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Two Upper Bounds for the Degree Distances of Four Sums of Graphs

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Abstract. The degree distance (DD), which is a weight version of the Wiener index, defined for a connected graph G as vertex-degree-weighted sum of the distances, that is, $DD(G) = \sum_{\{u,v\}\subseteq V(G)}[d_G(u) + d_G(v)]d(u, v|G)$, where $d_G(u)$ denotes the degree of a vertex u in G and $d(u, v|G)$ denotes the distance between two vertices u and v in G. In this paper, we establish two upper bounds for the degree distances of four sums of two graphs in terms of other indices of two individual graphs.

1. Introduction

All graphs considered in this paper are simple and connected. Let $G = (V, E)$ be a graph with vertex set V and edge set E. Let $d_G(v)$ be the degree of a vertex v in G and $d(u, v|G)$ be the distance between two vertices u and v in G.

One of the oldest and well-studied distance-based graph invariants associated with a connected graph G is the Wiener number $W(G)$, also termed as *Wiener index* in chemical or mathematical chemistry literature, which is defined [21] as the sum of distances over all unordered vertex pairs in G, namely,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v|G).$$

This equation was introduced by Haruo Hosoya [12], although the concept has been introduced by late Harry Wiener. However, the approach by Wiener is applicable only to acyclic structures, whilst Hosoya matrix definition allowed the Wiener index to be used for any structure.

In 1994, Dobrynin and Kochetova [6] and Gutman [10] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance* or *Schultz molecular topological index*, which is defined for a connected graph G as

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} [d_G(u) + d_G(v)]d(u, v|G).$$

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The interested readers may consult [5, 9, 11] for Wiener index and [2, 4, 7, 13, 16-20] for degree distance. The relation between the degree distance and the Wiener index was investigated in [10]. For other undefined terminology and notations from graph theory, the readers are referred to [1].

Wiener indices, hyper-Wiener indices and reverse Wiener indices of four sums of two graphs were computed in [8, 15], respectively. Vertex PI indices of four sums of two graphs were calculated in [14]. In this paper, we continue this program to compute the degree distances of four sums of two graphs and two upper bounds for them in terms of other indices of two individual graphs are given.

2. Preliminaries

We first recall some graph operations, see Fig. 1. More details on them may be found in [3].

For a graph $G = (V, E)$, we refer to each vertex of V as a *black vertex*. Denote by $S(G)$ the graph obtained from G by inserting an additional vertex which is referred to as the *white vertex* in each edge of G. Two black vertices in $S(G)$ are *related* if they are adjacent in G; and two white vertices in $S(G)$ are *related* if their corresponding edges in G are adjacent. Denote by $R(G)$ and $Q(G)$ the graphs obtained from $S(G)$ by joining every pair of related black vertices and every pair of related white vertices, respectively. Suppose that graphs X and Y have the same vertex set V, then their *union* is the graph $X \cup Y$ with vertex set V and edge set $E(X) \cup E(Y)$; in particular, we denote by $T(G)$ the union of $R(G)$ and $Q(G)$.

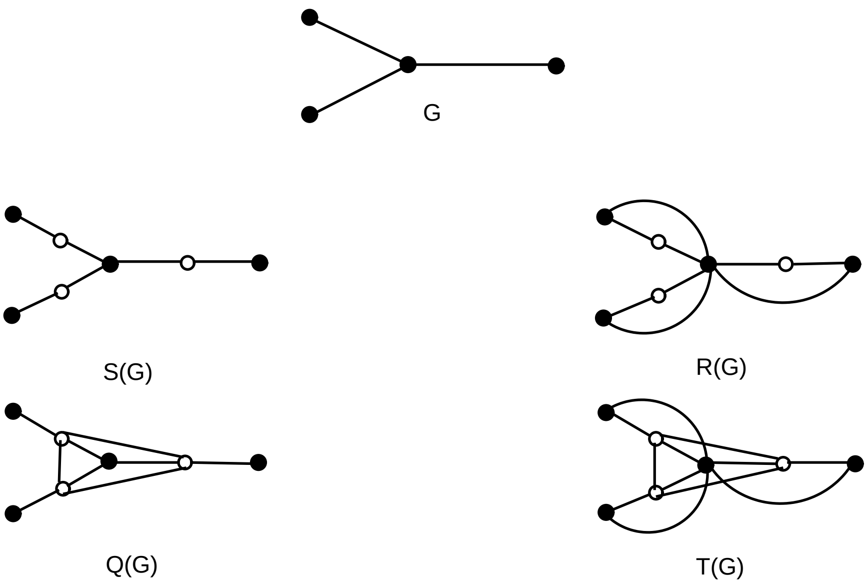

Fig. 1. A graph G and $S(G), R(G), Q(G)$ and $T(G)$.

