

# Two theorems of N. Wiener for solutions of quasilinear elliptic equations

by

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## 1. Introduction

Relatively little is known about boundary behavior of solutions of quasilinear elliptic partial differential equations as compared to that of harmonic functions. In this paper two results, which in the harmonic case are due to N. Wiener, are generalized to a non-linear situation. Suppose that  $G$  is a bounded domain in  $\mathbf{R}^n$ . We consider functions  $u: G \rightarrow \mathbf{R}$  which are free extremals of the variational integral

$$\int F(x, \nabla u(x)) \, dm(x)$$

in the conformally invariant or borderline case  $F(x, h) \approx |h|^n$ . For the precise assumptions on the kernel  $F: G \times \mathbf{R}^n \rightarrow \mathbf{R}$  see Section 2. Equivalently, the  $F$ -extremality of  $u$  means that  $u$  is a weak solution of the corresponding Euler equation

$$\nabla \cdot \nabla_h F(x, \nabla u(x)) = 0 \tag{1.1}$$

with measurable coefficients. Solutions  $u$ ,  $F$ -extremals, of the equation (1.1) form a similar basis for a non-linear potential theory as harmonic functions do for the classical potential theory. Especially, the Perron-Wiener-Brelot method can be applied, see Section 2.10. For each bounded function  $f: \partial G \rightarrow \mathbf{R}$  there exist two  $F$ -extremals, the upper Perron solution  $\bar{H}_f$  and the lower Perron solution  $\underline{H}_f$  with “boundary values”  $f$  in  $G$ . These functions are defined via sub- and super-solutions as in the classical harmonic case. In 1970, W. Mazja [M] proved, although his formulation was slightly less general, that if  $f$  is continuous and if the Wiener condition

$$\int_0^{\infty} \frac{\varphi(t)^{1/(n-1)}}{t} dt = \infty, \quad (1.2)$$

$\varphi(t) = \text{cap}_n(B^n(x_0, 2t), [G \cap \bar{B}^n(x_0, t)])$ , holds at  $x_0 \in \partial G$ , i.e.  $[G$  is not thin at  $x_0$ , then

$$\lim_{x \rightarrow x_0} H_f(x) = f(x_0) = \lim_{x \rightarrow x_0} \tilde{H}_f(x). \quad (1.3)$$

We call the point  $x_0$  *F-regular* if the condition (1.3) holds for each continuous  $f: \partial G \rightarrow \mathbf{R}$ . Our first theorem, proved in Section 3, shows that Mazja's result has a converse.

**THEOREM 1.4.** *If  $x_0 \in \partial G$  is F-regular, then  $[G$  is not thin at  $x_0$ .*

The proof for Theorem 1.4 shows that the *F-regularity* can be studied via restrictions of  $C^\infty(\mathbf{R}^n)$ -functions to the boundary.

The result of Mazja and Theorem 1.4 were proved by N. Wiener [W2] in the classical harmonic case in  $\mathbf{R}^n$ ,  $n \geq 2$ . His proof strongly employed linearity. Our proof is based on sharp *F-capacity* estimates and hence in the classical plane harmonic case our version gives a new and geometric proof for Theorem 1.4 based on energy considerations.

The variational interpretation of Mazja's result and Theorem 1.4 is the following. For a function  $f: \bar{G} \rightarrow \mathbf{R}$  in the Sobolev class  $W_n^1(G)$ , let  $u = u_f$  be the unique *F-extremal* with Sobolev boundary values  $f$ , i.e.  $u - f \in W_{n,0}^1(G)$ . Then

$$\lim_{x \rightarrow x_0} u(x) = f(x_0)$$

for all continuous functions  $f$  if and only if the condition (1.2) holds at  $x_0$ .

Theorem 1.4 implies the following result, which in the linear case is due to W. Littman, G. Stampacchia and H. F. Weinberger [LSW], see also [P].

**COROLLARY 1.5.** *F-regularity is independent of the variational kernel  $F$ .*

To formulate our second result we call a function  $f: \partial G \rightarrow \mathbf{R} \cup \{-\infty, \infty\}$  *F-resolutive*, if the corresponding upper and lower Perron solutions  $\tilde{H}_f$  and  $H_f$  are *F-extremals* in  $G$  and  $\tilde{H}_f = H_f$  there. In Section 4 we extend the well-known potential theoretic theorem of N. Wiener [W3].

**THEOREM 1.6.** *All continuous functions  $f: \partial G \rightarrow \mathbf{R}$  are F-resolutive.*

Proofs for Wiener's result are usually based on approximation of a continuous function by a difference of two subharmonic functions. Our proof for Theorem 1.6 rests

on an important obstacle method, which for  $n=2$  gives a new and direct proof for Wiener's result.

The results in Theorems 1.4 and 1.6 can be extended considerably. For instance, it is possible to consider solutions  $u$  of an elliptic quasilinear differential equation

$$\nabla \cdot A(x, \nabla u(x)) = 0$$

in divergence form where for a.e.  $x \in G$ ,  $|A(x, h)| \leq \gamma |h|^{p-1}$  and  $p > n-1$  (for the precise assumption on  $A$  see Remark 2.3 in Section 2). The assumption  $p > n-1$  is needed in the proof for Theorem 1.4 to guarantee that the  $p$ -capacity between two non-degenerate continua in  $\mathbf{R}^n$  is positive. However, we restrict our considerations to the case  $p=n$ .

## 2. Perron's method

In this section we first present the basic assumptions on variational kernels  $F$  and recall some properties of the corresponding  $F$ -extremals. The rest of the section is devoted to the Perron-Wiener-Brelot method in a non-linear situation.

2.1. *Variational integrals.* Let  $G$  be a bounded domain in  $\mathbf{R}^n$ . We consider weak solutions, called  $F$ -extremals, of an Euler equation

$$\nabla \cdot \nabla_h F(x, \nabla u(x)) = 0 \tag{2.2}$$

where the variational kernel  $F: G \times \mathbf{R}^n \rightarrow \mathbf{R}$  satisfies the assumptions

(a) For each  $\varepsilon > 0$  there is a closed set  $K$  in  $G$  such that  $m(G \setminus K) < \varepsilon$  and the restriction  $F|_{K \times \mathbf{R}^n}$  is continuous.

