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Two solutions for a singular elliptic equation by variational methods

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Abstract. We find two nontrivial solutions of the equation $-\Delta u = (-\frac{1}{u^\beta} + \lambda u^p)\chi_{\{u>0\}}$ in Ω with Dirichlet boundary condition, where $0 < \beta < 1$ and $0 < p < 1$. In the first approach we consider a sequence of ε-problems with $1/u^\beta$ replaced by $u^q/(u+\varepsilon)^{q+\beta}$ with $0 < q < p < 1$. When the parameter $\lambda > 0$ is large enough, we find two critical points of the corresponding ε-functional which, at the limit as $\varepsilon \to 0$, give rise to two distinct nonnegative solutions of the original problem. Another approach is based on perturbations of the domain Ω, we then find a unique positive solution for λ large enough. We derive gradient estimates to guarantee convergence of approximate solutions u_ε to a true solution u of the problem.

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1. Introduction

In this paper we prove that the problem

$$\begin{cases} -\Delta u = \left(-\dfrac{1}{u^\beta} + \lambda u^p\right)\chi_{\{u>0\}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has two nonnegative solutions when the parameter $\lambda > 0$ is large. The expression $\chi_{\{u>0\}}$ denotes the characteristic function corresponding to the set $\{u > 0\}$. Hereafter, $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain, $0 < \beta < 1$ and $0 < p < 1$. By a solution we mean a function $u \in H_0^1(\Omega)$ satisfying (1.1) in the weak sense, that is,

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega \cap \{u>0\}} \left(-\frac{1}{u^\beta} + \lambda u^p\right)\varphi$$

for every $\varphi \in C_c^1(\Omega)$.

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There are a few recent papers where a variational approach is pursued for treating an equation with a singular nonlinearity on the right hand side, namely

$$-\Delta u = \frac{1}{u^\beta} + \lambda u^p, \tag{1.2}$$

see [5, 6, 18, 22, 23], see also [11] for nonvariational techniques used to seek positive solutions of equation (1.2). Problem (1.1) has been studied in [7, 10, 13, 16, 19] with the aid of nonvariational techniques. Nonlinear singular boundary value problems arise in several physical models such as fluid mechanics, pseudoplastic flows, chemical heterogeneous catalysts, non-Newtonian fluids and biological pattern formation, for more details about these subjects, we quote the papers [4, 8, 9, 17, 20]. Equation (1.1) is also intimately related to free boundary problems, see [2, 24, 25].

We define the perturbation

$$g_\varepsilon(u) = \begin{cases} \dfrac{u^q}{(u+\varepsilon)^{q+\beta}} & \text{for } u \geq 0 \\ 0 & \text{for } u < 0, \end{cases} \tag{1.3}$$

where $0 < q < p < 1$ and the corresponding perturbed problem

$$\begin{cases} -\Delta u + g_\varepsilon(u) = \lambda u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Since $g_\varepsilon \geq 0$ and is continuous, then $G_\varepsilon(u) = \int_0^u g_\varepsilon(s)ds \geq 0$. We define the C^1 functional $I_\varepsilon : H_0^1(\Omega) \to \mathbb{R}$ corresponding to (1.4) by

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G_\varepsilon(u) - \frac{\lambda}{p+1}\int_\Omega (u^+)^{p+1}.$$

Our aim is to show that I_ε satisfy the assumptions of the Mountain Pass Theorem. This allows us to find two distinct nontrivial solutions of problem (1.4). Letting $\varepsilon \to 0$ these two solutions do not tend to zero neither collapse at the same limit, they tend to two distinct nontrivial solutions of (1.1). For that matter, the main ingredient is a gradient estimate for solutions u_ε of (1.4) that allows us to conclude that u_ε tend to a solution u of (1.1) as $\varepsilon \to 0$, according to Sections 2, 3 and 4. When taking the limit we need to be careful since the gradient estimate provided by Lemma 3.1 is local. We state our first existence result.

Theorem 1.1. *There is a* $\lambda_0 > 0$ *such that problem* (1.1) *has two distinct nontrivial nonnegative solutions for* $\lambda > \lambda_0$.

Instead of working with problem (1.4), we also develop an approach to study problem (1.1) based on perturbations of the domain Ω, in Section 5. We then find a positive solution for a large $\lambda > 0$. Moreover, this solution is bounded from below by $c\varphi_1^{2/(1+\beta)}$, that is, a constant $c > 0$ times the first positive eigenfunction $\varphi_1 \in H_0^1(\Omega)$ of $-\Delta$. This allows us to apply Hardy-Rellich inequality to show that the solution is in fact unique for a certain range of p. A reference about this inequality is the paper [3], where they prove

$$\int_\Omega \frac{\varphi^2}{\varphi_1^2} \leq \Lambda \int_\Omega |\nabla \varphi|^2 \tag{1.5}$$

for every $\varphi \in C_c^\infty(\Omega)$, where $\Lambda > 0$ is a constant (the best one) depending only on Ω.

In the domain perturbation approach we work directly with the limit functional

$$I(u) = \frac{1}{2}\int_{\Omega_k} |\nabla u|^2 + \int_{\Omega_k} \frac{1}{1-\beta}(u^+)^{1-\beta} - \frac{\lambda}{p+1}\int_{\Omega_k} (u^+)^{p+1}$$

over a sequence of nested subdomains $\emptyset \neq \Omega_1 \subset\subset \Omega_2 \ldots \subset\subset \Omega$ such that $\Omega = \bigcup_{k=1}^\infty \Omega_k$. Our second existence result reads as follows.

Theorem 1.2. *There is* $\lambda^0 > 0$ *such that problem* (1.1) *has a positive solution for* $\lambda > \lambda^0$*. Moreover, there is* $0 < p^0 < 1$ *small, such that if* $0 < p < p^0$*, then the solution is unique in the class of functions* $u > c\varphi_1^{2/(1+\beta)}$ *for some* $c > 0$.

We are unable to prove that the solutions of Theorem 1.1 are positive. We conjecture that one of them is positive and the other one vanishes somewhere in Ω. This would match the result of [21] in the radial setting. The existence of multiple solutions of problem (1.1) is not surprising since in [21] it is established the existence of a positive radial solution u of (1.1) on a ball $B_R(0)$ with $u(R) = u'(R) = 0$. This free boundary solution could be used to produce infinitely many solutions on a domain Ω with finitely many separated bumps supported on balls in the interior of Ω. One of the achievements of our results in the present paper is the variational characterization of the solutions. For results dealing with $p > 1$, see [15] and references therein. When $\lambda > 0$ is small enough, it is easy to see that there is no positive solution of (1.1), see [16].

In [13] the authors proved the existence of a maximal solution u_λ of problem (1.1), which is positive and has the lower bound $u_\lambda \geq c\varphi_1^{2/(1+\beta)}$ with $c > 0$ for $\lambda > 0$ large enough. Therefore the solution we found in Theorem 1.2 is precisely the maximal one when λ is large. This result is also related to the one in [12], where the author proved that for λ grater then a precise constant, than the maximal solution u_λ is a strict local minimizer of I in the convex subset of $H_0^1(\Omega)$ of non-negative functions in Ω. By results from [13] one sees that our solutions belong to $C^{1,(1-\beta)/(1+\beta)}_{\mathrm{loc}}(\Omega)$.

2. Two solutions of the perturbed problem

We proceed to show that the perturbed functional I_ε has two nontrivial critical points, a global minimum and a mountain pass, whenever $\lambda > 0$ is large and $\varepsilon > 0$

is sufficiently small. We need to prove estimates for the associated critical levels which are independent of the value of the parameter ε. Allowing us to show that weak limits of the critical points of the perturbed functional, obtained by making $\varepsilon \to 0$, converge to nontrivial and distinct functions in $H_0^1(\Omega)$.

