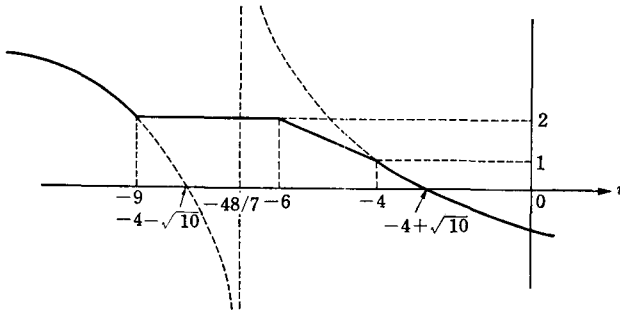


Fig. 1. Graph of $f(v)$.

As is seen in this example v_0 is the *only one* root of the equation (13). This fact will be proved later in Theorem 6.

3. Solution of the Game with $\beta=0$.

We shall consider in this section the case where $\beta=0$. From (3) and (1) the game in this case is a $m \times m$ matrix game with payoff matrix M given by (11). Let $a_i=(1-\alpha_i)R_i > 0, i=1, \dots, m$.

We assume without losing generality that the boxes are labeled such that

$$(14) \quad c_1 \leq \min_{2 \leq j \leq m} c_j$$

and

$$(15) \quad \frac{c_2 - c_1}{a_2} \leq \frac{c_3 - c_1}{a_3} \leq \dots \leq \frac{c_m - c_1}{a_m}.$$

First of all we note that the value $v(M)$ of the game satisfies

$$c_1 - (\sum_i a_i^{-1})^{-1} \leq v(M) \leq c_1,$$

since we have element-wise inequalities

$$c_1 E - \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \dots & \\ 0 & & & a_m \end{bmatrix} \leq M \leq \begin{bmatrix} c_1 & c_2 & \dots & c_m \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \dots & c_m \end{bmatrix}$$

where E denotes the $m \times m$ matrix whose elements are all unity.

We shall prove:

Theorem 4. The game with payoff matrix M is solved as follows:

(i) If $\sum_{i \geq 2} \frac{c_i - c_1}{a_i} \leq 1$, or equivalently,

$$(16) \quad \left(\sum_{i=1}^m c_i a_i^{-1} - 1 \right) / \sum_{i=1}^m a_i^{-1} \leq c_1$$

then the game has the value

$$(17) \quad v_0 = \left(\sum_{i=1}^m c_i a_i^{-1} - 1 \right) / \sum_{i=1}^m a_i^{-1},$$

and the unique simple solution

$$(18) \quad \begin{aligned} x_i^0 &= (c_i - v_0) a_i^{-1} \\ y_i^0 &= a_i^{-1} / \sum_i a_i^{-1}, \quad (i=1, \dots, m). \end{aligned}$$

(ii) If $\sum_{i \geq k+1} \frac{c_i - c_1}{a_i} \leq 1 < \sum_{i \geq k} \frac{c_i - c_1}{a_i}$, ($k=2, 3, \dots, m-1$), then the game has the value c_1 and $(m-k+1)$ basic solutions with the common $\mathbf{y}^0 = \langle 1, 0, \dots, 0 \rangle$ and $(m-k+1)$ \mathbf{x}^0 's, i.e.,

$$(19) \quad \begin{aligned} \mathbf{x}^0 &= \left\langle 0, \dots, 0, 1 - \sum_{i \geq k+1} \frac{c_i - c_1}{a_i}, \frac{c_{k+1} - c_1}{a_{k+1}}, \dots, \frac{c_m - c_1}{a_m} \right\rangle \\ &\text{or } \dots\dots\dots \\ &\text{or } \left\langle 0, \dots, 0, \frac{c_k - c_1}{a_k}, \dots, \frac{c_{m-1} - c_1}{a_{m-1}}, 1 - \sum_{i=k}^{m-1} \frac{c_i - c_1}{a_i} \right\rangle. \end{aligned}$$

(iii) If $1 \leq \frac{c_m - c_1}{a_m}$, then the payoff matrix has a saddle point at $(m, 1)$ and the saddle value is c_1 .

