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Two stability problems related to resistive magnetohydrodynamics

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Abstract

Two general problems related to resistive magnetohydrodynamic stability are addressed in this paper :

1. A general stability condition previously derived by the author for a class of real systems, occuring especially in plasma physics, is proved to persist to second order, despite the addition of several antisymmetric operators of first order in the linearized stability equation.
2. For a special but representative choice of the stability operators, a nonperturbative analysis demonstrates the existence of a critical density for the appearence of an overstability and the connected Hopf bifurcation, as suggested in a previous note of the author.

1 Persistence of a general stability condition in real systems

A challenging problem in linear stability theory is the stability of nonconservative mechanical systems [1] and fluids [2] possessing 'circulatory forces'. A simple reduction of the problem to a quadratic form and the analysis of its sign is possible only for the case without circulatory forces and is expressed by the Thompson-Tait theorem [3]. The theorem is easy to prove, using Lyapunov methods [4], for systems of the form

$$N_s\ddot{\xi} + (P_s + P_a)\dot{\xi} + Q_s\xi = 0, \tag{1}$$

where ξ is a column vector, N_s and P_s are symmetric positive definite matrices, P_a is an antisymmetric matrix and Q_s is a symmetric matrix. In case of fluids the vector ξ is replaced by functions and the matrices by continuous operators. Nevertheless, equation (1) can be considered as a discretization of a fluid stability problem. The theorem states that

$$(\xi, Q_s\xi) \geq 0 \tag{2}$$

for all ξ is necessary and sufficient for the stability of (1). Many problems in mechanics [1] and in plasma physics [5] were solved in this way.

If we add circulatory forces and a kind of antisymmetric inertia to (1), then we have to face the stability problem for a real equation of the form

$$(N_s + N_a)\ddot{\xi} + (P_s + P_a)\dot{\xi} + (Q_s + Q_a)\xi = 0. \tag{3}$$

In most cases ξ is a Lagrangean variable (see [2]), in which are expressed the friction $P_s\dot{\xi}$ and the circulatory force $Q_a\xi$, but $N_a = 0$. In some cases (e.g. dissipative magnetohydrodynamics with Hall term), however, an antisymmetric inertia $N_a\ddot{\xi}$ can also appear as in (3).

The purpose of this note is to extend to equation (3) statements proved by the author for stability equations without N_a (see [6]) and equations with neither N_a nor P_a but with a small Q_a (see [7]). The first statement concerns purely growing modes. In this case, the eigenfunctions related to (3) can be taken real, since (3) contains only real quantities, similarly to the situation in [6]. It follows that, making the ansatz $\xi = \Psi e^{\gamma t}$, and taking the scalar product of (3) with Ψ, we obtain

$$\gamma^2(\Psi, N_s\Psi) + \gamma(\Psi, P_s\Psi) + (\Psi, Q_s\Psi) = 0. \tag{4}$$

Since N_s and P_s are positive operators, then a general sufficient stability condition with respect to purely growing modes is

$$(\Psi, Q_s\Psi) \geq 0 \tag{5}$$

for all Ψ real.

The second point is to extend the perturbation procedure and the result of [7] to the situation where N_a, P_a and Q_a are all small of order ϵ. At zero order we know from [3] that (5) is also necessary, so that, in the absence of degeneracy in the zero order spectrum, the following holds: If (5) is violated, only purely growing modes exist (see [7]), with

$$\xi = \Psi_0 e^{\gamma t} \tag{6}$$

and a real Ψ_0 obeying

$$\gamma^2 N_s \Psi_0 + \gamma P_s \Psi_0 + Q_s \Psi_0 = 0. \tag{7}$$

To obtain the next order we set $\xi = \Psi e^{\omega t}$ and expand

$$\Psi = \Psi_0 + \epsilon \Psi_1 + ... \tag{8}$$

$$\omega = \gamma + \epsilon \omega_1 + ... \tag{9}$$

Inserting (8) and (9) in (3) leads, up to the first order in ϵ, to

$$\begin{aligned} \gamma^2 N_s \Psi_1 + \gamma P_s \Psi_1 + Q_s \Psi_1 + \\ 2\gamma\omega_1 N_s \Psi_0 + \omega_1 P_s \Psi_0 + \\ + \gamma^2 N_a \Psi_0 + \gamma P_a \Psi_0 + Q_a \Psi_0 &= 0. \end{aligned} \tag{10}$$

