Two substitutable perishable product disaster inventory systems

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Abstract A disaster inventory system is considered in which two substitutable items are stored for disaster management. In the event of disaster management, a particular product may become stock-out and the situation warrants that a demand for the particular product during its stock-out period may be substituted with another available similar product in the inventory. From the utility point of view, continuous review inventory models are quite appro-priate in disaster inventory management. In this paper, a continuous review two substitutable perishable product disaster inventory model is proposed and analyzed. Since the inventory is maintained for disaster management, an adjustable joint reordering policy for replenishment is adopted. There is no lead time and the replenishment is instantaneous. For this model, some measures of system performance are obtained. The stationary behavior of the model is also considered. Numerical examples are also provided to illustrate the results obtained.

1 Introduction

Perishable items are stored in disaster inventory systems to meet emergency situations. Some examples of perishable items are certain food items, blood-sachets in blood bank and medicines. Perishable items become unusable unless they are used before their expiry time. Because of the uncertainty in the occurrence of disastrous events such as earthquake, famine, tsunami and cyclone, disaster inventory has to be maintained with all essential items stored in.

Several reviews of the research work on perishable inventory systems are available (see, for example, Nahmias 1982; 2010; 2011; Parlar 1985). Inventory systems dealing with a single perishable product have been analysed by several authors (see for example, Kalpakam and Arivarignan 1988; Kalpakam and Sapna 1996; Gürler and Özkaya 2008; Baron et al. 2010; Parlar et al. 2011). In disaster inventory systems, we come across inventories dealing with multi perishable products. For example, in the blood bank, several groups of blood are stored. Demand interaction can occur in multi product inventory systems. In the disaster management, the situation warrants that a demand for a particular product during its stock-out period may be substituted with another available similar product in the inventory. If one group of blood is not available at the time of demand, the universal group may be accepted if it is available. Another aspect in disaster inventory systems dealing with perishable items is the movement of items from the inventory to the destination point (the disaster cite). Sometimes these inventory systems become unreliable as they operate in random environment and they are liable to be cut-off totally from the disaster cite by the failure of logistics and communication links. This aspect has been studied from a different point of view by Ozekici and Parlar (1999). When talking about the maintenance of humanitarian disaster inventory and the reliable flow of life-saving items to the victims located in the affected area, the paper of Iannoni et al. (2008) deserves special mention. In their paper, they propose a comprehensive methodology for the development of a humanitarian emergency management framework based on the real-time tracking of emergency supplies and demands through the use of radio frequency identification devices. Ozdamar et al. (2004) have analysed emergency logistics planning in natural disasters. Sebastian et al. (2012) have considered UK blood supply chain and demonstrated that managerial changes and training issues have a significant impact on waste reduction and inventory management performance in perishable supply chains. They have also developed six recommendations for how managers can improve perishable inventory performance. Ozguven and Ozbay (2014) have reviewed various aspects of emergency inventory management for disasters including the characteristic of storage and delivery options for emergency supplies. They use a novel classification scheme to distinguish between emergency inventories and conventional inventories. Recently Perlman and Levner (2014) have considered a problem of perishable inventory management in health-care and studied a multi-echelon, multi-supplier inventory system and unite together aspects of perishability and outsourcing under deterministic demand for medical products, which include both perishable and deteriorating goods. In their study, they have determined the optimal number of products to be purchased from regular and out-source suppliers so as to meet the required demand at the minimum operating cost. All these models and analysis have focused on deterministic aspects of inventory systems only. Further most of the models available in literature are periodic review inventory models only. In our opinion, not much focus has been thrown on the stochastic analysis of inventory systems dealing multi-perishable products with demand interaction in disaster management. Yadavalli et al. (2014) have considered a temporo-spatial stochastic model for optimal positioning of humanitarian inventories for disaster relief management. They have not considered perishable product inventory systems with demand interaction. However, from utility point of view of disaster inventory management, continuous review inventory models are quite appropriate. In view of this, we propose and analyze a continuous review two substitutable perishable product disaster inventory model. Since the inventory is maintained for disaster management, we adopt an adjustable joint reordering policy for replenishment. That is a reorder is placed for both the products when the sum of the inventory levels of both the products put together reaches a preassigned level and both the products are replenished up to their maximum level at the time of replenishment. There is no lead time and the replenishment is instantaneous. For the model, the following measures of system performance are obtained: The mean number of (i) the demands satisfied; (ii) demands substituted; (iii) demands lost; and (iv) replenishments in the interval $(0, t]$. The stationary behavior of the model is also considered. Numerical examples are also provided to illustrate the results obtained. The organization of the paper is as follows: In Sect. 2, we give the analysis of the model. The Sects. 2.1 and 2.2 provide the assumptions and notation used in the analysis of the model. The Sects. from 2.3 to 2.6 provide the analysis of the model. The measures of system performance of the model are provided in the Sect. 3. The steady-state analysis is provided in Sect. 4 and a numerical illustration is provided in Sect. 5. A Conclusion is provided in Sect. 6.

2 Model description

The assumptions and notation used in the model are given below:

2.1 Assumptions

1. The inventory system deals with two perishable products say product 1 and product 2. Separate inventories are maintained for the products.
2. The maximum inventory level of product i is S_i; $i = 1, 2$.
3. Demands for product i occur at rate λ_i, $i = 1, 2$.
4. Product i perishes with rate η_i, $i = 1, 2$.
5. An adjustable joint reordering policy is adopted for replenishment. That is, a reorder is placed for both the products when the sum of the inventory levels of both the products put together reaches the level s and both the products are replenished up to their maximum level S_i at the time of replenishment.
6. There is no lead time and hence the replenishment is instantaneous.
7. When ever a demand for product $i (i = 1, 2)$ occurs during its stock out period, the demand can be satisfied with the other product j with some probability p_{ij}, if that product j is available ($j = 1, 2$; $j \neq i$).

2.2 Notation

$X_i(t)$: The inventory level of product $i (i = 1, 2)$ at time t.
$Z(t)$: The vector process $(X_1(t), X_2(t))$ representing the state of the system at time t.
$'r'$: Event that represents a replenishment.
$'a_1'$: Event that a demand for product 1 is satisfied.
$'a_2'$: Event that a demand for product 1 is lost.
$'a_3'$: Event that a demand for product 1 is substituted by the other product.
$'b_1'$: Event that a demand for product 2 is satisfied.
$'b_2'$: Event that a demand for product 2 is lost.
$'b_3'$: Event that a demand for product 2 is substituted.