(b) For a.e.  $x \in G$  the mapping  $h \rightarrow F(x, h)$  is strictly convex and differentiable in  $\mathbf{R}^n$ ; for a fixed  $x$  the gradient of  $F$  with respect to  $h$  is denoted by  $\nabla_h F$ .

(c) There are constants  $0 < \alpha \leq \beta < \infty$  such that for a.e.  $x \in G$

$$\alpha |h|^n \leq F(x, h) \leq \beta |h|^n,$$

whenever  $h \in \mathbf{R}^n$ .

(d) For a.e.  $x \in G$

$$F(x, \lambda h) = |\lambda|^n F(x, h),$$

when  $h \in \mathbf{R}^n$  and  $\lambda \in \mathbf{R}$ .

A typical example is the  $n$ -Dirichlet kernel  $F(x, h) = |h|^n$ . For a thorough analysis of the above assumptions see [GLM1]. If the exponent  $n$  in (c) and (d) is replaced by an

exponent  $p > 1$ , most of our theory still holds; in the cases  $1 < p \leq n-1$  the method in Section 3 breaks down.

A function  $u$  in  $C(G) \cap \text{loc } W_n^1(G)$ , i.e.  $u$  is  $ACL^n$ , is called an  $F$ -extremal in  $G$ , if for all domains  $D \subset\subset G$

$$I_F(u, D) = \inf_{v \in \mathcal{F}_u} I_F(v, D)$$

where

$$I_F(v, D) = \int_D F(x, \nabla v(x)) \, dm(x)$$

is the variational integral with the kernel  $F$  and

$$\mathcal{F}_u = \{v \in C(\bar{D}) \cap W_n^1(D) : v = u \text{ in } \partial D\}.$$

A function  $u$  in the class  $C(G) \cap \text{loc } W_n^1(G)$  is an  $F$ -extremal if and only if it is a weak solution of (2.2), i.e.

$$\int_G \nabla_h F(x, \nabla u(x)) \cdot \nabla \eta(x) \, dm(x) = 0$$

for all  $\eta \in C_0^\infty(G)$ . For this result see [GLM1, Theorem 3.18].

**2.3 Remark.** In [M], see also the remark [MH] due to L.-I. Hedberg, Mazja considered quasilinear second order elliptic equations in divergence form

$$\nabla \cdot A(x, \nabla u(x)) = 0 \tag{2.4}$$

where the function  $A: G \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ , in addition to the usual measurability conditions, satisfies in the borderline case  $p=n$  the assumptions:

- (i) There are constants  $0 < \gamma, \sigma < \infty$  such that for a.e.  $x \in G$ ,  $|A(x, h)| \leq \gamma |h|^{n-1}$ , and  $h \cdot A(x, h) \geq \sigma |h|^n$ .
- (ii) For a.e.  $x \in G$ ,  $A(x, \lambda h) = \lambda^{n-1} A(x, h)$ ,  $\lambda > 0$ .
- (iii) For a.e.  $x \in G$ ,  $(A(x, h) - A(x, k)) \cdot (h - k) > 0$  for  $h \neq k$ .

It is easy to see, cf. [GLM1, pp. 48–49], that the function  $A(x, h) = \nabla_h F(x, h)$  satisfies (i)–(iii). Although it is not true that a vector function  $A$  satisfying (i)–(iii) is the gradient of a variational kernel  $F$ , it is possible in Section 3 to replace the concept of  $F$ -capacity by the so called  $A$ -potential, see [M, p. 47]. Thus Theorems 1.4 and 1.6 hold for equations (2.4) as well.

2.5. *Properties of F-extremals.* Although  $F$ -extremals do not in general form a linear space,  $\lambda u$  and  $u + \lambda$  are  $F$ -extremals whenever  $u$  is an  $F$ -extremal and  $\lambda \in \mathbf{R}$ . Moreover,  $F$ -extremals satisfy the important *F-comparison principle* [GLM3, Lemma 2.3]: If  $u$  and  $v$  are  $F$ -extremals in  $G$ ,

$$\overline{\lim}_{x \rightarrow y} u(x) \leq \underline{\lim}_{x \rightarrow y} v(x)$$

for all  $y \in \partial G$  and the left and right hand sides are neither  $\infty$  nor  $-\infty$  at the same time, then  $u \leq v$  in  $G$ . Also *Harnack's principle* is true for  $F$ -extremals [GLM1, Theorem 4.22]: If  $u_i$  is an increasing family of  $F$ -extremals in  $G$  then either  $u_i \rightarrow u_0$  uniformly on compact subsets of  $G$  and  $u_0$  is an  $F$ -extremal in  $G$  or  $u_i(x) \rightarrow \infty$  at each point  $x \in G$ .

2.6. *Sub-F-extremals.* An upper semi-continuous function  $u: G \rightarrow [-\infty, \infty]$  is called a *sub-F-extremal* in  $G$  if  $u$  satisfies the  $F$ -comparison principle in each domain  $D \subset\subset G$ , i.e. if  $h \in C(\bar{D})$  is an  $F$ -extremal in  $D$  and  $h \geq u$  in  $\partial D$ , then  $h \geq u$  in  $D$ .

A function  $v: G \rightarrow (-\infty, \infty]$  is called a *super-F-extremal*, if  $-v$  is sub- $F$ -extremal. For the properties of sub- and super- $F$ -extremals see [GLM1] and [GLM3].

2.7. *Existence of F-extremals.* Suppose that  $\varphi \in C(\bar{G}) \cap W_n^1(G)$  is given. Then it is well-known that there is a unique  $F$ -extremal  $u$  in  $C(G) \cap W_n^1(G)$  with "Sobolev boundary values"  $\varphi$ , i.e.  $u - \varphi \in W_{n,0}^1(G)$ . Here  $W_{n,0}^1(G)$  consists of all functions  $w \in W_n^1(G)$  which can be approximated in  $W_n^1(G)$  by  $C_0^\infty(G)$ -functions. The existence is easily established via a minimizing sequence as in the proof of Theorem 2.9 and the uniqueness is a consequence of the strict convexity (b), cf. [GLM1, 4.17]. Especially,

$$I_F(u, G) \leq I_F(v, G)$$

for all  $v$  with  $v - \varphi \in C(G) \cap W_{n,0}^1(G)$ .

