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Two-Time-Scale Wonham Filters

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***Abstract*— This paper is concerned with a two-time-scale approximation of Wonham filters. A main feature is that the underlying hidden Markov chain has a large state space. To reduce computational complexity, we develop two-time-scale approach. Under time scale separation, we divide the state space of the Markov chain into a number of groups such that the chain jumps rapidly within each group and switches occasionally from one group to another. Such structure yields a limit Wonham filter preserving the main features of the filtering process, but has a much smaller dimension and therefore is easier to compute. Using the limit filter enables us to develop efficient approximations for the filters for hidden Markov chains. One of the main advantages of our approach is the substantial reduction of dimensionality.**

***Index Terms*— Wonham filter, hidden Markov chain, two-time-scale Markov process**

I. INTRODUCTION

There has been a growing interest in using switching diffusion systems for emerging applications in wireless communication, signal processing, and financial engineering. Different from the usual diffusion models used in the traditional setup, both continuous dynamics and discrete events coexist in the regime-switching models. The hybrid formulation makes the models more versatile, but the analysis becomes more challenging.

In this work, we focus on hybrid diffusions or switching diffusions, which uses a continuous-time Markov chain to capture the discrete event features resulting in a set of diffusions modulated by the Markov chain. To carry out control and optimization tasks for regime-switching diffusions under partial observations, it is desirable to extract characteristics or features of the systems based on the limited information available, which brings us to the framework of hybrid filtering.

Optimal filtering of hybrid systems typically yields infinite dimensional stochastic differential equations. Efforts have been made to find finite dimensional approximations. Some of these approximation schemes can be simplified if the conditional probability of the Markov chain given observation over time is available. In this paper, we consider the model in which a function of the Markov chain is observed in additive white noise. We focus on the conditional probability of the chain given the observation. The corresponding filter, developed in [25], is known as the Wonham filter and is given by the solution of a system of stochastic differential equations.

A. Wonham Filter

In the literature, after the Kalman filter was developed in the 1960's, the first rigorous development of nonlinear filters for diffusion-type processes came into being; see Kushner [18]. Nonlinear filter problems soon attracted growing and continued attention. In contrast to the Kalman filter which is finite dimensional, it is well known that nonlinear filters are generally infinite dimensional. There are only a handful of finite dimensional nonlinear filters known to exist to date. The first of such finite-dimensional filters was developed by Wonham [25]. Owing to its importance, it has received much attention. In the new era, because of the use of hidden Markov models, filters involving jump processes have drawn resurgent attention. For example, to characterize stock price movements, one may use $\alpha(t)$, a continuous-time Markov chain, to represent the stock trends and stochastic volatility. Suppose that we can observe $y(t)$, the percentage change of the stock price represented by $f(\alpha(t))$ plus white noise, where $f(\cdot)$ is an appropriate function. For simplicity, suppose $\alpha(t) \in \mathcal{M} = \{1, 2\}$. For example, use $\alpha(t) = 2$ to represent an up trend with $f(2) > 0$ and $\alpha(t) = 1$ a down trend with $f(1) < 0$); see [33] for more details and related literature. In manufacturing applications, one may take $f(\alpha(t))$ as the discrete demand for a product that is corrupted with Gaussian white noise. In a recent work [24], Wang, Zhang, and Yin considered Kalman-type filters for the partially observed system

$$\begin{cases} dx(t) = A(\alpha(t))x(t)dt + \sigma(\alpha(t))dw(t), & x(0) = x_0, \\ dy(t) = H(\alpha(t))x(t)dt + \delta dv(t), & y(0) = 0, \end{cases}$$

where $x(t)$ is the continuous state variable, $\alpha(t)$ is a finite state Markov chain (a discrete-event state), and $y(t)$ is the observation process; a quadratic variation test was developed and near-optimal filters were obtained by examining the associated system of Riccati equations. In this paper, we concentrate on Wonham filters, in

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which only noisy corrupted observations of a Markov chain are available. So there are no Riccati equations we can utilize, and the methods developed in [24] are not applicable.

To proceed, we first summarize results about the Wonham filter. Let $\alpha(t)$ be a continuous-time Markov chain having a finite state space $\mathcal{M} = \{1, \ldots, m\}$ and generator $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$. A function of the Markov chain $\alpha(t)$ together with additive Gaussian noise is observed. Let $y(t)$ denote the observation given by

$$dy(t) = f(\alpha(t))dt + \sigma dw(t), \; y(0) = 0, \tag{1}$$

where σ is a positive constant and $w(\cdot)$ is a standard Brownian motion independent of $\alpha(\cdot)$.