$'E_0'$: Represents the initial condition that an $'r'$ event has occurred at $t = 0$.
p_{ij}: probability that a demand for product i is satisfied
with product j, $i, j = 1, 2; i \neq j$.
$q_{ij} := 1 - p_{ij}$
©: convolution symbol
$f^*(\theta) = \int_0^\infty e^{-\theta t} f(t) dt$ Laplace transform of an arbitrary function $f(t)$.
$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

$$H(i) = \begin{cases} 1 & \text{if } i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$N(\eta, t)$: Number of $'\eta'$ events in $(0, t]$, $\eta = a, b, c$ etc.,
$E[N(\eta, t)]$: Expected number of $'\eta'$ events in $(0, t]$.
$E[N(\eta, \infty)]$: Mean stationary rate of $'\eta'$ events.

2.3 Analysis

The stochastic process $Z(t) = (X_1(t), X_2(t))$ has the state space $E = \{(i, j) : i = 0, 1, 2, \ldots, S_1, j = 0, 1, 2, \ldots, S_2$ such that $i + j > s\}$. Since there is no lead time and the replenishment is instantaneous, the $r-$events form a renewal process. The factor that simplifies the analysis of the model is that

(i) the life of an item is exponentially distributed.
(ii) the demands for product 1 arrive according to a Poisson process with rate λ_1.
(iii) the demands for product 2 arrive according to a Poisson process with rate λ_2.

To describe the behavior of the process $Z(t)$ at any time t, it is enough to obtain the probability density function of the interval between any two successive $r-$events and the distribution of $Z(t)$ in a cycle (the interval between any two successive $r-$events). To this end, we consider certain auxiliary functions and restrict their usage to a cycle.

2.4 The auxiliary function $\tilde{p}(i, j, t|l, m)$

We define

$$\tilde{p}(i, j, t|l, m) = P\{Z(t) = (i, j)|Z(0) = (l, m)\},$$

where $(i, j), (l, m) \in E$. Then $\tilde{p}(i, j, t|l, m)$ gives the inventory level distribution in a renewal cycle. To derive an expression for this function, we note that a change in the inventory level may occur due to any one of the following possibilities.

1. A demand for a product 1 occurs and is satisfied by product 1.
2. Product 1 perishes.
3. A demand for a product 2 occurs and is satisfied by product 2.
4. Product 2 perishes.
5. A demand for product 1 occurs during its stock-out period and is substituted by the other product if available.
6. A demand for product 2 occurs during its stock-out period and is substituted by the other product if available.

Using probabilistic arguments, we derive the following equations:

Case 1 $l > 0, m > 0, s < i + j < l + m$:

$$\tilde{p}(i, j, t|l, m) = e^{-(\lambda_1+\lambda_2+l\nu_1+m\nu_2)t}(\lambda_1 + l\nu_1)©\tilde{p}(i, j, t|l - 1, m) + e^{-(\lambda_1+\lambda_2+l\nu_1+m\nu_2)t}(\lambda_2 + m\nu_2)©\tilde{p}(i, j, t|l, m - 1). \quad (2.1)$$

Case 2 $l > 0, m > 0, s < i + j = l + m$:

$$\tilde{p}(i, j, t|l, m) = e^{-(\lambda_1+\lambda_2+l\nu_1+m\nu_2)t}. \quad (2.2)$$

Case 3 $l = 0, s < j < m$:

$$\tilde{p}(0, j, t|0, m) = e^{-(\lambda_1 p_{12}+\lambda_2+m\nu_2)t}(\lambda_2 + \lambda_1 p_{12} + m\nu_2)©\tilde{p}(0, j, t|0, m - 1). \quad (2.3)$$

Case 4 $l = 0,\ j = m > s$:

$$\tilde{p}(0, m, t|0, m) = e^{-(\lambda_1 p_{12}+\lambda_2+m\nu_2)t}. \tag{2.4}$$

Case 5 $m = 0,\ s < i < l$:

$$\tilde{p}(i, 0, t|l, 0) = e^{-(\lambda_1+\lambda_2 p_{21}+l\nu_1)t}(\lambda_1 + \lambda_2 p_{21} + l\nu_1)©\tilde{p}(i, 0, t|l - 1, 0). \tag{2.5}$$

Case 6 $m = 0,\ i = l > s$:

$$\tilde{p}(l, 0, t|l, 0) = e^{-(\lambda_1+\lambda_2 p_{21}+l\nu_1)t}. \tag{2.6}$$

The Eqs. (2.1)–(2.6) can be solved by Laplace transform technique and the probabilities $\tilde{p}(i, j, t|l, m)$ are obtained.

2.5 The probability density function $\psi(t)$

Let $\psi(t)$ be defined by

$$\psi(t) = \lim_{\Delta\to 0} P[r - \text{event in } (t, t + \Delta), N(r, t) = 0|r - \text{event at } t = 0]/\Delta.$$

Then $\psi(t)$ is the probability density function of the interval between any two successive $r-$events. To derive an expression for $\psi(t)$, let $K(i, j, t) = \tilde{p}(i, j, t|E_0)$. We note that an $'r'$ event occurs when the system enters the state (i, j) such that $i + j = s$. This can occur in the following ways:

(i) The inventory level may be in $(i, s + 1 - i), i > 0$ and a demand for product 1 occurs or a demand for product 2 occurs, or one unit of product 1 or product 2 perishes.
(ii) The inventory level may be in $(0, s + 1)$ and a demand for product 1 occurs and is substituted by product 2 or a demand for product 2 occurs and is satisfied or one unit of product 2 perishes.
(iii) The inventory level may be in $(s + 1, 0)$ and a demand for product 1 occurs and is satisfied or a demand for product 2 occurs and is substituted by product 1 or one unit of product 1 perishes.

Then we get explicitly

$$\begin{aligned}\psi(t) = &\sum_{i=1}^{s} K(i, s - i + 1, t)(\lambda_1 + \lambda_2 + i\nu_1 + (s - i + 1)\nu_2)\\ &+ K(0, s + 1, t)[\lambda_2 + \lambda_1 p_{12} + (s + 1)\nu_2]\\ &+ K(s + 1, 0, t)[\lambda_1 + \lambda_2 p_{21} + (s + 1)\nu_1].\end{aligned} \tag{2.7}$$

2.6 The renewal density function $h_r(t)$

Let $h_r(t)$ be the renewal density of $r-$events. Then

$$h_r(t) = \lim_{\Delta\to 0} P[\text{an } r - \text{event in } (t, t + \Delta)|r - \text{event at } t = 0]/\Delta.$$

We note that

$$h_r(t) = \sum_{n=1}^{\infty} \psi^{(n)}(t). \tag{2.8}$$

3 Measures of system performance

3.1 Distribution of the inventory level

The joint distribution of the inventory levels of product 1 and product 2 at any time t is given by the function

$$P(i, j, t) = K(i, j, t) + h_r(t) \copyright K(i, j, t), (i, j) \in E. \tag{3.1}$$

3.2 Mean number of events

Let $'\eta'$ represent any one of the events $'r', 'a_i', 'b_i', i = 1, 2, 3$.

