TWO SIMILAR QUEUES IN PARALLEL

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1. Introduction. Haight [3] has considered a system consisting of two unbounded single server queues, in which a customer, on arrival, joins the shorter queue. In the present paper, we make the simplifying assumption of symmetry between the two queues, an assumption that enables us to use generating functions to study the behavior of the stationary solution.

Thus we assume that the two servers each have an exponential service time distribution with unit mean, and that the arrivals form a Poisson process with mean 2ρ. If an arriving customer finds that both queues have equal length, he joins either with probability $\frac{1}{2}$.

We first prove that, so long as $\rho < 1$, a state of statistical equilibrium is reached. Then the equilibrium equations are converted into an equation for a bivariate generating function, by which this function is given in terms of two univariate generating functions. These two functions are shown to be meromorphic, and the positions of, and residues at, their poles are found. This enables us to express the probabilities as an infinite sum of geometric distributions. It also provides us with approximations valid when ρ is near unity, such as the result that the waiting time distribution of a customer is the same as that for a single queue with traffic intensity ρ^2.

2. Limiting behavior of the system. The first problem to be decided is whether or not the queue will settle down into a stationary state. Under the assumptions that have been made, the lengths of the two queues form a continuous time Markov process, and we first prove a lemma referring to these processes in general, giving a sufficient condition for a valid limiting distribution to exist. This lemma, which is an extension of a theorem of Foster [2] on the discrete time case, is of wide applicability, and it is hoped to publish an account of further extensions elsewhere.

We consider an irreducible Markov process $X(t)$, taking a countable number of values i, and we assume that the limits

$$q_{ij} = \lim_{t\to 0} t^{-1}\{P(X(t) = j \mid X(0) = i) - \delta_{ij}\}$$

exist and satisfy the conservation conditions $\sum_j q_{ij} = 0$.

Lemma 1. *Let $-q_{ii}$ be bounded. Then the limits*

$$p_j = \lim_{t\to\infty} P(X(t) = j \mid X(0) = i)$$

exist and are independent of i. The $\{p_j\}$ form a probability distribution if and only

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if there exist non-negative y_i such that $\sum_{j\neq i} q_{ij}y_j < \infty$ for all i, and

$$\sum_{j\neq i} q_{ij}(y_i - y_j) \geqq 1 \tag{1}$$

for all but a finite number of i.

PROOF. Let $Y(n)$ be a discrete time Markov chain with transition probabilities q_{ij}/Q $(i \neq j)$, where $Q > -q_{ii}$ for each i. Then $Y(n)$ is irreducible, and hence a Césaro limit p_j of

$$p_{ij}^{(n)} = P(Y(n) = j \mid Y(0) = i)$$

exists as $n \to \infty$, i.e.,

$$p_j = \lim_{n\to\infty} \frac{1}{n}\left[\sum_0^n p_{ij}^{(n)}\right].$$

Now define a Poisson process $N(t)$ with $N(0) = 0$, and $E\{N(t)\} = Qt$, and let $X^*(t) = Y\{N(t)\}$. Then $X^*(t)$ is a Markov process with the same transition intensities q_{ij} as $X(t)$, and $X^*(t)$ has, with probability 1, only a finite number of discontinuities in every finite interval. Hence (see, for example, [1]),

$$\begin{aligned} P(X(t) = j \mid X(0) = i) &= P(X^*(t) = j \mid X^*(0) = i) \\ &= P(Y\{N(t)\} = j \mid Y(0) = i) \\ &= \sum_{k=0}^{\infty} \frac{e^{-Qt}(Qt)^k}{k!} p_{ij}^{(k)}. \end{aligned}$$

It follows without difficulty that $P(X(t) = j \mid X(0) = i) \to p_j$, as $t \to \infty$. The $\{p_j\}$ form a probability distribution if and only if $Y(n)$ is ergodic. By a result of Foster ([2], Theorem 2) this is so if, and only if, there exist non-negative y_j such that

$$\sum_{j\neq i} Q^{-1}q_{ij}y_j + (1 + Q^{-1}q_{ii})y_i < \infty, \qquad \text{all } i,$$

and

$$\sum_{j\neq i} Q^{-1}q_{ij}y_j + (1 + Q^{-1}q_{ii})y_i \leqq y_i - 1, \quad \text{all but finitely many } i.$$

This is easily seen to be equivalent to the conditions stated above, and the lemma is proved.