If G is a graph, then the line graph of G, denoted by $L(G)$, is the graph with $E(G)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Let G_1 and G_2 be two graphs. For convenience, throughout the paper we denote $V(G_i)$ and $E(G_i)$ by V_i and $E_i, i = 1, 2$, respectively.

Next we present the definition of F-sum.

Let F be one of the symbols S, R, Q or T. We denote by $G_1 +_F G_2$ the *F-sum* of G_1 and G_2 for which the set of vertices $V(G_1 +_F G_2) = (V_1 \cup E_1) \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $u_1 = v_1 \in V_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E(F(G_1))$.

Note that $G_1 +_F G_2$ has $|V_2|$ copies of the graph $F(G_1)$, and we may label these copies by vertices of G_2.

The vertices in each copy have two situations: the vertices in V_1 which are still referred to as black vertices and the vertices in E_1 which are still referred to as white vertices. Now we join only black vertices with the same name in $F(G_1)$ in which their corresponding labels are adjacent in G_2.

Moreover, we state three lemmas which are proved in [8] and will be used repeatedly in the proofs of our main results.

Lemma 2.1 (*[8]*). Let G_1 and G_2 be two graphs and $v = (v_1, v_2)$ be a vertex of $G_1 +_F G_2$. Then:
(a) If $v_1 \in V_1$ (that is v is a black vertex), then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$ we have

$$d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2).$$

(b) If $v_1 \in E_1$, then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$ with $u_2 \neq v_2, u_1 = u_1^1 v_1^1 \in E_1$ and $u_1^1, v_1^1 \in V_1$ (that is v and u are white vertices in different copies of $F(G_1)$), we have

$$d(u, v|G_1 +_F G_2) = 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, v_1|F(G_1), d(v_1^1, v_1|F(G_1))\}.$$

(c) If $v_1 \in E_1$, then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$, where $u_2 = v_2$ and $u_1 \in E_1$ (that is v and u are white vertices in the same copy of $F(G_1)$), we have

$$d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)).$$

Lemma 2.2 (*[8]*). Let G_1 and G_2 be two graphs, $u_1, v_1 \in E_1, u_2, v_2 \in V_2$ and $F = S$ or R. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $G_1 +_F G_2$ with $u_2 \neq v_2$, we have

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}$$

Lemma 2.3 (*[8]*). Let G_1 and G_2 be two graphs, $u_1, v_1 \in E_1, u_2, v_2 \in V_2$ and $F = Q$ or T. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $G_1 +_F G_2$ with $u_2 \neq v_2$, we have

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ 1 + d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}$$

The following two lemmas, which can be easily deduced from the definitions of F-sum and graph operations, respectively, are also crucial in the proofs of our main results.

Lemma 2.4. Let G_1 and G_2 be two graphs and $u = (u_1, u_2)$ be a vertex of $G_1 +_F G_2$. Then:
(a) If $u_1 \in V_1$ and $u_2 \in V_2$ (that is u is a black vertex), then we have

$$d_{G_1 +_F G_2}(u) = d_{F(G_1)}(u_1) + d_{G_2}(u_2).$$

(b) If $u_1 \in E_1$ and $u_2 \in V_2$ (that is u is a white vertex), then we have

$$d_{G_1 +_F G_2}(u) = d_{F(G_1)}(u_1).$$

Lemma 2.5. Let G be a graph. Then:
(a) If $u_1 \in V(G)$, then we have

$$d_{F(G)}(u_1) = k \cdot d_G(u_1),$$

where

$$k = \begin{cases} 1 & \text{if } F = S \text{ or } Q \\ 2 & \text{if } F = R \text{ or } T. \end{cases} \quad (1)$$

(b) If $u_1 = u_1'u_1'' \in E(G)$, then we have

$$d_{S(G)}(u_1) = d_{R(G)}(u_1) = 2;$$

$$d_{Q(G)}(u_1) = d_{T(G)}(u_1) = d_{L(G)}(u_1) + 2;$$

where $d_{L(G)}(u_1) = d_G(u_1') + d_G(u_1'') - 2$.

3. Main results

In this section, we give two upper bounds for the degree distance of $G_1 +_F G_2$. First, we present an upper bound for the degree distance of $G_1 +_F G_2$ in terms of degree distances of $F(G_1)$ and G_2, where $F = R$ or S.