Define $h_\eta(t) = \lim_{\Delta \to 0} Pr\{\text{an } '\eta' \text{ event in } (t, t + \Delta] | E_0\}/\Delta$. Then $h_\eta(t)$ represents the first order product density (Srinivasan 1974) of the $'\eta'$ events. Consequently, the mean number of $'\eta'$ events that have occurred up to time t is given by

$$E(N(\eta, t)) = \int_0^t h_\eta(u) du.$$

Hence we derive expressions for the first order product densities of the events to get the mean number in $(0, t]$.

3.3 Mean number of replenishments

We have already obtained the renewal density $h_r(t)$ in the Eq. (2.8). Hence we get

$$E[N(r, t)] = \int_0^t h_r(u) du. \tag{3.2}$$

3.4 Mean number of demands satisfied

A demand for a product will be satisfied with probability 1 if the inventory level of that product is greater than zero at the epoch of its occurrence and with probability p_{ij} with the other product if the inventory level of that product is zero and the other product is greater than zero. Hence we have

$$h_{a_1}(t) = \sum_{i=1}^{S_1} \sum_{j=0}^{S_2}{}_{i+j=s+1} P(i, j, t|E_0)\lambda_1 + \sum_{j=s+1}^{S_2} P(0, j, t|E_0) p_{12} \lambda_1, \tag{3.3}$$

$$h_{b_1}(t) = \sum_{i=0}^{S_1} \sum_{j=1}^{S_2}{}_{i+j=s+1} P(i, j, t|E_0)\lambda_2 + \sum_{i=s+1}^{S_1} P(i, 0, t|E_0) p_{21} \lambda_2. \tag{3.4}$$

3.5 Mean number of demands lost

A demand for a product will be lost with probability 1 if the inventory level of both the products is zero at the time of its occurrence and with probability q_{ij} if the inventory level of that product is zero and the other product is greater than zero. Hence we have

$$h_{a_2}(t) = \sum_{j=s+1}^{S_2} P(0, j, t|E_0) q_{12} \lambda_1, \tag{3.5}$$

$$h_{b_2}(t) = \sum_{i=s+1}^{S_1} P(i, 0, t|E_0) q_{21} \lambda_2. \tag{3.6}$$

3.6 Mean number of demands substituted

A demand for a product will be substituted with probability p_{ij} with the other product if the inventory level of that product is zero and the other product is greater than zero at the epoch of its occurrence. Hence we have

$$h_{a_3}(t) = \sum_{j=s+1}^{S_2} P(0, j, t|E_0) p_{12} \lambda_1, \tag{3.7}$$

$$h_{b_3}(t) = \sum_{i=s+1}^{S_1} P(i, 0, t|E_0) p_{21} \lambda_2. \tag{3.8}$$

4 Steady state analysis

Let us define

$$P(i, j) = \lim_{t \to \infty} P(i, j, t|E_0).$$

Then $P(i, j)$ is the stationary joint distribution of the inventory levels of both the products. Using the principle of flow balance, we derive the following equations satisfied by $P(i, j)$;

Case 1(a) $i = 0$ **and** $j > s$:

$$\begin{aligned}(\lambda_2 + j\nu_2 + \lambda_1 p_{12}) P(0, j) = {} & \lambda_1 q_{12} P(0, j) + (\lambda_1 + \nu_1) P(1, j) \\ & + (p_{12}\lambda_1 + (j+1)\nu_2 + \lambda_2) P(0, j+1).\end{aligned} \tag{4.1}$$

Case 1(b) $i > s$ **and** $j = 0$:

$$\begin{aligned}(\lambda_1 + i\nu_1 + \lambda_2 p_{21}) P(i, 0) = {} & \lambda_2 q_{21} P(i, 0) + (\lambda_2 + \nu_2) P(i, 1) \\ & + (p_{21}\lambda_2 + (i+1)\nu_1 + \lambda_1) P(i+1, 0).\end{aligned} \tag{4.2}$$

Case 1(c) $i, j > 0$ **and** $i + j > s$:

$$\begin{aligned}(\lambda_1 + \lambda_2 + i\nu_1 + j\nu_2) P(i, j) = {} & (\lambda_1 + (i+1)\nu_1) P(i+1, j) \\ & + (\lambda_2 + (j+1)\nu_2) P(i, j+1).\end{aligned} \tag{4.3}$$

The set of Eqs. (4.1)–(4.3) can be solved along with the constraint

$$\sum_{i=0}^{S_1} \sum_{j=0}^{S_2}{}_{i+j=s+1} P(i, j) = 1 \tag{4.4}$$

to obtain explicitly the steady-state distribution $P(i, j)$.

4.1 Mean stationary rate of events

The mean stationary rate of η events is given by $E[N(\eta,\infty)] = \lim_{t\to\infty} \frac{1}{t} E[N(\eta,t)]$.

4.1.1 Mean stationary rate of replenishments

$$\begin{aligned}E[N(r,\infty)] = &\sum_{i=1}^{s} P(i, s-i+1)(\lambda_1+\lambda_2+i\nu_1+(s-i+1)\nu_2)\\ &+P(0,s+1)[\lambda_2+\lambda_1 p_{12}+(s+1)\nu_2]\\ &+P(s+1,0)[\lambda_1+\lambda_2 p_{21}+(s+1)\nu_1]. \end{aligned}\tag{4.5}$$

4.1.2 Mean stationary rate of demands satisfied

$$E[N(a_1,\infty)] = \sum_{i=1}^{S_1}\sum_{\substack{j=0\\ i+j>s}}^{S_2} P(i,j)\lambda_1 + \sum_{j>s}^{S_2} P(0,j)p_{12}\lambda_1. \tag{4.6}$$

$$E[N(b_1,\infty)] = \sum_{i=0}^{S_1}\sum_{\substack{j=1\\ i+j>s}}^{S_2} P(i,j)\lambda_2 + \sum_{i>s}^{S_1} P(i,0)p_{21}\lambda_2. \tag{4.7}$$