In order to apply this lemma to the problem in hand, we have to consider the values of the q_{ij} for this process. The q_{ij} correspond to transitions involving only one event. Thus, if m, n are the lengths of the two queues, there is a transition of rate 2ρ corresponding to an arrival, which increases the smaller of m, n by 1. There is also a transition of rate 1 which decreases m by 1, and another which decreases n by 1. If we restrict y_{mn} to be symmetric, we have to satisfy the inequalities

$$2\rho(y_{mn} - y_{m,n+1}) + (y_{mn} - y_{m-1,n}) + (y_{mn} - y_{m,n-1})(1 - \delta_{n0}) \geqq 1 \ (m \geqq n)$$

for all but finitely many (m, n). It is easily seen that $y_{mn} = m^2 + n^2$ satisfies these inequalities for sufficiently large m, so long as $\rho < 1$. Hence we obtain

THEOREM 1. *There exists a unique limiting distribution* $\{p_{mn}\}$ *of the lengths of the two queues so long as* $\rho < 1$.

In all the analysis that follows, we shall confine attention to the case $\rho < 1$, and to the stationary distribution $\{p_{mn}\}$.

3. The equilibrium equations. These are derived from the Kolmogorov forward equations in exactly the same way as in Haight's paper [3], and we shall not, therefore, go into the details. We note that, by symmetry,

$$p_{mn} = p_{nm}. \tag{2}$$

With this simplification, the equations become, for all $m \geqq n$,

$$\begin{aligned}&\left\{\begin{matrix} 2\rho(m = n = 0) \\ 1 + 2\rho(m > n = 0) \\ 2 + 2\rho(m, n \neq 0)\end{matrix}\right\} p_{mn} \\ &\quad = \left\{\begin{matrix} 2\rho(m = n) \\ \rho(m = n + 1) \\ 0(m \geqq n + 2)\end{matrix}\right\} p_{m-1,n} + 2\rho p_{m,n-1} + p_{m,n+1} + p_{m+1,n}.\end{aligned} \tag{3}$$

Now define

$$F_r(x) = \sum_{n=0}^{\infty} p_{n+r,n} x^n \qquad (r \geqq 0, |x| \leqq 1). \tag{4}$$

Then equations (3) reduce to

$$\begin{cases} x(2\rho x + 1)F_1(x) - (1 + \rho)xF_0(x) = -p_{00}x \\ x(2\rho x + 1)F_2(x) - 2(1 + \rho)xF_1(x) + (1 + \rho x)F_0(x) = p_{00} - p_{10}x \\ x(2\rho x + 1)F_{r+1}(x) - 2(1 + \rho)xF_r(x) + F_{r-1}(x) = p_{r-1,0} - p_{r0}x \\ \qquad\qquad (r = 2, 3, \cdots) \end{cases} \tag{5}$$

LEMMA 2.

$$F(x, y) = \sum_{r=0}^{\infty} F_r(x) y^r \tag{6}$$

exists in $|x| \leqq 1$, $|y| < 1 + 2\rho$.

PROOF. Put $x = 1$ in (5) and add the first r equations.

$$(1 + 2\rho)F_{r+1}(1) - F_r(1) = -p_{r0} \leqq 0 \qquad (r \geqq 1)$$

so that $F_r(1) \leqq (1 + 2\rho)^{1-r}F_1(1)$. Hence $|F_r(x)| \leqq F_r(1) \leqq F_1(1)(1 + 2\rho)^{1-r}$, and the lemma follows.

