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TWO WAYS TO COMPACTNESS

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Abstract. **In this paper we give two different ways of proving the compactness of some linear operators between certain sequence spaces. One of them is based only on the theory of matrix transformations and the other uses the Hausdorff measure of noncompactness.**

1. Introduction

The theory of FK and BK spaces is of great importance in the characterization of matrix transformations between certain sequence spaces, so we will give some necessary definitions and notations which will be used in our work.

An FK space is a complete linear metric sequence space with the property that convergence implies coordinatewise convergence; a BK space is normed FK space.

By ϕ, we denote the set of all finite sequences and by e and $e^{(n)} (n \in N_0)$ we denote the sequences such that $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. An FK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$.

Let ω be the set of all complex sequences and X and Y be sequence spaces . By (X, Y) we denote the set of all matrices that map X into Y. If we denote by $A = (a_{nk})_{n,k=0}^{\infty}$ an infinite matrix with complex entries and by A_n its n-th row, we write

$$A_n x = \sum_{k=0}^{\infty} a_{nk} x_k \text{ and } Ax = (A_n x)_n,$$

$A \in (X, Y)$ if and only if $A_n x$ converges for all $x \in X$ and all n and $A(x) \in Y$

$$X^{\beta} = \{a \in \omega \mid \sum_k a_k x_k \text{ converges for all } x \in X\}.$$

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For our investigation we also need the next important results.

Theorem 1.1. ([3, Theorem 1.17]) *Any matrix map between FK spaces is continuous.*

Theorem 1.2. ([3, Theorem 1.23]) *Let X and Y be FK spaces. Then, $(X,Y) \subset B(X,Y)$, that is each $A \in (X,Y)$ defines an element $L_A \in B(X,Y)$ where $L_A x = Ax, x \in X$.*

(In this paper, we will write A instead of L_A)

In our work we consider the compact operators in the class (X,Y) and denote the class of such operators by $K(X,Y)$; that is, we try to find necessary and sufficient conditions for L_A to be a compact operator. Hence, let us recall that if X and Y are metric spaces and $f : X \to Y$, we say that f is a compact map if $f(Q)$ is a relatively compact subset of Y for every bounded subset Q of X (that is, for every bounded sequence $(x_n)_n$ in X, the sequence $(f(x_n))_n$ has a convergent subsequence in Y).

2. Matrix transformations and compactness

In this section we will consider matrix transformations between classical sequence spaces and give necessary and sufficient conditions for A to be a compact operator in the form of conditions for the entries of the infinite matrix A. The whole investigation is based on results from [6]. For further work, the next theorem will be very useful.

Theorem 2.1. ([6, Theorem 3]) *Let $A \in (X,Y)$ and A^T denote the transpose of A. Then $A \in K(X,Y)$ if and only if $A^T \in K(Y^\beta, X^\beta)$.*

We can conclude that if we find conditions for compactness of operators from $B(c_0, \ell_p)$ or $B(\ell_1, \ell_p)$, $1 \le p < \infty$, we will be able to find all the other conditions. Hence, let us find the conditions.

We need following notations.

Let $x_{[n]}$ denote the element of X whose first n coordinates coincide with those of x and whose remaining coordinates are zero;

$$A^{(n)}x = A(x^{[n]}) \text{ and } A_{(n)}x = (Ax)^{[n]}.$$

Theorem 2.2. ([6, Corollary]) *Let $A \in (X,Y)$ and let X^β have AK. Then $A \in K(X,Y)$ if and only if*

$$\lim_{n\to\infty} \|A - A^{(n)}\| = 0.$$

Corollary 2.3. *We have $A \in B(c_0, \ell_p)$, $1 \le p < \infty$ if and only if $A \in K(c_0, \ell_p)$, $1 \le p < \infty$, that is if A is given by an infinite matrix $A =$*

$(a_{nk})_{n,k=0}^{\infty}$, *then* $A \in K(c_0, \ell_p)$, $1 \le p < \infty$, *if and only if*

$$\sup_{K \subset \mathcal{F}} (\sum_n | \sum_{k \in K} a_{nk}|^p)^{\frac{1}{p}} < \infty,$$

where $\mathcal{F}$ *denotes the class of all finite sets of positive integers.*

It remains to find the conditions for $A \in (\ell_1, \ell_p)$, $1 \le p < \infty$ to be a compact transformation. Since $\ell_1^{\beta} = \ell_{\infty}$ and ℓ_{∞} is not an AK space, we cannot use Theorem 2.2. Hence, we need following result.