4.1.3 Mean stationary rate of demands lost

$$E[N(a_2,\infty)] = \sum_{j>s}^{S_2} P(0,j)q_{12}\lambda_1, \tag{4.8}$$

$$E[N(b_2,\infty)] = \sum_{i>s}^{S_1} P(i,0)q_{21}\lambda_2. \tag{4.9}$$

4.1.4 Mean stationary rate of demands substituted

$$E[N(a_3,\infty)] = \sum_{j>s}^{S_2} P(0,j)p_{12}\lambda_1, \tag{4.10}$$

$$E[N(b_3,\infty)] = \sum_{i>s}^{S_1} P(i,0)p_{21}\lambda_2. \tag{4.11}$$

5 A numerical illustration

As an illustration, let us consider the simple situation: $S_1 = 1, S_2 = 2, s = 1$. Then the state space is given by

$$E = \{(i,j) : i = 0,1,\ j = 0,1,2 \text{ such that } i+j > 1\} = \{(1,2),(1,1),(0,2)\}.$$

The state transition probabilities are

$$\tilde{p}(1,2,t|1,2) = e^{-(\lambda_1+\lambda_2+\nu_1+2\nu_2)t};\quad \tilde{p}(1,1,t|1,1) = e^{-(\lambda_1+\lambda_2+\nu_1+\nu_2)t};$$
$$\tilde{p}(0,2,t|0,2) = e^{-(\lambda_1 p_{12}+\lambda_2+2\nu_2)t};$$

$$\tilde{p}(1,1,t|1,2) = e^{-(\lambda_1+\lambda_2+\nu_1+2\nu_2)t}(\lambda_2+2\nu_2)©e^{-(\lambda_1+\lambda_2+\nu_1+\nu_2)t};$$
$$\tilde{p}(0,2,t|1,2) = e^{-(\lambda_1+\lambda_2+\nu_1+2\nu_2)t}(\lambda_1+\nu_1)©e^{-(\lambda_1 p_{12}+\lambda_2++2\nu_2)t}.$$

Then the probability density function $\psi(t)$ is given by

$$\psi(t) = K(1,1,t)(\lambda_1+\lambda_2+\nu_1+\nu_2) + K(0,2,t)(\lambda_2+\lambda_1 p_{12}+2\nu_2),$$

where we have

$$K(1,1,t) = \tilde{p}(1,1,t|1,2) = e^{-(\lambda_1+\lambda_2+\nu_1+2\nu_2)t}(\lambda_2+2\nu_2)©e^{-(\lambda_1+\lambda_2+\nu_1+\nu_2)t};$$
$$K(0,2,t) = \tilde{p}(0,2,t|1,2) = e^{-(\lambda_1+\lambda_2+\nu_1+2\nu_2)t}(\lambda_1+\nu_1)©e^{-(\lambda_1 p_{12}+\lambda_2+2\nu_2)t}.$$

Taking Laplace transform, we get

$$K^*(1,1,\theta) = \frac{(\lambda_2+2\nu_2)}{(\theta+\lambda_1+\lambda_2+\nu_1+2\nu_2)}\frac{1}{(\theta+\lambda_1+\lambda_2+\nu_1+\nu_2)};$$
$$K^*(0,2,\theta) = \frac{(\lambda_1+\nu_1)}{(\theta+\lambda_1+\lambda_2+\nu_1+2\nu_2)}\frac{1}{(\theta+\lambda_1 p_{12}+\lambda_2+2\nu_2)}.$$

Consequently, we obtain

$$\begin{aligned}\psi^*(\theta) &= K^*(1,1,\theta)(\lambda_1+\lambda_2+\nu_1+\nu_2) + K^*(0,2,\theta)(\lambda_2+\lambda_1 p_{12}+2\nu_2)\\ &= \frac{(\lambda_2+2\nu_2)}{(\theta+\lambda_1+\lambda_2+\nu_1+2\nu_2)}\frac{(\lambda_1+\lambda_2+\nu_1+\nu_2)}{(\theta+\lambda_1+\lambda_2+\nu_1+\nu_2)}\\ &\quad+\frac{(\lambda_1+\nu_1)}{(\theta+\lambda_1+\lambda_2+\nu_1+2\nu_2)}\frac{(\lambda_2+\lambda_1 p_{12}+2\nu_2)}{(\theta+\lambda_1 p_{12}+\lambda_2+2\nu_2)}.\end{aligned}$$

We note that $\psi^*(0) = 1$, confirming that $\psi(t)$ is indeed a probability density function. Inverting $\psi^*(\theta)$, we obtain

$$\begin{aligned}\psi(t) = &\frac{(\lambda_2+2\nu_2)(\lambda_1+\lambda_2+\nu_1+\nu_2)}{\nu_2}\left[e^{-(\lambda_1+\lambda_2+\nu_1+\nu_2)t} - e^{-(\lambda_1+\lambda_2+\nu_1+2\nu_2)t}\right]\\ &+\frac{(\lambda_1+\nu_1)(\lambda_2+\lambda_1 p_{12}+2\nu_2)}{\lambda_1 q_{12}+\nu_1}\left[e^{-(\lambda_1 p_{12}+\lambda_2+2\nu_2)t} - e^{-(\lambda_1+\lambda_2+\nu_1+2\nu_2)t}\right].\end{aligned}$$

For the above illustration, the steady-state probabilities are given by the following flow-balance equations:

$$(\lambda_2+2\nu_2+\lambda_1 p_{12}+\lambda_1 q_{12})P(0,2) = \lambda_1 q_{12}P(0,2) + (\lambda_1+\nu_1)P(1,2);$$
$$(\lambda_1+\lambda_2+\nu_1+2\nu_2)P(1,2) = (\lambda_2+2\nu_2+\lambda_1 p_{12})P(0,2) + (\lambda_1+\lambda_2+\nu_1+\nu_2)P(1,1);$$
$$(\lambda_1+\lambda_2+\nu_1+\nu_2)P(1,1) = (\lambda_2+2\nu_2)P(1,2).$$

Solving the above equations along with the constraint

$$P(0,2) + P(1,1) + P(1,2) = 1,$$

we obtain

$$P(1,1) = \frac{\lambda_2+2\nu_2}{\lambda_1+\lambda_2+\nu_1+\nu_2}P(1,2);$$
$$P(0,2) = \frac{\lambda_1+\nu_1}{\lambda_2+2\nu_2+\lambda_1 p_{12}}P(1,2);$$
$$P(1,2) = \left[1+\frac{\lambda_1+\nu_1}{\lambda_2+2\nu_2+\lambda_1 p_{12}}+\frac{\lambda_2+2\nu_2}{\lambda_1+\lambda_2+\nu_1+\nu_2}\right]^{-1}.$$