Theorem 3.1. Let G_1 and G_2 be two graphs and $F = S$ or R. Let $\Delta(G_2)$ be the maximum degree of G_2. Then

$$\begin{aligned}DD(G_1 +_F G_2) \leq& |V_1|^2 DD(G_2) + |V_2|^2 DD(F(G_1)) + 4|E_1|(|V_2|^2 - |V_2|) + 4(k+1)|E_1|(|E_1| + |V_1|)\cdot \\ & W(G_2) + 4|E_2||V_2|W(F(G_1)) + 2|E_1||V_1|\Delta(G_2)W(G_2),\end{aligned}$$

where k is defined in Eq. (1).

Proof. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vertices in $G_1 +_F G_2$. According to the colors of u and v we must consider the following three cases:

Case 1. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are black, that is $u, v \in V_1 \times V_2$. By Lemma 2.1(a),

$$d((u_1, u_2), (v_1, v_2)|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2).$$

Therefore, by Lemma 2.4, the vertex-degree-weighted summation of distances between black vertices is

$$\begin{aligned}A :=& \frac{1}{2}\sum \{[d_{G_1+_FG_2}(u) + d_{G_1+_FG_2}(v)]d((u_1, u_2), (v_1, v_2)|G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in V_1 \times V_1\} \\ =& \frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)} \left[d_{F(G_1)}(u_1) + d_{G_2}(u_2) + d_{F(G_1)}(v_1) + d_{G_2}(v_2)\right][d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2)] \\ =& \frac{1}{2}\sum_{u_2,v_2\in V_2}\sum_{u_1,v_1\in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)\right] d(u_1, v_1|F(G_1)) \\ &+ \frac{1}{2}\sum_{u_2,v_2\in V_2}\sum_{u_1,v_1\in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)\right] d(u_2, v_2|G_2) \\ &+ \frac{1}{2}\sum_{u_2,v_2\in V_2}\sum_{u_1,v_1\in V_1} [d_{G_2}(u_2) + d_{G_2}(v_2)]\, d(u_1, v_1|F(G_1)) \\ &+ \frac{1}{2}\sum_{u_1,v_1\in V_1}\sum_{u_2,v_2\in V_2} [d_{G_2}(u_2) + d_{G_2}(v_2)]\, d(u_2, v_2|G_2)).\end{aligned}$$

In what follows, each summation of A is computed, separately.

$$\begin{aligned}A_1 :=& \frac{1}{2}\sum_{u_2,v_2\in V_2}\sum_{u_1,v_1\in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)\right] d(u_1, v_1|F(G_1)) \\ =& \frac{1}{2}|V_2|^2 \sum_{u_1,v_1\in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)\right] d(u_1, v_1|F(G_1));\end{aligned}$$

by Lemma 2.5,

$$\begin{aligned} A_2 :=& \frac{1}{2} \sum_{u_2,v_2 \in V_2} \sum_{u_1,v_1 \in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1) \right] d(u_2, v_2 | G_2) \\ =& kW(G_2) \sum_{u_1,v_1 \in V_1} \left[d_{G_1}(u_1) + d_{G_1}(v_1) \right] \\ =& 4k|E_1||V_1|W(G_2), \end{aligned}$$

where k is illustrated in Eq. (1); and

$$\begin{aligned} A_3 :=& \frac{1}{2} \sum_{u_2,v_2 \in V_2} \left[d_{G_2}(u_2) + d_{G_2}(v_2) \right] \sum_{u_1,v_1 \in V_1} d(u_1, v_1 | F(G_1)) \\ =& 2|E_2||V_2| \sum_{u_1,v_1 \in V_1} d(u_1, v_1 | F(G_1)); \end{aligned}$$

$$\begin{aligned} A_4 :=& \frac{1}{2} \sum_{u_1,v_1 \in V_1} \sum_{u_2,v_2 \in V_2} \left[d_{G_2}(u_2) + d_{G_2}(v_2) \right] d(u_2, v_2 | G_2)) \\ =& |V_1|^2 DD(G_2). \end{aligned}$$

Thus,

$$\begin{aligned} A =& A_1 + A_2 + A_3 + A_4 \\ =& \frac{1}{2} |V_2|^2 \sum_{u_1,v_1 \in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1) \right] d(u_1, v_1 | F(G_1)) + 4k|E_1||V_1|W(G_2) \\ & + 2|E_2||V_2| \sum_{u_1,v_1 \in V_1} d(u_1, v_1 | F(G_1)) + |V_1|^2 DD(G_2). \end{aligned}$$