Let $p_i(t)$ be the conditional probability of $\{\alpha(t) = i\}$ given the observations up to time t, i.e.,

$$p_i(t) = P(\alpha(t) = i | y(s) : s \leq t),$$

for $i = 1, \ldots, m$. Let $p(t) = (p_1(t), \ldots, p_m(t)) \in \mathbb{R}^{1\times m}$. Then the Wonham filter is given by

$$\begin{aligned} dp(t) &= p(t)Qdt - \frac{1}{\sigma^2}\left(\sum_{i=1}^m f(i)p_i(t)\right) p(t)A(t)dt \\ &\quad + \frac{1}{\sigma^2} p(t)A(t)dy(t), \\ p(0) &= p_0, \quad \text{initial probability vector}, \end{aligned} \tag{2}$$

where

$$A(t) = \text{diag}(f(1), \ldots, f(m)) - \sum_{i=1}^m f(i)p_i(t)I.$$

Here and from now on, we use I as an identity matrix of appropriate dimension and use K as a generic constant with the convention $K + K = K$ and $KK = K$.

B. *Brief Review of Literature*

Because of its importance, filtering problems have received much attention. For example, Caines and Chen [4] derived an optimal filter when it involves a random variable but with no switching; see also Hijab [12]. Haussmann and Zhang [11] used two statistical hypothesis tests, the quadratic variation test and the likelihood ratio test, to estimate the value of the random variable and to choose among competing filters on successive time intervals. These results are generalized in Zhang [30] to incorporate unobservable Markov chains.

For related work on filtering, see Dey and Moore [6] and Moore and Baras [21] for risk sensitive filtering; Wang, Zhang, and Yin [24] and Yin and Dey [26] for reduction of complexity of filtering problems involving large-scale Markov chains; Zhang [32], [31] for the most probable estimates in discrete-time and continuous-time models, respectively; and Liu and Zhang [20] for numerical experiments involving piecewise approximation of nonlinear systems; and Yin, Zhang, and Liu [29] for numerical methods of Wonham filters.

A survey of results on filtering can be found in the books by Anderson and Moore [1] on classical linear filtering. Results concerning hidden Markov models and related filtering problems can be found in Elliott, Aggoun, and Moore [8]. For general nonlinear filtering, see Kallianpur [16] and Liptser and Shiryayev [15]; see also the books by Bensoussan [2] and Kushner [19] for related topics on partially observed systems.

The primary concern of this paper is on constructing Wonham filters for Markov chains with a large state space. Note that related results may be found in Tweedie [23] for quasi-stationary distributions of Markov processes for a general state space, Huisinga, Meyn, and Schutte [13] for a spectral theory approach to approximation of a complex Markov process with a simpler one, and Jerrum [14] for further discussion. When the state space of the Markov chain is large, the number of the filter equations will be large comprising a switching diffusion system with large dimension. We focus on developing good approximations for large dimensional filters. The main idea is to use time-scale separation and hierarchy within the Markov chain to reduce the computation complexity. In applications of manufacturing (see [22, Section 5.9]) and in system reliability (see [27, Section 3.2]), the state space of the Markov chain can be partitioned to a number of groups so that the Markov chain jumps rapidly within a group of states and less frequently (or occasionally) among different groups. Under such a setup, due to the fast variation of the Markov chain, it is difficult to pinpoint the exact location of the Markov chain. Nevertheless, it is much easier to identify if the chain belongs to certain groups. This leads to a formulation involving states having weak and strong interactions.

In this paper, we present a two-time-scale filter, its corresponding limit filter and related convergence. We show that the original filter can be approximated in a two stage procedure under different topologies. Before proceeding further, we point out: First, the time-scale separation in this paper is formulated by using a small parameter $\varepsilon > 0$. The asymptotic results to follow require that ε approaches zero. However, in applications, ε can be a fixed constant. For example, given the magnitude of other parameters being of order 1, ε can take the value 0.01 or 0.1. Mainly, the small parameter brings out the different scales of the jump rates in different states of the Markov chain. Second, in the formulation, the Markov chain has a particular structure. Since any finite state Markov chain has at least one recurrent state, conversion to such a "canonical form" is always possible; see for example, [27, Section 3.6] and references therein. Due to page limitations, proofs of results are omitted. A reader is referred to [34] for the detailed development.