If  $G$  is a *regular domain*, i.e.  $\partial G$  contains no point components, and if  $\varphi \in C(\bar{G}) \cap W_n^1(G)$ , then there is a unique  $F$ -extremal  $h$  in  $G$  such that  $h \in C(\bar{G}) \cap W_n^1(G)$  and  $h|_{\partial G} = \varphi|_{\partial G}$ . Especially,

$$I_F(h, G) \leq I_F(v, G)$$

for all similar  $v$  and  $h = u$  where  $u$  is the  $F$ -extremal with Sobolev boundary values  $\varphi$ . See [GLM1, Theorem 3.24 and Remark 3.21] for these results.

2.8. *Obstacles.* A constructive approach to the theory of sub- $F$ -extremals is provided by a variational obstacle consideration. Let  $\varphi \in C^\infty(\mathbb{R}^n)$  and set

$$\mathcal{F}_\varphi = \{u \in C(G) \cap W_n^1(G) : u \leq \varphi, u - \varphi \in W_{n,0}^1(G)\}.$$

THEOREM 2.9. *There is a unique  $u_\varphi$  in  $\mathcal{F}_\varphi$  such that*

$$I_F(u_\varphi, G) \leq I_F(u, G)$$

for all  $u \in \mathcal{F}_\varphi$ . Moreover,  $u_\varphi$  is a sub- $F$ -extremal.

*Proof.* Choose a minimizing sequence  $u_i$  in  $\mathcal{F}_\varphi$ . Using [GLM1, Lemma 5.9] in each subdomain  $D$  of  $G$  we may assume that every  $u_i$  satisfies

$$\omega(u_i, D) \leq \max(\omega(\varphi, \partial D), \omega(u_i, \partial D))$$

where  $\omega(v, A) = \sup_A v - \inf_A v$  denotes the oscillation of a function  $v$  on a set  $A$ . From the proof of [GLM1, Lemma 2.10] it follows that the family  $u_i$  is equicontinuous in  $G$  and passing to a subsequence, if necessary, it is easy to see that  $u_i \rightarrow u_\varphi$  uniformly in compact subsets of  $G$ , where  $u_\varphi \in C(G) \cap W_n^1(G)$  minimizes  $I_F$  in  $\mathcal{F}_\varphi$ , cf. [GLM1, p. 53 and p. 62]. Since  $W_{n,0}^1(G)$  is weakly closed in  $W_n^1(G)$ ,  $u_\varphi - \varphi \in W_{n,0}^1(G)$  and thus  $u_\varphi \in \mathcal{F}_\varphi$ . The uniqueness of  $u_\varphi$  follows from the strict convexity of  $F$ , the assumption (b) in Section 2.1. Finally, the sub- $F$ -extremality of  $u_\varphi$  follows from [GLM1, Theorem 5.17 (ii)].

2.10. *Perron's method.* Let  $f: \partial G \rightarrow [-\infty, \infty]$  be any function. The lower class  $L_f$  consists of all functions  $u: G \rightarrow [-\infty, \infty)$  for which

- (i)  $u$  is a sub- $F$ -extremal,
- (ii)  $u$  is bounded above and
- (iii)  $\overline{\lim}_{x \rightarrow y} u(x) \leq f(y)$  for every  $y \in \partial G$ .

The upper class  $U_f$  is defined analogously via super- $F$ -extremals. The lower and upper solutions

$$H_f = \sup_{u \in L_f} u, \quad \bar{H}_f = \inf_{u \in U_f} u$$

are the main objects in Perron's method. Note that  $H_f \leq \bar{H}_f$ , see [GLM3, Lemma 2.5], and that  $f \leq g$  implies  $H_f \leq H_g$  and  $\bar{H}_f \leq \bar{H}_g$ .

**THEOREM 2.11** [GLM3, Theorem 2.2]. *The lower solution  $H_f$  is either an  $F$ -extremal, or identically  $\infty$ , or identically  $-\infty$  in  $G$ . One of these three alternatives is also true for the upper solution  $\bar{H}_f$ .*

The proof for Theorem 2.11 is based on the Poisson-modification of sub- $F$ -extremals, which will be employed in Section 4 in a special case. Suppose that  $u \in C(G) \cap W_n^1(G)$  is sub- $F$ -extremal and that  $D \subset\subset G$  is a regular domain. Let  $h \in C(\bar{D}) \cap W_n^1(D)$  denote the unique  $F$ -extremal in  $D$  with boundary values  $h|_{\partial D} = u|_{\partial D}$ , cf. Section 2.7. Then the *Poisson-modification*

$$P(u, D) = \begin{cases} h & \text{in } D \\ u & \text{in } G \setminus D \end{cases}$$

of  $u$  is a sub- $F$ -extremal in  $G$ , see [GLM3, Lemma 2.9]. Especially,  $P(u, D) \geq u$  by the  $F$ -comparison principle.

### 3. Wiener's criterion and boundary regularity

**3.1. Condensers and  $F$ -capacity.** Let  $F: G \times \mathbf{R}^n \rightarrow \mathbf{R}$  be a variational kernel satisfying the assumptions (a)–(d) of Section 2.1. If we set  $F(x, h) = |h|^n$  for  $x \in \mathbf{R}^n \setminus G$ , then  $F$  is defined in  $\mathbf{R}^n \times \mathbf{R}^n$  and satisfies the same assumptions in any bounded domain of  $\mathbf{R}^n$  as  $F$  in  $G$  possibly with different  $\alpha$  and  $\beta$  but with the same  $\beta/\alpha$ . Hence we may assume that  $F$  is defined everywhere.