Case 2. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ have different colors, that is $u \in E_1 \times V_2$ and $v \in V_1 \times V_2$ or $u \in V_1 \times V_2$ and $v \in E_1 \times V_2$. In this case, by Lemma 2.1(a),

$$d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) = d(u_1, v_1 | F(G_1)) + d(u_2, v_2 | G_2).$$

Therefore, by Lemma 2.4, the vertex-degree-weighted summation of distances between vertices u and v, where u is black and v is white, is

$$\begin{aligned} B' :=& \frac{1}{2} \sum \{ [d_{G_1 +_F G_2}(u) + d_{G_1 +_F G_2}(v)] d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2) \in V_1 \times V_2, \\ & (v_1, v_2) \in E_1 \times V_2 \} \\ =& \frac{1}{2} \sum_{u_2,v_2 \in V_2} \sum_{v_1 \in E_1} \sum_{u_1 \in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1) \right] d(u_1, v_1 | F(G_1)) \\ & + \frac{1}{2} \sum_{u_2,v_2 \in V_2} \sum_{v_1 \in E_1} \sum_{u_1 \in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1) \right] d(u_2, v_2 | G_2) \\ & + \frac{1}{2} \sum_{u_2,v_2 \in V_2} \sum_{v_1 \in E_1} \sum_{u_1 \in V_1} d_{G_2}(u_2) d(u_1, v_1 | F(G_1)) \\ & + \frac{1}{2} \sum_{u_2,v_2 \in V_2} \sum_{v_1 \in E_1} \sum_{u_1 \in V_1} d_{G_2}(u_2) d(u_2, v_2 | G_2). \end{aligned}$$

In what follows, each summation of B' is computed, separately.

$$\begin{aligned}B'_1 :=&\frac{1}{2}\sum_{u_2,v_2\in V_2}\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]d(u_1,v_1|F(G_1))\\ =&\frac{1}{2}|V_2|^2\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]d(u_1,v_1|F(G_1));\\ B'_2 :=&\frac{1}{2}\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]\sum_{u_2,v_2\in V_2}d(u_2,v_2|G_2)\\ =&W(G_2)\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right];\\ B'_3 :=&\frac{1}{2}\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)\sum_{v_1\in E_1}\sum_{u_1\in V_1}d(u_1,v_1|F(G_1))\\ =&|V_2||E_2|\sum_{v_1\in E_1}\sum_{u_1\in V_1}d(u_1,v_1|F(G_1));\\ B'_4 :=&\frac{1}{2}\sum_{v_1\in E_1}\sum_{u_1\in V_1}\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)d(u_2,v_2|G_2))\\ =&\frac{1}{2}|E_1||V_1|\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)d(u_2,v_2|G_2)).\end{aligned}$$

Thus, the vertex-degree-weighted summation of distances between vertices with different colors is:

$$\begin{aligned}B =&2B' = 2(B'_1+B'_2+B'_3+B'_4)\\ =&|V_2|^2\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]d(u_1,v_1|F(G_1))+2W(G_2)\sum_{v_1\in E_1}\sum_{u_1\in V_1}[d_{F(G_1)}(u_1)\\ &+d_{F(G_1)}(v_1)]+2|V_2||E_2|\sum_{v_1\in E_1}\sum_{u_1\in V_1}d(u_1,v_1|F(G_1))+|E_1||V_1|\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)d(u_2,v_2|G_2)).\end{aligned}$$

Case 3. Suppose that $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are white, that is $u\in E_1\times V_2$ and $v\in E_1\times V_2$. Let

$$C :=\frac{1}{2}\sum\left\{[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)]d((u_1,u_2),(v_1,v_2)|G_1+_F G_2):(u_1,u_2),(v_1,v_2)\in E_1\times V_2\right\}.$$

We break down this summation into two sums $C=C_1+C_2$, where

$$\begin{aligned}C_1 :=&\ \frac{1}{2}\sum\{[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)]d((u_1,u_2),(v_1,v_2)|G_1+_F G_2):(u_1,u_2),(v_1,v_2)\in E_1\times V_2,\\ &u_1=v_1,u_2\neq v_2\};\\ C_2 :=&\ \frac{1}{2}\sum\{[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)]d((u_1,u_2),(v_1,v_2)|G_1+_F G_2):(u_1,u_2),(v_1,v_2)\in E_1\times V_2,\\ &u_1\neq v_1\}.\end{aligned}$$