Denoting by $\varphi_1 > 0$ the first normalized eigenfunction of the operator $-\Delta$ in $H_0^1(\Omega)$, we may state our first preliminary result.

Lemma 2.1. *There exist* $\lambda_0 > 0$ *and* $a_1,\ b_1 > 0$ *such that, for every* $\lambda \geq \lambda_0$ *and every* $\varepsilon > 0$*, we have*

$$\max_{0\leq s\leq 1} I_\varepsilon(s\varphi_1) \leq a_1 < \infty \tag{2.1}$$

and

$$I_\varepsilon(\varphi_1) \leq -b_1 < 0. \tag{2.2}$$

Proof. From (1.3), we obtain

$$g_\varepsilon(t) \leq |t|^{-\beta} \text{ for every } t \neq 0. \tag{2.3}$$

Therefore, since $0 < \beta < 1$, we obtain $|G_\varepsilon(t)| \leq |t|^{1-\beta}/(1-\beta)$, for every $t \in \mathbb{R}$. Consequently, for $0 \leq s \leq 1$,

$$I_\varepsilon(s\varphi_1) \leq \frac{s^2}{2} + \frac{s^{1-\beta}}{(1-\beta)} \int_\Omega \varphi_1^{1-\beta} - \frac{\lambda s^{p+1}}{p+1} \int_\Omega \varphi_1^{p+1},$$

since $\|u\|_{H_0^1} = 1$. The estimates (2.1) and (2.2) follow immediately from the above inequality. □

Next lemma implies, in particular, that the functional I_ε is coercive and bounded from below. Combined with Lemma 2.1, it will be used to show that this functional has two nontrivial critical points.

Lemma 2.2. *Given* $\lambda > 0$*, there exist* $a_2,\ b_2 > 0$ *and* $0 < \rho < 1$ *such that, for every* $0 < \varepsilon < 1$,

$$I_\varepsilon(u) \geq a_2 > 0 \quad \textit{for every} \quad u \in \partial B_\rho(0) \quad \textit{such that} \quad \|u\|_{H_0^1} = \rho, \tag{2.4}$$

$$I_\varepsilon(u) \to \infty \quad \textit{as} \quad \|u\|_{H_0^1} \to \infty \tag{2.5}$$

and

$$I_\varepsilon(u) \geq -b_2 > -\infty \quad \textit{for every} \quad u \in H_0^1(\Omega). \tag{2.6}$$

Proof. Given $0 < \varepsilon < 1$, from (1.3), we have that $g_\varepsilon(t) \geq t^q/(t+1)^{q+\beta}$ for every $t \geq 0$. Since $0 < q < p$, we may find $\delta = \delta(\lambda) > 0$ such that

$$g_\varepsilon(t) \geq \lambda t^p \quad \text{for every} \quad 0 \leq t \leq \delta. \tag{2.7}$$

Since $p + 1 < 2$, there exist $C_1 > 0$ and $2 < \sigma < 2N/(N-2)$ if $N \geq 3$ ($2 < \sigma < \infty$ if $N = 2$) so that

$$t^{p+1} \leq C_1 t^{\sigma} \text{ for every } t \geq \delta. \tag{2.8}$$

Recalling that $G_\varepsilon \geq 0$, we may use (2.7), (2.8) and the Sobolev Imbedding Theorem to obtain $C_2 > 0$ such that, for every $0 < \varepsilon < 1$,

$$\begin{aligned} I_\varepsilon(u) &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \frac{\lambda C_1}{p+1}\int_{\{u>0\}} u^{\sigma} \\ &\geq \|u\|^2_{H_0^1(\Omega)}\left(\frac{1}{2} - C_2\|u\|^{\sigma-2}_{H_0^1(\Omega)}\right) \text{ for every } u \in H_0^1(\Omega). \end{aligned}$$

Hence, taking $0 < \rho < 1$ sufficiently small, we obtain $I_\varepsilon(u) \geq a_2 := \rho^2/4$ for every u such that $\|u\|_{H_0^1} = \rho$. The relation (2.4) is proved.

Next we use the fact that $G_\varepsilon \geq 0$ and the Sobolev Imbedding Theorem once more to find $C_3 > 0$ such that, for every $\varepsilon > 0$,

$$I_\varepsilon(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - C_3\|u\|^{p+1}_{H_0^1(\Omega)} \text{ for every } u \in H_0^1(\Omega).$$

The above estimate and $0 < p < 1$ imply that (2.5) and (2.6) are true. □

Given a Banach space E, we recall that a functional $\Phi \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale (PS) condition if every sequence $(u_n) \subset E$, satisfying $\Phi(u_n) \to c$ and $\|\Phi'(u_n)\| \to 0$ as $n \to \infty$, has a convergent subsequence.

From now on in this Section we fix λ, where $\lambda \geq \lambda_0 > 0$ and λ_0 is given by Lemma 2.1. Next proposition provides the existence of two critical points for the functional I_ε.

Proposition 2.3. *Suppose* $0 < \varepsilon < 1$. *Then the functional* I_ε *possesses a global minimum* u_ε^1 *and a mountain pass critical point* u_ε^2 *satisfying*

$$-\infty < -b_2 \leq c_\varepsilon^1 := I_\varepsilon(u_\varepsilon^1) \leq -b_1 < 0 \tag{2.9}$$

and

$$0 < a_2 \leq c_\varepsilon^2 := I_\varepsilon(u_\varepsilon^2) \leq a_1 < \infty; \tag{2.10}$$

where a_1, b_1 *and* a_2, b_2 *are given by Lemmas* 2.1 *and* 2.2, *respectively, and do not depend on* $0 < \varepsilon < 1$.

Proof. First we claim that the functional I_ε satisfies the (PS) condition. Indeed, given a sequence u_n in $H_0^1(\Omega)$ satisfying $I_\varepsilon(u_n) \to c$ and $\|I_\varepsilon'(u_n)\| \to 0$, as $n \to \infty$, by Lemma 2.2-(2.5), we assert that u_n is a bounded sequence. Observing that the nonlinear term $f(t) = \lambda(t^+)^p - g_\varepsilon(t)$ is continuous and has subcritical growth at infinity, we use the Sobolev Imbedding Theorem (see [26]) to derive that u_n possesses a convergent subsequence. The claim is proved.

By the above claim, Lemma 2.1-(2.2) and Lemma 2.2-(2.6), we conclude that the functional I_ε has a global minimum u_ε^1 satisfying (2.9). Defining

$$c_\varepsilon^2 := \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I_\varepsilon(\gamma(t)), \tag{2.11}$$

where

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega));\ \gamma(0) = 0,\ \gamma(1) = \varphi_1\}. \tag{2.12}$$

By Lemma 2.1, Lemma 2.2-(2.4) and the fact that I_ε satisfies the (PS) condition, we may invoke the Mountain Pass Theorem [1] to conclude that c_ε^2 is a critical level of the functional I_ε and that the associated critical point u_ε^2 satisfies (2.10). □

Let u be a critical point of the functional I_ε. Then, setting $u^- := u^+ - u$, we have

$$0 = I_\varepsilon'(u)u^- = -\int_\Omega |\nabla u^-|^2 + \int_\Omega g_\varepsilon(u)u^- - \lambda \int_\Omega (u^+)^p u^- = -\|u^-\|^2_{H_0^1(\Omega)}.$$

Thus $u^- \equiv 0$, and one concludes that $u \ge 0$. Consequently, u is a nonnegative weak solution of the perturbed problem (1.4). We also note that by standard regularity argument, the weak solutions of (1.4) are classical solutions. Next lemma provides an a priori bound in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$ for the solutions u of (1.4)