Let  $E = (A, C)$  be a *condenser* in  $\mathbf{R}^n$ , i.e.  $A$  is open in  $\mathbf{R}^n$  and  $C$  is a compact subset of  $A$ . The  $F$ -capacity of the condenser  $E$  is

$$\text{cap}_F E = \inf_{u \in W(E)} I_F(u, A)$$

where  $W(E)$  is the set of all continuous *ACL*-functions  $u: A \rightarrow \mathbf{R}$  such that

$$\overline{\lim}_{x \rightarrow y} u(x) \leq 0$$

for all  $y \in \partial A$  and  $u|_C \geq 1$ . It is well-known that  $W(E)$  can be replaced by  $W^\infty(E) = C_0^\infty(A) \cap W(E)$  in the definition for  $\text{cap}_F E$ . In the case  $F(x, h) = |h|^n$  for all  $x \in \mathbf{R}^n$  we write  $\text{cap}_F E = \text{cap } E$  and call  $\text{cap } E$  the ( $n$ -)capacity of the condenser  $E$ . If  $A \setminus C$  is a regular open set, i.e. each component of  $A \setminus C$  is a regular domain, see Section 2.7, then the condenser  $E = (A, C)$  is called *regular*. Let  $E = (A, C)$  be a regular condenser and fix a variational kernel  $F$ . By Section 2.7 there exists a unique  $u = u_E \in W(E)$  such that

- (i)  $u$  has a continuous extension to  $\bar{A}$  and  $u|_{\partial A} = 0$ .
- (ii)  $u|_C = 1$  if  $C \neq \emptyset$ .
- (iii)  $\text{cap}_F E = I_F(u, A) = I_F(u, A \setminus C)$ .

The function  $u$  is called the  $F$ -capacity function for  $E$ .

The next four lemmata are needed for the proof of Theorem 1.4. The first lemma immediately follows from the assumption (c) in Section 2.1.

LEMMA 3.2. *If  $E$  is a condenser, then*

$$\alpha \text{cap} E \leq \text{cap}_F E \leq \beta \text{cap} E$$

where  $\alpha, \beta$  are the structure constants for  $F$ .

LEMMA 3.3. *Let  $C$  be closed in  $\mathbf{R}^n$  and  $0 < r < s \leq t$ . If  $C_r = C \cap \bar{B}^n(r)$ , then*

$$\text{cap}(B^n(t), C_r) \leq \text{cap}(B^n(s), C_r) \leq a^{n-1} \text{cap}(B^n(t), C_r)$$

where  $a = (\log t/r)/(\log s/r)$ .

*Proof.* The left hand side inequality is trivial. To prove the right hand side inequality let  $f: B^n(s) \rightarrow B^n(t)$  be the mapping

$$f(x) = \begin{cases} x, & x \in \bar{B}^n(r) \\ rg(x/r), & x \in B^n(s) \setminus \bar{B}^n(r) \end{cases}$$

where  $g(z) = |z|^{a-1}z$ . Then  $f$  is a quasiconformal mapping and

$$K(f) = K(g) = a^{n-1}.$$

For the calculation of the dilatation  $K(g)$  see [V, 16.2]. Since the change of the  $n$ -capacity of a condenser under a quasiconformal mapping  $f$  is controlled by  $K(f)$ , we obtain

$$\begin{aligned} \text{cap}(B^n(s), C_r) &\leq K(f) \text{cap}(f(B^n(s)), f(C_r)) \\ &= a^{n-1} \text{cap}(B^n(t), C_r) \end{aligned}$$

as desired.

3.4 *Remark.* The second inequality of Lemma 3.2 is sharp; to see this choose  $C = \mathbf{R}^n$ . The lemma holds as well for the  $F$ -capacity but this is not needed in the sequel.



The next lemma follows from the fact that each open set in  $\mathbf{R}^n$  can be approximated from inside by regular open sets, cf. [S, Lemma 5.5].

LEMMA 3.5. *Let  $E=(A, C)$  be a condenser such that  $A$  is a regular domain. Then for each  $\varepsilon>0$  there exists a compact set  $C_\varepsilon$  in  $A$  such that  $C\subset C_\varepsilon$ ,  $E_\varepsilon=(A, C_\varepsilon)$  is a regular condenser and*

$$\text{cap}_F E_\varepsilon \leq \text{cap}_F E + \varepsilon.$$

Suppose that  $E=(A, C)$  is a regular condenser and let  $u$  be the  $F$ -capacity function for  $E$ . For  $0<\gamma\leq 1$  write

$$E_\gamma = (A, \{x \in A: u(x) \geq \gamma\}).$$

The following lemma gives the basic estimate for the main result.

LEMMA 3.6. *If  $b=\text{cap}_F E$ , then*

$$\text{cap}_F E_\gamma \leq \frac{nb}{\gamma^{n-1}}.$$

*Proof.* If  $\gamma=1$ , then the result follows immediately. Assume  $0<\gamma<1$ . The function  $\min(u/\gamma, 1)$  belongs to  $W(E_\gamma)$ . Hence

$$\text{cap}_F E_\gamma \leq \frac{1}{\gamma^n} I_F(u, \{u < \gamma\}) = \frac{1}{\gamma^n} (b - I_F(u, \{u > \gamma\})).$$

On the other hand

$$I_F(u, \{u > \gamma\}) = (1-\gamma)^n I_F\left(\frac{u-\gamma}{1-\gamma}, \{u > \gamma\}\right) \geq (1-\gamma)^n \text{cap}_F E'_\gamma$$

where the assumption (d) of Section 2.1 has been used and  $E'_\gamma$  is the condenser

$$(\{x \in A: u(x) > \gamma\}, C).$$

Since  $\text{cap}_F E'_\gamma \geq \text{cap}_F E = b$ , the above inequalities yield

$$\text{cap}_F E_\gamma \leq \frac{1}{\gamma^n} (b - (1-\gamma)^n b) = \frac{b}{\gamma^n} (1 - (1-\gamma)^n)$$

and because  $(1-\gamma)^n \geq 1 - n\gamma$ , we obtain the desired inequality.