In the above illustration, the inventory level of product 2 never reaches the level 0 and hence the demand for product 2 is always satisfied. So we consider another illustration where both the inventories may reach level 0 before replenishment. For the purpose of numerical illustration, we assume the following values for the parameters:

$$S1 = 2; \quad S2 = 3; \quad s = 1.$$

Then the state space is

$$E = \{(2,3), (2,2), (1,3), (2,1), (1,2), (0,3), (1,1), (2,0), (0,2)\}.$$

5.1 Stationary distribution of the inventory level

Using the principle of flow balance, we obtain the following system of linear equations:

$$\begin{aligned}
&a_{111}P(1,1) + a_{120}P(2,0) + a_{102}P(0,2) - a_{123}P(2,3) &&= 0,\\
&a_{213}P(1,3) - a_{223}P(2,3) &&= 0,\\
&a_{322}P(2,2) - a_{323}P(2,3) &&= 0,\\
&a_{412}P(1,2) - a_{422}P(2,2) - a_{413}P(1,3) &&= 0,\\
&a_{521}P(2,1) - a_{522}P(2,2) &&= 0,\\
&a_{603}P(0,3) - a_{613}P(1,3) &&= 0,\\
&a_{711}P(1,1) - a_{712}P(1,2) - a_{721}P(2,1) &&= 0,\\
&a_{820}P(2,0) - a_{821}P(2,1) &&= 0,\\
&a_{902}P(0,2) - a_{912}P(1,2) - a_{903}P(0,3) &&= 0,
\end{aligned}$$

where

$$\begin{array}{lll}
a_{111} = \lambda_1 + \nu_1 + \lambda_2 + \nu_2, & a_{120} = \lambda_1 + 2\nu_1 + \lambda_2 p_{21}, & a_{102} = \lambda_2 + 2\nu_2 + \lambda_1 p_{12},\\
a_{123} = \lambda_1 + 2\nu_1 + \lambda_2 + 3\nu_2; & a_{213} = \lambda_1 + \nu_1 + \lambda_2 + 3\nu_2, & a_{223} = \lambda_1 + 2\nu_1;\\
a_{322} = \lambda_1 + 2\nu_1 + \lambda_2 + 2\nu_2, & a_{323} = \lambda_2 + 3\nu_2; & a_{412} = \lambda_1 + \nu_1 + \lambda_2 + 2\nu_2,\\
a_{422} = \lambda_1 + 2\nu_1; & a_{413} = \lambda_2 + 3\nu_2; & a_{521} = \lambda_1 + 2\nu_1 + \lambda_2 + \nu_2,\\
a_{522} = \lambda_2 + 2\nu_2; & a_{603} = \lambda_2 + 3\nu_2 + \lambda_1 p_{12}, & a_{613} = \lambda_1 + \nu_1;\\
a_{711} = \lambda_1 + \nu_1 + \lambda_2 + \nu_2, & a_{712} = \lambda_2 + 2\nu_2; & a_{721} = \lambda_1 + 2\nu_1;\\
a_{820} = \lambda_1 + 2\nu_1 + \lambda_2 p_{21}, & a_{821} = \lambda_2 + \nu_2; & a_{902} = \lambda_2 + 2\nu_2 + \lambda_1 p_{12},\\
a_{912} = \lambda_1 + \nu_1; & a_{903} = \lambda_2 + 3\nu_2 + \lambda_1 p_{12}. &
\end{array}$$

The above system of linear equations can be solved along with the constraint

$$\sum_{i=0}^{2}\sum_{j=0_{i+j>1}}^{3} P(i,j) = 1.$$

We fix the following values for the parameters:

$$\nu_1 = 3.000; \quad \nu_2 = 4.000; \quad \lambda_1 = 1.100; \quad \lambda_2 = 2.000.$$

Table 1 gives the distribution of the joint inventory level for values of $p_{12} = 0.25, 0.5, 0.75$ and $p_{21} = 0.25, 0.5, 0.75$.

5.2 Mean stationary rate of replenishment versus perishing rate for product 1

We fix $\nu_2 = 4.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_1. Table 2/Fig. 1 gives the mean stationary rate of replenishments for values ν_1 increasing from 1 to 10. We observe that as ν_1 increases (that is, the mean perishing time for product 1 decreases), the mean stationary rate of replenishments increases as expected.

Table 1 Joint inventory level distribution

(p_{12}, p_{21})	$P(2,3)$	$P(1,3)$	$P(2,2)$	$P(1,2)$	$P(2,1)$	$P(0,3)$	$P(1,1)$	$P(2,0)$	$P(0,2)$
(0.25, 0.25)	0.1663	0.0735	0.1361	0.1415	0.1039	0.0211	0.2132	0.0649	0.0794
(0.25, 0.50)	0.1668	0.0737	0.1366	0.1420	0.1043	0.0212	0.2139	0.0619	0.0797
(0.25, 0.75)	0.1673	0.0739	0.1370	0.1424	0.1046	0.0212	0.2145	0.0592	0.0799
(0.50, 0.25)	0.1663	0.0735	0.1362	0.1416	0.1040	0.0207	0.2133	0.0650	0.0795
(0.50, 0.50)	0.1669	0.0738	0.1366	0.1420	0.1043	0.0208	0.2139	0.0620	0.0797
(0.50, 0.75)	0.1674	0.0740	0.1370	0.1425	0.1046	0.0208	0.2146	0.0592	0.0799
(0.75, 0.25)	0.1664	0.0736	0.1362	0.1416	0.1040	0.0203	0.2133	0.0650	0.0795
(0.75, 0.50)	0.1669	0.0738	0.1367	0.1421	0.1043	0.0204	0.2140	0.0620	0.0797
(0.75, 0.75)	0.1674	0.0740	0.1371	0.1425	0.1046	0.0205	0.2147	0.0592	0.0800

Table 2 Mean Stationary rate of replenishment versus perishing rate for product 1

ν_1	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(r,\infty))$	2.2575	2.9744	3.5036	3.9123	4.2365	4.4980	4.7116	4.8876	5.0338	5.1559

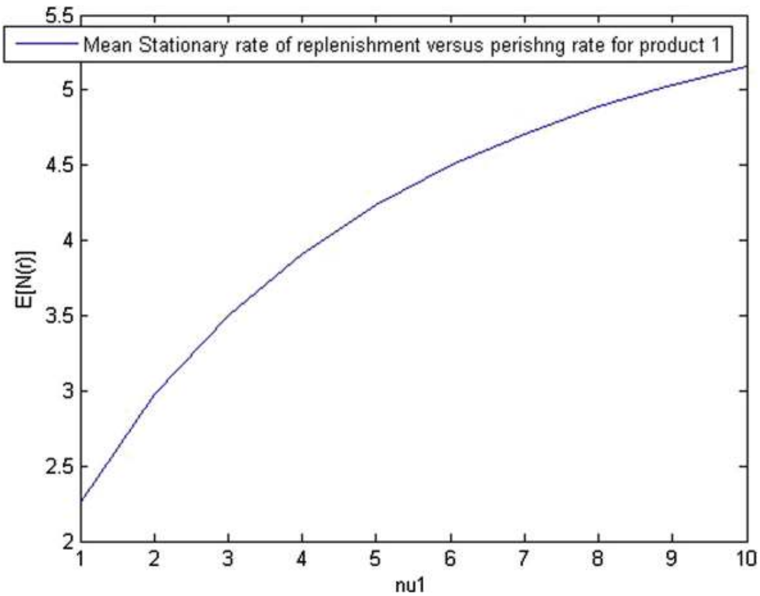

Fig. 1 Mean stationary rate of replenishment versus perishing rate for product 1

5.3 Mean stationary rate of replenishment versus perishing rate for product 2

We fix $\nu_1 = 3.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_2. Table 3/Fig. 2 gives the mean stationary rate of replenishments for values ν_2 increasing from