II. Singularly Perturbed Markov Chains

Suppose that $\alpha(t)$ is a continuous-time Markov chain whose generator is Q. We say that the Markov chain or the generator Q is weakly irreducible if the system of equations

$$\nu Q = 0, \quad \text{and} \quad \sum_{i=1}^{m} \nu_i = 1$$

has a unique solution satisfying $\nu_i \geq 0$ for $i = 1, \ldots, m$. The solution (row-vector-valued function) $\nu = (\nu_1, \ldots, \nu_m)$ is termed a quasi-stationary distribution. Note that the definitions were used in our work [17] and [27]. They are different from the usual definitions of irreducibility and stationary distribution in that we do not require all the components $\nu_i > 0$; they are also different than that of [23].

A. Time-Scale Separation in Markov Chains

In this work, we focus on Markov chains that have large state spaces with complex structures. Suppose that the states of the underlying Markov chain are divisible to a number of weakly irreducible classes such that the Markov chain fluctuates rapidly among different states within a weakly irreducible class, but jumps less frequently from one weakly irreducible class to another. To highlight the different rates of variation, introduce a small parameter $\varepsilon > 0$ and assume the generator of the Markov chain to be of the form

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q}. \tag{3}$$

Throughout the paper, we assume both $\widetilde{Q}$ and $\widehat{Q}$ to be generators. As a result, the Markov chain becomes an ε-dependent singularly perturbed Markov chain. An averaging approach requires aggregating the states in each weakly irreducible class into a single state, and replacing the original complex system by its limit, an average with respect to the quasi-stationary distributions. In this and the following three sections, we concentrate on the case that the underlying Markov chain has weakly irreducible classes with no transient states, which specifies the form of $\widetilde{Q}$ as

$$\widetilde{Q} = \operatorname{diag}\left(\widetilde{Q}^1, \ldots, \widetilde{Q}^l\right). \tag{4}$$

Here, for each $k = 1, \ldots, l$, $\widetilde{Q}^k$ is the weakly irreducible generator corresponding to the states in $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$, for $k = 1, \ldots, l$. The state space is decomposed as

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \\ &= \{s_{11}, \ldots, s_{1m_1}\} \cup \cdots \cup \{s_{l1}, \ldots, s_{lm_l}\}. \end{aligned} \tag{5}$$

Note that $\widetilde{Q}$ governs the rapidly changing part and $\widehat{Q}$ describes the slowly varying components. Lumping the states in $\mathcal{M}_k$ into a single "state," an aggregated process, containing l states, is obtained, in which these l states interact through the matrix $\widehat{Q}$ resulting in transitions from $\mathcal{M}_k$ to $\mathcal{M}_j$. Thus by aggregation, we obtain a process with considerably smaller state space. To be more specific, the aggregated process $\{\overline{\alpha}^\varepsilon(\cdot)\}$ is defined by

$$\overline{\alpha}^\varepsilon(t) = k \quad \text{when} \quad \alpha^\varepsilon(t) \in \mathcal{M}_k. \tag{6}$$

Note that $\overline{\alpha}^\varepsilon(\cdot)$ is not necessarily Markovian. However, using certain probabilistic arguments and assuming $\widetilde{Q}^k$ to be weakly irreducible, we have shown in [27, Section 7.5] that

(a) $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$, which is a continuous-time Markov chain generated by

$$\begin{aligned} &\overline{Q} = \nu \widehat{Q} \widetilde{\mathbb{1}}, \\ &\nu = \operatorname{diag}(\nu^1, \ldots, \nu^l), \;\; \widetilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}), \end{aligned} \tag{7}$$

where ν^k is the quasi-stationary distribution of $\widetilde{Q}^k$, $k = 1, \ldots, l$, $\mathbb{1}_\ell = (1, \ldots, 1)' \in \mathbb{R}^\ell$ is an ℓ-dimensional column vector with all components being equal to 1, $\operatorname{diag}(D^1, \ldots, D^r)$ is a block-diagonal matrix with appropriate dimensions.

(b) For any bounded deterministic $\beta(\cdot)$, then

$$\begin{aligned} &E\left(\int_0^T \left(I_{\{\alpha^\varepsilon(t)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(t)=k\}}\right)\beta(t)dt\right)^2 \\ &\quad = O(\varepsilon), \end{aligned} \tag{8}$$

where I_A is the indicator function of a set A.

(c) Let $\overline{P}(t) = \widetilde{\mathbb{1}}(\exp \overline{Q}t)\nu \in \mathbb{R}^{m\times m}$. Then

$$|\exp(Q^\varepsilon t) - \overline{P}(t)| = O(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}), \tag{9}$$

for some $\kappa > 0$.

Note that for the process $\overline{\alpha}(\cdot)$, the state space is given by $\overline{\mathcal{M}} = \{1, \ldots, l\}$. For a complete treatment of two-time-scale Markov chains in continuous time, see the book by Yin and Zhang [27]. In addition, we would like to point out that the aggregation process depends on the decomposition of $\widetilde{Q}$ in (4). Substantial reduction of dimensionality can be achieved when $l \ll m$.