Lemma 2.4. *There exists $S > 0$, independent of $0 < \varepsilon < 1$, such that, for every solution u of (1.4),*

$$\|u\|_{H_0^1(\Omega)} \le S \tag{2.13}$$

and

$$\|u\|_{L^\infty(\Omega)} \le S. \tag{2.14}$$

Proof. Let $u \in H_0^1(\Omega)$ be a solution of (1.4). By the Sobolev Imbedding Theorem and the fact that $g_\varepsilon(t)t \ge 0$, we find $C_1 > 0$ such that

$$\|u\|^2_{H_0^1(\Omega)} \le \int_\Omega |\nabla u|^2 + \int_\Omega g_\varepsilon(u)u = \lambda \int_\Omega u^{p+1} \le C_1 \|u\|^{p+1}_{H_0^1(\Omega)}.$$

The estimate (2.13) is a direct consequence of the above inequality. Now, applying a version of Brezis-Kato argument for singular problems [5, 6] (see also [23] for a related result) we obtain estimate (2.14). □

Given a sequence ε_n in the interval $(0, 1)$, we denote by u_n^1 and u_n^2, respectively, the two solutions $u_{\varepsilon_n}^1$ and $u_{\varepsilon_n}^2$ of (1.4) provided by Proposition 2.3.

Proposition 2.5. *Suppose $\varepsilon_n \subset (0, 1)$ is a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. Then u_n^1 and u_n^2 have subsequences which converge weakly in $H_0^1(\Omega)$ to u^1 and u^2, respectively. Moreover, u^1 and u^2 are nontrivial and distinct.*

Proof. From the estimate (2.13) provided by Lemma 2.4 we may find subsequences (still denoted by u_n^i, $i = 1, 2$) such that, for $i = 1, 2$,

$$\begin{cases} u_n^i \rightharpoonup u^i \text{ weakly in } H_0^1(\Omega); \\ u_n^i \to u^i \text{ strongly in } L^r(\Omega); \\ u_n^i \to u^i \text{ a.e in } \Omega; \\ |u_n^i| \le h_r \text{ a.e in } \Omega \text{ for some } h_r \in L^r(\Omega), \end{cases} \tag{2.15}$$

where $1 \le r < 2N/(N-2)$ if $N \ge 3$ ($1 \le r < \infty$ if $N = 2$). Since u_n^i is a critical point of $I_n := I_{\varepsilon_n}$, we have that $u_n^i \ge 0$ and

$$\int_\Omega |\nabla u_n^i|^2 + \int_\Omega g_{\varepsilon_n}(u_n^i)u_n^i = \lambda \int_\Omega (u_n^i)^{p+1}, \ i = 1, 2.$$

The above relations and Proposition 2.3, imply

$$I_n(u_n^1) = \int_\Omega \left[G_{\varepsilon_n}(u_n^1) - \frac{1}{2} g_{\varepsilon_n}(u_n^1)u_n^1 \right] + \frac{\lambda(p-1)}{2(p+1)} \int_\Omega (u_n^1)^{p+1} \le -b_1 < 0, \tag{2.16}$$

and

$$I_n(u_n^2) = \int_\Omega \left[G_{\varepsilon_n}(u_n^2) - \frac{1}{2} g_{\varepsilon_n}(u_n^2)u_n^2 \right] + \frac{\lambda(p-1)}{2(p+1)} \int_\Omega (u_n^2)^{p+1} \ge a_2 > 0. \tag{2.17}$$

We claim that, for $i = 1, 2$,

$$\int_\Omega g_{\varepsilon_n}(u_n^i)u_n^i \to \int_\Omega (u^i)^{1-\beta}, \ \text{as } n \to \infty, \tag{2.18}$$

and

$$\int_\Omega G_{\varepsilon_n}(u_n^i) \to \frac{1}{1-\beta} \int_\Omega (u^i)^{1-\beta}, \ \text{as } n \to \infty. \tag{2.19}$$

Assuming the above claim and taking into account (2.15)–(2.17), we obtain

$$\frac{1+\beta}{2(1-\beta)} \int_\Omega (u^1)^{1-\beta} + \frac{\lambda(p-1)}{2(p+1)} \int_\Omega (u^1)^{p+1} \le -b_1 < 0,$$

$$\frac{1+\beta}{2(1-\beta)} \int_\Omega (u^1)^{1-\beta} + \frac{\lambda(p-1)}{2(p+1)} \int_\Omega (u^1)^{p+1} \ge a_2 > 0.$$

The above inequalities imply that u^1 and u^2 are nontrivial and distinct. To accomplish, it suffices to verify (2.18) and (2.19).

Without loss of generality we assume $i = 1$. By (2.3) and (2.15), we have

$$|g_{\varepsilon_n}(u_n^1)(u_n^1)| \le |u_n^1|^{1-\beta} \le 1 + h_1 \in L^1(\Omega) \text{ a.e. in } \Omega.$$

The above inequality and (2.15) imply $g_{\varepsilon_n}(u_n^1)(u_n^1) \to (u^1)^{1-\beta}$ a.e. in Ω. Thus, relation (2.18) is a direct consequence of these facts and the Lebesgue Dominated Convergence Theorem. Invoking (2.3) and (2.15) one more time, we get

$$|G_{\varepsilon_n}(u_n^1)| \leq \frac{(u_n^1)^{1-\beta}}{1-\beta} \leq \frac{1+h_1}{1-\beta} \in L^1(\Omega), \text{ a.e. in } \Omega. \tag{2.20}$$

We also assert that

$$G_{\varepsilon_n}(u_n^1) \to \frac{(u^1)^{1-\beta}}{1-\beta} \text{ a.e. in } \Omega. \tag{2.21}$$

Indeed, by (2.15) we may assume that $u_n^1 \to u^1$ a.e. as $n \to \infty$. Moreover, setting $\xi_n(t) := g_{\varepsilon_n}(t)\chi_{\{0<t<u_{\varepsilon_n}^1\}}$, where χ_A is the characteristic function of the set $A \subset \mathbb{R}$, we have

$$G_{\varepsilon_n}(u_n^1) = \int_{\mathbb{R}} \xi_n(t)\, dt \text{ for every } n \in \mathbb{N}.$$

Since $u_n^1 \to u^1$ a.e in Ω, we obtain $\xi_n(t) \to t^{-\beta}\chi_{\{0<t<u^1\}}$ for every $t \in \mathbb{R} \setminus \{0, u^1\}$. Notice that by (2.14), there exists $C_1 > 0$ such that $\xi_n(t) \leq |t|^{-\beta}\chi_{\{0<t<C_1\}} \in L^1(\mathbb{R})$ for every $t \in \mathbb{R}$.

By virtue of (2.20) and (2.21), we may apply the Dominated Convergence Theorem to derive (2.19). The claim is proved. □

Our aim now is to get estimates for solutions of (1.4) and prove that in the limit as $\varepsilon \to 0$ the functions u_1 and u_2, given by Proposition 2.5, are indeed solutions of (1.1).

3. Gradient estimates

In this section we shall obtain a local gradient estimate for solutions u_ε to the perturbed equation (1.4).

Let ψ be such that

$$\psi \in C^2(\overline{\Omega}),\ \psi > 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega \text{ and } \frac{|\nabla\psi|^2}{\psi} \text{ is bounded in } \Omega. \tag{3.1}$$

Observe that an example is $\psi = \varphi_1^2$. Another remark is that a solution u_ε of (1.4) is nontrivial, nonnegative and by elliptic regularity belongs to $C^3(\Omega) \cap C^1(\overline{\Omega})$. We need these informations in the estimates below. The approach is similar to the one in [14]. Here we cannot use the maximum principle to ensure that u_ε is positive or identically zero, since $u^{q-1}/(u+\varepsilon)^{q+\beta}$ is singular when u is close to 0.