In $|x| \leqq 1$, $|y| < 1 + 2\rho$, the equations (5) may be combined to give

$$\begin{aligned} x(2\rho x + 1)\{[F(x, y) - F(x, 0)]/y\} - 2(1 + \rho)xF(x, y) + (1 + \rho)xF(x, 0) \\ + yF(x, y) + \rho xyF(x, 0) = (y - x)F(0, y), \end{aligned}$$

or

$$(7)\quad \begin{aligned}&\{x(2\rho x + 1) - 2(1 + \rho)xy + y^2\}F(x, y)\\ &\qquad = y(y - x)F(0, y) + \{x(2\rho x + 1) - (1 + \rho)xy - \rho xy^2\}F(x, 0).\end{aligned}$$

It follows that, whenever x and y satisfy $|x| \leqq 1$, $|y| < 1 + 2\rho$, and

$$(8)\qquad x(2\rho x + 1) - 2(1 + \rho)xy + y^2 = 0,$$

then $y(y - x)F(0, y) = -\{x(2\rho x + 1) - (1 + \rho)xy - \rho xy^2\}F(x, 0)$, which may be reduced to

$$(9)\qquad y(y - x)F(0, y) = -x(2\rho x + 1)\{1 + \rho x - (1 + \rho)y\}F(x, 0).$$

4. The fundamental correspondence. We may define a symmetric $(2 - 2)$ correspondence S as follows:

DEFINITION. $Y = Sy$ if, and only if, there exists an x such that the pairs (x, y), (x, Y) both satisfy (8). Then $y + Y = 2(1 + \rho)x$, $yY = x(2\rho x + 1)$, and, eliminating x, we obtain

$$(10)\qquad \rho Y^2 - 2\{(1 + \rho + \rho^2)y - (1 + \rho)\}Y + y(1 + \rho + \rho y) = 0$$

For a given y_0, we define an "S-sequence"

$$\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots$$

such that $y_{n+1} = Sy_n$, $y_{n-1} = Sy_n$.

LEMMA 3. *Any S-sequence $\{y_n\}$ is of the form*

$$(11)\qquad y_n = A + \mu(a\lambda^n + a^{-1}\lambda^{-n})$$

where

$$(12)\qquad A = (1 + \rho)/2(1 + \rho^2),$$

λ is the real, positive root, less than unity, of

$$(13)\qquad \lambda + \lambda^{-1} = 2(1 + \rho + \rho^2)/\rho,$$

$$(14)\qquad \mu = 2^{-\frac{3}{2}}\rho^{\frac{1}{2}}/(1 + \rho^2),$$

and a is an arbitrary complex number.

PROOF. From (10)

$$\rho(y_{n+1} + y_{n-1}) = 2(1 + \rho + \rho^2)y_n - (1 + \rho).$$

Now $2\rho A = 2(1 + \rho + \rho^2)A - (1 + \rho)$, so that

$$(y_{n+1} - A) - [(1 + \rho + \rho^2)/\rho](y_n - A) + (y_{n-1} - A) = 0.$$

Hence $y_n = A + B\lambda^n + C\lambda^{-n}$ for some B, C. However, since $y_1 = Sy_0$, we may put

$$y = A + B + C, \qquad Y = A + B\lambda + C\lambda^{-1}$$

in (10), which yields an equation simplifying to

$$BC = \mu^2.$$

Writing $B = \mu a$, $C = \mu a^{-1}$ proves the lemma.

Two other results which will be used later are:

(i) Since

$$\begin{aligned} \lambda + \lambda^{-1} + 2 &= 2 + 2\frac{1+\rho+\rho^2}{\rho} \\ &= 2(1+\rho)^2/\rho = A^2/\mu^2, \\ \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} &= A/\mu \end{aligned} \tag{15}$$

(ii) If $|y| \leqq 1 + \rho$, $|Y| \leqq 1 + \rho$, the corresponding value of x given by $x = y + Y)/2(1 + \rho)$ satisfies

$$|x| \leqq \frac{|y| + |Y|}{2(1+\rho)} \leqq 1.$$

5. The univariate generating function $F(0, y)$. Suppose that $Y = Sy$, and that $|y|$, $|Y| \leqq 1 + \rho$. Then the corresponding x has $|x| \leqq 1$, and we may eliminate $F(x, 0)$ from (9) to give