Proof. Let $\mathbf{x}^0, \mathbf{y}^0$ and v respectively denote a pair of optimal strategies and the value of the game with payoff matrix M , then we must have

$$M(\mathbf{x}, \mathbf{y}^0) \leq v \leq M(\mathbf{x}^0, \mathbf{y})$$

for all \mathbf{x} and \mathbf{y} .

(i): By rearranging and collecting terms this relation can be replaced by the system of linear inequalities

$$c_i - x_i^0 a_i \geq v, \quad \text{for all } i$$

$$\sum_i c_i y_i^0 - a_i y_i^0 \leq v, \quad \text{for all } i.$$

These inequalities, together with $\sum x_i^0 = \sum y_i^0 = 1$ can be satisfied as equalities, resulting (18) where $v = v_0$ satisfies the equation

$$\sum_{i=1}^m \frac{c_i - v}{a_i} = 1$$

which evidently has a unique root v_0 given by (17). If

$$\sum_{i \geq 2} \frac{c_i - c_1}{a_i} \leq 1,$$

then $c_1 \geq v_0$ from (16) and (17) and therefore, by the assumption (14) \mathbf{x}^0 given by (18) is certainly a mixed strategy. It readily follows that \mathbf{x}^0 and \mathbf{y}^0 are a pair of optimal strategies and v_0 is the value of the game. This proves the part (i) of the theorem.

(ii): For $\mathbf{y}^0 = \langle 1, 0, \dots, 0 \rangle$, we have $M(\mathbf{x}, \mathbf{y}^0) \leq c_1$, for all \mathbf{x} . For

$$\mathbf{x}^0 = \left\langle 0, \dots, 0, 1 - \sum_{i \geq k+1} \frac{c_i - c_1}{a_i}, \frac{c_{k+1} - c_1}{a_{k+1}}, \dots, \frac{c_m - c_1}{a_m} \right\rangle,$$

we have

$$\mathbf{x}^0 \mathbf{M} = \left[c_1, c_2, \dots, c_{k-1}, c_k - a_k \left(1 - \sum_{i \geq k+1} \frac{c_i - c_1}{a_i} \right), \underbrace{c_1, \dots, c_1}_{m-k} \right]$$

$$\geq [c_1, \dots, c_1],$$

since

$$c_k - a_k \left(1 - \sum_{i \geq k+1} \frac{c_i - c_1}{a_i} \right) = a_k \left\{ \sum_{i \geq k} \frac{c_i - c_1}{a_i} - 1 \right\} > c_1.$$

Thus \mathbf{x}^0 and \mathbf{y}^0 are a pair of optimal strategies and c_1 is the value of the game. Moreover \mathbf{x}^0 and \mathbf{y}^0 constitute a simple solution of the sub-matrix game of \mathbf{M} which is composed of the last $m - k + 1$ rows, the first column and the last $m - k$ columns. The proof that \mathbf{y}^0 and the other \mathbf{x}^0 given in (19) constitute a basic solution of \mathbf{M} is similar, and so it will be omitted. (See, for example, McKinsey [5], Chapter 3).

(iii): Since $c_1 \leq c_m - a_m$ in this case, $(m, 1)$ is a saddle point of the

matrix M .

Thus we have completed all of the proof.

A striking fact which is found by this theorem is that the optimal strategy \mathbf{y}^0 for the searcher is either completely mixed or concentrates to the box (or boxes) with the smallest c_i . We also note that

$$(20) \quad v(\mathbf{M}) \begin{cases} > \\ = \\ < \end{cases} 0, \text{ if } \sum_{i=1}^m \frac{c_i(1-\alpha_i)^{-1}}{R_i} \begin{cases} > \\ = \\ < \end{cases} 1$$

i.e., (10) is again valid.

An example will be given.