3.7 Remark. The upper bound of Lemma 3.6 for  $\text{cap}_F E_\gamma$  is essentially sharp, since

$$\text{cap}_F E_\gamma \geq \frac{b}{(n/(n-1))^{n-1} \gamma^{n-1}}. \tag{3.8}$$

To prove (3.8) first observe that if  $E_i=(A_i, C_i)$ ,  $i=1, 2, 3$ , are three condensers with

$$A_1 \supset A_2 \supset C_2 \supset A_3 \supset C_3 \supset C_1,$$

then

$$\text{cap}_F E_1^{-1/(n-1)} \geq \text{cap}_F E_2^{-1/(n-1)} + \text{cap}_F E_3^{-1/(n-1)}. \quad (3.9)$$

This is a well known inequality for the usual  $n$ -capacity, see [G, Lemma 2], and the proof for the general case is similar: Choose any functions  $u_i \in W^\infty(E_i)$ ,  $i=2, 3$ , such that  $u_i=0$  on  $\mathbf{R}^n \setminus A_i$  and  $u_i=1$  in  $C_i$ . Set

$$u_1 = a_2 u_2 + a_3 u_3$$

where  $a_2, a_3 \geq 0$  and  $a_2 + a_3 = 1$ . Then  $u_1 \in W(E_1)$  and hence

$$\text{cap}_F E_1 \leq I_F(u_1, A_1) = a_2^n I_F(u_2, A_2 \setminus C_2) + a_3^n I_F(u_3, A_3 \setminus C_3).$$

Taking infimums over all such  $u_2, u_3$  yields

$$\text{cap}_F E_1 \leq a_2^n \text{cap}_F E_2 + a_3^n \text{cap}_F E_3. \quad (3.10)$$

If  $\text{cap}_F E_i > 0$  for  $i=2, 3$ , then set

$$a_i = \text{cap}_F E_i^{-1/(n-1)} (\text{cap}_F E_2^{-1/(n-1)} + \text{cap}_F E_3^{-1/(n-1)})^{-1}$$

and (3.10) implies (3.9). If  $\text{cap}_F E_i = 0$  for  $i=2$  or  $3$ , then  $\text{cap}_F E_1 = 0$  and (3.9) again follows.

To complete the proof for (3.8) write for  $0 < \gamma < 1$

$$E'_\gamma = (\{x \in A: u(x) > \gamma\}, C).$$

The function  $(u-\gamma)/(1-\gamma)$  belongs to  $W(E'_\gamma)$  and hence

$$\text{cap}_F E'_\gamma \leq \frac{1}{(1-\gamma)^n} I_F(u, \{u > \gamma\}) \leq \frac{1}{(1-\gamma)^n} b.$$

By (3.9)

$$b^{-1/(n-1)} \geq \text{cap}_F E_\gamma^{-1/(n-1)} + \text{cap}_F E'_\gamma^{-1/(n-1)},$$

and we obtain

$$\text{cap}_F E_\gamma \geq \frac{b}{(1-(1-\gamma)^{n/(n-1)})^{n-1}}. \quad (3.11)$$

The elementary inequality

$$(1-\gamma)^{n/(n-1)} \geq 1 - \frac{n}{n-1} \gamma$$

now shows that (3.8) follows from (3.11).

3.12. *Wiener's criterion.* Suppose that  $G$  is a bounded domain in  $\mathbf{R}^n$  and  $F: \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$  is the continuation, see Section 3.1, of a variational kernel  $F$  satisfying (a)–(d) in Section 2.1. A point  $x_0 \in \partial G$  is called  $F$ -regular if for all continuous functions  $f: \partial G \rightarrow \mathbf{R}$

$$\lim_{x \rightarrow x_0} H_f(x) = f(x_0) = \lim_{x \rightarrow x_0} \tilde{H}_f(x). \quad (3.13)$$

For the definition of  $H_f$  and  $\tilde{H}_f$  see Section 2.10. Write  $C = \mathbf{R}^n \setminus G$  and for  $t > 0$  let

$$\varphi(t) = \text{cap}(B^n(x_0, 2t), \tilde{B}^n(x_0, t) \cap C).$$

The point  $x_0$  satisfies *Wiener's criterion* if

$$W(x_0, C) = \int_0^1 \frac{\varphi(t)^{1/(n-1)}}{t} dt = \infty. \quad (3.14)$$

If  $W(x_0, C) < \infty$ , then  $C$  is said to be *thin* at  $x_0$ .

For the proof of Theorem 1.4 we still need two lemmata. In the first lemma the Wiener integral  $W(x_0, C)$  is simply estimated from below by a Wiener sum.

LEMMA 3.15. *Let  $r_1 > 0$  and  $r_{i+1} = r_i/2$ ,  $i = 1, 2, \dots$ . Then*

$$\sum_{i=2}^{\infty} a_i^{1/(n-1)} \leq \frac{2}{\log 2} \int_0^{r_1} \frac{\varphi(t)^{1/(n-1)}}{t} dt$$

where  $a_i = \varphi(r_i)$ ,  $i = 2, 3, \dots$

*Proof.* We may assume  $x_0 = 0$ . For  $t \in [r_{i+1}, r_i]$  set  $C_t = C \cap \tilde{B}^n(t)$ . Then Lemma 3.3 yields

$$\begin{aligned} \varphi(t) &= \text{cap}(B^n(2t), C_t) \geq \text{cap}(B^n(2r_i), C_t) \\ &\geq \text{cap}(B^n(2r_i), C_{r_{i+1}}) \geq 2^{1-n} \text{cap}(B^n(2r_{i+1}), C_{r_{i+1}}) \\ &= 2^{1-n} \varphi(r_{i+1}) = 2^{1-n} a_{i+1}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{r_1} \frac{\varphi(t)^{1/(n-1)}}{t} dt &= \sum_{i=1}^{\infty} \int_{r_{i+1}}^{r_i} \frac{\varphi(t)^{1/(n-1)}}{t} dt \\ &\geq \frac{1}{2} \sum_{i=1}^{\infty} a_{i+1}^{1/(n-1)} \int_{r_{i+1}}^{r_i} \frac{dt}{t} \\ &= \frac{\log 2}{2} \sum_{i=1}^{\infty} a_{i+1}^{1/(n-1)} \end{aligned}$$

as required.