$$\begin{aligned} \frac{YF(0,Y)}{yF(0,y)} &= \frac{1+\rho x - (1+\rho)Y}{1+\rho x - (1+\rho)y}\,\frac{y-x}{Y-x} = \frac{1+\rho x-(1+\rho)Y}{1+\rho x-(1+\rho)y}\,\frac{(1+2\rho)x-Y}{(1+2\rho)x-y} \\ &= \frac{(1+\rho x)(1+2\rho)x-(1+\rho)(1+2\rho)xY-(1+\rho x)Y+(1+\rho)Y^2}{(1+\rho x)(1+2\rho)x-(1+\rho)(1+2\rho)xy-(1+\rho x)y+(1+\rho)y^2} \\ &= \frac{\rho x(1-x)-(1-x)Y}{\rho x(1-x)-(1-x)y} = \frac{Y-\rho x}{y-\rho x} = \frac{(2+\rho)Y-\rho y}{(2+\rho)y-\rho Y}. \end{aligned}$$

According to Lemma 3, we may write

$$\begin{aligned} y &= A + \mu(z + z^{-1}), \\ Y &= A + \mu(\lambda z + \lambda^{-1}z^{-1}), \end{aligned} \tag{16}$$

and we may define

$$g(z) = yF(0, y). \tag{17}$$

Then

$$\frac{g(\lambda z)}{g(z)} = \frac{(2+\rho)Y-\rho y}{(2+\rho)y-\rho Y} = \frac{2A/\mu + \{(2+\rho)\lambda-\rho\}z + \{(2+\rho)\lambda^{-1}-\rho\}z^{-1}}{2A/\mu + \{2+\rho-\rho\lambda\}z + \{2+\rho-\rho\lambda^{-1}\}z^{-1}}$$

Equation (15) implies that $z + \lambda^{-\frac{1}{2}}$ is a factor of both numerator and denominator, so that

$$\frac{g(\lambda z)}{g(z)} = \frac{\gamma - \lambda^{\frac{1}{2}}z}{\lambda^{\frac{1}{2}}\gamma z - 1}, \tag{18}$$

where

$$\gamma = \frac{2 + \rho - \rho\lambda}{\lambda - (2 + \rho)} > 1. \tag{19}$$

Equation (18) is valid in $|A + \mu(z + z^{-1})|$, $|A + \mu(\lambda z + \lambda^{-1}z^{-1})| \leqq 1 + \rho$. Now, if $\lambda^{\frac{1}{2}} \leqq |z| \leqq \lambda^{-\frac{1}{2}}$, then

$$|A + \mu(z + z^{-1})| \leqq A + \mu|z| + \mu|z|^{-1} \leqq A + \mu(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}) = 2A = \frac{1+\rho}{1+\rho^2} < 1 + \rho.$$

Hence (18) is valid in

$$\lambda^{\frac{1}{2}} - \delta_1 < |\lambda z| < |z| < \lambda^{-\frac{1}{2}} + \delta_2, \qquad \text{for some } \delta_1, \delta_2 > 0,$$

and hence in

$$\lambda^{-\frac{1}{2}} - \delta < |z| < \lambda^{-\frac{1}{2}} + \delta, \qquad \text{for some } \delta > 0.$$

$g(z)$ is regular in this annulus, and may therefore be expanded in a Laurent series

$$g(z) = \sum_{-\infty}^{\infty} a_n z^n \tag{20}$$

Hence $(\lambda^{\frac{1}{2}}\gamma z - 1) \sum a_n \lambda^n z^n = (\gamma - \lambda^{\frac{1}{2}} z) \sum a_n z^n$,

$$a_n(\lambda^n + \gamma) = a_{n-1}(\lambda^{n-1}\gamma + 1)\lambda^{\frac{1}{2}},$$

$$a_n = a_{-n} = a_0(\lambda^{\frac{1}{2}}\gamma^{-1})^n \prod_{j=1}^{n} \frac{1 + \lambda^{j-1}\gamma}{1 + \lambda^j\gamma^{-1}}. \tag{21}$$

