Two-Time-Scale Approximation for Wonham Filters*

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Abstract

This paper is concerned with approximation of Wonham filters. A focal point is that the underlying hidden Markov chain has a large state space. To overcome the difficulties and to reduce the computational complexity, a two-time-scale approach is developed. Under the time-scale separation, the state space of the underlying Markov chain is divided into a number of groups so that the chain jumps rapidly within each group and switches occasionally from one group to another. Such structure gives rise to a limit filter for the Wonham filter. The limit filter preserves the main features of the filtering process, but it has a much smaller dimension and therefore is easier to compute. Using such a limit filter enables us to develop efficient approximations and useful filters for hidden Markov chains.

Index terms. Wonham filter, hidden Markov model, two-time-scale Markov chain

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1 Introduction

There has been a growing interest in control and optimization using switching diffusion systems. This arises in emerging applications such as wireless communication, signal processing, and financial engineering. Unlike the pure diffusion models used in the traditional setup, both continuous dynamics and discrete events coexist in the regime-switching models. Such a hybrid formulation makes the models more versatile, but the analysis becomes more challenging.

A class of promising models uses a continuous-time Markov chain to capture the discrete event features resulting in a set of diffusions modulated by the Markov chain. Dealing with such systems and carrying out control and optimization tasks under partial observations, it is desirable to extract characteristics or features of the system based on the limited information available, which brings us to the framework of hybrid filtering.

Optimal filtering of hybrid systems typically gives rise to infinite dimensional stochastic differential equations. Various efforts have been made to find finite dimensional approximations. Some of these approximation schemes can be simplified if the conditional probability of the Markov chain given observation overtime is available. In this paper, instead of dealing with optimal or finite dimensional approximations, we consider the model in which a function of the Markov chain plus a white noise is observable. We focus on the conditional probability of the chain given the observation. In this case, the filter developed by Wonham [19], is referred to as Wonham filter, which is given by the solution of a system of stochastic differential equations.

1.1 Wonham Filter

Next we summarize the result about the Wonham filter. Let $\alpha(t)$ be a continuous-time Markov chain having finite state space $\mathcal{M} = \{1, \ldots, m\}$ and generator $Q = (q_{ij}) \in \mathbb{R}^{m \times m}$. Consider a function of the Markov chain $\alpha(t)$ that is observable with additive Gaussian noise. Let $y(t)$ denote the observation measurement given by

$$dy(t) = f(\alpha(t))dt + \sigma dw(t), \; y(0) = 0, \tag{1}$$

where σ is a positive constant and $w(\cdot)$ is a standard Brownian motion.

Let $p_i(t)$ denote the conditional probability of $\{\alpha(t) = i\}$ given the observations up to time t, i.e.,

$$p_i(t) = P(\alpha(t) = i | y(s) : s \le t),$$

for $i = 1, \dots, m$. Let $p(t) = (p_1(t), \dots, p_m(t)) \in \mathbb{R}^{1\times m}$. Then the Wonham filter is given by

$$\begin{aligned} dp(t) &= p(t)Qdt - \frac{1}{\sigma^2}\left(\sum_{i=1}^{m} f(i)p_i(t)\right)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dy(t), \\ p(0) &= p_0, \quad \text{being the initial probability,} \end{aligned} \tag{2}$$

where

$$A(t) = \mathrm{diag}(f(1), \dots, f(m)) - \sum_{i=1}^{m} f(i)p_i(t)I.$$

In this paper, we use I as an identity matrix of appropriate dimension and use K as a generic constant with the convention $K + K = K$ and $KK = K$.

1.2 Brief Review of Literature

Owing to its importance, filtering problems have received much attention, and various efforts have been made. For example, Caines and Chen [4] derived an optimal filter when it involves a random variable but with no switching; see also Hijab [11]. Haussmann and Zhang [10] used two statistical hypothesis tests: the quadratic variation test (QVT) and the likelihood ratio test (LRT), to estimate the value of the random variable and to choose among competing filters on successive time intervals. These results are generalized in Zhang [24] to incorporate the case when the underlying Markov chain is not observable.

Concerning nonlinear filtering of a hybrid system in discrete time, Blom and Bar-Shalom [3] proposed a numerical algorithm to compute the conditional expectation of the state given observation up to time t. The algorithm seems to perform well numerically. However, there is no theoretical justification for optimality (or near optimality) of these filters; see Li [15] for further discussions.

For other related work on filtering, see Dey and Moore [6] and Moore and Baras [17] for risk sensitive filtering; Wang et al. [18] and Yin and Dey [20] for the reduction of complexity for filtering problem involving large-scale Markov chain; Zhang [26, 25] for the most probable estimates in discrete-time and continuous-time models, respectively; and Liu and Zhang [16] for numerical experiments involving piecewise approximation of nonlinear systems; and Yin et al. [23] for numerical methods for Wonham filters.

For survey of results on filtering, we refer to the books by Anderson and Moore [1] on classical linear filtering. For hidden Markov models and related filtering problems see Elliott et al. [7]. For general nonlinear filtering, see Kallianpur [13] and Liptser and Shiryayev [12]. For recent developments and review of the literature on partially observed systems, we refer the reader to the books by Bensoussan [2], Kushner [14], and references therein.

1.3 Contributions of This Paper

The primary concern of this paper is on constructing Wonham filters for Markov chains with large state space. When the state space of the Markov chain is large, the number of the filter equations will be large as well, resulting in the need of solving a large number of diffusion equations. We focus on developing good approximation of the Wonham filters. The main idea is to use time-scale separation and the hierarchy of the Markov chain to reduce the computation complexity. In many applications, the state space of the Markov chain can be partitioned to a number of groups so that the Markov chain jumps rapidly among a group of states and less frequently (or occasionally) among different groups. In this case, it is difficult to pinpoint the exact location of the chain and any estimation errors can lead to misleading results. Nevertheless, it is much easier to identify if the chain belongs to certain groups. This leads to a two-time-scale formulation involving states having weak and strong interactions. Our contributions in this paper includes:

1. Present a two-time-scale formulation;

2. construct a limit filter;

3. prove its convergence to the desired Wonham filter in the limit as the rate of fluctuations of the Markov chain goes to infinity in each group of irreducible states;

4. construct an approximation scheme based on the limit filter which is easier to compute;

5. prove that the original filter can be approximated in a two stage procedure under different topologies. Hence establish the asymptotic optimality of the approximate

filter.

To proceed, there are a couple of points that we wish to point out. First, the time-scale separation in this formulated by use of a small parameter $\varepsilon > 0$. The asymptotic results require the small parameter go to 0. In applications, it is simply a fixed constant, however. For example, it may be $\varepsilon = 0.01$ or $\varepsilon = 0.1$. The main point this small parameter brings out the different scale of the jump rates in different states of the Markov chain. Second, in the formulation, the Markov chain is of a particular structure. Since any finite state Markov chain has at least one recurrent states, reduction to such a "canonical form" is always possible; see for example, [21, Chapter 3.6] and the references therein. One of the main observations is that in a large-scale system, not all states change at the same rate. As a result, the two-time scale is natural and ubiquitous.