Lemma 3.1. *If u_ε is a solution of* (1.4)*, then there is a constant $M > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\psi(x)|\nabla u_\varepsilon(x)|^2 \leq M(u_\varepsilon(x)^{1-\beta} + u_\varepsilon(x)) \textit{ for every } x \in \Omega, \tag{3.2}$$

where M depends only on Ω, N, β, ψ and S. Notice that from (2.14) *we have* $\|u_\varepsilon\|_{L^\infty(\Omega)} \le S$.

Proof. The solutions of (1.4) are a priori bounded by (2.14), so the constant M does not depend on ε. In the course of the prove we shall denote u_ε by simply u. Write

$$h_\varepsilon(u) = \frac{u^q}{(u+\varepsilon)^{q+\beta}} - \lambda u^p. \tag{3.3}$$

Define

$$Z(u) = u^{1-\beta} + u + \delta \qquad \text{for a small } \delta > 0, \tag{3.4}$$

and the functions

$$w = \frac{|\nabla u|^2}{Z(u)}, \qquad v = w\psi. \tag{3.5}$$

At the end of the proof we will let $\delta \to 0$. For that matter we need to derive estimates with constants independent of δ. The strategy is to prove the estimate by contradiction, so we assume that (3.2) fails, i.e. that

$$\sup_\Omega v > \tilde{M}, \tag{3.6}$$

where $\tilde{M} > 0$ will be chosen later independently of ε.

Since v is continuous in $\overline{\Omega}$, therefore it attains its maximum at some point $x_0 \in \overline{\Omega}$. Hence, by (3.6)

$$v(x_0) > \tilde{M}. \tag{3.7}$$

Then $x_0 \in \Omega$, because $v = 0$ on $\partial\Omega$. Hence

$$\nabla v(x_0) = 0 \tag{3.8}$$

and

$$\Delta v(x_0) \le 0. \tag{3.9}$$

We are going to compute Δv and evaluate at x_0. As we shall see this leads to the absurd $\Delta v(x_0) > 0$ if one fixes M large enough. We proceed with the computations:

$$\Delta v = \psi \Delta w + w \Delta \psi + 2\nabla w \nabla \psi. \tag{3.10}$$

The derivatives of w are (where the convention of summation over repeated indices is adopted)

$$\partial_i w = \frac{2\partial_j u\, \partial_{ij} u\, Z(u) - |\nabla u|^2 Z'(u) \partial_i u}{Z(u)^2} \tag{3.11}$$

and

$$\Delta w = \partial_{ii} w = \frac{2(\partial_{ij}u)^2 Z(u) + 2\partial_j u\, \partial_j(\Delta u)\, Z(u) - |\nabla u|^4 Z''(u) - |\nabla u|^2 Z'(u)\Delta u}{Z(u)^2} - 2\frac{Z'(u)}{Z(u)}\partial_i u \partial_i w.$$

Using equation (3.11) we obtain

$$\Delta w = \frac{2(\partial_{ij}u)^2 Z(u) + 2|\nabla u|^2 Z(u) h'_\varepsilon(u) - |\nabla u|^4 Z''(u) - |\nabla u|^2 Z'(u) h_\varepsilon(u)}{Z(u)^2} - 2\frac{Z'(u)}{Z(u)}\partial_i u \partial_i w. \tag{3.12}$$

From now on all functions appearing in the expressions below are evaluated at the point x_0. Relation (3.8) provides

$$\psi \nabla w + w \nabla \psi = 0$$

and hence

$$\nabla w \nabla \psi = -w \frac{|\nabla \psi|^2}{\psi}.$$

Replacing in (3.10),

$$\Delta v = \psi \Delta w + w \left(\Delta \psi - 2\frac{|\nabla \psi|^2}{\psi} \right). \tag{3.13}$$

Inserting (3.12) into (3.13),

$$\begin{aligned} \Delta v &= \psi \Delta w + w \left(\Delta \psi - 2\frac{|\nabla \psi|^2}{\psi} \right) \\ &= \psi \Bigg[\frac{2(\partial_{ij}u)^2 Z(u) + 2|\nabla u|^2 h'_\varepsilon(u) Z(u) - |\nabla u|^4 Z''(u) - |\nabla u|^2 Z'(u) h_\varepsilon(u)}{Z(u)^2} \\ &\qquad - 2\frac{Z'(u)}{Z(u)}\partial_i u \partial_i w \Bigg] + w \left(\Delta \psi - 2\frac{|\nabla \psi|^2}{\psi} \right) \end{aligned}$$

which is equivalent to

$$\begin{aligned} \Delta v = \frac{1}{Z(u)} \Big[& 2\psi(\partial_{ij}u)^2 + 2\psi Z(u) h'_\varepsilon(u) w - \psi Z(u) Z''(u) w^2 \\ & - \psi w h_\varepsilon(u) Z'(u) - 2\psi Z'(u) \partial_i u \partial_i w \Big] + w \left(\Delta \psi - 2\frac{|\nabla \psi|^2}{\psi} \right). \end{aligned} \tag{3.14}$$

We are going to assume, without loss of generality, that $\nabla u(x_0)$ is parallel to the first coordinate axis. Then from (3.8) we have

$$\partial_1 v(x_0) = 0. \tag{3.15}$$

By virtue of (3.11) we obtain the following expression

$$\partial_{11} u = \frac{1}{2} w \left(Z'(u) - \frac{\partial_1 \psi}{\psi \partial_1 u} Z(u) \right)$$

which combined with (3.14) yields

$$\begin{aligned}\Delta v \geq \frac{1}{Z(u)} \Bigg[& \frac{1}{2} \psi w^2 \left(Z'(u)^2 + \frac{(\partial_1 \psi)^2}{\psi^2 (\partial_1 u)^2} Z(u)^2 - 2 Z(u) Z'(u) \frac{\partial \psi}{\psi \partial_1 u} \right) \\ & + 2\psi Z(u) h'_\varepsilon(u) w - \psi Z(u) Z''(u) w^2 - \psi w h_\varepsilon(u) Z'(u) - 2\psi Z'(u) \partial_1 u \partial_1 w \Bigg] \\ & + w \left(\Delta \psi - 2 \frac{|\nabla \psi|^2}{\psi} \right).\end{aligned} \tag{3.16}$$

We estimate now some of the terms appearing in the above expression. From (3.15) and (3.5) we obtain the relation

$$\psi \partial_1 w = -w \partial_1 \psi$$

and therefore

$$\begin{aligned} 2\psi Z'(u) \partial_1 u \partial_1 w &= -2 Z'(u) \partial_1 u w \partial_1 \psi \\ &\leq 2 Z'(u) Z(u)^{1/2} \psi^{1/2} w^{3/2} \sup_\Omega \frac{|\nabla \psi|}{\psi^{1/2}}. \end{aligned} \tag{3.17}$$

On the other hand

$$\begin{aligned} \frac{1}{2} w^2 \frac{(\partial_1 \psi)^2}{\psi (\partial_1 u)^2} Z(u)^2 &= \frac{1}{2} \frac{(\partial_1 \psi)^2}{\psi} Z(u) w \\ &\geq -\frac{1}{2} \left(\sup_\Omega \frac{|\nabla \psi|^2}{\psi} \right) Z(u) w. \end{aligned} \tag{3.18}$$

We also have

$$-w^2 Z(u) Z'(u) \frac{\partial_1 \psi}{\partial_1 u} \geq -\left(\sup_\Omega \frac{|\nabla \psi|}{\psi^{1/2}} \right) Z'(u) Z(u)^{1/2} \psi^{1/2} w^{3/2}. \tag{3.19}$$

The last term to estimate is

$$w\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right) \geq -w\sup_{\Omega}\left(\Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right). \tag{3.20}$$

Putting together (3.16) with (3.17)–(3.20) we obtain the following expression evaluated at point x_0

$$\begin{aligned}\Delta v \geq \frac{1}{Z(u)}\bigg[&\psi w^2\left(\frac{1}{2}Z'(u)^2 - Z(u)Z''(u)\right)\\ &+ w\left(2\psi Z(u)h'_\varepsilon(u) - \psi h_\varepsilon(u)Z'(u) - KZ(u)\right)\\ &- KZ'(u)Z(u)^{1/2}\psi^{1/2}w^{3/2}\bigg],\end{aligned} \tag{3.21}$$

where $K > 0$ is a constant. More precisely, $K = \max\left(\sup_\Omega \frac{|\nabla\psi|}{\psi^{1/2}}, \sup_\Omega \Delta\psi - 2\frac{|\nabla\psi|^2}{\psi}\right)$.