Example 2. Let $c_1=1, c_2=2, c_3=3; a_i=4i$ ($i=1, 2, 3$). Then since

$$\frac{c_2-c_1}{a_2} = \frac{1}{8}, \quad \frac{c_3-c_1}{a_3} = \frac{1}{6},$$

part (i) of Theorem 4 gives the value $v_0 = -\frac{6}{11}$ of the game and a unique simple solution

$$\mathbf{x}^0 = \left\langle \frac{17}{44}, \frac{14}{44}, \frac{13}{44} \right\rangle, \quad \mathbf{y}^0 = \left\langle \frac{6}{11}, \frac{3}{11}, \frac{2}{11} \right\rangle.$$

Also let $c_1=1, c_2=2, c_3=3; a_i=i$ ($i=1, 2, 3$). Then since

$$\frac{c_2-c_1}{a_2} = \frac{1}{2}, \quad \frac{c_3-c_1}{a_3} = \frac{2}{3},$$

we have from part (ii) of Theorem 4, two basic solutions

$$\left\{ \begin{array}{l} \mathbf{x}^0 = \left\langle 0, \frac{1}{3}, \frac{2}{3} \right\rangle \\ \mathbf{y}^0 = \left\langle 1, 0, 0 \right\rangle \end{array} \right\}, \quad \left\{ \begin{array}{l} \mathbf{x}^0 = \left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle \\ \mathbf{y}^0 = \left\langle 1, 0, 0 \right\rangle \end{array} \right\}$$

and the value 1 of the game. By a routine work in finding all solutions of a game we can easily check that these are the only two basic solutions.

Corollary 5. If $R_1 = \dots = R_m = R$ and $\sum_{i \geq 2} \frac{c_i - c_1}{1 - \alpha_i} \leq R$, then we obtain as the solution of the game:

$$\begin{aligned}
 v_0 &= \left\{ \sum_i c_i (1-\alpha_i)^{-1} - R \right\} / \sum_i (1-\alpha_i)^{-1}, \\
 (21) \quad x_i^0 &= \frac{c_i - v_0}{(1-\alpha_i)R}, \\
 y_i^0 &= (1-\alpha_i)^{-1} / \sum_i (1-\alpha_i)^{-1}, \quad (i=, 1, \dots, m).
 \end{aligned}$$

Again we get a striking fact, *i.e.*, the independence of y^0 of all parameters but the α_i 's in the identical-reward case.

4. A Sequential Game Involving a Moving Target

In [2] Dobbie suggested that the two-sided search problems deserved more attention than they had received in the development of search theory. In [4] Klein indicated how certain search problems involving a moving, but blind target can be formulated by the use of appropriate Markovian decision models. Although he treated one-sided models, a general description of such search problems was given.

In this section we shall treat a sequential two-sided search model which is characterized by the following two assumptions:

- (A₁): Target movements are dependent of the searcher's location, that is, the target is "non-blind" and even considered to be intellectual. In fact, the target is the maximizing player in our game model.
- (A₂): The searcher is "noisy". Hence, the target can base his movements on knowledge of the searcher's location at the end of each period.

Other sequential search games have been discussed using closely related models. Norris [8] studied the structure of some special "search games for a conscious evader." Neuts [7] examined a sequential search game in which a stationary (*i.e.*, [motionless) target and a "memoryless" searcher were involved. Recently Sweat [10] analysed a search model for stationary target by an intellectual searcher who

can learn from past history. Also Meinardi [6] studied a special two-sided sequential game in which the number of boxes increases as long as unsuccessful searches are repeated.

We now give a more precise description of the problem and its formulation within the sequential-game framework. A class of states will be used for the Markov chain to be constructed. Let the state i ($i=1, \dots, m$) indicate that the i th box has just been searched unsuccessfully. Since the target is "non-blind" and the searcher is "noisy" as stated above by the assumptions (A_1) and (A_2) , both players know correctly the present state of the Markov chain, at each stage of the sequential game.

The sequential play is terminated as soon as the target is found by the searcher. Examination cost $c_i (>0)$, overlooking probability α_i , $0 \leq \alpha_i < 1$, and reward $R_i (>0)$ of finding the target are involved as in the previous sections. Moreover travel costs between successive search locations will be counted for the searcher only. Let t_{ij} ($i, j = 1, \dots, m$) be the travel cost needed for the searcher to go from box i to box j ($\neq i$).