1.4 Outline

The rest of the paper is organized as follows. In the next section, we give notation needed for two-time-scale Markov chains and summarize relevant results to be used in this work. In Section 3, we consider limit filters and two-time-scale approximations and verification of these results. In Section 4, we provide a numerical example illustrating the main results of this paper. In Section 5, extension of results to Markov chains with transient states are considered. The paper is concluded with a few remarks followed by a short appendix.

2 Singularly Perturbed Markov Chains

2.1 Time-Scale Separation in Markov Chains

In this work, we focus on Markov chains that have large state spaces with complex structures. We consider the case that the states of the underlying Markov chain are divisible to a number of weakly irreducible classes such that it fluctuates rapidly among different states within a weakly irreducible class, but jumps less frequently from one weakly irreducible class to another. We introduce a small parameter $\varepsilon > 0$ into the problem and assume the generator of the Markov chain to be of the form:

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q}. \tag{3}$$

Throughout the paper, we assume both $\widetilde{Q}$ and $\widehat{Q}$ to be generators. As a result, the Markov chain becomes $\alpha^\varepsilon(\cdot)$, an ε-dependent singularly perturbed Markov chain. An averaging approach requires aggregating the states in each weakly irreducible class into a single state, and replacing the original complex system by its limit, an average with respect to the quasi-stationary distributions. In this and the following three sections, we concentrate on the case that the underlying Markov chain has only weakly irreducible classes, which specifies the form of $\widetilde{Q}$ as

$$\widetilde{Q} = \operatorname{diag}\left(\widetilde{Q}^1, \ldots, \widetilde{Q}^l\right). \tag{4}$$

Here, for each $k = 1, \ldots, l$, $\widetilde{Q}^k$ is the weakly irreducible generator corresponding to the states in $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$, for $k = 1, \ldots, l$. The state space is decomposable as

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \\ &= \{s_{11}, \ldots, s_{1m_1}\} \cup \cdots \cup \{s_{l1}, \ldots, s_{lm_l}\}. \end{aligned} \tag{5}$$

Note that $\widetilde{Q}$ governs the rapidly changing part and $\widehat{Q}$ describes the slowly varying components. The slow and fast components are coupled through weak and strong interactions in the sense that the underlying Markov chain fluctuates rapidly within a single group $\mathcal{M}_k$ and jumps less frequently among groups $\mathcal{M}_k$ and $\mathcal{M}_j$ for $k \neq j$. Lumping the states in $\mathcal{M}_k$ into a single "state" (aggregating the states s_{kj} in $\mathcal{M}_k$ as one state k), an aggregated process, containing l states, can be obtained, in which these l states interact through the matrix $\widehat{Q}$ resulting in transitions from $\mathcal{M}_k$ to $\mathcal{M}_j$. By lumping all the states in each weakly irreducible class into one state results in a process with considerably smaller state space. To be more specific, the aggregated process $\{\overline{\alpha}^\varepsilon(\cdot)\}$ is defined by

$$\overline{\alpha}^\varepsilon(t) = k \quad \text{when} \quad \alpha^\varepsilon(t) \in \mathcal{M}_k. \tag{6}$$

The process $\overline{\alpha}^\varepsilon(\cdot)$ is not necessarily Markovian. However, using certain probabilistic arguments, in [21, Section 7.5], assuming $\widetilde{Q}^k$ to be weakly irreducible, we have shown:

(a) $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$, which is a continuous-time Markov chain generated by

$$\begin{aligned} &\overline{Q} = \nu \widehat{Q} \widetilde{\mathbb{1}}, \\ &\nu = \operatorname{diag}(\nu^1, \ldots, \nu^l), \;\; \widetilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}), \end{aligned} \tag{7}$$

where ν^k is the quasi-stationary distribution of $\widetilde{Q}^k$, $k = 1, \ldots, l$, $\mathbb{1}_\ell = (1, \ldots, 1)' \in \mathbb{R}^\ell$ is an ℓ-dimensional column vector with all components being equal to 1, $\mathrm{diag}(D^1, \ldots, D^r)$ is a block-diagonal matrix with appropriate dimensions.

(b) For any bounded deterministic $\beta(\cdot)$,

$$E\left(\int_0^T \left(I_{\{\alpha^\varepsilon(t)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(t)=k\}}\right)\beta(t)dt\right)^2 = O(\varepsilon), \tag{8}$$

where I_A is the indicator function of a set A.

(c) Let $\overline{P}(t) = \widetilde{\mathbb{1}}(\exp \overline{Q}t)\nu \in \mathbb{R}^{m\times m}$. Then

$$|\exp(Q^\varepsilon t) - \overline{P}(t)| = O(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}),$$

for some $\kappa > 0$.

Note that for the process $\overline{\alpha}(\cdot)$, the state space is given by $\overline{M} = \{1, \ldots, l\}$. For complete treatment of two-time-scale Markov chains in continuous time, see the book by Yin and Zhang [21].

2.2 Two-Time-Scale Wonham Filters

Using the above notation, let $y^\varepsilon(t)$ denote the observation measurement given by

$$dy^\varepsilon(t) = f(\alpha^\varepsilon(t))dt + \sigma dw(t),\ y^\varepsilon(0) = 0, \tag{9}$$

where σ is a positive constant and $w(\cdot)$ is a standard Brownian motion. We assume that $\alpha^\varepsilon(\cdot)$ and $w(\cdot)$ are independent.

Let $p_{ij}^\varepsilon(t)$ denote the conditional probability of $\{\alpha^\varepsilon(t) = s_{ij}\}$ given the observation up to time t, i.e.,

$$p_{ij}^\varepsilon(t) = P(\alpha^\varepsilon(t) = s_{ij} | y^\varepsilon(s) : s \le t),$$

for $i = 1, \ldots, l$ and $j = 1, \ldots, m_i$. Let

$$p^\varepsilon(t) = (p_{11}^\varepsilon, \ldots, p_{1m_1}^\varepsilon, \ldots, p_{l1}^\varepsilon, \ldots, p_{lm_l}^\varepsilon) \in \mathbb{R}^{1\times m}.$$

Let

$$\widehat{\alpha}^\varepsilon(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i} f(s_{ij})p_{ij}^\varepsilon(t),$$

and

$$A^{\varepsilon}(t) = \mathrm{diag}(f(s_{11}), \ldots, f(s_{1m_1}), \ldots, f(s_{l1}), \ldots, f(s_{lm_l})) - \widehat{\alpha}^{\varepsilon}(t)I. \tag{10}$$

Then the corresponding Wonham filter can be rewritten as

$$dp^{\varepsilon}(t) = p^{\varepsilon}(t)Q^{\varepsilon}dt - \frac{1}{\sigma^2}\widehat{\alpha}^{\varepsilon}(t)p^{\varepsilon}(t)A^{\varepsilon}(t)dt + \frac{1}{\sigma^2}p^{\varepsilon}(t)A^{\varepsilon}(t)dy^{\varepsilon}(t), \tag{11}$$

with given initial condition

$$p^{\varepsilon}(0) = p_0 = (p_{0,11}, \ldots, p_{0,1m_1}, \ldots, p_{0,l1}, \ldots, p_{0,lm_l}).$$

3 Limit Filter and Two-Time-Scale Approximation

3.1 Limit Filter

Suggested by applications of two-time-scale Markov chains in manufacturing and elsewhere, the conditional probability should converge to a limit filter. In this section, we first derive formally the limit filter and then provide a verification theorem that shows that the limit filter is indeed the limit of the original filter as $\varepsilon \to 0$ in some sense.