Theorem 2.4. ([6, Theorem 2]) *Let* $A \in (X, Y)$ *and* Y *have AK. Then* $A \in K(X, Y)$ *if and only if*

$$\lim_{n \to \infty} \|A - A_{(n)}\| = 0.$$

Corollary 2.5. *Let* A *be given by an infinite matrix* $A = (a_{nk})_{n,k=0}^{\infty}$. *Then* $A \in K(\ell_1, \ell_p)$, $1 \le p < \infty$ *if and only if*

$$\sup_{K \subset \mathcal{F}} (\sum_n | \sum_{k \in K} a_{nk}|^p)^{\frac{1}{p}} < \infty$$

and

$$\lim_{n \to \infty} \sup_k (\sum_{j=n+1}^{\infty} |a_{jk}|^p)^{1/p} = 0.$$

Now, using the previous results,we can obtain all the other conditions. For example, we are interested in transformations A in (c, ℓ_{∞}).

Corollary 2.6. *We have* $A \in K(c, \ell_{\infty})$ *if and only if*

$$\lim_{n \to \infty} \sup_k \sum_{j=n+1}^{\infty} |a_{kj}| = 0.$$

3. The Hausdorff measure of noncompactness and matrix transformations

Now, applying the Hausdorff measure of noncompactness, we solve the problem from the previous section. Let us recall some definitions and well-known results.

Definition 3.1. Let (X, d) be a metric space and Q a bounded subset of X. Then the Hausdorff measure of noncompactness of Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\{\epsilon > 0 \mid Q \subset \cup_{i=1}^{n} K(x_i, r_i), x_i \in X, r_i < \epsilon, i = 1, \dots, n, n \in N\}.$$

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X, d), then

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is a totally bounded set}$$

$$\chi(Q) = \chi(\overline{Q}),$$

$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leq \chi(Q_2),$$

$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\},$$

$$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}.$$

If Q, Q_1 andQ_2 are bounded subsets of the normed space X, then

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$

$$\chi(Q + x) = \chi(Q), x \in X,$$

$$\chi(\lambda Q) = |\lambda|\chi(Q), \forall\lambda \in C.$$

Let X and Y be Banach spaces, $S = \{x \in X \mid \|x\| = 1\}$, $K = \{x \in X \mid \|x\| \leq 1\}$ and $A \in B(X, Y)$. Then, the Hausdorff measure of noncompactness of an operator A, denoted by $\|A\|_\chi$, can be obtained by

$$\|A\|_\chi = \chi(AK) = \chi(AS).$$

Furthermore, A is a compact if and only if $\|A\|_\chi = 0$. It holds that $\|A\|_\chi \leq \|A\|$.

Theorem 3.2 (Goldenštein, Gohberg, Markus)**.** ([3, Theorem 2.23]) *Let X be a Banach space with Schauder basis $\{e_1, e_2, ...\}$, Q a bounded subset of X, and $P_n : X \to X$ the projector onto the linear span of $\{e_1, e_2, ..., e_n\}$. Then we have*

$$\frac{1}{a} \limsup_{n\to\infty} (\sup_{x\in Q} \|(I - P_n)x\|) \leq \chi(Q) \leq \limsup_{n\to\infty} (\sup_{x\in Q} \|(I - P_n)x\|)$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

Theorem 3.3. *Let $A \in B(c_0, \ell_p)$, $1 \leq p < \infty$ be given by an infinite matrix $A = (a_{nk})_{n,k=0}^\infty$. Then we have $\|A\|_\chi = 0$.*

Corollary 3.4. *Let $A \in B(c_0, \ell_p)$, $1 \leq p < \infty$ be given by an infinite matrix $A = (a_{nk})_{n,k=0}^\infty$. Then, $A \in B(c_0, \ell_p)$, $1 \leq p < \infty$ if and only if $A \in K(c_0, \ell_p)$, $1 \leq p < \infty$.*