B. Two-Time-Scale Wonham Filters

Let $y^\varepsilon(t)$ be the observation given by

$$dy^\varepsilon(t) = f(\alpha^\varepsilon(t))dt + \sigma dw(t), \; y^\varepsilon(0) = 0, \tag{10}$$

where σ is a positive constant and $w(\cdot)$ is a standard Brownian motion. We assume that $\alpha^\varepsilon(\cdot)$ and $w(\cdot)$ are independent. Let $p_{ij}^\varepsilon(t)$ denote the conditional probability of $\{\alpha^\varepsilon(t) = s_{ij}\}$ given the observation up to time t, i.e.,

$$p_{ij}^\varepsilon(t) = P(\alpha^\varepsilon(t) = s_{ij} | y^\varepsilon(s) : s \leq t),$$

for $i = 1, \ldots, l$ and $j = 1, \ldots, m_i$. Let $p^\varepsilon(t) \in \mathbb{R}^{1\times m}$ such that

$$p^\varepsilon(t) = (p^\varepsilon_{11}(t), \ldots, p^\varepsilon_{1m_1}(t), \ldots, p^\varepsilon_{l1}(t), \ldots, p^\varepsilon_{lm_l}(t)).$$

Let

$$\widehat{\alpha}^\varepsilon(t) = \sum_{i=1}^{l}\sum_{j=1}^{m_i} f(s_{ij}) p^\varepsilon_{ij}(t),$$

and

$$\begin{aligned} A^\varepsilon(t) = \operatorname{diag}(&f(s_{11}), \ldots, f(s_{1m_1}),\\ &\ldots, f(s_{l1}), \ldots, f(s_{lm_l})) - \widehat{\alpha}^\varepsilon(t) I. \end{aligned} \tag{11}$$

Then the corresponding Wonham filter can be rewritten as

$$\begin{aligned} dp^\varepsilon(t) = p^\varepsilon(t) Q^\varepsilon dt - \frac{1}{\sigma^2}\widehat{\alpha}^\varepsilon(t) p^\varepsilon(t) A^\varepsilon(t) dt\\ + \frac{1}{\sigma^2} p^\varepsilon(t) A^\varepsilon(t) dy^\varepsilon(t), \end{aligned} \tag{12}$$

with given initial condition

$$p^\varepsilon(0) = p_0 = (p_{0,11}, \ldots, p_{0,1m_1}, \ldots, p_{0,l1}, \ldots, p_{0,lm_l}).$$

III. Limit Filter and Two-Time-Scale Approximation

A. Limit Filter

Intuitively, similar to the probability distributions of two-time-scale Markov chains, as $\varepsilon \to 0$, the conditional probability in Wonham filter for $\alpha^\varepsilon(t)$ should converge to a limit filter. In this section, we first derive formally the limit filter and then provide a verification theorem that shows that the limit filter is indeed the limit of the original filter as $\varepsilon \to 0$ in an appropriate sense.

Write (12) in the its integral form and note that the boundedness of $p^\varepsilon(t)$ and $A^\varepsilon(t)$. It follows that

$$E\left|\int_0^t p^\varepsilon(u) Q^\varepsilon du\right|^2 \le K$$

for some finite K for all $\varepsilon > 0$. Therefore, we have

$$\begin{aligned} &\frac{1}{\varepsilon} E\left|\int_0^t p^\varepsilon(u)\widetilde{Q} du\right|^2\\ &\le 2E\left|\int_0^t p^\varepsilon(u) Q^\varepsilon du\right|^2 + 2E\left|\int_0^t p^\varepsilon(u)\widehat{Q} du\right|^2\\ &\le K. \end{aligned}$$

This implies that

$$E\left|\int_0^t p^\varepsilon(u)\widetilde{Q} du\right|^2 \le \varepsilon K.$$

Moreover, note that $p^\varepsilon_{ij}(t)$ are conditional probability measures that are uniformly bounded between 0 and 1. If $p^\varepsilon(t) \to p^0(t)$ as $\varepsilon \to 0$ for some $p^0(t)$ and $t > 0$, then necessarily

$$E\left|\int_0^t p^0(u)\widetilde{Q} du\right|^2 = 0, \text{ for } t > 0.$$

This implies $p^0(t)\widetilde{Q} = 0$. In view of the block-diagonal structure of $\widetilde{Q}$, the vector $p^0(t)$ must have the following form

$$p^0(t) = (\nu^1 \overline{p}_1(t), \ldots, \nu^l \overline{p}_l(t)) = \overline{p}(t)\nu,$$

where $\overline{p}(t) = (\overline{p}_1(t), \ldots, \overline{p}_l(t)) \in \mathbb{R}^{1\times l}$ is to be determined later. Recall the definition of $\widetilde{\mathbb{1}}$ in (7). It follows that

$$p^\varepsilon(t)\widetilde{\mathbb{1}} \to p^0(t)\widetilde{\mathbb{1}} = \overline{p}(t)(\nu\widetilde{\mathbb{1}}) = \overline{p}(t).$$