Note that $p^{\varepsilon}_{ij}(t)$ are conditional probability measures. Therefore, they are uniformly bounded between 0 and 1. It follows from (11) that

$$E\left|\int_0^t p^{\varepsilon}(u)Q^{\varepsilon}du\right|^2$$

is bounded for all $\varepsilon > 0$. So if $p^{\varepsilon}(t) \to p^0(t)$ as $\varepsilon \to 0$ for some $p^0(t)$ and $t > 0$, then necessarily

$$E\left|\int_0^t p^0(u)\widetilde{Q}du\right|^2 = 0, \text{ for } t > 0.$$

This implies $p^0(t)\widetilde{Q} = 0$. In view of the block-diagonal structure of $\widetilde{Q}$, the vector $p^0(t)$ must have the following form

$$p^0(t) = (\nu^1\overline{p}_1(t), \ldots, \nu^l\overline{p}_l(t)) = \overline{p}(t)\nu,$$

where $\overline{p}(t) = (\overline{p}_1(t), \ldots, \overline{p}_l(t)) \in \mathbb{R}^{1\times l}$ is to be determined later. Recall the definition of $\widetilde{\mathbb{1}}$ in in (7). It follows that

$$p^{\varepsilon}(t)\widetilde{\mathbb{1}} \to p^0(t)\widetilde{\mathbb{1}} = \overline{p}(t)(\nu\widetilde{\mathbb{1}}) = \overline{p}(t).$$

We next derive the equation for $\overline{p}(t)$. As in Wang et al. [18], we can show that the weak limit of $y^\varepsilon(\cdot)$ is given by

$$dy(t) = \overline{f}(\overline{\alpha}(t))dt + \sigma dw(t),\ y(0) = 0,$$

where

$$\overline{f}(i) = \sum_{j=1}^{m_i} f(s_{ij})\nu_j^i.$$

Recall that $\widetilde{Q}\widetilde{\mathbb{1}} = 0$. In (11), multiplying from the right by $\widetilde{\mathbb{1}}$ and sending $\varepsilon \to 0$, we obtain

$$\overline{p}(t) = \overline{p}(0) + \int_0^t \overline{p}(u)\overline{Q}du - \frac{1}{\sigma^2}\int_0^t \widetilde{\alpha}(u)\overline{p}(u)\overline{A}(u)du + \frac{1}{\sigma^2}\int_0^t \overline{p}(u)\overline{A}(u)dy(u), \tag{12}$$

with initial condition

$$\overline{p}(0) = p_0\widetilde{\mathbb{1}} = (p_{0,11} + \cdots + p_{0,1m_1}, \ldots, p_{0,l1} + \cdots + p_{0,lm_l}) \in \mathbb{R}^{1\times l},$$

where

$$\widetilde{\alpha}(t) = \sum_{i=1}^{l} \overline{f}(i)\overline{p}_i(t),$$

and

$$\overline{A}(t) = \operatorname{diag}(\overline{f}(1), \ldots, \overline{f}(l)) - \widetilde{\alpha}(t)I. \tag{13}$$

We will show in what follows that, for each $t > 0$, $p^\varepsilon(t)$ converges to $p^0(t) = \overline{p}(t)\nu$ in a two stage procedure as $\varepsilon \to 0$.

3.2 Two-Time-Scale Approximation

Note that the noise driving the limit filter is the weak limit of $y^\varepsilon(\cdot)$. In order to use the filter in real time applications, one needs to feed the filter by the actual observation $y^\varepsilon(\cdot)$ in (12). Let $\widetilde{p}^\varepsilon(t) = \overline{p}^\varepsilon(t)\nu$ denote such a filter with $\overline{p}^\varepsilon(t)$ given by

$$\overline{p}^\varepsilon(t) = \overline{p}^\varepsilon(0) + \int_0^t \overline{p}^\varepsilon(u)\overline{Q}du - \frac{1}{\sigma^2}\int_0^t \widetilde{\alpha}^\varepsilon(u)\overline{p}^\varepsilon(u)\overline{A}^\varepsilon(u)du + \frac{1}{\sigma^2}\int_0^t \overline{p}^\varepsilon(u)\overline{A}^\varepsilon(u)dy^\varepsilon(u), \tag{14}$$

with initial $\overline{p}^\varepsilon(0) = p_0\widetilde{\mathbb{1}}$, where

$$\widetilde{\alpha}^\varepsilon(t) = \sum_{i=1}^{l} \overline{f}(i)\overline{p}_i^\varepsilon(t),$$

and

$$\overline{A}^{\varepsilon}(t) = \mathrm{diag}(\overline{f}(1), \ldots, \overline{f}(l)) - \widetilde{\alpha}^{\varepsilon}(t)I.$$

Then we have the following theorem.

Theorem 3.1. *The following assertions hold.*

(a) $\widetilde{p}^{\varepsilon}(\cdot)$ *is an approximation to* $p^{\varepsilon}(\cdot)$ *for small* ε. *More precisely,*

$$E|p^{\varepsilon}(t) - \widetilde{p}^{\varepsilon}(t)|^2 = O\left(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}\right),$$

for some constant $\kappa > 0$.

(b) $\overline{p}^{\varepsilon}(\cdot)$ *converges weakly to* $\overline{p}(\cdot)$ *in* $C([0,T];\mathbb{R}^m)$, *where* $C([0,T];\mathbb{R}^m)$ *denotes the space of* $\mathbb{R}^m$*-valued continuous functions defined on* $[0,T]$.

Remark 3.2. This theorem shows the two stage approximation of $p^{\varepsilon}(t) \in \mathbb{R}^m$ by the limit $p^0(t) = \overline{p}(t)\nu$ with $\overline{p}(t) \in \mathbb{R}^l$. The Stage 1 approximation provides a practical way for computing $p^{\varepsilon}(t)$ using $\overline{p}^{\varepsilon}(t)\nu$ which is governed by a system of SDEs of much smaller dimension. The Stage 2 approximation leads to a theoretical weak limit for completeness of the two-time-scale analysis.