As we devised before, if $v(x_0)$ is large enough then the right hand side of (3.21) must be positive, which would contradict (3.9). For this purpose we need to establish the following estimates uniformly for all $0 < \varepsilon < 1$ and $0 < \delta < 1$:

$$Z'(u)Z(u)^{1/2} \leq C(\tfrac{1}{2}Z'(u)^2 - Z''(u)Z(u)), \tag{3.22}$$

$$Z(u)|h'_\varepsilon(u)| \leq C(\tfrac{1}{2}Z'(u)^2 - Z''(u)Z(u)), \tag{3.23}$$

$$Z'(u)h_\varepsilon(u) \leq C(\tfrac{1}{2}Z'(u)^2 - Z''(u)Z(u)), \tag{3.24}$$

$$Z(u) \leq C(\tfrac{1}{2}Z'(u)^2 - Z''(u)Z(u)), \tag{3.25}$$

for all $0 \leq u \leq S$, see (2.14). Notice that by (2.14), the constant C depends only on β, S, λ, but not on ε. The constant C does not depend on $0 < \delta < 1$ as it can be seen in the explicit computation of constants in the proof of (3.22)–(3.25).

Suppose for a moment that (3.22)-(3.25) have been proved. Then inequality (3.21) implies that

$$\begin{aligned}\Delta v &\geq \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)}\left(\psi w^2 - C(w + \psi^{1/2}w^{3/2})\right)\\ &= \frac{\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)}{Z(u)\psi}\left(v^2 - C(v + v^{3/2})\right).\end{aligned}$$

Thus if $v(x_0) = \sup v > \tilde{M}$ for some large enough $\tilde{M}$ independent of ε we obtain a contradiction with (3.9).

We now turn to the proof of (3.22)–(3.25). First note that for $u > 0$, expression (3.4) furnishes

$$\begin{aligned}
&\frac{1}{2}Z'(u)^2 - Z''(u)Z(u)\\
&= \frac{1}{2}((1-\beta)u^{-\beta}+1)^2 + \beta(1-\beta)u^{-1-\beta}(u^{1-\beta}+u) + \delta\beta(1-\beta)u^{-\beta-1} \qquad (3.26)\\
&\geq \frac{1}{2}(1-\beta)^2(u^{-2\beta}+1) + \delta\beta(1-\beta)u^{-\beta-1}.
\end{aligned}$$

To verify (3.22) observe that since $\delta < 1$,

$$\begin{aligned}
Z'(u)Z(u)^{1/2} &\leq C'(u^{-\beta}+1)(u^{(1-\beta)/2}+u^{1/2}+\delta^{1/2}) \qquad (3.27)\\
&\leq C'(u^{-2\beta}+1) + C'\delta^{1/2}(u^{-\beta}+1)\\
&\leq C'(u^{-2\beta}+1) + C'(u^{-\beta}+1),
\end{aligned}$$

where $C' > 0$ is a constant independent of δ. Notice that by (3.26)

$$\frac{1}{2}Z'(u)^2 - Z(u)''Z(u) \geq \frac{1}{2}(1-\beta)^2(u^{-2\beta}+1) \text{ for } u > 0, \qquad (3.28)$$

then (3.22) follows for $u < 1$, since $u^{-2\beta} \geq u^{-\beta}$ for $u < 1$. In the compact interval $1 \leq u \leq S$, we use (3.27) and (3.28) to obtain a constant C independent of δ such that (3.22) is true.

We proceed with (3.23). Returning to (3.3), observe that

$$h'_\varepsilon(u) = u^{q-1}\frac{(\varepsilon q - \beta u)}{(u+\varepsilon)^{q+\beta+1}} - \lambda p u^{p-1}.$$

We distinguish two cases. Assume $0 < u \leq \alpha\varepsilon$, where $0 < \alpha < q/2$. We claim that $h'_\varepsilon(u) \geq 0$. Indeed, if $h'_\varepsilon(u) < 0$, which is equivalent to

$$\varepsilon q u^{q-1} < \beta u^q + \lambda p u^{p-1}(u+\varepsilon)^{q+\beta+1}.$$

Hence

$$\varepsilon q u^{q-1} < \beta u^q + \lambda p(\alpha+1)^{q+\beta+1}u^{p-1}\varepsilon^{q+\beta+1}.$$

There exists an $\varepsilon_0 > 0$ such that

$$\lambda p(\alpha+1)^{q+\beta+1}\varepsilon^{q+\beta+1} < \frac{\varepsilon q}{2} \text{ for every } 0 < \varepsilon < \varepsilon_0.$$

Thus

$$\varepsilon q u^{q-1} < \beta u^q + \frac{\varepsilon q}{2}u^{p-1} < \beta u^q + \frac{\varepsilon q}{2}u^{q-1}$$

for $0 < u < 1$ and $0 < q < p < 1$, which implies

$$\frac{\varepsilon q}{2} < \beta u \leq \beta\alpha\varepsilon.$$

Therefore $0 < \varepsilon < \frac{2\beta\alpha\varepsilon}{q} < \beta\varepsilon < \varepsilon$, a contradiction. Since $h'_\varepsilon(u) > 0$, then

$$h'_\varepsilon(u) \le q\frac{\varepsilon}{u}\frac{u^q}{(u+\varepsilon)^q}\frac{1}{(u+\varepsilon)^{1+\beta}} \le q\frac{u+\varepsilon}{u}\frac{1}{u^{1+\beta}} \le \frac{q}{u^{\beta+1}},$$

since $\varepsilon \le u+\varepsilon$ and $u \le u+\varepsilon$. Hence

$$\begin{aligned} |h'_\varepsilon(u)|Z(u) &\le q(u^{-2\beta} + u^{-\beta} + \delta u^{-\beta-1}) \\ &\le 2qu^{-2\beta} + \delta u^{-\beta-1}, \end{aligned} \tag{3.29}$$

and (3.23) follows for $0 < u \le \alpha\varepsilon$ by comparing (3.26) and (3.29). The constant C in (3.23) does not depend on δ, since the powers appearing in (3.26) and (3.29) are the same.