We next derive the equation for $\overline{p}(t)$. First, recall (8) and the convergence of $\overline{\alpha}^\varepsilon(\cdot) \to \overline{\alpha}(\cdot)$. Intuitively, we have

$$\begin{aligned} \int_0^t f(\alpha^\varepsilon(s)) ds = &\int_0^t \sum_{i,j} f(s_{ij}) I_{\{\alpha^\varepsilon(s) = s_{ij}\}} ds\\ \sim &\int_0^t \sum_{i=1}^{l} f(s_{ij}) \nu^i_j I_{\{\overline{\alpha}^\varepsilon(s) = i\}} ds\\ \sim &\int_0^t \overline{f}(i) I_{\{\overline{\alpha}(s) = i\}} ds\\ = &\int_0^t \overline{f}(\overline{\alpha}(s)) ds. \end{aligned}$$

Therefore, we expect the weak limit of $y^\varepsilon(\cdot)$ to have the form

$$dy(t) = \overline{f}(\overline{\alpha}(t)) dt + \sigma dw(t), \; y(0) = 0,$$

where

$$\overline{f}(i) = \sum_{j=1}^{m_i} f(s_{ij}) \nu^i_j.$$

The proof of this can be found in Wang, Zhang, and Yin [24].

Recall that $\widetilde{Q}\widetilde{\mathbb{1}} = 0$. In (12), multiplying from the right by $\widetilde{\mathbb{1}}$ and sending $\varepsilon \to 0$, we obtain

$$\begin{aligned} \overline{p}(t) = \overline{p}(0) + \int_0^t \overline{p}(u)\overline{Q} du - \frac{1}{\sigma^2}\int_0^t \widetilde{\alpha}(u)\overline{p}(u)\overline{A}(u) du\\ + \frac{1}{\sigma^2}\int_0^t \overline{p}(u)\overline{A}(u) dy(u), \end{aligned} \tag{13}$$

with initial condition

$$\overline{p}(0) = p_0\widetilde{\mathbb{1}} = (p_{0,11} + \cdots + p_{0,1m_1}, \ldots, p_{0,l1} + \cdots + p_{0,lm_l}) \in \mathbb{R}^{1\times l},$$

where

$$\widetilde{\alpha}(t) = \sum_{i=1}^{l} \overline{f}(i)\overline{p}_i(t),$$

and

$$\overline{A}(t) = \operatorname{diag}(\overline{f}(1), \ldots, \overline{f}(l)) - \widetilde{\alpha}(t) I. \tag{14}$$

We will show in what follows that, for each $t > 0$, $p^\varepsilon(t)$ can be approximated by $p^0(t) = \overline{p}(t)\nu$ in a two stage procedure as $\varepsilon \to 0$.

B. Two-Time-Scale Approximation

Note that the noise driving the limit filter is the weak limit of $y^\varepsilon(\cdot)$. In order to use the filter in real time applications, one needs to feed the filter by the actual observation $y^\varepsilon(\cdot)$ in (13). Let $\widetilde{p}^\varepsilon(t) = \overline{p}^\varepsilon(t)\nu$ denote such a filter with $\overline{p}^\varepsilon(t)$ given by

$$\begin{aligned}\overline{p}^\varepsilon(t) = \overline{p}^\varepsilon(0) + \int_0^t \overline{p}^\varepsilon(u)\overline{Q}du \\ -\frac{1}{\sigma^2}\int_0^t \widetilde{\alpha}^\varepsilon(u)\overline{p}^\varepsilon(u)\overline{A}^\varepsilon(u)du \\ +\frac{1}{\sigma^2}\int_0^t \overline{p}^\varepsilon(u)\overline{A}^\varepsilon(u)dy^\varepsilon(u),\end{aligned} \tag{15}$$

and $\overline{p}^\varepsilon(0) = p_0\widetilde{\mathbb{1}}$, where

$$\widetilde{\alpha}^\varepsilon(t) = \sum_{i=1}^{l} \overline{f}(i)\overline{p}_i^\varepsilon(t),$$

and

$$\overline{A}^\varepsilon(t) = \mathrm{diag}(\overline{f}(1),\ldots,\overline{f}(l)) - \widetilde{\alpha}^\varepsilon(t)I.$$

Then we have the following theorem.