Taking the scalar product of (10) with Ψ_0, we obtain

$$\omega_1 (2\gamma(\Psi_0, N_s \Psi_0) + (\Psi_0, P_s \Psi_0)) = 0. \tag{11}$$

The terms in Ψ_1 vanish because of (7) and the symmetry of the operators. Those with N_a, P_a and Q_a vanish also because of the reality of Ψ_0. Since N_s and P_s are positive, it follows that

$$\omega_1 = 0. \tag{12}$$

This means that the unstable spectrum of (3) is affected by N_a, P_a and Q_a (all of order ϵ) only at the order ϵ^2. In that sense, condition (5) can be said to be necessary and sufficient for stability. In other words the condition is persistent to second order in ϵ.

One may be tempted to check the persistency of (2) by considering (1), which has a finite P_a, as a zero order equation and adding to it small N_a and Q_a. Indeed the first statement concerning the sufficient part of (2) with respect to purely growing modes is persitent. It is not possible, however, to prove the second statement i.e. $\omega_1 = 0$, since, in case of violation of (2), the unstable modes of (1) do not need to be purely growing and Ψ_0 becomes complex, so that the contributions coming from N_a and P_a do not vanish anymore. The persistency of (2) would be to first order in ϵ, which is trivial.

Let us finally note that the stability of fluid motion with moderate and large Reynolds numbers cannot be approached in this way because the corresponding P_a and Q_a are finite. This explains why the persistent condition applies naturally to configurations in plasma physics [7], where the mass flows are not the dominant cause of instabilities.

2 Nonperturbative procedure for an ad hoc choice of operators

In a previous note [8] the linearized equations of resistive magnetohydrodynamics were shown to be of the form

$$N\ddot{\Psi} + P\dot{\Psi} + (Q_s + Q_a)\Psi = 0, \tag{13}$$

with N and P real symmetric and positive operators, Q_s and Q_a being real but symmetric and antisymmetric respectively. A sufficient condition for stability with respect to purely growing modes was found and can be written as

$$\delta W = (\Psi, Q_s\Psi) \geq 0, \tag{14}$$

for all Ψ.

Several considerations concerning the up-grading of condition (14) were given in [7] . In essence, they lead first, to the sufficiency of (14) w. r. to all modes, in case N is neglected, and second, to the necessity of (14) if Q_a is small. Finally, a Hopf bifurcationwas demonstrated [9], for (14) fulfilled and N increaincreased, until an overstability occurs.

In this section, we want to illustrate the general results of [7],[8] and [9] through a problem related to (13), in which the operators N and P are both proportional to the idendity. For this choice, (13) becomes

$$nI\ddot{\Psi} + pI\dot{\Psi} + (Q_s + Q_a)\Psi = 0, \tag{15}$$

where n and p are positive numbers.

If the eigenvalue problem for $Q_s + Q_a$ can be solved, we have

$$(Q_s + Q_a)\Psi_m = \lambda_m \Psi_m \tag{16}$$

with

$$\lambda_m = \lambda_{mR} + i\lambda_{mI}. \tag{17}$$

λ_m and Ψ_m are, in general, complex, though (15) involves only real quantities.

If we make the ansatz $\Psi = \Psi_m e^{\omega_m t}$, the eigenvalues of (15) ω_m are related to the λ_m by the following equation

$$n\omega^2 + p\omega + \lambda_R + i\lambda_I = 0, \tag{18}$$

valid for each pair of eigenvalues λ_m and ω_m.

Let us split ω in real and imaginary parts, then (18) can be written as a system

$$n(\omega_R^2 - \omega_I^2) + p\omega_R + \lambda_R = o, \tag{19}$$

$$\omega_I(2n\omega_R + p) + \lambda_I = 0. \tag{20}$$

Inserting in (19) the value of ω_I obtained from (20), we have

$$n\omega_R^2 + p\omega_R + \lambda_R = \frac{n\lambda_I^2}{(2n\omega_R + p)^2}. \tag{21}$$

Since n and p are positive, it is easy to make schematic plots of the left hand side (lhs) and right hand side (rhs) of (20) (see Figures 1 and 2). Let us consider two cases : first, some $\lambda_R < 0$ (Fig. 1), and second, all $\lambda_R > 0$ (Fig. 2), which follows from the validity of (14).