This defines $g(z)$ uniquely except for a multiplying factor. Since $a_n \sim C(\lambda^{\frac{1}{2}}\gamma^{-1})^{|n|}$ as $|n| \to \infty$, $g(z)$ is regular in $\lambda^{\frac{1}{2}}\gamma^{-1} < |z| < \lambda^{-\frac{1}{2}}\gamma$. Equation (18) may be written as

$$g(\lambda z) = \frac{\gamma - \lambda^{\frac{1}{2}} z}{\lambda^{\frac{1}{2}}\gamma z - 1} g(z),$$

which may be regarded as defining a function regular in $\lambda^{\frac{3}{2}}\gamma^{-1} < |z| < \lambda^{\frac{1}{2}}\gamma$, except for a pole at $z = \lambda^{\frac{1}{2}}\gamma^{-1}$, and coinciding with $g(z)$ in $\lambda^{\frac{1}{2}}\gamma^{-1} < |z| < \lambda^{\frac{1}{2}}\gamma$. Hence $g(z)$ can be continued into

$$\lambda^{\frac{3}{2}}\gamma^{-1} < |z| < \lambda^{\frac{1}{2}}\gamma^{-1}$$

except for a pole at $z = \lambda^{\frac{1}{2}}\gamma^{-1}$. Repeating this procedure, we can continue $g(z)$ over the whole unit disc, excluding $z = 0$, as a regular function except for poles at

$$z = \lambda^{n+\frac{1}{2}}\gamma^{-1}, \qquad (n = 0, 1, 2, \cdots).$$

This proves

THEOREM 2. *$F(0, y)$ can be continued to a meromorphic function over the whole* *y-plane. Its poles are at the points* Y_n, $n = 0, 1, 2, \cdots$, *where*

$$Y_n = A + \mu(\lambda^{n+\frac{1}{2}}\gamma^{-1} + \lambda^{-n-\frac{1}{2}}\gamma) \tag{22}$$

It is an easy matter to show that Y_n takes its smallest value at $Y_0 = (2 + \rho)/\rho^2$. Then, from the fact that, for some C, $F(0, y) - [C/(y - Y_0)]$ is regular in $|y| < Y_1$, we obtain the

COROLLARY

$$p_{0n} \sim C'[\rho^2/(2 + \rho)]^n \quad \text{as} \quad n \to \infty. \tag{23}$$

Let the residue of $g(z)$ at $\lambda^{n+\frac{1}{2}}\gamma^{-1}$ be g_n. Then $g(\lambda^{n+\frac{1}{2}}\gamma^{-1} + \zeta) = g_n/\zeta + O(1)$. From (18)

$$\frac{g(\lambda^{n+\frac{3}{2}}\gamma^{-1} + \lambda\zeta)}{g(\lambda^{n+\frac{1}{2}}\gamma^{-1} + \zeta)} = \frac{\gamma - \lambda^{n+\frac{1}{2}}\gamma^{-1}}{\lambda^{n+1} - 1} + Q(\zeta).$$

Thus

$$\frac{g_{n+1}}{\lambda g_n} = -\gamma\frac{1 - \lambda^{n+1}\gamma^{-2}}{1 - \lambda^{n+1}},$$

so that

$$g_n = g_0(-\lambda\gamma)^n \prod_{j=1}^{n} \frac{1 - \lambda^j\gamma^{-2}}{1 - \lambda^j}. \tag{24}$$

From this it is easy to see that the residue at $y = Y_n$ of $F(0, y)$ is ϕ_n, where

$$\phi_n = \phi_0 \frac{1 - \lambda^{2n+1}\gamma^{-2}}{1 - \lambda\gamma^{-2}} \frac{Y_0}{Y_n} (-\lambda^{-1}\gamma)^n \prod_{j=1}^{n} \frac{1 - \lambda^j\gamma^{-2}}{1 - \lambda^j}. \tag{25}$$

LEMMA 4. *Let* C_n *be the contour in the y-plane corresponding to* $|z| = \lambda^{n+\frac{1}{2}}$. *Then*

$$\sup_{C_n} |y^{-1}F(0, y)| \to 0, \qquad \textit{as } n \to \infty.$$

PROOF. Since $y \sim \mu/z$ as $z \to 0$ it is sufficient to prove that

$$G_n = \sup_\theta |(\lambda^{n+\frac{1}{2}}e^{i\theta})^2 g(\lambda^{n+\frac{1}{2}}e^{i\theta})| \to 0.$$