The other case is when $\alpha\varepsilon \le u < 1$ for some $0 < \alpha < \frac{q}{2}$. In this way

$$\lambda p u^{p-1}(u+\varepsilon)^{q+\beta+1} \le \lambda p(\frac{\alpha+1}{\alpha})^{q+\beta+1}u^{p-1}u^{q+\beta+1}.$$

Hence

$$\lambda p u^{p-1}(u+\varepsilon)^{q+\beta+1} \le \lambda p c^{q+\beta+1}u^{p+q+\beta} \text{ for } c = \frac{\alpha+1}{\alpha}.$$

There exists $0 < s_0 < 1$ such that

$$\lambda p c^{q+\beta+1}u^{p+\beta} < \beta \text{ for } 0 < u < s_0 < 1.$$

Thus

$$\lambda p u^{p-1}(u+\varepsilon)^{q+\beta+1} \le \beta u^q \text{ for } 0 < u < s_0.$$

And we get the estimate

$$|h'_\varepsilon(u)| \le \frac{2\beta u^q}{(u+\varepsilon)^{q+\beta+1}} \le \frac{2\beta}{u^{\beta+1}} \text{ for } 0 < u < s_0.$$

In the range $0 < \min\{1, s_0\} \le u \le S$ (where S is the constant that appears in the Lemma 2.4) then, we have $|h'_\varepsilon(u)| \le C'$ for some constant $C' > 0$ independent of δ. Hence in this case (3.23) follows from (3.25). To prove (3.25), observe that $0 < \delta < 1$ and for $0 < u < 1$ we have $u^{1-\beta} < u^{-2\beta}$, then $Z(u) \le C'(u^{-2\beta} + 1)$ for some constant C' independent of δ. Since $u^{1-\beta}$ and $u^{-2\beta}$ are comparable in the compact interval $1 \le u \le S$, the same estimate $Z(u) \le C'(u^{-2\beta} + 1)$ holds for some constant C' independent of δ. In both cases, we use (3.28) to obtain (3.25) with a constant C independent of δ.

To prove inequality (3.24) we observe that for every $u > 0$,

$$h_\varepsilon(u) \le \frac{1}{u^\beta}.$$

Thus, for $0 < u \le S$,

$$Z'(u)h_\varepsilon(u) \le (1-\beta)u^{-2\beta} + u^{-\beta} \le C'u^{-2\beta},$$

where C' is independent of δ. Therefore, (3.24) follows from (3.28) with a constant C independent of δ.

In synthesis, we have obtained the estimate $\psi|\nabla u|^2 \le M(u^{1-\beta} + u + \delta)$ in Ω, where M is independent of δ. To get estimate (3.2) we let $\delta \to 0$. □

4. Taking the limit

In this section we prove Theorem 1.1 by letting $\varepsilon \to 0$. We use the results of Section 3 to prove that an arbitrary solution u_ε of (1.4) converges to a solution of (1.1). With this, we obtain that u^1 and u^2 are distinct solutions of (1.1).

Proof of Theorem 1.1. Let u_ε be a solution of problem (1.4) and $\varphi \in C_c^1(\Omega)$, hence

$$\int_\Omega \nabla u_\varepsilon \nabla\varphi = \int_\Omega (-g_\varepsilon(u_\varepsilon) + \lambda u_\varepsilon^p)\varphi.$$

Let $\eta \in C^\infty(\mathbb{R})$, $0 \le \eta \le 1$, $\eta(s) = 0$ for $s \le 1/2$, $\eta(s) = 1$ for $s \ge 1$. For $m > 0$ the function $\varrho := \varphi\eta(u_\varepsilon/m)$ belongs to $C_c^1(\Omega)$.

Since u_ε is a critical point of I_ε, we obtain

$$\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega g_\varepsilon(u_\varepsilon)u_\varepsilon = \int_\Omega \lambda u_\varepsilon^{p+1}.$$

Thus $\|u_\varepsilon\|_{H_0^1}$ is bounded independently of ε, see the proof of Lemma 2.1. For a sequence ε_n which for the sake of notation we will continue to denote by ε, we have

$$\begin{cases} u_\varepsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega); \\ u_\varepsilon \to u \text{ strongly in } L^\sigma(\Omega); \\ u_\varepsilon \to u \text{ a.e in } \Omega; \\ |u_\varepsilon| \le h \text{ a.e in } \Omega \text{ for some } \in L^\sigma(\Omega), \end{cases} \tag{4.1}$$

where $1 \le \sigma < 2N/(N-2)$ if $N \ge 3$ ($1 \le \sigma < \infty$ if $N = 2$).

By Lemma 3.1, $|\nabla u_\varepsilon|$ is locally bounded independently of ε. Thus for a sequence ε_n which we keep denoting by ε, we have $u_\varepsilon \to u$ in $C^0_{\text{loc}}(\Omega)$, and the set $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ is open. Let $\tilde\Omega$ be an open set such that $\overline{support(\varphi)} \subset \tilde\Omega$ and $\overline{\tilde\Omega} \subset \Omega$. Let $\Omega_0 = \Omega_+ \cap \tilde\Omega$. For every $m > 0$ there is an $\varepsilon_0 > 0$ such that

$$u_\varepsilon(x) \le m/2 \text{ for every } x \in \tilde\Omega \setminus \Omega_0 \text{ and } 0 < \varepsilon \le \varepsilon_0. \tag{4.2}$$

Replacing φ by ϱ we obtain

$$\int_\Omega \nabla u_\varepsilon \nabla(\varphi\eta(u_\varepsilon/m)) = \int_{\tilde\Omega} (-g_\varepsilon(u_\varepsilon) + \lambda u_\varepsilon^p)\varphi\eta(u_\varepsilon/m). \tag{4.3}$$

We split the previous integral as

$$P_\varepsilon := \int_{\Omega_0} (-g_\varepsilon(u_\varepsilon) + \lambda u_\varepsilon^p)\varphi\eta(u_\varepsilon/m)$$

and

$$Q_\varepsilon := \int_{\tilde{\Omega}\setminus\Omega_0} (-g_\varepsilon(u_\varepsilon) + \lambda u_\varepsilon^p)\varphi\eta(u_\varepsilon/m).$$

Clearly, $Q_\varepsilon = 0$, whenever $0 < \varepsilon \le \varepsilon_0$ by (4.2) and the definition of η. Notice that

$$P_\varepsilon \to \int_{\Omega_0} (-u^{-\beta} + \lambda u^p)\varphi\eta(u/m) \quad \text{as } \varepsilon \to 0. \tag{4.4}$$

Indeed, $u_\varepsilon \to u$ uniformly in Ω_0. If $u \le m/4$, for $\varepsilon > 0$ sufficiently small, we have $u_\varepsilon \le m/2$. So the integral P_ε restricted to this set is zero. For $u > m/4$, then $u_\varepsilon \ge m/8$ for $\varepsilon > 0$ small enough. We then apply the Dominated Convergence Theorem as $\varepsilon \to 0$ to get (4.4).

We now take the limit in m to conclude that

$$\int_{\Omega_0} (-u^{-\beta} + \lambda u^p)\varphi\eta(u/m) \to \int_{\Omega_0} (-u^{-\beta} + \lambda u^p)\varphi \quad \text{as } m \to 0, \tag{4.5}$$

since $\eta(u/m) \le 1$ and $-u^{-\beta} + \lambda u^p \in L^1(\tilde{\Omega})$, according to Lemma 4.1 below.