Table 3 Mean Stationary rate of replenishment versus perishing rate for product 2

ν_2	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(r, \infty))$	1.98977	2.6464	3.1293	3.5036	3.8061	4.0582	4.2730	4.4592	4.6228	4.7680

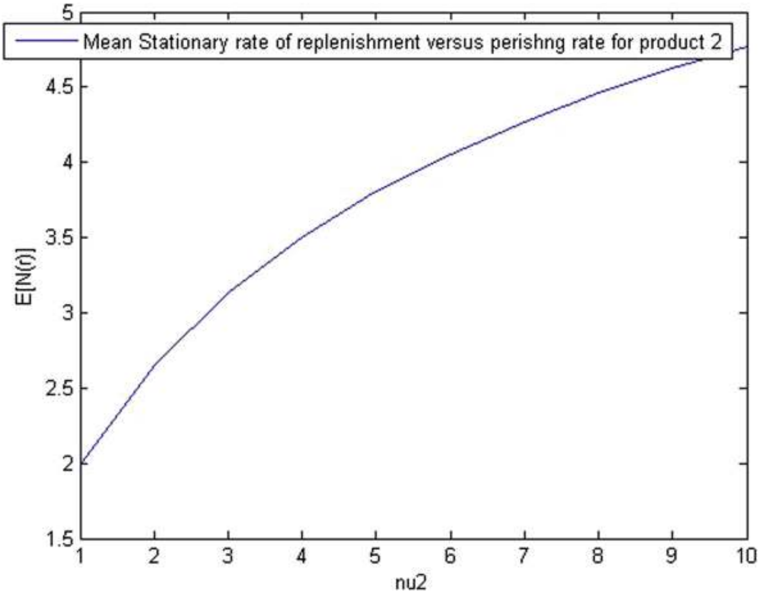

Fig. 2 Mean stationary rate of replenishment versus perishing rate for product 2

Table 4 Mean Stationary rate of demands satisfied versus perishing rate of product 1

ν_1	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(a_1, \infty))$	1.0829	1.0653	1.0447	1.0226	1.0000	0.9777	0.9561	0.9355	0.9160	0.8978
$E(N(b_1, \infty))$	2.1147	2.0827	2.0620	2.0477	2.0376	2.0301	2.0245	2.0201	2.0168	2.0141

1 to 10. We observe that as ν_2 increases (that is, the mean perishing time for product 2 decreases), the mean stationary rate of replenishments increases as expected.

5.4 Mean stationary rate of demands satisfied versus perishing rate of product 1

We fix $\nu_2 = 4.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_1. Table 4/Fig. 3 gives the mean stationary rate of demands for product 1 and product 2 satisfied for values ν_1 increasing from 1 to 10. We observe that as ν_1 increases (that is, the mean time for product 1 to perish decreases), the mean stationary rate of demands satisfied increases as expected. Further, the demand rate for product 2 ($\lambda_2 = 2$) is higher than the demand rate of product 1 ($\lambda_1 = 1$) the mean stationary rate of demands satisfied by product 2 is more than the corresponding value of product 1.

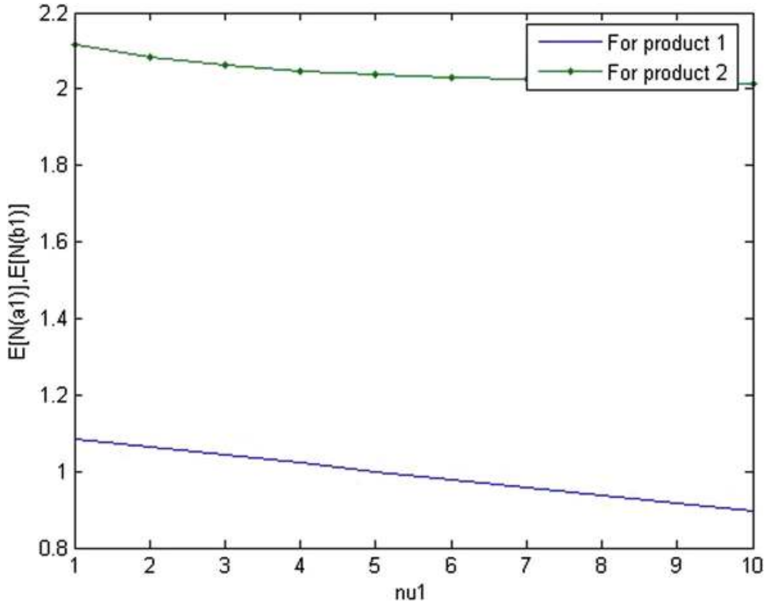

Fig. 3 Mean stationary rate of demands satisfied versus perishing rate of product 1

Table 5 Mean Stationary rate of demands satisfied versus perishing rate of product 2

ν_2	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(a_1,\infty))$	0.9185	0.9858	1.0228	1.0447	1.0586	1.0679	1.0745	1.0792	1.0827	1.0854
$E(N(b_1,\infty))$	2.0204	2.0360	2.0499	2.0620	2.0724	2.0814	2.0894	2.0964	2.1026	2.1082

5.5 Mean stationary rate of demands satisfied versus perishing rate of product 2

We fix $\nu_1 = 3.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_2. Table 5/Fig. 4 gives the mean stationary rate of demands for product 1 and product 2 satisfied for values ν_2 increasing from 1 to 10. We observe that as ν_2 increases (that is, the mean time for product 2 to perish decreases), the mean stationary rate of demands satisfied increases as expected. Further, the demand rate for product 2 ($\lambda_2 = 2$) is higher than the demand rate of product 1 ($\lambda_1 = 1$) the mean stationary rate of demands satisfied by product 2 is more than the corresponding value of product 1.