From (18),

$$\frac{G_n}{G_{n-1}} \leqq \lambda^2\gamma \sup_\theta \left|\frac{1 - \lambda^n\gamma^{-1}e^{i\theta}}{1 - \lambda^n\gamma e^{i\theta}}\right| = \lambda^2\gamma\frac{1 + \lambda^n\gamma^{-1}}{1 - \lambda^n\gamma},$$

$$G_n \leqq G_0(\lambda^2\gamma)^n \prod_{j=1}^{n} \frac{1 + \lambda^j\gamma^{-1}}{1 - \lambda^j\gamma} \to 0, \qquad \text{if } \lambda^2\gamma < 1.$$

But, if $\lambda^2\gamma \geqq 1$, then

$$\lambda^2(2 + \rho - \rho\lambda) \geqq \rho - (2 + \rho)\lambda, \qquad (2 + \rho)\lambda(1 + \lambda) \geqq \rho(1 + \lambda^3),$$

$$[(2 + \rho)/\rho] \geqq \lambda + \lambda^{-1} - 1 = 2[(1 + \rho + \rho^2)/\rho] - 1, \qquad 0 \geqq \rho.$$

The contradiction establishes the lemma.

THEOREM 3.

$$F(0, y) = F(0,0) + y \sum_{r=0}^{\infty} \frac{\phi_r}{Y_r(y - Y_r)}. \tag{26}$$

PROOF. $y^{-1}F(0, y)$ is meromorphic with poles at $y = 0$ (with residue $F(0, 0)$) and at $y = Y_r$ (with residue $\phi_r Y_r^{-1}$). By virtue of Lemma 4 we may apply Cauchy's partial fraction theorem to give (26).

COROLLARY

$$p_{n0} = -\sum_{r=0}^{\infty} \phi_r / Y_r^{n+1} \qquad (n \geqq 1). \tag{27}$$

Putting $m = n = 0$ in (3) gives

$$p_{00} = \rho^{-1} p_{10}. \tag{28}$$

6. The univariate generating function $F(x, 0)$. Equation (9) gives

$$x(2\rho x + 1)F(x, 0) = -y(y - x)F(0, y)/\{1 + \rho x - (1 + \rho)y\} \tag{29}$$

where (x, y) satisfy (8). Hence $F(x, 0)$ is an analytic function except when $y = Y_n$, or when $1 + \rho x = (1 + \rho)y$. This last equation is satisfied only when $x = 0$ or $x = 1/\rho^2$. Now define X_n as the value of x such that (X_n, Y_{n-1}) and (X_n, Y_n) both satisfy (8). Then $X_0 = 1/\rho^2$, and it follows that the poles of $F(x, 0)$ are exactly at

$$x = X_n = (Y_{n-1} + Y_n)/2(1 + \rho) = A(2 + \lambda^n \gamma^{-1} + \lambda^{-n}\gamma)/2(1 + \rho), \qquad (n = 0, 1, \cdots). \tag{30}$$

Now (29) enables us to find the residue ψ_n of $F(x, 0)$ at $x = X_n$, namely

$$\psi_n = \frac{A\lambda^{\frac{1}{2}}(X_n - Y_n)(1 - \lambda^{-2n}\gamma^2)g_n}{2(1 + \rho)X_n(2\rho X_n + 1)\{1 + \rho X_n - (1 + \rho)Y_n\}}. \tag{31}$$

It is also clear that the supremum of $F(x, 0)$ on the contour in the x-plane corresponding to $|z| = \lambda^{n+\frac{1}{2}}$ tends to zero as $n \to \infty$. Hence, as in Theorem 3, we have

THEOREM 4.

$$F(x,0) = \sum_{r=0}^{\infty} \frac{\psi_r}{x - X_r} \tag{32}$$

$$p_{nn} = -\sum_{r=0}^{\infty} \frac{\psi_r}{X_r^{n+1}} \tag{33}$$

$$\sim C\rho^{2n} \tag{34}$$

as $n \to \infty$, for some C.

7. The bivariate generating function $F(x, y)$. By equation (7), $F(x, y)$ has singularities only on the planes $x = X_n$ and $y = Y_n$. We may therefore prove

THEOREM 5.