Theorem 3.1:. The following assertions hold.

(a) $\widetilde{p}^\varepsilon(\cdot)$ is an approximation to $p^\varepsilon(\cdot)$ for small ε. More precisely,

$$E|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)|^2 = O\left(\varepsilon + e^{-\frac{\kappa t}{\varepsilon}}\right),$$

for some constant $\kappa > 0$.

(b) $\overline{p}^\varepsilon(\cdot)$ converges weakly to $\overline{p}(\cdot)$ in $C([0,T];\mathbb{R}^m)$, where $C([0,T];\mathbb{R}^m)$ denotes the space of $\mathbb{R}^m$-valued continuous functions defined on $[0,T]$.

Remark 3.2:. This theorem reveals that the two-stage approximation of $p^\varepsilon(t) \in \mathbb{R}^m$ leads to the limit $p^0(t) = \overline{p}(t)\nu$ with $\overline{p}(t) \in \mathbb{R}^l$. Stage 1 approximation provides a practical way for computing $p^\varepsilon(t)$ using $\overline{p}^\varepsilon(t)\nu$ that is governed by a system of SDEs of much smaller dimension. Stage 2 approximation leads to a theoretical weak limit for completeness of the two-time-scale analysis.

Remark 3.3:. The conditional probability vector $p^\varepsilon(t)$ does not converge in a neighborhood (of size $O(\varepsilon)$) of $t = 0$ due to an initial layer with thickness $O(\varepsilon)$ near the origin. Note that $p^\varepsilon(0)$ need not be the same as $p^0(0)$ and $p^\varepsilon(t)$ approaches $p^0(t)$ for $t > 0$ away from the initial layer. These observations are summarized in the next corollary.

Corollary 3.4:. The following assertions hold:

(a) $E\int_0^T |p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)|^2 dt = O(\varepsilon)$.

(b) For any $\delta > 0$, $\sup_{t\in[\delta,T]} E|p^\varepsilon(t) - \widetilde{p}^\varepsilon(t)|^2 = O(\varepsilon)$.

(c) For each $t > 0$, $p^\varepsilon(t) \to p^0(t)$ in distribution.

ε	0.5	0.1	0.05	0.01	0.005
$\|\|p^\varepsilon - \widetilde{p}^\varepsilon\|\|_T^2$	0.0335	0.0090	0.005	0.00117	0.00063

TABLE I
DEMONSTRATION OF ERROR BOUNDS.

IV. A NUMERICAL EXAMPLE

In this section, we consider a simple example involving a four state Markov chain. Let

$$Q^\varepsilon = \frac{1}{\varepsilon}\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The corresponding state space is $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 = \{s_{11}, s_{12}\} \cup \{s_{21}, s_{22}\}$. In this case, $\widetilde{Q}^1 = \widetilde{Q}^2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. The quasi-stationary distributions ν^1 and ν^2 (in fact, the stationary distributions) are given by $\nu^1 = \nu^2 = (1/2, 1/2)$ as solutions to $\nu^1\widetilde{Q}^1 = \nu^2\widetilde{Q}^2 = 0$. Note that the stationary distributions depend only on the block matrices in $\widetilde{Q}$.

Let $(p_{11}^\varepsilon(t), p_{12}^\varepsilon(t), p_{21}^\varepsilon(t), p_{22}^\varepsilon(t))$ denote the conditional probability vector and its approximation by $(\widetilde{p}_{11}^\varepsilon(t), \widetilde{p}_{12}^\varepsilon(t), \widetilde{p}_{21}^\varepsilon(t), \widetilde{p}_{22}^\varepsilon(t))$. Define the norm

$$\begin{aligned}&||p^\varepsilon(\cdot) - \widetilde{p}^\varepsilon(\cdot)||_T^2 \\ &= E\int_0^T\Big(|p_{11}^\varepsilon(t) - \widetilde{p}_{11}^\varepsilon(t)|^2 + |p_{12}^\varepsilon(t) - \widetilde{p}_{12}^\varepsilon(t)|^2 \\ &\quad + |p_{21}^\varepsilon(t) - \widetilde{p}_{21}^\varepsilon(t)|^2 + |p_{22}^\varepsilon(t) - \widetilde{p}_{22}^\varepsilon(t)|^2\Big)dt.\end{aligned}$$

In this example, we take

$$\begin{aligned} f(s_{11}) = 1,\ f(s_{12}) = 1.5, \\ f(s_{21}) = -1.5,\ f(s_{22}) = -1, \end{aligned}$$