Let  $F_0$  be a closed set in a ball  $B^n(2r)$ ,  $r > 0$ , and suppose that the condenser  $E = (B^n(2r), F_0)$  is regular, see Section 3.1. Let  $u$  be the  $F$ -capacity function for the condenser  $E$ . For  $\gamma > 0$  let

$$A_\gamma = \{x \in B^n(2r) : u(x) < \gamma\}.$$

LEMMA 3.16. *There is a constant  $c_1$  depending only on  $n$  and  $\alpha$  such that the set  $A_\gamma$  contains some sphere  $S^{n-1}(t)$ ,  $t \in (r/4, r/2)$ , whenever*

$$\gamma \geq c_1 \operatorname{cap} E^{1/(n-1)}.$$

*Proof.* For  $0 < \gamma < 1$  let  $E_\gamma$  be the condenser

$$(B^n(2r), \{x : u(x) \geq \gamma\}).$$

Suppose that  $A_\gamma$  does not contain any  $S^{n-1}(t)$ ,  $r/4 < t < r/2$ . Then the set  $\{x : u(x) \geq \gamma\}$  meets  $S^{n-1}(t)$  for all  $t \in (r/4, r/2)$  and the spherical symmetrization yields, cf. [G, Theorem 1] or [S],

$$\operatorname{cap} E_\gamma \geq \operatorname{cap} E' = c(n) > 0$$

where  $E'$  is the condenser

$$E' = (B^n(2r), \{s e_1 : r/4 \leq s \leq r/2\})$$

and  $c(n)$  depends only on  $n$ , see [L] for  $n=3$  and [V, Theorem 11.9 and Remark 11.11] for  $n \geq 2$ . Observe that the borderline case  $p=n$  has been used in the above estimate. For  $p < n-1$  the  $p$ -capacity of  $E'$  vanishes. Thus by Lemma 3.2

$$\operatorname{cap}_F E_\gamma \geq \alpha \operatorname{cap} E_\gamma \geq \alpha c(n)$$

and Lemma 3.6 gives the estimate

$$\text{cap}_F E_\gamma \leq \frac{nb}{\gamma^{n-1}}$$

where  $b = \text{cap}_F E$ . The last two inequalities imply

$$\gamma \leq b^{1/(n-1)} \left( \frac{n}{\alpha c(n)} \right)^{1/(n-1)} = b^{1/(n-1)} c' \tag{3.17}$$

and if we choose  $c_1 = c' + 1$ , then for  $\gamma \geq c_1 b^{1/(n-1)}$  the inequality (3.17) is a contradiction. Consequently  $A_\gamma$  contains  $S^{n-1}(t)$  for some  $t \in (r/4, r/2)$  as desired.

3.18. *Proof for Theorem 1.4.* We may assume  $x_0 = 0$ . Suppose that  $W(x_0, C) < \infty$ . Let  $\varepsilon > 0$ . We shall fix  $\varepsilon$  later. Choose  $r_1 > 0$  such that

$$\int_0^{r_1} \frac{\varphi(t)^{1/(n-1)}}{t} dt \leq \varepsilon$$

and let  $r_{i+1} = r_i/2$ ,  $i = 1, 2, \dots$ . Set

$$a_i = \text{cap}(B^n(2r_i), C_{r_i})$$

where  $C_i = \bar{B}^n(t) \cap C$ . Lemma 3.15 yields

$$\sum_{i=2}^{\infty} a_i^{1/(n-1)} \leq \frac{2\varepsilon}{\log 2}$$

and hence

$$a_i^{1/(n-1)} \leq 2\varepsilon/\log 2$$

for all  $i = 2, 3, \dots$ . Next for each  $i$  choose a regular condenser  $E_i = (B^n(2r_i), F_i)$  such that  $F_i \supset C_{r_i}$  and

$$\text{cap } E_i^{1/(n-1)} \leq a_i^{1/(n-1)} + \varepsilon 2^{-i}.$$

See Lemma 3.5. Set  $b_i = \text{cap}_F E_i$  and let  $u_i$  be the  $F$ -capacity function for the condenser  $E_i$ ,  $i = 2, 3, \dots$

By Lemma 3.16 for  $\gamma_i = c_1 b_i^{1/(n-1)}$  the set

$$A_i = \{x \in B^n(2r_i); u_i(x) < \gamma_i\}$$

contains  $S^{n-1}(t_i)$  for some  $t_i \in (r_i/4, r_i/2)$ . On the other hand

$$\begin{aligned} \sum_{i=2}^{\infty} b_i^{1/(n-1)} &\leq \beta^{1/(n-1)} \sum_{i=2}^{\infty} \text{cap } E_i^{1/(n-1)} \\ &\leq \beta^{1/(n-1)} \left( \sum_{i=2}^{\infty} a_i^{1/(n-1)} + \varepsilon \right) \\ &\leq \beta^{1/(n-1)} \left( \frac{2\varepsilon}{\log 2} + \varepsilon \right) < 5\beta^{1/(n-1)} \varepsilon \end{aligned} \quad (3.19)$$

and hence especially

$$b_i^{1/(n-1)} < 5\beta^{1/(n-1)} \varepsilon$$

for all  $i=2, 3, \dots$ . If we now choose

$$\varepsilon = (\beta^{1/(n-1)} 5c_1)^{-1},$$

then  $\gamma_i < 1$  for  $i=2, 3, \dots$ . In particular,  $S^{n-1}(t_2)$  does not meet  $F_2$  and hence not  $C$ . Thus  $S^{n-1}(t_2)$  is contained in  $G$ .