5.6 Mean stationary rate of demands lost versus ν_1

We fix $\nu_2 = 4.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_1. Table 6/Fig. 5 gives the mean stationary rate of demands for product 1 and product 2 lost for values of ν_1 increasing from 1 to 10. We observe that as ν_1 increases (that is the mean time for product 1 to perish decreases), the mean stationary rate of demands lost decreases as expected. Further, the demand rate for product 2 ($\lambda_2 = 2$) is higher than the demand rate of product 1 ($\lambda_1 = 1$) the mean stationary rate of demands for product 2 lost is more than the corresponding value of product 1.

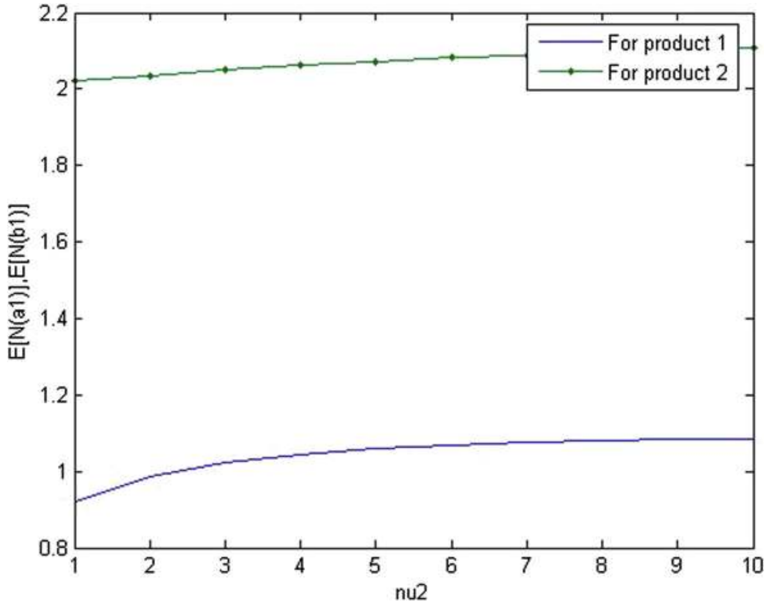

Fig. 4 Mean stationary rate of demands satisfied versus perishing rate of product 2

Table 6 Mean Stationary rate of demands lost versus ν_1

ν_1	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(a_2, \infty))$	0.0171	0.0347	0.0553	0.0774	0.1000	0.1223	0.1439	0.1645	0.1840	0.2022
$E(N(b_2, \infty))$	0.1147	0.0827	0.0620	0.0477	0.0376	0.0301	0.0245	0.0201	0.0168	0.0141

5.7 Mean stationary rate of demands lost versus ν_2

We fix $\nu_1 = 3.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_2. Table 7/Fig. 6 gives the mean stationary rate of demands for product 1 and product 2 lost for values of ν_2 increasing from 1 to 10. We observe that as ν_2 increases (that is the mean time for product 2 to perish decreases), the mean stationary rate of demands lost decreases as expected. Further, the demand rate for product 2 ($\lambda_2 = 2$) is higher than the demand rate of product 1 ($\lambda_1 = 1$) the mean stationary rate of demands for product 2 lost is more than the corresponding value of product 1.

5.8 Mean stationary rate of demands substituted versus ν_1

We fix $\nu_2 = 4.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_1. Table 8/Fig. 7 gives the mean stationary rate of demands for product 1 substituted with product 2 and product 2 with product 1 for values of ν_1 increasing from 1 to 10. We observe that as ν_1 increases (that is the mean time for product 1 to perish decreases), the mean stationary rate of demands for product 1 substituted with product 2 increases and the mean stationary rate of demands for product 2 substituted with product 1 decreases.

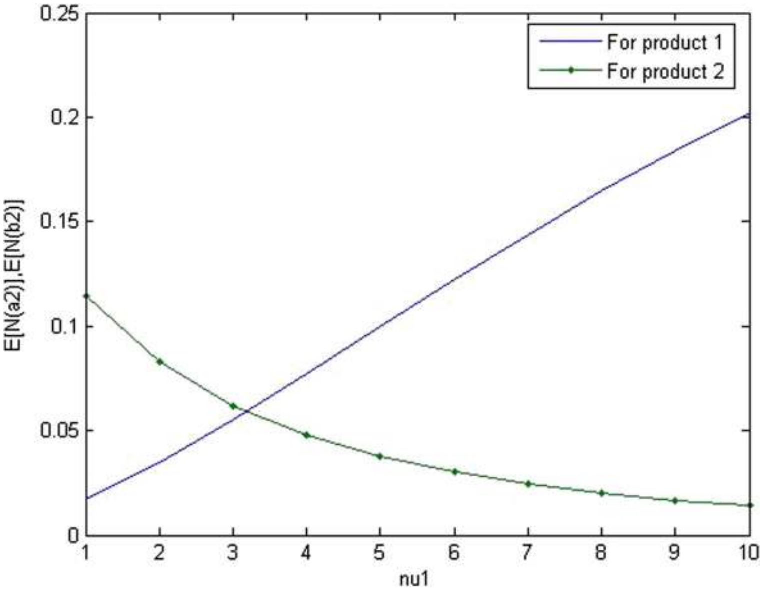

Fig. 5 Mean stationary rate of demands lost versus ν_1

Table 7 Mean Stationary rate of demands lost versus ν_2

ν_2	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(a_2, \infty))$	0.1815	0.1142	0.0772	0.0553	0.0414	0.0321	0.0255	0.0208	0.0173	0.0146
$E(N(b_2, \infty))$	0.0204	0.0360	0.0499	0.0620	0.0724	0.0814	0.0894	0.0964	0.1026	0.1082

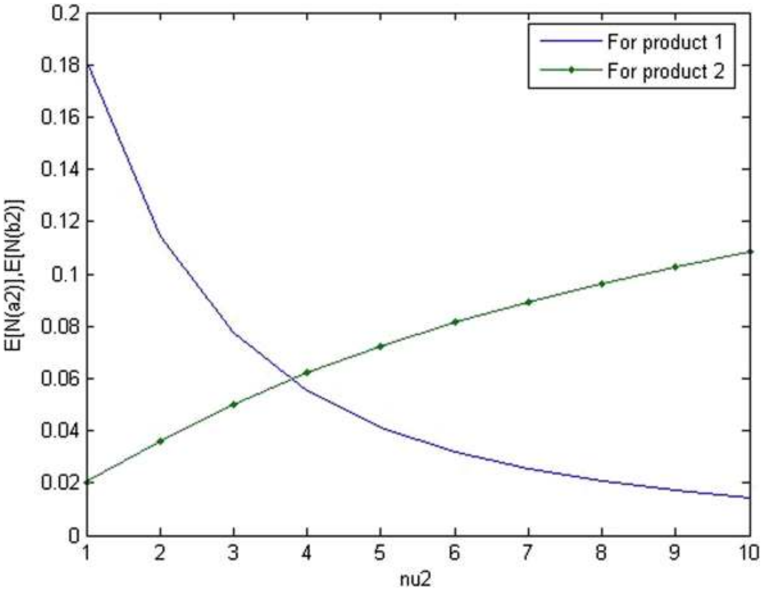

Fig. 6 Mean stationary rate of demands lost versus ν_2

Table 8 Mean Stationary rate of demands substituted versus ν_1

ν_2	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(a_3, \infty))$	0.0171	0.0347	0.0553	0.0774	0.1000	0.1223	0.1439	0.1645	0.1840	0.2022
$E(N(b_3, \infty))$	0.1147	0.0827	0.0620	0.0477	0.0376	0.0301	0.0245	0.0201	0.0168	0.0141