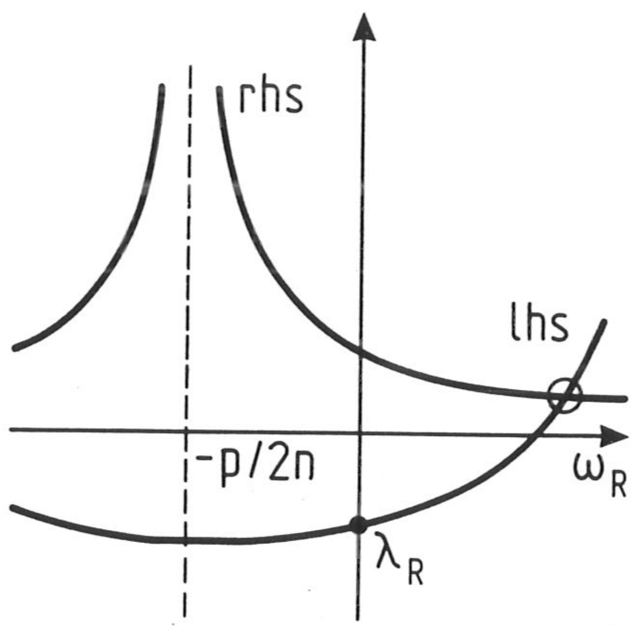

Fig. 1 $\lambda_R < 0$

We see that, if some $\lambda_R < 0$, the system (19, 20) has always a positive root ω_R (see Fig. 1), which means instability.

Let us note that a violation of (14) for a Ψ which is not representative of the eigenfunction, does not necessarily imply that $\lambda_R < 0$.

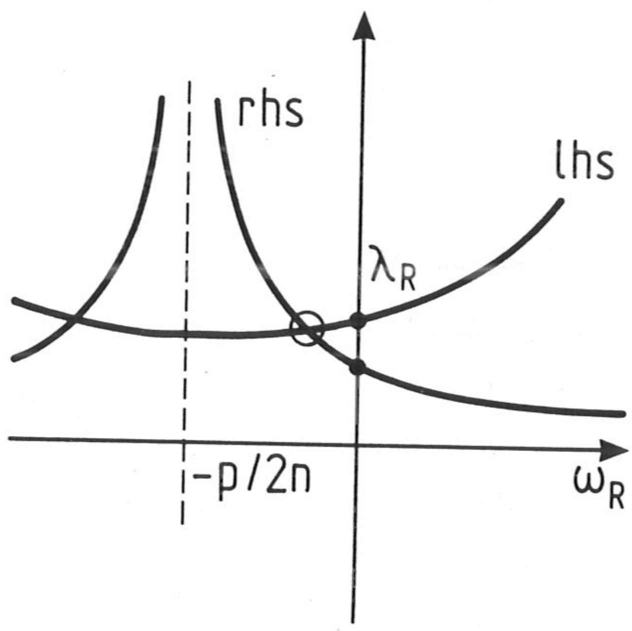

Fig. 2 $\lambda_R > 0$

In case $\lambda_R > 0$, the crossing point leads to an instability only if $\lambda_R < \frac{n\lambda_I^2}{p^2}$ (see Fig. 2). As mentioned above and in [7] the unstable crossing cannot occur for $n \approx 0$ or λ_I very small or p large. Starting from a small n, for $\lambda_R > 0$ and λ_I and p fixed, we obtain an overstability by increasing the value of n until

$$\lambda_R < \frac{n\lambda_I^2}{p^2}, \tag{22}$$

as explained in [9] .

The study of this example suggests a two steps procedure for investigating the stability of a resistive MHD system of the type (13):

1. Check the validity of (14), which can be done numerically for real con-

figurations, using standard procedures for Hermitean eigenvalue problems, as suggested in [7] and [8] .

2. Increasing slowly the value of the operator N, search for a pure imaginery eigenvalue $\omega = i\omega_I$ for problem (13). The first appearence of such an eigenvalue gives the critical value of N for overstability and, possibly, for a Hopf bifurcation as described in [9] .

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