$\sigma = 0.5$, $T = 5$ and the discretization step size $\delta = 0.0005$. A sample path of $\alpha^\varepsilon(\cdot)$ (with $\varepsilon = 0.05$) and the corresponding conditional probabilities are given in the first 5 rows in Figure 1. In Figure 1, the states are labelled as $1 = s_{11}$, $2 = s_{12}$, $3 = s_{21}$, and $4 = s_{22}$. The differences between $p^\varepsilon(\cdot)$ and $\widetilde{p}^\varepsilon(\cdot)$ are plotted in the last 4 rows. As can be seen in Figure 1, $\alpha^\varepsilon(\cdot)$ stays in group $\mathcal{M}_1$ from $t = 0.2$ to 1.5, jumps to group $\mathcal{M}_2$ at $t = 1.5$, goes back to $\mathcal{M}_1$ at $t = 3.4$, then to $\mathcal{M}_2$, and finally lands in $\mathcal{M}_1$ from $t = 4.4$ to 5. The approximation filter $\widetilde{p}_{ij}^\varepsilon(t)$ tracks the corresponding conditional probabilities $p^\varepsilon(t)$ pretty well on these time intervals.

In addition, we vary ε and run 1000 samples for each ε. The results are recorded in Table I. As can be seen in Table 1, the differences between the exact conditional probabilities and their approximations $\widetilde{p}^\varepsilon(\cdot)$ are fairly

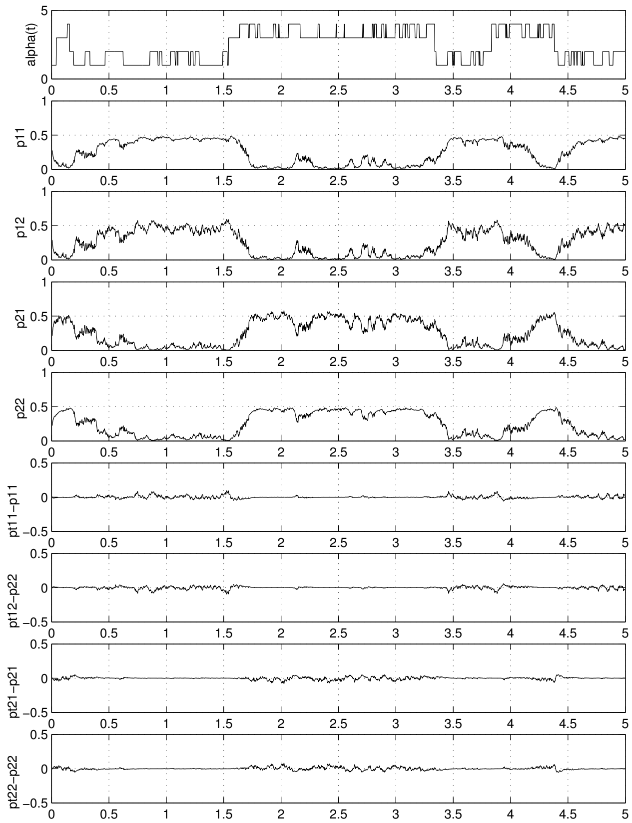

Fig. 1. Sample paths of $\alpha^\varepsilon(t)$, $p^\varepsilon(t)$, and $\tilde{p}^\varepsilon(t) - p^\varepsilon(t)$ with $\varepsilon = 0.05$.

small. This example validates the effectiveness of our approach.

References

[1] B.D.O. Anderson and J.B. Moore, Optimal Filtering, Prentice-Hall, 1978.

[2] A. Bensoussan, *Stochastic Control of Partially Observed Systems*, Cambridge University Press, Cambridge, 1992.

[3] P. Billingsley, *Convergence of Probability Measures*. J. Wiley, New York, 1968.

[4] P. E. Caines and H. F. Chen, Optimal adaptive LQG control for systems with finite state process parameters, *IEEE, Trans. Automat. Control*, AC-30, pp 185-189, 1985.

[5] J.M. Clark, The design of robust approximation to the stochastic differential equation for nonlinear filtering, in J.K. Skwirzynski Ed., *Communications Systems and Random Process Theory*, Alphen aan den Rijn, the Netherlands, 1978, 721-734.

[6] S. Dey and J.B. Moore, Risk-sensitive filtering and smoothing for hidden Markov models, *Systems and Control Letters*, Vol 25, no 5, pp 361-366, 1995.

[7] S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence*, J. Wiley, New York, NY, 1986.