Observing the integral on the left side of (4.3), we set

$$\int_\Omega \nabla u_\varepsilon \nabla(\varphi\eta(u_\varepsilon/m)) := \int_{\tilde{\Omega}} (\nabla u_\varepsilon \nabla\varphi)\eta(u_\varepsilon/m) + J_\varepsilon. \tag{4.6}$$

Clearly,

$$\int_{\tilde{\Omega}} (\nabla u_\varepsilon \nabla\varphi)\eta(u_\varepsilon/m) \to \int_{\tilde{\Omega}} (\nabla u \nabla\varphi)\eta(u/m) \quad \text{as } \varepsilon \to 0,$$

since $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_\varepsilon \to u$ uniformly in $\tilde{\Omega}$. Consequently, by the Dominated Convergence Theorem,

$$\int_{\tilde{\Omega}} (\nabla u \nabla\varphi)\eta(u/m) \to \int_{\tilde{\Omega}} \nabla u \nabla\varphi \quad \text{as } m \to 0. \tag{4.7}$$

We claim that

$$J_\varepsilon := \int_{\tilde{\Omega}} \frac{|\nabla u_\varepsilon|^2}{m}\eta'(u_\varepsilon/m)\varphi \to 0 \quad \text{as } \varepsilon \to 0 \quad \text{(and then as } m \to 0). \tag{4.8}$$

By the estimate $|\nabla u_\varepsilon|^2 \le M(u_\varepsilon^{1-\beta} + u_\varepsilon)$ in $\tilde{\Omega}$ provided by Lemma 3.1, the fact that $\eta(u/m) \le 1$, Lemma 2.4 and the Dominated Convergence Theorem, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0} |J_\varepsilon| &\le M \lim_{\varepsilon\to 0} \int_{\tilde{\Omega}\cap\{\frac{m}{2}\le u_\varepsilon \le m\}} \frac{(u_\varepsilon^{1-\beta} + u_\varepsilon)}{m} |\eta'(u_\varepsilon/m)\varphi| \\ &= M \int_{\tilde{\Omega}\cap\{\frac{m}{2}\le u \le m\}} \frac{(u^{1-\beta} + u)}{m} |\eta'(u/m)\varphi|.\end{aligned}$$

Letting $m \to 0$ in the above estimate, we may invoke Lemma 4.1, the fact that $\eta'(u/m)$ is uniformly bounded and the Dominated Convergence Theorem to conclude that (4.8) must hold. The claim is proved.

As a direct consequence of (4.3), (4.5),(4.6), (4.7) and (4.8), we have

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Omega\cap\{u>0\}} \left(-\frac{1}{u^\beta} + \lambda u^p\right)\varphi$$

for every $\varphi \in C_c^1(\Omega)$. This concludes the proof of Theorem 1.1. □

We need the following lemma to justify a calculation in the proof of Theorem 1.1.

Lemma 4.1. *Let* $\Omega_+ = \{x \in \Omega : u(x) > 0\}$. *The function* $\frac{1}{u^\beta}\chi_{\Omega_+}$ *belongs to* $L^1_{\mathrm{loc}}(\Omega)$.

Proof. Let $K \subset \Omega$ be a compact set. Take $\zeta \in C_c^1(\Omega)$ such that $0 \le \zeta \le 1$ and $\zeta \equiv 1$ in K. Since u_ε is a critical point of I_ε, we obtain

$$\int_\Omega \nabla u_\varepsilon \nabla \zeta + \int_\Omega g_\varepsilon(u_\varepsilon)\zeta = \int_\Omega \lambda u_\varepsilon^p \zeta.$$

The information provided by (4.1) can be used again here. Thus

$$\int_\Omega u_\varepsilon^p \zeta \to \int_\Omega u^p \zeta$$

and

$$\int_\Omega \nabla u_\varepsilon \nabla \zeta \to \int_\Omega \nabla u \nabla \zeta \quad \text{as } \varepsilon \to 0.$$

Therefore,

$$\int_\Omega g_\varepsilon(u_\varepsilon)\zeta \to \lambda \int_\Omega u^p \zeta - \int_\Omega \nabla u \nabla \zeta \quad \text{as } \varepsilon \to 0. \tag{4.9}$$

Notice that

$$\int_K \frac{u_\varepsilon^q}{(u_\varepsilon + \varepsilon)^{q+\beta}}\zeta \le \int_\Omega \frac{u_\varepsilon^q}{(u_\varepsilon + \varepsilon)^{q+\beta}}\zeta = \int_\Omega g_\varepsilon(u_\varepsilon)\zeta$$

and define the set $\Omega_\rho = \{x \in \Omega : u(x) \geq \rho\}$ for $\rho > 0$, then

$$\int_{K\cap\Omega_\rho} \frac{u_\varepsilon^q}{(u_\varepsilon + \varepsilon)^{q+\beta}}\zeta \leq \int_K \frac{u_\varepsilon^q}{(u_\varepsilon + \varepsilon)^{q+\beta}}\zeta \leq \int_\Omega g_\varepsilon(u_\varepsilon)\zeta.$$

It follows from Fatou Lemma and (4.9) that

$$\int_K \frac{1}{u^\beta}\chi_{\Omega_\rho} = \int_{K\cap\Omega_\rho} \frac{1}{u^\beta} \leq \liminf_{\varepsilon\to 0} \int_\Omega g_\varepsilon(u_\varepsilon)\zeta = \lambda \int_\Omega u^p\zeta - \int_\Omega \nabla u \nabla\zeta.$$

Taking $\rho \to 0$ and applying Fatou Lemma once more, we conclude that

$$\int_K \frac{1}{u^\beta}\chi_{\Omega^+} < \infty$$

for every compact subset $K \subset \Omega$. □

5. The domain perturbation approach

In correspondence with problem (1.1) we write the functional $I : H_0^1(\Omega) \to \mathbb{R}$ as

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u^+),$$

where $f(u) = -\frac{1}{u^\beta} + \lambda u^p$ and $F(u) = \int_0^u f(s)ds$.

Proof of Theorem 1.2. The first part of the proof is devoted to prove the uniqueness of the positive solution u of (1.1) for large values of λ. For a moment we assume $u \geq \underline{u}$ and $u \not\equiv \underline{u}$, where $\underline{u} = c\varphi_1^{\frac{2}{1+\beta}}$ is a subsolution. Later we will see that in fact $u \geq \underline{u}$ and $u \not\equiv \underline{u}$. It is known that $\underline{u}$ is a subsolution for large λ of problem (1.1) (see [13] or [16]), which in our new notation is

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

The constant $c > 0$ appearing in $\underline{u}$ can be taken large if one chooses λ large enough, say, grater than some $\lambda^0 > 0$. Indeed from [13], c and λ satisfy $c_0 \leq \lambda c^{p-1}$ where the constant $c_0 > 0$ does not depend on c, λ and p. Recall $\Lambda > 0$ the best constant of Hardy inequality (1.5). Thus we can fix c large enough and p small enough less than some $0 < p_0 < p$ in order to satisfy $\beta c^{-1-\beta} + p\lambda c^{p-1}\varphi_1^{2(p+\beta)/(1+\beta)} < \Lambda$ if one takes λ sufficiently large.

If u and v are both solutions of problem (1.1) (or (5.1)) which are bigger than $\underline{u}$, define $w = u - v$. By convexity of $u \mapsto u^{-\beta}$ and concavity of $u \mapsto u^p$ we obtain

$$\begin{aligned} f(u) - f(v) &= v^{-\beta} - u^{-\beta} + \lambda u^p - \lambda v^p \\ &\leq \beta v^{-1-\beta} w + p\lambda v^{p-1} w. \end{aligned}$$

Since $v > c\varphi_1^{\frac{2}{1+\beta}}$, then we obtain

$$f(u) - f(v) \leq (\beta c^{-1-\beta} \varphi_1^{-2} + p\lambda c^{p-1} \varphi_1^{2(p-1)/(1+\beta)}) w.$$

Hence

$$-\Delta w - (\beta c^{-1-\beta} + p\lambda c^{p-1} \varphi_1^{2(p+\beta)/(1+\beta)}) \varphi_1^{-2} w \leq -\Delta w - f(u) + f(v) = 0. \quad (5.2)$$

Assume $w^+ \not\equiv 0$, then

$$\begin{aligned} 0 &\leq \int \left(\frac{\Lambda}{\varphi_1^2} - \frac{\beta c^{-1-\beta} + p\lambda c^{p-1} \varphi_1^{2(p+\beta)/(1+\beta)}}{\varphi_1^2} \right) (w^+)^2 \\ &\leq \int |\nabla w^+|^2 - \frac{\beta c^{-1-\beta} + p\lambda c^{p-1} \varphi_1^{2(p+\beta)/(1+\beta)}}{\varphi_1^2} (w^+)^2 \leq 0. \end{aligned}$$

We have used (5.2) in the last inequality. Hence, $w^+ = (u - v)^+ \equiv 0$. By the same reasoning $(v - u)^+ \equiv 0$. Therefore, $u = v$.