Proof of Part (a). Let

$$\phi^{\varepsilon}(t) = p^{\varepsilon}(t) - \widetilde{p}^{\varepsilon}(t) = (\phi^{\varepsilon}_{11}(t), \ldots, \phi^{\varepsilon}_{1m_1}(t), \ldots, \phi^{\varepsilon}_{l1}(t), \ldots, \phi^{\varepsilon}_{lm_l}(t)).$$

Recall that $\widetilde{p}^{\varepsilon}(t) = \overline{p}^{\varepsilon}(t)\nu$. It follows that $\widetilde{p}^{\varepsilon}(t)Q^{\varepsilon} = \widetilde{p}^{\varepsilon}(t)\widehat{Q}$. Let $\widehat{\phi}^{\varepsilon}(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i} f(s_{ij})\phi^{\varepsilon}_{ij}(t)$ and

$$\widehat{A}^{\varepsilon}(t) = \mathrm{diag}(f(s_{11}), \ldots, f(s_{1m_1}), \ldots, f(s_{l1}), \ldots, f(s_{lm_l})) - \widetilde{\alpha}^{\varepsilon}(t)I. \qquad (15)$$

Note that

$$\begin{aligned}
\widehat{\alpha}^{\varepsilon}(t) &= \sum_{i=1}^{l}\sum_{j=1}^{m_i} f(s_{ij})p^{\varepsilon}_{ij}(t) \\
&= \sum_{i=1}^{l}\sum_{j=1}^{m_i} f(s_{ij})(p^{\varepsilon}_{ij}(t) - \nu^i_j\overline{p}^{\varepsilon}_i(t)) + \sum_{i=1}^{l}\sum_{j=1}^{m_i} f(s_{ij})\nu^i_j\overline{p}^{\varepsilon}_i(t) \\
&= \widehat{\phi}^{\varepsilon}(t) + \widetilde{\alpha}^{\varepsilon}(t).
\end{aligned}$$

In view of this relation, (10) and (15) yields that

$$A^\varepsilon(t) = \widehat{A}^\varepsilon(t) - \widehat{\phi}^\varepsilon(t)I.$$

Moreover, owing to (11) and (14), $\phi^\varepsilon(t)$ satisfies the following equation

$$\begin{aligned}
\phi^\varepsilon(t) = \;& \phi^\varepsilon(0) + \int_0^t \phi^\varepsilon(u)Q^\varepsilon du - \frac{1}{\sigma^2}\int_0^t \widehat{\alpha}^\varepsilon(u)\phi^\varepsilon(u)A^\varepsilon(u)du + \frac{1}{\sigma^2}\int_0^t \phi^\varepsilon(u)A^\varepsilon(u)dy^\varepsilon(u)\\
& -\widetilde{p}^\varepsilon(t) + \widetilde{p}^\varepsilon(0) + \int_0^t \widetilde{p}^\varepsilon(u)\widehat{Q}du - \frac{1}{\sigma^2}\int_0^t \widetilde{\alpha}^\varepsilon(u)\widetilde{p}^\varepsilon(u)\widehat{A}^\varepsilon(u)du\\
& +\frac{1}{\sigma^2}\int_0^t \widetilde{p}^\varepsilon(u)\widehat{A}^\varepsilon(u)dy^\varepsilon(u)\\
& -\frac{1}{\sigma^2}\int_0^t [\widehat{\phi}^\varepsilon(u)\widetilde{p}^\varepsilon(u)\widehat{A}^\varepsilon(u) - (\widehat{\phi}^\varepsilon(u))^2\widetilde{p}^\varepsilon(u) - \widetilde{\alpha}^\varepsilon(u)\widetilde{p}^\varepsilon(u)\widehat{\phi}^\varepsilon(u)]du\\
& -\frac{1}{\sigma^2}\int_0^t \widetilde{p}^\varepsilon(u)\widehat{\phi}^\varepsilon(u)dy^\varepsilon(u).
\end{aligned}$$

Write

$$\phi^\varepsilon(t) = \phi^\varepsilon(0) + \int_0^t \phi^\varepsilon(u)Q^\varepsilon du + \int_0^t F_1(u)du + \int_0^t F_2(u)dw(u) + B(t),$$

where

$$\begin{aligned}
B(t) = \;& -\widetilde{p}^\varepsilon(t) + \widetilde{p}^\varepsilon(0) + \int_0^t \widetilde{p}^\varepsilon(u)\widehat{Q}du - \frac{1}{\sigma^2}\int_0^t \widetilde{\alpha}^\varepsilon(u)\widetilde{p}^\varepsilon(u)\widehat{A}^\varepsilon(u)du\\
& +\frac{1}{\sigma^2}\int_0^t \widetilde{p}^\varepsilon(u)\widehat{A}^\varepsilon(u)dy^\varepsilon(u),\\
F_1(t) = \;& \frac{1}{\sigma^2}\Big[- \widehat{\alpha}^\varepsilon(t)\phi^\varepsilon(t)A^\varepsilon(t) + \phi^\varepsilon(t)A^\varepsilon(t)\alpha^\varepsilon(t)\\
& -\widehat{\phi}^\varepsilon(t)\widetilde{p}^\varepsilon(t)\widehat{A}^\varepsilon(t) + (\widehat{\phi}^\varepsilon(t))^2\widetilde{p}^\varepsilon(t) + \widetilde{\alpha}^\varepsilon(t)\widetilde{p}^\varepsilon(t)\widehat{\phi}^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\widehat{\phi}^\varepsilon(t)\alpha^\varepsilon(t)\Big],\\
F_2(t) = \;& \frac{1}{\sigma}\Big[\phi^\varepsilon(t)A^\varepsilon(t) - \widetilde{p}^\varepsilon(t)\widehat{\phi}^\varepsilon(t)\Big],
\end{aligned} \tag{16}$$

or in differential form

$$d\phi^\varepsilon(t) = \phi^\varepsilon(t)Q^\varepsilon dt + F_1(t)dt + F_2(t)dw(t) + dB(t).$$

Right multiply both sides by the matrix $\exp(-Q^\varepsilon t)$ to yield

$$d[\phi^\varepsilon(t)\exp(-Q^\varepsilon t)] = [F_1(t)dt + F_2(t)dw(t) + dB(t)]\exp(-Q^\varepsilon t).$$

Integrating both sides from 0 to t leads to

$$\phi^\varepsilon(t) = \phi^\varepsilon(0)\exp(Q^\varepsilon t) + \int_0^t [F_1(u)du + F_2(u)dw(u) + dB(u)]\exp(Q^\varepsilon(t-u)).$$