Next let  $f: \partial G \rightarrow \mathbf{R}$  be the continuous boundary function

$$f(x) = \begin{cases} 1, & x \in B^n(t_2) \cap \partial G, \\ 0, & x \in \partial G \setminus B^n(t_2). \end{cases}$$

Now  $H_f \equiv 1$ , since  $G$  is bounded and hence there is  $r > 0$  such that  $G \subset B^n(r)$  and if we let  $f_0(x) = 0$ ,  $x \in S^{n-1}(r)$ , and  $f_0(x) = 1$ ,  $x \in B^n(t_2) \cap \partial G$ , then the  $F$ -extremal  $u = H_{f_0}$  in  $G' = B^n(r) \setminus (B^n(t_2) \cap \bar{G})$  takes the boundary values 0 in  $\partial B^n(r)$ , see e.g. [GLM2, Remark 2.20], and thus  $u < 1$  in  $B^n(r) \setminus \bar{B}^n(t_2)$ . The  $F$ -comparison principle yields  $H_f \leq u$  in  $G$  and thus  $H_f \equiv 1$ . On the other hand

$$\max_{S^{n-1}(t_2)} H_f = M < 1$$

since the Harnack's inequality [GLM1, Theorem 4.15] holds for the non-negative  $F$ -extremal  $1 - H_f$ . Set

$$v = \frac{H_f - M}{1 - M}.$$

Then the  $F$ -extremal  $v$  is  $\leq 0$  in  $S^{n-1}(t_2)$  and we shall show that  $W(x_0, C) < \infty$  implies

$$\lim_{\substack{x \rightarrow x_0 \\ x \in G}} v(x) < 1. \quad (3.20)$$

This will complete the proof.

To this end observe that  $\bar{B}^n(r_3) \supset S^{n-1}(t_2)$  and  $u_3(x) = 1$  for  $x$  in  $F_3$  implies via the  $F$ -comparison principle, see Section 2.5, that

$$v(x) \leq u_3(x), \quad x \in B^n(t_3) \setminus F_3.$$

Now  $u_3(x) < \gamma_3$  for  $x \in S^{n-1}(t_3)$  by the selection of  $\varepsilon$  and hence the  $F$ -extremal  $v - \gamma_3$  satisfies

$$v - \gamma_3 \leq 0$$

in  $S^{n-1}(t_3)$ . But now  $t_3 < r_3/2 = r_4$  and the  $F$ -comparison principle again yields

$$v - \gamma_3 \leq u_4 \quad \text{in } B^n(t_3) \setminus F_4$$

and since  $u_4 < \gamma_4$  in  $S^{n-1}(t_4)$  and  $t_4 < t_3$ , we obtain  $v - \gamma_3 < \gamma_4$  in  $S^{n-1}(t_4)$ . Continuing we have

$$v - \sum_{i=3}^k \gamma_i \leq \gamma_{k+1} \quad \text{in } S^{n-1}(t_{k+1}),$$

$k = 3, 4, \dots$ , and thus

$$v \leq \sum_{i=3}^{\infty} \gamma_i \tag{3.21}$$

on each  $S^{n-1}(t_i)$ ,  $i = 3, 4, \dots$ . The definition of  $\gamma_i$  and (3.19) yield

$$\sum_{i=3}^{\infty} \gamma_i = c_1 \sum_{i=3}^{\infty} b_i^{1/(n-1)} < 5c_1 \beta^{1/(n-1)} \varepsilon = 1.$$

Hence (3.21) implies (3.20) and  $x_0 = 0$  cannot be an  $F$ -regular point of  $\partial G$ .

**3.22 Remark.** From the proof for Theorem 1.4 it follows that if  $W(x_0, C) < \infty$ , then there exists a  $C^\infty$ -function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  such that

$$\lim_{\substack{x \rightarrow x_0 \\ x \in G}} H_f(x) \neq f(x_0).$$

Moreover,  $H_f$  coincides with the  $F$ -extremal with "boundary values"  $f$  in Sobolev's sense; see Remark 4.6.

3.23. *Corollaries to Theorem 1.4.* As in the classical harmonic case a point  $x_0 \in \partial G$  is said to have an  $F$ -barrier if there exists a sub- $F$ -extremal  $w: G \rightarrow \mathbb{R}$  such that

- (a)  $\overline{\lim}_{x \rightarrow y} w(x) < 0$  for all  $y \in \partial G$ ,  $y \neq x_0$ ,
- (b)  $\lim_{x \rightarrow x_0} w(x) = 0$ .

See [GLM3, Section 3].

**THEOREM 3.24.** *Suppose that  $G$  is a bounded domain in  $\mathbb{R}^n$ , that  $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies the assumptions (a)–(d) of Section 2.1 and that  $x_0 \in \partial G$ . Then the following conditions are equivalent:*

- (i)  $x_0$  is  $F$ -regular.
- (ii)  $W(x_0, \mathbb{R}^n \setminus G) = \infty$ .
- (iii)  $x_0$  has an  $F$ -barrier.

*Proof.* The equivalence (i)  $\Leftrightarrow$  (iii) follows from [GLM3, Theorem 3.7]. Theorem 1.4 implies (i)  $\Rightarrow$  (ii). Mazja [M, p. 51] proved (ii)  $\Rightarrow$  (i), although his formulation was slightly less general; he assumed that the boundary function belongs to  $W_n^1(G) \cap C(\bar{G})$ . For the general case observe that if  $f \in W_n^1(G) \cap C(\bar{G})$ , then Remark 4.6 yields  $H_f = \bar{H}_f = u$  where  $u$  is the  $F$ -extremal with Sobolev-boundary values  $f$ , i.e.  $u - f \in W_{n,0}^1(G)$ . The  $F$ -regularity of  $x_0$  now follows by a simple approximation method via the estimate [M, (26)\*, p. 51].

Theorem 3.24 immediately implies Corollary 1.5 and the variational interpretation of Theorem 3.24 presented in Introduction can now be handled as follows. If  $W(x_0, \mathbb{R}^n \setminus G) = \infty$ , then for all functions  $f: \bar{G} \rightarrow \mathbb{R}$  in the class  $C(\bar{G}) \cap W_n^1(G)$  the aforementioned result of Mazja implies  $\lim_{x \rightarrow x_0} u(x) = f(x_0)$  where  $u$  is the  $F$ -extremal with Sobolev boundary values  $f$ , i.e.  $u - f \in W_{n,0}^1(G)$ . The converse follows from Remark 3.22.

#### 4. Resolutivity

4.1. *Preliminaries.* Let  $G$  be a bounded domain and let a variational kernel  $F$  satisfy the assumptions (a)–(d) in Section 2.1. Let  $f: \partial G \rightarrow [-\infty, \infty]$  be any function. We recall that the function  $f$  is  $F$ -resolutive if  $H_f$  and  $\bar{H}_f$  are  $F$ -extremals and  $H_f = \bar{H}_f$  in  $G$ .