We now prove the existence of a solution of problem (1.1) (or (5.1)). Let

$$\emptyset \neq \Omega_1 \subset\subset \Omega_2 \ldots \subset\subset \Omega \quad (5.3)$$

be a sequence of smooth domains such that $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$.

Define the truncated function

$$\hat{f}(x, u) = \begin{cases} f(\underline{u}(x)) & \text{for } s \leq \underline{u}(x) \\ f(s) & \text{for } s \geq \underline{u}(x), \end{cases} \quad (5.4)$$

where $\underline{u}$ is the subsolution defined above. Consider the truncated problems on each domain Ω_k,

$$\begin{cases} -\Delta u_k = \hat{f}(x, u_k) & \text{in } \Omega_k \\ u_k = \underline{u}(x) & \text{on } \partial\Omega_k. \end{cases} \quad (5.5)$$

In order to find a solution to (5.5) we translate it with $v_k = u_k - \underline{u}$ to the following homogeneous boundary condition problem

$$\begin{cases} -\Delta v_k = \hat{f}(x, v_k + \underline{u}) + \Delta \underline{u} & \text{in } \Omega_k \\ v_k = 0 & \text{on } \partial\Omega_k. \end{cases} \quad (5.6)$$

Define the functional $\tilde{I}_k : H_0^1(\Omega_k) \to \mathbb{R}$ by

$$\tilde{I}_k(v) = \int_{\Omega_k} \frac{1}{2}|\nabla v|^2 - \tilde{F}(x, v) - \nabla \underline{u} \nabla v,$$

here

$$\tilde{F}(x, v) = \int_0^v \hat{f}(x, t^+ + \underline{u})dt.$$

Notice that

$$\tilde{F}(x, v) = \begin{cases} f(\underline{u}(x))v & \text{for } v \le 0 \\ \hat{F}(x, v + \underline{u}) - \hat{F}(x, \underline{u}) & \text{for } v > 0 \end{cases} \tag{5.7}$$

where $\hat{F}(x, s) = \int_0^s \hat{f}(x, t)dt$, then $|\hat{F}(x, v)| \le c' + \eta |v|^2$ on $\Omega_k \times \mathbb{R}$ for some constants $c' > 0$ and $\eta > 0$ (depending on k), with η small. Thus $\tilde{I}_k$ is coercive and satisfies the so-called Palais-Smale condition. Hence there is $v_k \in H_0^1(\Omega_k)$ such that

$$\tilde{I}_k(v_k) = \inf_{v \in H_0^1(\Omega_k)} \tilde{I}_k(v).$$

Since v_k is a solution of (5.6), then $u_k = v_k + \underline{u}$ is a solution of (5.5). Notice that by the maximum principle and since $\underline{u} > 0$ is a subsolution, we get $v_k \ge 0$ in Ω_k. Indeed, since

$$-\Delta u_k = \hat{f}(x, u_k) \quad \text{in} \quad \Omega_k,$$

and

$$(\underline{u} - u_k)^+ \in H_0^1(\Omega_k) \subset H_0^1(\Omega).$$

Then,

$$\begin{aligned} \int_{\Omega_k} \nabla \underline{u} \nabla (\underline{u} - u_k)^+ &\le \int_{\Omega_k} f(\underline{u}(x))(\underline{u} - u_k)^+ \\ &= \int_{\Omega_k} \hat{f}(x, \underline{u}(x))(\underline{u} - u_k)^+ = \int_{\Omega_k} \nabla u_k \nabla (\underline{u} - u_k)^+. \end{aligned}$$

Hence

$$\int_{\Omega_k} |\nabla (\underline{u} - u_k)^+|^2 \le 0$$

implying $\underline{u} \le u_k$ in Ω_k.

In the sequel we need to establish that $\|u_k\|_{L^{p+1}(\Omega_k)} \le C$ where $C > 0$ is a constant independent of the domain Ω_k. Multiplying (5.6) by v_k, using the Sobolev inequality and the fact that $-\Delta u \le \lambda u^p$ we obtain

$$\overline{C} \left(\int_{\Omega_k} v_k^{p+1} \right)^{2/(p+1)} \le \lambda \int_{\Omega_k} (v_k + \underline{u})^p v_k + \underline{u}^p v_k$$

where $\overline{C}$ is the Sobolev constant which can be taken independent of Ω_k by (5.3), but it depends on Ω. By Hölder inequality, there is a constant C depending only on Ω, p, N, λ such that

$$\|v_k\|^2_{L^{p+1}(\Omega_k)} \leq C(\|v_k\|^{p+1}_{L^{p+1}(\Omega_k)} + \|v_k\|_{L^{p+1}(\Omega_k)}).$$

Therefore $\|v_k\|_{L^{p+1}(\Omega_k)} \leq C$ and $\|u_k\|_{L^{p+1}(\Omega_k)} \leq C$.

We affirm that there is k_0 such that $\|u_k\|_{H^1(\Omega_{k_0})}$ is bounded for every $k \geq k_0$. We need to show that the integral $\int_{\Omega_{k_0}} |\nabla u_k|^2$ remains bounded in Ω_{k_0} for every sufficiently large $k > k_0$. In fact, for that matter take Ω_{k_0} and $\delta > 0$ such that $0 < \delta < \inf_{\Omega_{k_0}} \underline{u}$. Notice that $(u_k - \delta)^+ \in H_0^1(\Omega_k)$, because $(u_k - \delta)^+ = (\underline{u} - \delta)^+ = 0$ on $\partial\Omega_k$, since by hypothesis $\underline{u}(x) \to 0$ whenever $x \to \partial\Omega$. For every $k \geq k_0$ one has

$$\begin{aligned}\int_{\Omega_{k_0}} |\nabla u_k|^2 &\leq \int_{\Omega_k} |\nabla(u_k - \delta)^+|^2 = \int_{\Omega_k} f(u_k)(u_k - \delta)^+ \leq \int_{\Omega_k} \lambda u_k^p (u_k - \delta)^+ \\ &\leq \int_{\Omega_k} \lambda u_k^{p+1} \leq C.\end{aligned}$$

Arguing with a subsequence, we obtain $u_k \rightharpoonup u$ in $H^1_{\text{loc}}(\Omega)$, $u_k \to u$ in $L^2_{\text{loc}}(\Omega)$, $u_k \to u$ a.e in Ω, similarly to (4.1). Hence $\underline{u} \leq u_k$ in Ω.

Let φ be a test function in $C_0^\infty(\Omega)$. There is a $k' > 0$ and a bounded domain Ω' such that $support(\varphi) \subset\subset \Omega' \subset\subset \Omega_k$ for every $k \geq k'$. Thus,

$$\int_{\Omega'} \nabla u_k \nabla\varphi = \int_{\Omega'} f(u_k)\varphi \quad \text{for every } k \geq k'.$$

Letting $k \to \infty$ we obtain

$$\int_{\Omega'} \nabla u \nabla\varphi = \int_{\Omega'} f(u)\varphi.$$

This last integral also holds in Ω, and u is a weak solution such that $u \geq \underline{u}$ and $u \not\equiv \underline{u}$ in Ω. $\square$

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