By Lemmas 2.2, 2.4 and 2.5, we have

$$\begin{aligned}C_1 =&\frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)][2+d(u_2,v_2|G_2)]\\ =&\frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)][2+d(u_2,v_2|G_2)]\\ =&\frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[2+2][2+d(u_2,v_2|G_2)]\\ =&4|E_1|(|V_2|^2-|V_2|)+4|E_1|W(G_2);\end{aligned}$$

and

$$\begin{aligned}
C_2 =& \frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2}[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)][d(u_1,v_1|F(G_1))+d(u_2,v_2|G_2)]\\
=& \frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2}[[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]]d(u_1,v_1|F(G_1))\\
&+\frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]d(u_2,v_2|G_2)\\
=& \frac{1}{2}|V_2|^2\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)]d(u_1,v_1|F(G_1))\\
&+4(|E_1|^2-|E_1|)W(G_2).
\end{aligned}$$

So,

$$\begin{aligned}
C =& C_1+C_2\\
=& \frac{1}{2}|V_2|^2\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]d(u_1,v_1|F(G_1))+4|E_1|(|V_2|^2-|V_2|)\\
&+4|E_1|^2W(G_2).
\end{aligned}$$

Therefore, by the above computation and the definition of degree distance,

$$\begin{aligned}
DD(G_1+_FG_2) =& A+B+C\\
=& |V_1|^2DD(G_2)+4|E_1|(|V_2|^2-|V_2|)+4|E_1|(|E_1|+k|V_1|)W(G_2)\\
&+\frac{|V_2|^2}{2}[\sum_{u_1,v_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]d(u_1,v_1|F(G_1))\\
&+2\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]d(u_1,v_1|F(G_1))\\
&+\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)]d(u_1,v_1|F(G_1))]\\
&+2|E_2||V_2[\sum_{u_1,v_1\in V_1}d(u_1,v_1|F(G_1))+\sum_{v_1\in E_1}\sum_{u_1\in V_1}d(u_1,v_1|F(G_1))]\\
&+2W(G_2)\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]\\
&+|E_1||V_1|\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)d(u_2,v_2|G_2)
\end{aligned}$$

$$\begin{aligned}
=& |V_1|^2DD(G_2)+4|E_1|(|V_2|^2-|V_2|)+4|E_1|(|E_1|+k|V_1|)W(G_2)\\
&+|V_2|^2DD(F(G_1))+2W(G_2)\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]\\
&+2|E_2||V_2[\sum_{u_1,v_1\in V_1}d(u_1,v_1|F(G_1))+\sum_{v_1\in E_1}\sum_{u_1\in V_1}d(u_1,v_1|F(G_1))]\\
&+|E_1||V_1|\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)d(u_2,v_2|G_2). \qquad (2)
\end{aligned}$$

On the other hand, by Lemma 2.5,

$$\sum_{v_1\in E_1}\sum_{u_1\in V_1}\left[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\right]=\sum_{v_1\in E_1}\sum_{u_1\in V_1}[k\cdot d_{G_1}(u_1)+2]$$
$$=2k|E_1|^2+2|E_1||V_1|; \tag{3}$$

by the definition of Wiener index,

$$\sum_{u_1,v_1\in V_1}d(u_1,v_1|F(G_1))+\sum_{v_1\in E_1}\sum_{u_1\in V_1}d(u_1,v_1|F(G_1))=2W(F(G_1))-\sum_{u_1,v_1\in E_1;u_1\neq v_1}d(u_1,v_1|F(G_1))$$
$$-\sum_{v_1\in V_1}\sum_{u_1\in E_1}d(u_1,v_1|F(G_1))$$
$$\leq 2W(F(G_1)); \tag{4}$$

clearly,

$$\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)d(u_2,v_2|G_2)\leq\Delta(G_2)\sum_{u_2,v_2\in V_2}d(u_2,v_2|G_2)$$
$$=2\Delta(G_2)W(G_2). \tag{5}$$

Thus, combining Eq. (3) and inequalities (4) and (5) with Eq. (2), we obtain

$$\begin{aligned}DD(G_1+_F G_2)=&|V_1|^2DD(G_2)+|V_2|^2DD(F(G_1))+4|E_1|(|V_2|^2-|V_2|)+4(k+1)|E_1|(|E_1|+|V_1|)\cdot\\&W(G_2)+2|E_2||V_2[\sum_{u_1,v_1\in V_1}d(u_1,v_1|F(G_1))+\sum_{v_1\in E_1}\sum_{u_1\in V_1}d(u_1,v_1|F(G_1))]\\&+|E_1||V_1|\sum_{u_2,v_2\in V_2}d_{G_2}(u_2)d(u_2,v_2|G_2)\\\leq&|V_1|^2DD(G_2)+|V_2|^2DD(F(G_1))+4|E_1|(|V_2|^2-|V_2|)+4(k+1)|E_1|(|E_1|+|V_1|)\cdot\\&W(G_2)+4|E_2||V_2|W(F(G_1))+2|E_1||V_1|\Delta(G_2)W(G_2).\end{aligned}$$

This completes the proof. ■

Now we give the other upper bound for the degree distance of $G_1+_F G_2$ in terms of degree distances of $F(G_1)$ and G_2, where $F=Q$ or T.