For the next lemma observe that if  $f$  is  $F$ -resolutive, then also  $\lambda f + \mu$ ,  $\lambda, \mu \in \mathbb{R}$ , is  $F$ -resolutive.



LEMMA 4.2. *Suppose that the functions  $f_i: \partial G \rightarrow \mathbf{R}$ ,  $i=1, 2, \dots$ , are  $F$ -resolutive. If  $\lim f_i = f$  uniformly on  $\partial G$ , then  $f$  is  $F$ -resolutive.*

*Proof.* Given any  $\varepsilon > 0$ , there is  $i_\varepsilon$  such that

$$f_i - \varepsilon < f < f_i + \varepsilon$$

for  $i > i_\varepsilon$ . Thus

$$\tilde{H}_{f_i - \varepsilon} = H_{f_i - \varepsilon} = \underline{H}_{f_i - \varepsilon} \leq H_f \leq \tilde{H}_f \leq \tilde{H}_{f_i + \varepsilon} = \tilde{H}_f + \varepsilon$$

for  $i > i_\varepsilon$ . This yields  $H_f = \tilde{H}_f$  and since  $\tilde{H}_f$  is finite, it is also an  $F$ -extremal.

4.3 *Remark.* Although the  $F$ -regularity of a boundary point  $x_0 \in \partial G$  is, by Corollary 1.5, independent of  $F$ , this does not hold for the  $F$ -resolutivity even in the linear case. To see this let  $G$  be the unit disk  $B^2$  in  $\mathbf{R}^2$  and let  $F(x, h) = |h|^2$  be the classical Dirichlet kernel. Choose a homeomorphism  $h$  of  $B^2$  onto itself such that  $h|_{B^2}$  is quasiconformal and  $h|_{\partial B^2}$  is not absolutely continuous, cf. [BA]. Let  $F_1 = h^*F: B^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}$  be the kernel induced by the mapping  $h$ . For the construction of  $F_1$  see [GLM1, 6.1]. In this case  $F$ -extremals are harmonic functions and since the Euler equation (1.1) for  $F_1$  is linear,  $F_1$ -extremals also form a linear space. The corresponding harmonic measures  $\mu$  and  $\mu_1$  on  $\partial B^2$  are singular, see [GLM2, Remark 5.6(b)]. Choose a set  $C$  in  $\partial B^2$  such that  $\mu(C) > 0$  but  $\mu_1(C) = 0$ . Now  $C$  contains a subset  $E$  which is not  $\mu$ -measurable. By a theorem of M. BreLOT [B1, p. 152] the characteristic function  $\chi_E$  of  $E$  is not  $F$ -resolutive although, by the same theorem, it is  $F_1$ -resolutive.

4.4. *Proof for Theorem 1.6.* Let  $f: \partial G \rightarrow \mathbf{R}$  be continuous. By Section 2.10

$$\min_{\partial G} f \leq H_f \leq \tilde{H}_f \leq \max_{\partial G} f,$$

and hence, by Theorem 2.11,  $H_f$  and  $\tilde{H}_f$  are  $F$ -extremals. Approximating  $f$  with smooth functions we can, according to Lemma 4.2, reduce the proof to the following case:  $H_\varphi = \tilde{H}_\varphi$ , whenever  $\varphi \in C^\infty(\mathbf{R}^n)$ . Fix such a  $\varphi$  and let  $v \in C(G) \cap W_{n,0}^1(G)$  be the unique  $F$ -extremal with Sobolev boundary values  $v - \varphi \in W_{n,0}^1(G)$ , see Section 2.7. We claim that  $H_\varphi \geq v$  and  $\tilde{H}_\varphi \leq v$ .

To this end let  $u = u_\varphi$  be the sub- $F$ -extremal of Theorem 2.9. Especially,  $u \leq \varphi$  in  $G$ ,  $u - \varphi \in W_{n,0}^1(G)$  and

$$I_F(u, G) \leq I_F(\varphi, G) = I. \tag{4.5}$$

Choose regular domains  $D_1 \subset\subset D_2 \subset\subset \dots$  such that  $G = \bigcup D_i$  and consider the Poisson-modifications

$$u_i = P(u, D_i), \quad i = 1, 2, \dots,$$

see Section 2.10. Clearly,  $u_i - u \in W_{n,0}^1(G)$  and thus  $u_i - \varphi \in W_{n,0}^1(G)$ . Note also that  $u_i \in L_\varphi$ , because  $u_i \leq \varphi$  in  $G \setminus D_i$ . Now  $u_{i+1} = P(u_i, D_{i+1})$  yields

$$u_1 \leq u_2 \leq \dots$$

and by Harnack's principle, see Section 2.5,  $h = \lim u_i$  is an  $F$ -extremal in  $G$ . Observe that the case  $h \equiv \infty$  is excluded by the bound  $u_i \leq \sup_{\partial G} \varphi$ .

By the construction  $I_F(u_i, G) \leq I_F(u, G)$  and thus (4.5) yields

$$\int_G |\nabla u_i|^n dm \leq \frac{\beta}{\alpha} I, \quad i = 1, 2, \dots$$

This implies that  $h \in W_n^1(G)$  and  $h - \varphi \in W_{n,0}^1(G)$  since  $W_{n,0}^1(G)$  is weakly closed. By uniqueness,  $v = h$ . On the other hand, the inequality  $u_i \leq H_\varphi$  gives

$$v = h = \lim u_i \leq H_\varphi.$$

A corresponding construction from above yields  $v \geq \bar{H}_\varphi$ . This implies the desired result  $H_\varphi = \bar{H}_\varphi$ .

**4.6 Remark.** If  $f: \bar{G} \rightarrow \mathbb{R}$  is sufficiently regular, i.e.  $f \in C(\bar{G}) \cap W_n^1(G)$ , then the  $F$ -extremal  $u$  with Sobolev boundary values  $f$ ,  $u - f \in W_{n,0}^1(G)$ , coincides with Perron's solution  $H_f = \underline{H}_f = \bar{H}_f$ . This result, mentioned by BreLOT [B2, X.2] in the classical harmonic case, is a byproduct of the previous proof.

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