A stationary strategy for player II (searcher) may be represented by an m -tuple of probability m -vectors

$$\vec{y} = [y^1, \dots, y^m], \text{ each } y^i = \langle y_1^i, \dots, y_m^i \rangle,$$

and similarly, for player I (target),

$$\vec{x} = [x^1, \dots, x^m], \text{ each } x^i = \langle x_1^i, \dots, x_m^i \rangle.$$

By stationary strategies, we mean the situation in which for each of m distinct states, a probability distribution is specified for the player for use every time that state is reached, by whatever route.

Let M^k ($k=1, \dots, m$) be the matrix M defined by (11) with each c_i , $i \neq k$, replaced by $t_{ki} + c_i$. Let, for $l=1, \dots, m$.

$$(22) \quad P^{1l} = \dots = P^{ml} = \begin{bmatrix} & \overset{l}{1} & \\ & \vdots & \\ 0 & \alpha_l & 0 \\ & \vdots & \\ & 1 & \end{bmatrix} (l$$

in which $P^{kl} \equiv [p_{ij}^{kl} | 1 \leq i, j \leq m]$ and p_{ij}^{kl} denotes the transition probability from state k to state l if player I hides himself in box i and player II examines box j .

Given a pair (\vec{x}, \vec{y}) of stationary strategies for the two players, the stationary transition matrix of the Markov chain, and the payoff to player I in state k are respectively:

$$P(\vec{x}, \vec{y}) = [x^k P^{kl} y^l | 1 \leq k, l \leq m]$$

$$M^k(\vec{x}, \vec{y}) = x^k M^k y^k, \quad k = 1, \dots, m$$

if x^k and y^k are written as row and column vectors, respectively.

Also let

$$S^k = [s_{ij}^k | 1 \leq i, j \leq m] = E - \sum_{l=1}^m P^{kl}, \quad k = 1, \dots, m$$

be the matrix of stop probabilities at state k . Then we easily find that $S^1 = \dots = S^m = S$, given by (12). Therefore our sequential game belongs to "stochastic games with zero stop probabilities", discussed by Gillette [3].

For $0 < \beta < 1$, we define, following Gillette, β -discounted payoff for stationary strategies (\vec{x}, \vec{y}) as:

$$(23) \quad \Gamma_{\beta^k}(\vec{x}, \vec{y}) = \sum_{n=0}^{\infty} \beta^n \sum_{l=1}^m [P^{(n)}(\vec{x}, \vec{y})]^{kl} M^l(\vec{x}, \vec{y}), \quad k = 1, \dots, m$$

where $P^{(n)}(\vec{x}, \vec{y})$ ($n \geq 1$) and $P^{(0)}(\vec{x}, \vec{y})$, respectively, denote the n th power of the matrix $P(\vec{x}, \vec{y})$ and the identity matrix. $[P^{(n)}(\vec{x}, \vec{y})]^{kl}$ means the (k, l) element of the matrix $P^{(n)}(\vec{x}, \vec{y})$.

From results by Shapley [9] and Gillette [3] there exist stationary

strategies (\vec{x}^*, \vec{y}^*) such that for all strategies (\vec{x}, \vec{y}) and all $k=1, \dots, m$

$$\Gamma_{\beta^k}(\vec{x}, \vec{y}^*) \leq \Gamma_{\beta^k}(\vec{x}^*, \vec{y}^*) \leq \Gamma_{\beta^k}(\vec{x}^*, \vec{y}) .$$

Denote $\Gamma_{\beta^k}(\vec{x}^*, \vec{y}^*)$, simply by v^k . This represents the value of the stochastic game when the play starts with state k . We readily find that an application of Bellman's principle of optimality [1] implies that the following equations must be satisfied

$$(24) \quad v^k = \text{val} \left(M^k + \beta \sum_{l=1}^m P^{kl} v^l \right), \quad k=1, \dots, m .$$

By the use of the "value-transformation" technique (Shapley [9]) we can easily prove that (24) has a unique solution, provided that $0 < \beta < 1$.