[8] R.J. Elliott, L. Aggoun, and J.B. Moore, *Hidden Markov Models: Estimation and Control*, Springer-Verlage, New York, NY, 1995.

[9] R.J. Elliott, W.P. Malcolm, and J.B. Moore, Robust dynamics and control of a partially observed Markov chain,

[10] J. Hale, *Ordinary Differential Equations*, 2nd Ed., R.E. Krieger Pub. Co., Malabar, FL, 1980.

[11] U.G. Haussmann and Q. Zhang, Stochastic adaptive control with small observation noise, *Stochastics Stochastics Rep.*, Vol. 32, pp. 109-144, 1990.

[12] O. Hijab, The adaptive LQG problem -Part I, *IEEE, Trans. Automat. Control*, AC-28, pp. 171-178, (1983).

[13] W. Huisinga, S.P. Meyn, and C. Schutte, Phase transitions and metastability in Markovian and molecular systems, *Ann. Appl. Probability*, Vol. 14, pp. 419-458, 2004.

[14] M. Jerrum, On the approximation of one Markov chain by another, Working paper, School of Informatics, University of Edinburgh, 2006.

[15] R.S. Lipster and A.N. Shiryayev, *Statistics of Random Processes*, I and II, 2nd Ed., Springer, Berlin, 2001.

[16] G. Kallianpur, *Stochastic filtering theory*, Springer-Verlag, New York, 1980.

[17] R.Z. Khasminskii, G. Yin and Q. Zhang, Asymptotic expansions of singularly perturbed systems involving rapidly fluctuating Markov chains, *SIAM J. Appl. Math.*, **56** (1996), 277-293.

[18] H.J. Kushner, On the differential equations satisfied by conditional probability densities of Markov processes with applications, *SIAM J. Control*, **2** (1964), 106-119.

[19] H.J. Kushner, *Approximation and Weak Convergence Methods for Random Processes with Applications to Stochastic Systems Theory*, MIT Press, Cambridge, MA, 1984.

[20] R.H. Liu and Q. Zhang, Nonlinear filtering: A hybrid approximation scheme, *IEEE Trans. Aerospace Electronic Systems*, 37, pp. 470-480, (2001).

[21] J.B. Moore and J.S. Baras, Finite-dimensional optimal controllers for nonlinear plants, *Systems and Control Letters*, Vol 26, pp 223-230, 1995.

[22] S.P. Sethi and Q. Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems*, in Systems & Control: Foundations & Applications series, Birkhäuser, Boston, 1994.

[23] R.L. Tweedie, Quasi-stationary distributions for Markov chains on a general state space, *J. Appl. Probab.*, Vol. 11, pp. 726-741, 1974.

[24] J.W. Wang, Q. Zhang, and G. Yin, Two-time-scale hybrid filters: Near optimality, *SIAM J. Control Optim.*, Vol. 45, pp. 298-319, 2006.

[25] W.M. Wonham, Some applications of stochastic differential equations to optimal nonlinear filtering, *SIAM J. Control*, Vol. 2, 347-369, (1965).

[26] G. Yin and S. Dey, Weak convergence of hybrid filtering problems involving nearly completely decomposable hidden Markov chains, *SIAM J. Control Optim.*, Vol. 41, 1820–1842. (2003).

[27] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, Springer-Verlag, New York, NY, 1998.

[28] G. Yin, Q. Zhang, and G. Badowski, Asymptotic properties of a singularly perturbed Markov chain with inclusion of transient states, *Ann. Appl. Probab.*, Vol. 10, pp. 549–572, (2000).

[29] G. Yin, Q. Zhang, and Y.J. Liu, Discrete-time approximation of Wonham filters, *Journal of Control Theory and Applications*, **2** (2004), 1-10.

[30] Q. Zhang, Nonlinear filtering and control of a switching diffusion with small observation noise, *SIAM J. Control Optim.*, Vol. 36, pp. 1738-1768, (1998).

[31] Q. Zhang, Optimal filtering of discrete-time hybrid systems, *J. Optim. Theory Appl.*, Vol. 100, pp. 123-144, (1999).

[32] Q. Zhang, Hybrid filtering for linear systems with non-Gaussian disturbances, *IEEE Trans. Automat. Control*, Vol. 45, pp. 50-61, (2000).

[33] Q. Zhang and G. Yin, Nearly optimal asset allocation in hybrid stock-investment models, *Journal of Optimization Theory and Applications*, Vol. 121, pp. 419-444, 2004.

[34] Q. Zhang, G. Yin, and J.B. Moore, Two-time scale approximation for Wonham filters, *IEEE Trans. Information Theory*, Vol. 53, pp. 1706-1715, (2007).