Theorem 3.2. Let G_1 and G_2 be two graphs and $F=Q$ or T. Let $\Delta(G_2)$ be the maximum degree of G_2. Then

$$\begin{aligned}DD(G_1+_F G_2)\leq&|V_1|^2DD(G_2)+|V_2|^2DD(F(G_1))+2(|E_1|+|V_1|)M_1(G_1)W(G_2)+4k|E_1|(|E_1|\\&+|V_1|)W(G_2)+(|E_1+1|)(|V_2|^2-|V_2|)M_1(G_1)+4|E_2||V_2|W(F(G_1))\\&+2|E_1||V_1|\Delta(G_2)W(G_2),\end{aligned}$$

where k is defined in Eq. (1).

Proof. Let A,B and C be as in the proof of the Theorem 3.1. The values of A and B do not change here. So we must only calculate the value of C. Let

$$C:=\frac{1}{2}\sum\{[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)]d((u_1,u_2),(v_1,v_2)|G_1+_F G_2):(u_1,u_2),(v_1,v_2)\in E_1\times V_2\}.$$

We break down this summation into three sums $C=C_1+C_2+C_3$, where

$$C_1:=\frac{1}{2}\sum\{[d_{G_1+_FG_2}(u)+d_{G_1+_FG_2}(v)]d((u_1,u_2),(v_1,v_2)|G_1+_F G_2):(u_1,u_2),(v_1,v_2)\in E_1\times V_2,$$
$$u_1=v_1,u_2\neq v_2\};$$

$$C_2 := \frac{1}{2}\sum\{[d_{G_1+_FG_2}(u) + d_{G_1+_FG_2}(v)]d((u_1,u_2),(v_1,v_2)|G_1 +_F G_2) : (u_1,u_2),(v_1,v_2) \in E_1 \times V_2,\ u_1 \neq v_1, u_2 = v_2\};$$

$$C_3 := \frac{1}{2}\sum\{[d_{G_1+_FG_2}(u) + d_{G_1+_FG_2}(v)]d((u_1,u_2),(v_1,v_2)|G_1 +_F G_2) : (u_1,u_2),(v_1,v_2) \in E_1 \times V_2,\ u_1 \neq v_1, u_2 \neq v_2\}.$$

By Lemmas 2.3, 2.4 and 2.5, we have

$$\begin{aligned} C_1 =& \frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{G_1+_FG_2}(u) + d_{G_1+_FG_2}(v)][2 + d(u_2,v_2|G_2)] \\ =& \frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{L(G_1)}(u_1) + d_{L(G_1)}(v_1) + 4][2 + d(u_2,v_2|G_2)] \\ =& \frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{L(G_1)}(u_1) + d_{L(G_1)}(v_1)]d(u_2,v_2|G_2) + 4\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}1 \\ &+ \sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{L(G_1)}(u_1) + d_{L(G_1)}(v_1)] + 2\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}d(u_2,v_2|G_2) \\ =& 2[W(G_2) + |V_2|^2 - |V_2|]\sum_{u_1\in E_1}d_{L(G_1)}(u_1) + 4|E_1|(|V_2|^2 - |V_2|) + 4|E_1|W(G_2). \end{aligned} \tag{6}$$

It is easy to know that:

$$\begin{aligned} \sum_{u_1\in E_1}d_{L(G_1)}(u_1) =& \sum_{u_1=u_1'u_1''\in E_1}[d_{G_1}(u_1') + d_{G_1}(u_1'') - 2] \\ =& M_1(G_1) - 2|E_1|. \end{aligned} \tag{7}$$

By putting Eq. (7) into Eq. (6), we obtain

$$\begin{aligned} C_1 =& 2[W(G_2) + |V_2|^2 - |V_2|][M_1(G_1) - 2|E_1|] + 4|E_1|(|V_2|^2 - |V_2|) + 4|E_1|W(G_2) \\ =& 2M_1(G_1)W(G_2) + 2(|V_2|^2 - |V_2|)M_1(G_1). \end{aligned}$$