For $m=2$, (24) becomes

$$(25) \quad \begin{aligned} v_1 &= \text{val} \begin{bmatrix} c_1 + \beta v_1 - (1 - \alpha_1)(R_1 + \beta v_1) & c_2' + \beta v_2 \\ c_1 + \beta v_1 & c_2' + \beta v_2 - (1 - \alpha_2)(R_2 + \beta v_2) \end{bmatrix} \\ v_2 &= \text{val} \begin{bmatrix} c_1' + \beta v_1 - (1 - \alpha_1)(R_1 + \beta v_1) & c_2 + \beta v_2 \\ c_1' + \beta v_1 & c_2 + \beta v_2 - (1 - \alpha_2)(R_2 + \beta v_2) \end{bmatrix} \end{aligned}$$

where $c_1' = t_{21} + c_1$, $c_2' = t_{12} + c_2$ and v^i are written as v_i .

Let A_1 and A_2 , respectively, be the matrix in the righthand side of each equation of (25). Then we obtain from Theorem 4

$$\begin{aligned} v_1 &= \left\{ \frac{c_1 + \beta v_1}{(1 - \alpha_1)(R_1 + \beta v_1)} + \frac{c_2' + \beta v_2}{(1 - \alpha_2)(R_2 + \beta v_2)} - 1 \right\} \\ &\quad / \left\{ \frac{1}{(1 - \alpha_1)(R_1 + \beta v_1)} + \frac{1}{(1 - \alpha_2)(R_2 + \beta v_2)} \right\} \\ v_2 &= \left\{ \frac{c_1' + \beta v_1}{(1 - \alpha_1)(R_1 + \beta v_1)} + \frac{c_2 + \beta v_2}{(1 - \alpha_2)(R_2 + \beta v_2)} - 1 \right\} \\ &\quad / \left\{ \frac{1}{(1 - \alpha_1)(R_1 + \beta v_1)} + \frac{1}{(1 - \alpha_2)(R_2 + \beta v_2)} \right\} \end{aligned}$$

provided that $R_i + v_i > 0$, $i=1, 2$, and both of A_1 and A_2 do not have saddle points. These give

$$\begin{cases} \frac{c_1 - (1 - \beta)v_1}{(1 - \alpha_1)(R_1 + \beta v_1)} + \frac{c_2' + \beta v_2 - v_1}{(1 - \alpha_2)(R_2 + \beta v_2)} = 1, \\ \frac{c_1' + \beta v_1 - v_2}{(1 - \alpha_1)(R_1 + \beta v_1)} + \frac{c_2 - (1 - \beta)v_2}{(1 - \alpha_2)(R_2 + \beta v_2)} = 1, \end{cases}$$

or equivalently

$$(26) \quad \begin{cases} \sum_{i=1}^2 \frac{c_i - (1-\beta)v_i}{(1-\alpha_i)(R_i + \beta v_i)} + \frac{t_{12} + t_{21}}{(1-\alpha_1)(R_1 + \beta v_1) + (1-\alpha_2)(R_2 + \beta v_2)} = 1, \\ v_1 - v_2 = \frac{t_{12}(1-\alpha_1)(R_1 + \beta v_1) - t_{21}(1-\alpha_2)(R_2 + \beta v_2)}{(1-\alpha_1)(R_1 + \beta v_1) + (1-\alpha_2)(R_2 + \beta v_2)}. \end{cases}$$

If, for example, $c_1 = c_2 = c$, $t_{12} = t_{21} = t$, $\alpha_1 = \alpha_2 = \alpha$ and $R_1 = R_2 = R$, then (26) gives $v_1 = v_2 (= v^*$, say) and

$$(27) \quad v^* = \frac{c + \frac{1}{2}\{t - (1-\alpha)R\}}{1 - \beta\left\{1 - \frac{1}{2}(1-\alpha)\right\}}.$$