Recall that $\overline{P}(t) = \widetilde{\mathbb{1}}(\exp \overline{Q}t)\nu$. It is easy to check that

$$\phi^\varepsilon(0)\overline{P}(t) = 0 \text{ and } \nu \widehat{A}^\varepsilon(t)\widetilde{\mathbb{1}} = \overline{A}^\varepsilon(t),$$

with $\overline{A}^\varepsilon(t)$ defined in (13). Recall uniform boundedness of $p^\varepsilon(t)$ and $\overline{p}^\varepsilon(t)$. We can show

$$|F_1(t)| \le K|\phi^\varepsilon(t)|,$$

$$|F_2(t)| \le K|\phi^\varepsilon(t)|.$$

Moreover, in view of (14) and (16), we have

$$B(t)\widetilde{\mathbb{1}} = 0.$$

Using these and $|\exp(Q^\varepsilon t) - \overline{P}(t)| = O(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}})$, we have

$$\begin{aligned} E|\phi^\varepsilon(t)|^2 \le E|\phi^\varepsilon(0)(\exp(Q^\varepsilon t) - \overline{P}(t))|^2 + K\int_0^t E|\phi^\varepsilon(u)|^2 du + KE\left|\int_0^t \phi^\varepsilon(u)du\right|^2 \\ +KE\left|\int_0^t (dB(u))\exp(Q^\varepsilon(t-u))\right|^2, \end{aligned} \tag{17}$$

for some constant K. Using Cauchy-Schwarz inequality and because of $T < \infty$, we have

$$E\left|\int_0^t \phi^\varepsilon(u)du\right|^2 \le T\int_0^t E|\phi^\varepsilon(u)|^2 du. \tag{18}$$

Moreover,

$$\begin{aligned} E\left(\int_0^t (dB(u))\exp(Q^\varepsilon t)\right)^2 &= E\left(\int_0^t (dB(u))(\exp(Q^\varepsilon(t-u)) - \overline{P}(t-u))\right)^2 \\ &\le KE\left(\int_0^t (\varepsilon + e^{-\frac{\kappa(t-u)}{\varepsilon}})dB(u)\right)^2 \\ &= K\left(\int_0^t (\varepsilon + e^{-\frac{\kappa(t-u)}{\varepsilon}})du\right)^2 \\ &\quad +K\int_0^t (\varepsilon + e^{-\frac{\kappa(t-u)}{\varepsilon}})^2 du = O(\varepsilon). \end{aligned} \tag{19}$$

Let $h(t) = E|\phi^\varepsilon(t)|^2$. It follows from (17)-(19) that

$$h(t) \le K(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}) + K\int_0^t h(u)du,$$

for some positive constants κ and K. Finally, Gronwall's inequality implies that

$$h(t) \le K(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}),$$

i.e.,

$$E|p^\varepsilon(t)-\widetilde{p}^\varepsilon(t)|^2 = E|p^\varepsilon(t)-\overline{p}^\varepsilon(t)\nu|^2 = O(\varepsilon+e^{-\frac{\kappa t}{\varepsilon}}).$$

This proves Part (a).

Proof of Part (b). The proof for Part (b) is divided into two steps. First, we introduce an intermediate process $\check{p}^\varepsilon(t)$ defined by

$$\check{p}^\varepsilon(t) = \check{p}^\varepsilon(0) + \int_0^t \check{p}^\varepsilon(u)\overline{Q}du - \frac{1}{\sigma^2}\int_0^t \check{\alpha}^\varepsilon(u)\check{p}^\varepsilon(u)\check{A}^\varepsilon(u)du + \frac{1}{\sigma^2}\int_0^t \check{p}^\varepsilon(u)\check{A}^\varepsilon(u)d\check{y}^\varepsilon(u), \quad (20)$$

with initial $\check{p}^\varepsilon(0) = p_0\widetilde{\mathbb{1}}$, where

$$d\check{y}^\varepsilon(t) = \overline{f}(\overline{\alpha}^\varepsilon(t))dt + \sigma dw(t),\ \check{y}^\varepsilon(0) = 0,$$

$$\check{\alpha}^\varepsilon(t) = \sum_{i=1}^{l} \overline{f}(i)\check{p}_i^\varepsilon(t),$$

and

$$\check{A}^\varepsilon(t) = \text{diag}(\overline{f}(1),\ldots,\overline{f}(l)) - \check{\alpha}^\varepsilon(t)I.$$

Step 1. We show that $\check{p}^\varepsilon(\cdot)$ converges to $\overline{p}(\cdot)$ weakly.

Recall that $\overline{\alpha}^\varepsilon(\cdot)\to\overline{\alpha}(\cdot)$ in distribution. By the Skorohod Representation Theorem, there exists a probability space $(\Omega_0,\mathcal{F}_0,P_0)$ and processes $\overline{\alpha}_0^\varepsilon(\cdot)$ and $\overline{\alpha}_0(\cdot)$ such that

$$P_0(\overline{\alpha}_0^\varepsilon(\cdot)\in\cdot) = P(\overline{\alpha}^\varepsilon(\cdot)\in\cdot),$$

$$P_0(\overline{\alpha}_0(\cdot)\in\cdot) = P(\overline{\alpha}(\cdot)\in\cdot),$$

and $\overline{\alpha}_0^\varepsilon(\cdot)\to\overline{\alpha}_0(\cdot)$ a.s. in $D([0,T];\overline{\mathcal{M}})$, where $\overline{M}=\{1,\ldots,l\}$. Let $(\Omega_w,\mathcal{F}_w,P_w)$ be a probability space upon which $w(\cdot)$ is a standard Brownian motion. Then, on the product space $(\Omega_0\times\Omega_w,\mathcal{F}_0\times\mathcal{F}_w,P_0\times P_w)$,

$$\check{p}^\varepsilon(\cdot)\to\overline{p}(\cdot) \text{ a.s. in } D([0,T];\mathbb{R}^m).$$

Step 2. We show that, for each t, $E|\overline{p}^\varepsilon(t)-\check{p}^\varepsilon(t)|\to 0$, as $\varepsilon\to 0$.

Let $\psi(t)=(\psi_1(t),\ldots,\psi_l(t))=\overline{p}^\varepsilon(t)-\check{p}^\varepsilon(t)$. Then $\psi(0)=0$. Moreover, in view of (14) and (20) and by re-ordering terms, we have

$$\begin{aligned} d\psi(t) = {} & \psi(t)\overline{Q}dt - \frac{1}{\sigma^2}[\check{\psi}(t)\overline{p}^\varepsilon(t)\overline{A}^\varepsilon(t) + \check{\alpha}^\varepsilon(t)\psi(t)\overline{A}^\varepsilon(t) + \check{\alpha}^\varepsilon(t)\check{p}^\varepsilon(t)\check{\psi}(t)I]dt \\ & + \frac{1}{\sigma^2}[(\psi(t)\overline{A}^\varepsilon(t) + \check{p}^\varepsilon(t)\check{\psi}(t))dy^\varepsilon(t) + \check{p}^\varepsilon(t)\check{\psi}(t)(f(\alpha^\varepsilon(t)) - \overline{f}(\overline{\alpha}^\varepsilon(t)))dt], \end{aligned}$$

where $\check{\psi}(t) = \widetilde{\alpha}^\varepsilon(t) - \check{\alpha}^\varepsilon(t) = \sum_{i=1}^l \overline{f}(i)\psi_i(t)$. Write

$$\psi(t) = \int_0^t G_1(u)du + \int_0^t G_2(u)dw(u) + H(t),$$

where

$$\begin{aligned}
G_1(t) = \quad & \frac{1}{\sigma^2}\Big[\check{\psi}(t)\overline{p}^\varepsilon(t)\overline{A}^\varepsilon(t) + \check{\alpha}^\varepsilon(t)\psi(t)\overline{A}^\varepsilon(t) + \check{\alpha}^\varepsilon(t)\check{p}^\varepsilon(t)\check{\psi}(t)I + [\psi(t)\overline{A}^\varepsilon(t) \\
& +\check{p}^\varepsilon(t)\check{\psi}(t))]f(\alpha^\varepsilon(t))\Big], \\
G_2(t) = \quad & \frac{1}{\sigma}[\psi(t)\overline{A}^\varepsilon(t) + \check{p}^\varepsilon(t)\check{\psi}(t)], \\
H(t) = \quad & \frac{1}{\sigma^2}\int_0^t \check{p}^\varepsilon(u)\check{\psi}(u)(f(\alpha^\varepsilon(u)) - \overline{f}(\overline{\alpha}^\varepsilon(u)))du.
\end{aligned}$$