(35) $$p_{mn} \sim C[\rho^{2m}/(2 + \rho)^{m-n}].$$

as $m, n \to \infty$ in $m \geqq n$, for some C.

PROOF. Since the nearest singularities to the origin are at $x = X_0$, and $y = Y_0$,

$$p_{n+r,n} \sim CX_0^{-n}Y_0^{-r}.$$

Putting $X_0 = 1/\rho^2$, $Y_0 = (2 + \rho)/\rho^2$ proves the result.

As in the two previous sections, we could make a detailed investigation of the properties of $F(x, y)$. However, much of the interest in a queueing system lies with the waiting time distribution, and it will be shown in the next section that this may be determined simply from a knowledge of $F(x, 0)$.

8. The waiting time distribution. The waiting time of a customer depends on the length of the queue he joins, i.e., on

(36) $$l = \min (m, n).$$

Now $E(z^l) = \sum_{l=0}^{\infty} z^l\{p_{ll} + 2\sum_{n=0}^{l-1} p_{ln}\} = 2F(z, z) - F(z, 0)$. In (7) put $x = y = z$. Then

$$z(1 - z)F(z, z) = z(1 - z)(1 + \rho z)F(z, 0),$$

so that $F(z, z) = (1 + \rho z)F(z, 0)$, and $E(z^l) = (1 + 2\rho z)F(z, 0)$. Hence the distribution $\{p_l\}$ of l is given by

(37) $$p_l = p_{ll} + 2\rho p_{l-1,l-1},$$

and is determined from Theorem 4.

The distribution of waiting time is then made up of a component of zero waiting time with probability p_0, together with an absolutely continuous component with density

(38) $$f(W) = \sum_{l=1}^{\infty} p_l \frac{W^{l-1}e^{-W}}{(l-1)!} \qquad (W = 0).$$

9. The one-pole approximation. It follows from (22) that, for n large, $Y_r \sim \mu\gamma\lambda^{-r-\frac{1}{2}}$, and from (26) that $\phi_r \sim C(-\lambda^{-1}\gamma)^r Y_r^{-1}$. Hence the rth term in the series (27) for p_{n0} is of order

$$(\lambda^{n+1}\gamma)^r \qquad \text{as } r \to \infty.$$

For all ρ, $\lambda + \lambda^{-1} \geqq 6$, and hence $\lambda \leqq 3 - 2\sqrt{2} \simeq 0.17$. Hence, for n fairly large, $\lambda^{n+1}\gamma$ will be very small, and we can safely neglect all but a few terms of the series. Even for $n = 1$ (when $\lambda^2\gamma$ decreases from 1 to 0.17 as ρ increases from 0 to 1) this will be valid so long as ρ is not too small.

Hence, in fairly heavy traffic, we may obtain a reasonable approximation by taking only the first term of the series for p_{n0}. Similar remarks hold for the other series, so that

$$p_{mn} \simeq C \frac{\rho^{2m}}{(2 + \rho)^{m-n}} \qquad (m \geqq n) \tag{39}$$
$$p_{00} \simeq C \frac{\rho}{2 + \rho}$$

for some C. Equation (37) then shows that

$$p_l \simeq C(1 + 2\rho)\rho^{2l} \qquad (l > 0). \tag{40}$$

Thus we are led to

THEOREM 6. *In heavy traffic the distribution of waiting time is approximately the same as for a single queue with traffic intensity* ρ^2.

10. Related problems. Haight [3] also considered the case in which a customer is permitted to change queues if by so doing he could improve his position. Under the symmetry conditions that have been imposed in this paper, this process is equivalent, from the point of view of the total number queueing, to a single queue with two servers. The determination of the waiting time distribution is, however, no longer a simple matter, since the order in either queue is not necessarily the order of arrival.

The problem considered in this paper is an example of a random walk on positive integer pairs, with rather complicated boundary conditions. The method of attack used may be generalized to deal with other problems of this sort, and it is hoped to publish an account of this work elsewhere.

This same method, together with the use of the Laplace transform, may also be used to study the transient behavior of the double queue and of other random walks.

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