We can easily check that $R + \beta v^* > 0$ and that both of A_1 , and A_2 do not have saddle points if

$$(28) \quad t < t_0 \equiv (1-\alpha)\left(R + \frac{\beta c}{1-\beta}\right).$$

Thus, for this choice of t , the optimal strategy of each player in state k is given, from (18), by

$$(29) \quad \vec{y}^* = [y^{1*}, y^{2*}], \text{ where } y^{1*} = y^{2*} = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$(30) \quad \vec{x}^* = [x^{1*}, x^{2*}], \text{ where}$$

$$x^{1*} = x^{2*} = \frac{1}{2} \left[1 - \frac{t \left\{ 1 - \beta \left(1 - \frac{1-\alpha}{2} \right) \right\}}{(1-\alpha)\{\beta(c+t/2) + (1-\beta)R\}} \right].$$

If $t \geq t_0$ then we have

$$(31) \quad \begin{aligned} v_1 = v_2 &= c/(1-\beta) \\ y^{1*} &= \langle 1, 0 \rangle, \quad y^{2*} = \langle 0, 1 \rangle, \\ x^{1*} &= \langle 0, 1 \rangle, \quad x^{2*} = \langle 1, 0 \rangle. \end{aligned}$$

The target will not be captured by the searcher, and hence the search process continues forever.

Example 3. Let $c=1$, $\alpha=\frac{1}{2}$, $R=10$ and $t=1$. Since

$$t < t_0 = 5 + \frac{1}{2} \left(\frac{\beta}{1-\beta} \right),$$

(27) gives $v^* = \frac{-1}{1 - \frac{3}{4}\beta}$. In fact, the first equation of (25) becomes in

this case

$$(1-\beta)v = \text{val} \begin{bmatrix} -4 - \frac{1}{2}\beta v & 2 \\ 1 & -3 - \frac{1}{2}\beta v \end{bmatrix}$$

$$= \begin{cases} -1 - \beta v/4, & \text{if } v < -12/\beta \text{ or } v > -8/\beta \\ -4 - \beta v/2, & \text{if } -12/\beta < v < -10/\beta \\ 1, & \text{if } -10/\beta < v < -8/\beta \end{cases}$$

yielding a unique root $v^* = -(1 - 3\beta/4)^{-1}$ (see Fig. 2).

Let c, α and R be as before, and let $t=10, \beta=8/9$. Then since $t > t_0=9$, (31) gives $v_1=v_2 = \frac{c}{1-\beta} = 9$. In fact, the first equation of (25) becomes in this case

$$\frac{1}{9}v = \text{val} \begin{bmatrix} -4 - \frac{4}{9}v & 11 \\ 1 & 6 - \frac{4}{9}v \end{bmatrix}$$

$$= \begin{cases} (7/2) - (2/9)v, & \text{if } v < -\frac{135}{4} \text{ or } v > \frac{45}{4} \\ -4 - (4/9)v, & \text{if } -\frac{135}{4} < v < -\frac{45}{4} \\ 1, & \text{if } -\frac{45}{4} < v < \frac{45}{4} \end{cases}$$

yielding a unique root $v^*=9$ (see Fig. 3).

Expressions (27)~(30) and the first half of Example 3 may suggest that the extreme case in which $\beta=1$ would also be possible. We have $\lim_{\beta \rightarrow 1} t_0 = +\infty$, and

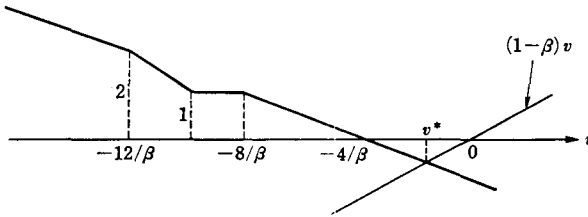


Fig. 2. Unique root v^* when $t < t_0$.