Also by Lemmas 2.3 and 2.4,

$$\begin{aligned} C_2 =& \frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2\in V_2}[d_{G_1+_FG_2}(u) + d_{G_1+_FG_2}(v)]d(u_1,v_1|F(G_1)) \\ =& \frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2\in V_2}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)]d(u_1,v_1|F(G_1)) \\ =& \frac{1}{2}|V_2|\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)]d(u_1,v_1|F(G_1)), \end{aligned}$$

and

$$\begin{aligned} C_3 =& \frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)][1 + d(u_1,v_1|F(G_1)) + d(u_2,v_2|G_2)] \\ =& \frac{1}{2}(|V_2|^2 - |V_2|)\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)] \\ &+ \frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)]d(u_1,v_1|F(G_1)) \\ &+ \frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)]d(u_2,v_2|G_2). \end{aligned}$$

In what follows, each part of C_3 is calculated, separately. We first compute the following summation which will be used later.

$$\begin{aligned}\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{L(G_1)}(u_1)+d_{L(G_1)}(v_1)] &= \sum_{u_1,v_1\in E_1}[d_{L(G_1)}(u_1)+d_{L(G_1)}(v_1)] - \sum_{u_1,v_1\in E_1;u_1=v_1}[d_{L(G_1)}(u_1)+d_{L(G_1)}(v_1)]\\ &=2(|E_1|-1)\sum_{u_1\in E_1}d_{L(G_1)}(u_1)\\ &=2(|E_1|-1)(M_1(G_1)-2|E_1|).\end{aligned}$$

Then by Lemma 2.5,

$$\begin{aligned}C_3' :=& \sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]\\ =& \sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{L(G_1)}(u_1)+d_{L(G_1)}(v_1)+4]\\ =& \sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{L(G_1)}(u_1)+d_{L(G_1)}(v_1)]+4\sum_{u_1,v_1\in E_1;u_1\neq v_1}1\\ =&2(|E_1|-1)(M_1(G_1)-2|E_1|)+4(|E_1|^2-|E_1|)\\ =&2(|E_1|-1)M_1(G_1);\end{aligned}$$

obviously

$$\begin{aligned}C_3'' :=&\frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\ \sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]d(u_1,v_1|F(G_1))\\ =&\frac{1}{2}(|V_2|^2-|V_2|)\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]d(u_1,v_1|F(G_1));\end{aligned}$$

and

$$\begin{aligned}C_3''' :=&\frac{1}{2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}\ \sum_{u_2,v_2\in V_2;u_2\neq v_2}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]d(u_2,v_2|G_2)\\ =&W(G_2)\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]\\ =&W(G_2)\cdot 2(|E_1|-1)M_1(G_1)\\ =&2(|E_1|-1)M_1(G_1)W(G_2).\end{aligned}$$

Thus, the value of C_3 is obtained:

$$\begin{aligned}C_3 =&\frac{1}{2}(|V_2|^2-|V_2|)C_3'+C_3''+C_3'''\\ =&(|V_2|^2-|V_2|)(|E_1|-1)M_1(G_1)+2(|E_1|-1)M_1(G_1)W(G_2)\\ &+\frac{1}{2}(|V_2|^2-|V_2|)\sum_{u_1,v_1\in E_1;u_1\neq v_1}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]d(u_1,v_1|F(G_1)).\end{aligned}$$

Therefore, we finally obtain the value of C as follows:

$$
\begin{aligned}
C =& C_1 + C_2 + C_3 \\
=& 2M_1(G_1)W(G_2) + 2(|V_2|^2 - |V_2|)M_1(G_1) \\
&+ \frac{1}{2}|V_2|^2 \sum_{u_1,v_1\in E_1; u_1\neq v_1} [d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)]d(u_1, v_1|F(G_1)) \\
&+ (|V_2|^2 - |V_2|)(|E_1 - 1|)M_1(G_1) + 2(|E_1 - 1|)M_1(G_1)W(G_2) \\
=& \frac{1}{2}|V_2|^2 \sum_{u_1,v_1\in E_1; u_1\neq v_1} [d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)]d(u_1, v_1|F(G_1)) \\
&+ 2|E_1|M_1(G_1)W(G_2) + (|E_1| + 1)(|V_2|^2 - |V_2|)M_1(G_1).
\end{aligned}
$$