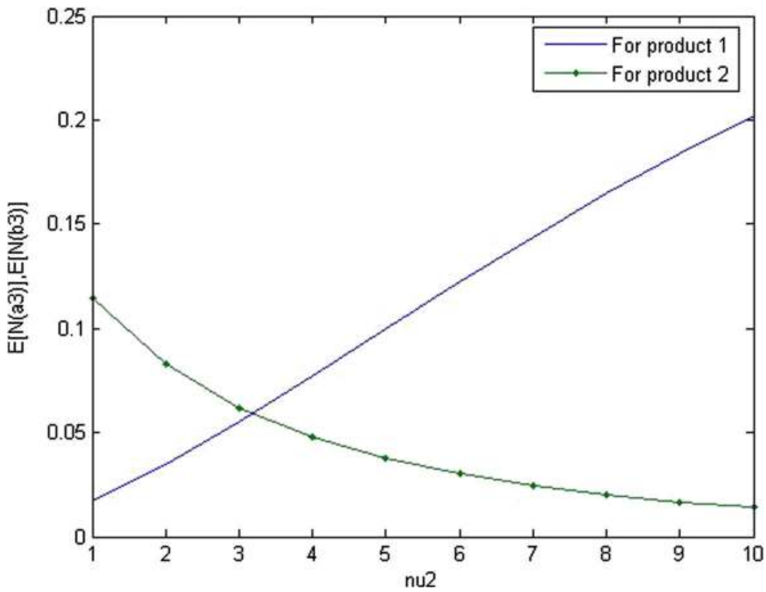

Fig. 7 Mean stationary rate of demands substituted versus ν_1

Table 9 Mean Stationary rate of demands substituted versus ν_2

ν_2	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
$E(N(a_3, \infty))$	0.1815	0.1142	0.0772	0.0553	0.0414	0.0321	0.0255	0.0208	0.0173	0.0146
$E(N(b_3, \infty))$	0.0204	0.0360	0.0499	0.0620	0.0724	0.0814	0.0894	0.0964	0.1026	0.1082

5.9 Mean stationary rate of demands substituted versus ν_2

We fix $\nu_1 = 3.000$; $\lambda_1 = 1.100$; $\lambda_2 = 2.000$; $p_{12} = 0.50$; $p_{21} = 0.50$ and vary ν_2. Table 9/Fig. 8 gives the mean stationary rate of demands for product 1 substituted with product 2 and product 2 with product 1 for values of ν_2 increasing from 1 to 10. We observe that as ν_2 increases (that is the mean time for product 2 to perish decreases), the mean stationary rate of demands for product 1 substituted with product 2 decreases and the mean stationary rate of demands for product 2 substituted with product 1 increases.

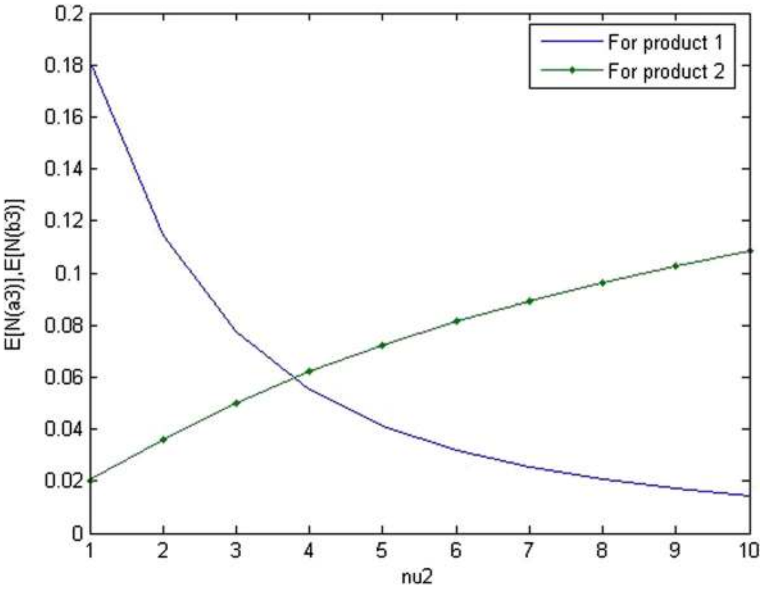

Fig. 8 Mean stationary rate of demands substituted versus ν_2

6 Conclusion

We considered a disaster inventory system which stores two substitutable perishable product. Formulating a two-dimensional stochastic process, we made a continuous-review analysis of the disaster inventory. Since the inventory is maintained for disaster management, we adopted an adjustable joint reordering policy for replenishment. Assuming no lead time for the replenishment, we obtained the mean number of (i) the demands satisfied; (ii) demands substituted; (iii) demands lost; and (iv) replenishments in the interval $(0, t]$. We also obtained the stationary distribution for the state of the system and validated the performance measures by a numerical example. From the numerical illustration, we observed that the system performed as expected:

(a) As the mean perishing time for product 1 decreases, the mean stationary rate of replenishments increases.
(b) As the mean perishing time for product 2 decreases, the mean stationary rate of replenishments increases.
(c) As the mean time for product 1 to perish decreases, the mean stationary rate of demands satisfied increases.
(d) As the mean time for product 2 to perish decreases, the mean stationary rate of demands satisfied increases.
(e) As the mean time for product 1 to perish decreases), the mean stationary rate of demands lost decreases.
(f) As the mean time for product 2 to perish decreases, the mean stationary rate of demands lost decreases.
(g) As the mean time for product 1 to perish decreases, the mean stationary rate of demands for product 1 substituted with product 2 increases and the mean stationary rate of demands for product 2 substituted with product 1 decreases.

(h) As the mean time for product 2 to perish decreases, the mean stationary rate of demands for product 1 substituted with product 2 decreases and the mean stationary rate of demands for product 2 substituted with product 1 increases.

Consequently, we conclude that the mean perishing times of the products control the flow of the demands for the products.

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