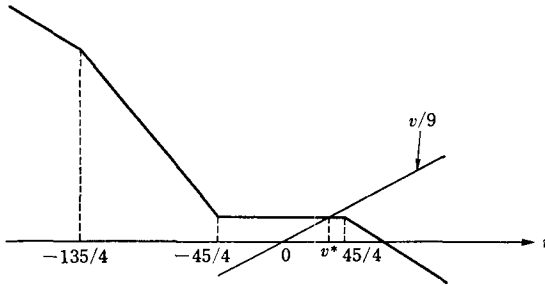


Fig. 3. Unique root v^* when $t > t_0$.

$$\lim_{\beta \rightarrow 1} v^* = (2c+t)(1-\alpha)^{-1} - R,$$

$$\lim_{\beta \rightarrow 1} x_1^{1*} = \lim_{\beta \rightarrow 1} x_2^{2*} = c/(2c+t),$$

which are consistent with (9) if $t=0$.

Although we were unable to prove or disprove the unique existence of the solution of (24) for $\beta=1$, we obtained the result that (24) can have a unique solution in some special cases. We conclude this section by showing this result in the following.

If travel costs are all zero, i.e., $t_{ij} \equiv 0$ ($i \neq j$), (22) substituted into (24) with $\beta=1$ gives $v^1 = \dots = v^m$ ($=v$, say), and v satisfies the equation (13), i.e.,

$$(13) \quad \text{val} (M - vS) = 0.$$

We shall prove:

Theorem 6. *The equation (13) has the only one real root v_0 , i.e., the*

largest real root of the equation (5).

Proof. Since

$$M - vS = \begin{bmatrix} c_1 - (1 - \alpha_1)(R_1 + v) & \cdots & c_m \\ \vdots & \ddots & \vdots \\ c_1 & \cdots & c_m - (1 - \alpha_m)(R_m + v) \end{bmatrix},$$

$f(v) \equiv \text{val}(M - vS)$ is continuous and non-increasing in v .

We also have $f(-\min_i R_i) = c_1 > 0$, if $c_1 \leq \min_{2 \leq j \leq m} c_j$, and that $\lim_{v \rightarrow \infty} f(v) = -\infty$, for

$$\begin{aligned} f(v) &= \min_{\mathbf{y}} \max \{ (M - vS)\mathbf{y} \} \leq \max \left\{ (M - vS) \begin{bmatrix} m^{-1} \\ \vdots \\ m^{-1} \end{bmatrix} \right\} \\ &= \max_{1 \leq i \leq m} \frac{1}{m} \left\{ \sum_j c_j - (1 - \alpha_i)(R_i + v) \right\} \xrightarrow{(v \rightarrow \infty)} -\infty. \end{aligned}$$

Now by Corollary 3 we have

$$f(v_0) = 0, \mathbf{x}^0(M - v_0S) \geq 0, (M - v_0S)\mathbf{y}^0 \leq 0$$

where \mathbf{x}^0 and \mathbf{y}^0 are a pair of optimal strategies given by (6) for the matrix game $M - v_0S$, and are written as a row and column vector respectively.

Since \mathbf{x}^0 is completely mixed we have, for $v < v_0$, $\mathbf{x}^0 M \geq v_0 \mathbf{x}^0 S > v \mathbf{x}^0 S$, and hence

$$f(v) = \max_{\mathbf{x}} \min \{ \mathbf{x}(M - vS) \} \geq \min \{ \mathbf{x}^0(M - vS) \} > 0.$$

Similarly, since \mathbf{y}^0 is also completely mixed we have, for $v > v_0$, $M\mathbf{y}^0 \leq v_0 S\mathbf{y}^0 < vS\mathbf{y}^0$, and hence

$$f(v) = \min_{\mathbf{y}} \max \{ (M - vS)\mathbf{y} \} \leq \max \{ (M - vS)\mathbf{y}^0 \} < 0.$$

Thus we have proved the theorem.

Theorem 6, combined with Corollary 3, gives

Corollary 7. Suppose that travel costs are all zero, i.e., $t_{ij} \equiv 0$. Then $v_1 = \cdots = v_m (\equiv v_0, \text{ say})$, where v_0 is the largest real root of the equation (5). The optimal strategy for each player in state k is independent of k , and is given by (6).

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