Hence, by the above calculation and the definition of degree distance,

$$
\begin{aligned}
DD(G_1 +_F G_2) =& A + B + C \\
=& |V_1|^2 DD(G_2) + 4k|E_1||V_1|W(G_2) + (|E_1| + 1)(|V_2|^2 - |V_2|)M_1(G_1) \\
&+ 2|E_1|M_1(G_1)W(G_2) + |E_1||V_1| \sum_{u_2,v_2\in V_2} d_{G_2}(u_2)d(u_2, v_2|G_2) \\
&+ \frac{|V_2|^2}{2}[\sum_{u_1,v_1\in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)\right] d(u_1, v_1|F(G_1)) \\
&+ 2\sum_{v_1\in E_1}\sum_{u_1\in V_1} \left[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)\right] d(u_1, v_1|F(G_1)) \\
&+ \sum_{u_1,v_1\in E_1, u_1\neq v_1} [d_{G_1+_F G_2}(u) + d_{G_1+_F G_2}(v)]d(u_1, v_1|F(G_1))] \\
&+ 2|E_2||V_2[\sum_{u_1,v_1\in V_1} d(u_1, v_1|F(G_1)) + \sum_{v_1\in E_1}\sum_{u_1\in V_1} d(u_1, v_1|F(G_1))] \\
&+ 2W(G_2)\sum_{v_1\in E_1}\sum_{u_1\in V_1}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)] \\
=& |V_1|^2 DD(G_2) + |V_2|^2 DD(F(G_1)) + 4k|E_1||V_1|W(G_2) + 2|E_1|M_1(G_1)W(G_2) \\
&+ (|E_1| + 1)(|V_2|^2 - |V_2|)M_1(G_1) + |E_1||V_1| \sum_{u_2,v_2\in V_2} d_{G_2}(u_2)d(u_2, v_2|G_2) \\
&+ 2|E_2||V_2[\sum_{u_1,v_1\in V_1} d(u_1, v_1|F(G_1)) + \sum_{v_1\in E_1}\sum_{u_1\in V_1} d(u_1, v_1|F(G_1))] \\
&+ 2W(G_2)\sum_{v_1\in E_1}\sum_{u_1\in V_1}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)]. \qquad (8)
\end{aligned}
$$

On the other hand, since $F = Q$ or S, by Lemma 2.5 and Eq. (7),

$$
\begin{aligned}
\sum_{v_1\in E_1}\sum_{u_1\in V_1}[d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)] =& \sum_{v_1\in E_1}\sum_{u_1\in V_1}[k\cdot d_{G_1}(u_1) + d_{L(G_1)}(v_1) + 2] \\
=& k\sum_{v_1\in E_1}\sum_{u_1\in V_1} d_{G_1}(u_1) + \sum_{u_1\in V_1}\sum_{v_1\in E_1} d_{L(G_1)}(v_1) + 2\sum_{v_1\in E_1}\sum_{u_1\in V_1} 1 \\
=& 2k|E_1|^2 + |V_1|(M_1(G_1) - 2|E_1|) + 2|E_1||V_1| \\
=& 2k|E_1|^2 + |V_1|M_1(G_1), \qquad (9)
\end{aligned}
$$

where k is stated in Eq. (1).

Thus, combining Eq. (9) and inequalities (4) and (5) with Eq. (8), we get

$$\begin{aligned}
DD(G_1 +_F G_2) =&|V_1|^2 DD(G_2) + |V_2|^2 DD(F(G_1)) + 2(|E_1| + |V_1|)M_1(G_1)W(G_2)\\
&+ 4k|E_1|(|E_1| + |V_1|)W(G_2) + (|E_1 + 1|)(|V_2|^2 - |V_2|)M_1(G_1)\\
&+ 2|E_2||V_2[\sum_{u_1,v_1\in V_1} d(u_1, v_1|F(G_1)) + \sum_{v_1\in E_1}\sum_{u_1\in V_1} d(u_1, v_1|F(G_1))]\\
&+ |E_1||V_1|\sum_{u_2,v_2\in V_2} d_{G_2}(u_2)d(u_2, v_2|G_2)\\
\leq&|V_1|^2 DD(G_2) + |V_2|^2 DD(F(G_1)) + 2(|E_1| + |V_1|)M_1(G_1)W(G_2) + 4k|E_1|(|E_1|\\
&+ |V_1|)W(G_2) + (|E_1 + 1|)(|V_2|^2 - |V_2|)M_1(G_1) + 4|E_2||V_2|W(F(G_1))\\
&+ 2|E_1||V_1|\Delta(G_2)W(G_2).
\end{aligned}$$

This completes the proof. ■

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