It is easy to see that

$$|G_1(t)| \le K|\psi(t)| \text{ and } |G_2(t)| \le K|\psi(t)|,$$

for some constant K. This implies that

$$E|\psi(t)|^2 \le E|H(t)|^2 + K\int_0^t E|\psi(u)|^2du.$$

In view of Gronwall's inequality, it suffices to show $E|H(t)|^2 \to 0$. Let

$$V(t) = \int_0^t (I_{\{\alpha^\varepsilon(u)=s_{ij}\}} - \nu_j^i I_{\{\overline{\alpha}^\varepsilon(u)=i\}})du.$$

Then (8) implies $E|V(t)|^2 \to 0$, as $\varepsilon \to 0$. For a diffusion $dg = g_1dt + g_2dw$ with bounded g_1 and g_2, let

$$H_0(t) = \int_0^t g(u)(I_{\{\alpha^\varepsilon(u)=s_{ij}\}} - \nu_j^i I_{\{\overline{\alpha}^\varepsilon(u)=i\}})du.$$

Then, by integration by parts, we have

$$H_0(t) = g(t)V(t) - \int_0^t g_1(u)V(u)du - \int_0^t g_2(u)V(u)dw(u).$$

It follows that

$$E|H_0(t)|^2 \le K|V(t)|^2 + K\int_0^t E|V(u)|^2du \to 0.$$

Take $g(t) = \check{p}^\varepsilon(t)\check{\psi}(t)$ and write

$$f(\alpha^\varepsilon(t)) - \overline{f}(\overline{\alpha}^\varepsilon(t)) = \sum_{i=1}^l\sum_{j=1}^{m_i} f(s_{ij})(I_{\{\alpha^\varepsilon(t)=s_{ij}\}} - \nu_j^i I_{\{\overline{\alpha}^\varepsilon(t)=i\}})$$

to obtain $E|H(t)|^2 \to 0$.

Step 3. Finally, using Kushner's tightness criterion [14, p. 47], it is easy to see that $\overline{p}^\varepsilon(\cdot)$ is tight. To complete the proof, it suffices to show the weak convergence of finite dimensional distributions. To this end, note that following Steps 1 and 2, we have, for any $f \in C_b^2$ (the space of functions whose second derivatives are bounded),

$$\begin{aligned}
&|Ef(a_1\overline{p}^\varepsilon(t_1) + \cdots + a_n\overline{p}^\varepsilon(t_n)) - Ef(a_1\overline{p}(t_1) + \cdots + a_n\overline{p}(t_n))| \\
&\quad \leq |Ef(a_1\overline{p}^\varepsilon(t_1) + \cdots + a_n\overline{p}^\varepsilon(t_n)) - Ef(a_1\check{p}^\varepsilon(t_1) + \cdots + a_n\check{p}^\varepsilon(t_n))| \\
&\qquad +|Ef(a_1\check{p}^\varepsilon(t_1) + \cdots + a_n\check{p}^\varepsilon(t_n)) - Ef(a_1\overline{p}(t_1) + \cdots + a_n\overline{p}(t_n))| \to 0,
\end{aligned}$$

for any constants $a_1, \ldots, a_n$ and $t_1, \ldots, t_n \in [0, T]$. This implies $(\overline{p}^\varepsilon(t_1), \ldots, \overline{p}^\varepsilon(t_n))$ converges to $(\overline{p}(t_1), \ldots, \overline{p}(t_n))$.

Finally, note that both $p^\varepsilon(\cdot)$ and $\overline{p}(\cdot)$ have continuous sample paths a.s., so the convergence, in fact, takes place on the space $C([0, T]; \mathbb{R}^m)$. □

Remark 3.3. The conditional probability vector $p^\varepsilon(t)$ behaves similarly to a regular probability vector for $\alpha^\varepsilon(t)$ in the sense that itself is not tight and therefore does not converge in the neighborhood of $t = 0$ due to a boundary layer near the origin. It takes a small amount of time for $p^\varepsilon(t)$ to correct if necessary from its initial $p^\varepsilon(0)$.

Corollary 3.4.

(a) $E\int_0^T |p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)|^2 dt = O(\varepsilon)$.

(b) *For any* $\delta > 0$, $\sup_{t\in[\delta,T]} E|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)|^2 = O(\varepsilon)$.

(c) *For each* $t > 0$, $p^\varepsilon(t) \to p^0(t)$ *in distribution.*

Proof. Parts (a) and (b) are immediate from Theorem 3.1. To see Part (c), note that for all $f \in C_b^2$, we have

$$|Ef(p^\varepsilon(t)) - Ef(p^0(t))| \leq |Ef(p^\varepsilon(t)) - Ef(\widetilde{p}^\varepsilon(t))| + |Ef(\widetilde{p}^\varepsilon(t)) - Ef(p^0(t))| \to 0. \qquad \square$$

4 Numerical Examples

In this section, we consider a simple example involving a four state Markov chain. Let

$$Q^\varepsilon = \frac{1}{\varepsilon}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The corresponding state space is $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 = \{s_{11}, s_{12}\} \cup \{s_{21}, s_{22}\}$. In this case, $\nu^1 = \nu^2 = (1/2, 1/2)$.

Let $(p^\varepsilon_{11}(t), p^\varepsilon_{12}(t), p^\varepsilon_{21}(t), p^\varepsilon_{22}(t))$ denote the conditional probability vector and its approximation by $(\widetilde{p}^\varepsilon_{11}(t), \widetilde{p}^\varepsilon_{12}(t), \widetilde{p}^\varepsilon_{21}(t), \widetilde{p}^\varepsilon_{22}(t))$. Define the norm

$$\begin{aligned} &||p^\varepsilon(\cdot) - \widetilde{p}^\varepsilon(\cdot)||^2_T \\ =&E\int_0^T \left(|p^\varepsilon_{11}(t) - \widetilde{p}^\varepsilon_{11}(t)|^2 + |p^\varepsilon_{12}(t) - \widetilde{p}^\varepsilon_{12}(t)|^2 + |p^\varepsilon_{21}(t) - \widetilde{p}^\varepsilon_{21}(t)|^2 + |p^\varepsilon_{22}(t) - \widetilde{p}^\varepsilon_{22}(t)|^2\right) dt. \end{aligned}$$

In this example, we take

$$f(s_{11}) = 1,\ f(s_{12}) = 1.5,\ f(s_{21}) = -1.5,\ f(s_{22}) = -1,$$

$\sigma = 0.5$, $T = 5$ and the discretization step size $\delta = 0.0005$. A sample path of $\alpha^\varepsilon(\cdot)$ (with $\varepsilon = 0.05$) and the corresponding condition probabilities are given in the first 5 rows in Figure 1. In Figure 1, the states are labelled as $1 = s_{11}$, $2 = s_{12}$, $3 = s_{21}$, and $4 = s_{22}$. The differences between $p^\varepsilon(\cdot)$ and $\widetilde{p}^\varepsilon(\cdot)$ are plotted in the last 4 rows. As can be seen in Figure 1, $\alpha^\varepsilon(\cdot)$ stays in group $\mathcal{M}_1$ from $t = 0.2$ to 1.5, jumps to group $\mathcal{M}_2$ at $t = 1.5$, goes back to $\mathcal{M}_1$ at $t = 3.4$, then to $\mathcal{M}_2$, and finally landed in $\mathcal{M}_1$ from $t = 4.4$ to 5. The approximation filter $\widetilde{p}^\varepsilon_{ij}(t)$ tracks corresponding conditional probabilities $p^\varepsilon(t)$ pretty well on these time intervals.

In addition, we vary ε and run 1000 samples for each ε. The results are recorded in Table 1.

As it can be seen in Table 1, the differences between the exact conditional probabilities and their approximations $\widetilde{p}^\varepsilon(\cdot)$ are fairly small. The result validates the effectiveness of our approach in this simple example.

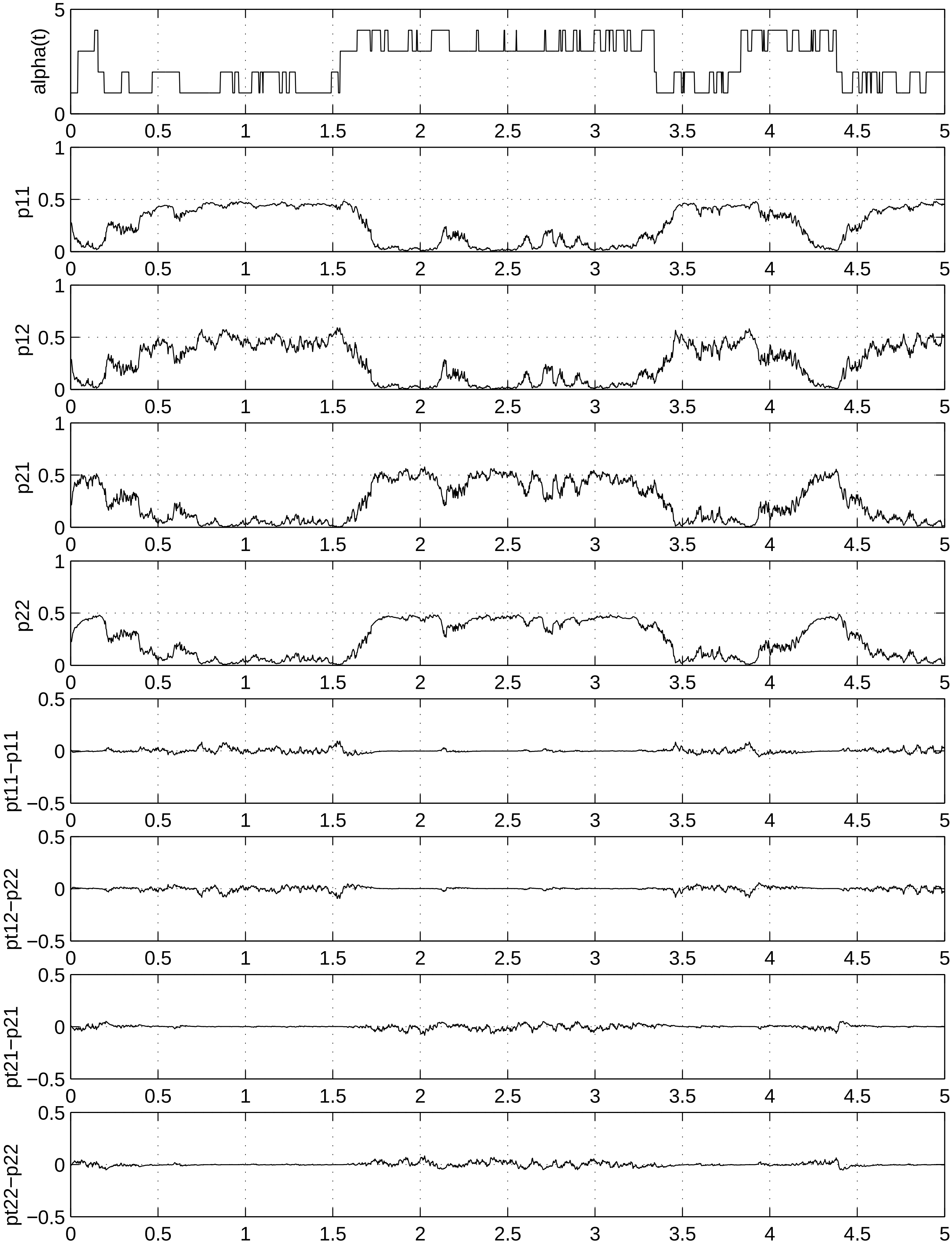

Figure 1: Sample paths of $\alpha^{\varepsilon}(t)$, $p^{\varepsilon}(t)$, and $\tilde{p}^{\varepsilon}(t) - p^{\varepsilon}(t)$ with $\varepsilon = 0.05$.

ε	0.5	0.1	0.05	0.01	0.005
$\|\|p^\varepsilon - \tilde{p}^\varepsilon\|\|_T^2$	0.0335	0.0090	0.005	0.00117	0.00063

Table 1: Demonstration of error bounds.

5 Inclusion of Transient States

In the previous sections, we have concentrated on the case that the underlying Markov chain consists of weakly irreducible classes only. This section takes up the issue that the underlying Markov chain contains weakly irreducible classes as well as transient states. In this case, the state space $\mathcal{M}$ is partitioned as:

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_* \\ &= \{s_{11}, \ldots, s_{1m_1}\} \cup \cdots \cup \{s_{l1}, \ldots, s_{lm_l}\} \cup \{s_{*1}, \ldots, s_{*m_*}\}, \end{aligned} \tag{21}$$

where $\mathcal{M}_* = \{s_{*1}, \ldots, s_{*m_*}\}$ is the collection of the transient states, and the generator is still of the form (3), but

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}^1 & & & & \\ & \widetilde{Q}^2 & & & \\ & & \ddots & & \\ & & & \widetilde{Q}^l & \\ \widetilde{Q}_*^1 & \widetilde{Q}_*^2 & \cdots & \widetilde{Q}_*^l & \widetilde{Q}_* \end{pmatrix}. \tag{22}$$

To distinguish the transient states with that of the states in weakly irreducible classes, we use $*$ as an index.

We assume that $\widetilde{Q}_*$ is asymptotically stable, i.e., all of its eigenvalues belong to the left half of the complex plane; To proceed, define

$$\nu_* = \mathrm{diag}(\nu, 0_{m_* \times m_*}),$$

and

$$\widetilde{\mathbb{1}}_* = \begin{pmatrix} \mathbb{1}_{m_1} & & & \\ & \ddots & & \\ & & \mathbb{1}_{m_l} & \\ a_{m_1} & \ldots & a_{m_l} & 0_{m_* \times m_*} \end{pmatrix}. \tag{23}$$

where $0_{m_* \times m_*}$ is an $m_* \times m_*$ zero matrix, and

$$a_{m_i} = -\widetilde{Q}_*^{-1} \widetilde{Q}_*^i \mathbb{1}_{m_i} \ \text{ for } \ i = 1, \ldots, l.$$

Let U be a random variable uniformly distributed on $[0,1]$ that is independent of $\alpha^\varepsilon(\cdot)$. For each $j=1,\ldots,m_*$, define an integer-valued random variable U_j by

$$U_j = I_{\{0\le U\le a_{m_1,j}\}} + 2I_{\{a_{m_1,j}<U\le a_{m_1,j}+a_{m_2,j}\}} + \cdots + lI_{\{a_{m_1,j}+\cdots+a_{m_{l-1},j}<U\le 1\}}.$$

We proceed to define the aggregated process. Note, however, the aggregation is taken over each weakly irreducible class only. Define

$$\overline{\alpha}^\varepsilon(t) = \begin{cases} i, & \text{if } \alpha^\varepsilon(t)\in\mathcal{M}_i, \\ U_j, & \text{if } \alpha^\varepsilon(t) = s_{*j}. \end{cases} \tag{24}$$

Using the partition

$$\widehat{Q} = \begin{pmatrix} \widehat{Q}^{11} & \widehat{Q}^{12} \\ \widehat{Q}^{21} & \widehat{Q}^{22} \end{pmatrix},$$

where

$$\widehat{Q}^{11}\in\mathbb{R}^{(m-m_*)\times(m-m_*)},\quad \widehat{Q}^{12}\in\mathbb{R}^{(m-m_*)\times m_*},$$
$$\widehat{Q}^{21}\in\mathbb{R}^{m_*\times(m-m_*)},\quad \text{and } \widehat{Q}^{22}\in\mathbb{R}^{m_*\times m_*},$$

Write

$$\overline{Q} = \operatorname{diag}(\nu^1,\ldots,\nu^l)(\widehat{Q}^{11}\widetilde{\mathbb{1}} + \widehat{Q}^{12}(a_{m_1},\ldots,a_{m_l})). \tag{25}$$

Define

$$\overline{Q}_* = \operatorname{diag}(\overline{Q}, 0_{m_*\times m_*}).$$

We proved in [22, Theorem 4.3] that $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$, a continuous-time Markov chain generated by $\overline{Q}$ given in (25). We also obtained similar mean squares estimates for the occupation measures as in the previous case. In fact,

$$\sup_{0\le t\le[0,T]} E\left(\int_0^t [I_{\{\alpha^\varepsilon(s)=s_{ij}\}} - \nu_j^i I_{\{\alpha^\varepsilon(s)\in\mathcal{M}_i\}}]ds\right)^2 = O(\varepsilon) \quad \text{for} \quad i=1,\ldots,l,\ j=1,\ldots,m_i,$$
$$\sup_{0\le t\le[0,T]} E\left(\int_0^t I_{\{\alpha^\varepsilon(s)=s_{ij}\}}\right)^2 = O(\varepsilon^2), \quad \text{for} \quad i=*, j=1,\ldots,m_*.$$

Moreover, let $\overline{P}_*(t) = \widetilde{\mathbb{1}}_*(\exp\overline{Q}_*t)\nu_*$. Then

$$|\exp(Q^\varepsilon t) - \overline{P}_*(t)| = O(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}).$$

Define $\overline{p}(t)$ as follows:

$$\overline{p}(t) = \overline{p}(0) + \int_0^t \overline{p}(u)\overline{Q}_* du - \frac{1}{\sigma^2}\int_0^t \widetilde{\alpha}(u)\overline{p}(u)\overline{A}(u)du + \frac{1}{\sigma^2}\int_0^t \overline{p}(u)\overline{A}(u)dy(u),$$

with initial condition

$$\overline{p}(0) = p_0 \widetilde{\mathbb{1}}_*.$$

Similarly, define $\overline{p}^\varepsilon(t)$ as above with $y(t)$ replaced by $y^\varepsilon(t)$.

We can prove the following results similarly to the proof of Theorem 3.1.

Theorem 5.1. *Let* $\widetilde{p}^\varepsilon(t) = (\overline{p}^\varepsilon(t)\nu_*, 0_{m_*})$ *where* $0_{m_*} = (0, \ldots, 0) \in \mathbb{R}^{1\times m_*}$. *Then, we have*

(a) $\sup_{t\in[0,T]} E|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)|^2 = O\left(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}\right)$, *for some* $\kappa > 0$

(b) $\overline{p}^\varepsilon(\cdot)$ *converges to* $\overline{p}(\cdot)$ *weakly on* $C([0,T];\mathbb{R}^m)$.

6 Concluding Remarks

In this paper, we developed approximate Wonham filters under the framework of two-time-scale Markov chains. The advantage of this approach is the reduction of dimensionality in the sense that the approach leads to a limit filter that is close to the original Wonham filter but of much smaller dimension. Such a filtering scheme is desirable in state estimation involving Markov chains with a large number of states such as in production planning of stochastic manufacturing systems.

Recently, there has been resurgent interest in constructing robust filters following the original work of Clark [5]. In [8], unnormalized densities are considered. It will be interesting to see if the two-time-scale approach can be used to treat such problems. Nevertheless, care must be taken, since although the counter part of $p^\varepsilon(\cdot)$ satisfies an ordinary differential equation, it is not bounded. More work is required.

7 Appendix

For convenience of reference, we recall here the definition of irreducibility of Markov chains [21] and Gronwall's inequality [9, p. 36].

Definition 7.1. The Markov chain or the generator Q is weakly irreducible if the system of equations

$$\nu Q = 0, \quad \text{and} \quad \sum_{i=1}^{m} \nu_i = 1$$

has a unique nonnegative solution. The nonnegative solution (row-vector-valued function) $\nu = (\nu_1, \ldots, \nu_m)$ is termed a quasi-stationary distribution. In addition, if ν is strictly positive, then we say the generator Q is irreducible.

Lemma 7.2 (*Gronwall's inequality.*) *Given a bounded measurable function* $c(t)$*, if*

$$0 \leq h(t) \leq c(t) + K \int_0^t h(u) du,$$

then

$$h(t) \leq c(t) + K \int_0^t c(u) e^{K(t-u)} du.$$

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