Theorem 3.5. *Let $A \in B(\ell_1, \ell_p)$, $1 \leq p < \infty$ be given by an infinite matrix $A = (a_{nk})_{n,k=0}^\infty$. Then we have*

$$\|A\|_\chi = \lim_{n\to\infty} \sup_k \{\sum_{j=n+1}^\infty |a_{jk}|^p\}^{\frac{1}{p}}.$$

Corollary 3.6. *Let $A \in B(\ell_1, \ell_p)$, $1 \leq p < \infty$ be given by an infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$. Then, A is compact if and only if*

$$\lim_{n\to\infty} \sup_k \{ \sum_{j=n+1}^{\infty} |a_{jk}|^p \}^{\frac{1}{p}} = 0.$$

Applying the properties of the Hausdorff measure of noncompactness in the investigation of matrix transformations in the class (c, ℓ_∞), we obtain following result

$$0 \leq \|A\|_\chi \leq \limsup_{n\to\infty} \sum_k |a_{nk}|.$$

Hence, in this case, we cannot give necessary and sufficient conditions for the compactness of the operator. Actually, we have following corollary.

Corollary 3.7. *Let $A \in B(c, \ell_\infty)$ be given by an infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$. Then A is compact if*

$$\limsup_{n\to\infty} \sum_k |a_{nk}| = 0.$$

Equivalence does not hold in general.

Example 3.8. This example illustrates that equivalence does not hold in the previous corollary. Let $A = (a_{nk})$ be an infinite matrix such that $a_{nk(0)} = 1$ and $a_{nk} = 0$ for $k \neq k(0)$. Using the known characterization of matrix transformations, we conclude $A \in (c, \ell_\infty)$. In this example, we obtain $\limsup_{n\to\infty} \sum_k |a_{nk}| \neq 0$. Also, if we put $x = e = (1, 1, \dots)$, we obtain that A is a compact operator.

In the previous corollary, we could not find necessary and sufficient conditions for the compactness of the operator by only applying the Hausdorff measure of noncompactness. That was not the case in the previous section where we had a more powerful tool, but in the all other cases we have got the same conditions, hence here is only the matter of choice.

Proof. Now we prove Theorem 3.3. Set $S = \{x \in c_0 \mid \|x\| = 1\}$. By Theorem 3.2 we have

$$0 \leq \|A\|_\chi = \chi(AS) = \lim_{n\to\infty} \sup_{x\in S} \|(I - P_n)Ax\| = \lim_{n\to\infty} \sup_{x\in S} \{\sum_{j>n} |A_j x|^p\}^{\frac{1}{p}} = 0,$$

hence $\|A\|_\chi = 0$. $\square$

Proof. Now we prove Theorem 3.5. Set $S = \{x \in \ell_1 \mid \|x\| = 1\}$. By Theorem 3.2, we have

$$\begin{aligned}\|A\|_\chi &= \chi(AS) = \lim_{n\to\infty}\sup_{x\in S}\|(I-P_n)Ax\| = \lim_{n\to\infty}\sup_{x\in S}\{\sum_{j>n}|A_jx|^p\}^{\frac{1}{p}}\\ &= \lim_{n\to\infty}\sup_{x\in S}\{\sum_{j=n+1}^{\infty}|\sum_k a_{jk}x_k|^p\}^{\frac{1}{p}} \le \sum_k\{\sum_{j=n+1}^{\infty}|a_{jk}x_k|^p\}^{\frac{1}{p}}\\ &= \sum_k |x_k|\cdot\{\sum_{j=n+1}^{\infty}|a_{jk}|^p\}^{\frac{1}{p}} \le \|x\|\cdot\sup_k\{\sum_{j=n+1}^{\infty}|a_{jk}|^p\}^{\frac{1}{p}}\\ &= \sup_k\{\sum_{j=n+1}^{\infty}|a_{jk}|^p\}^{\frac{1}{p}}.\end{aligned}$$

It remains to prove the converse inequality.

Let Q be the set defined as $Q = \{Ae_i, i = 1, 2, \dots\}$. Since $A(Q) \subset \ell_p$, we have

$$\begin{aligned}\chi(Q) &= \lim_{n\to\infty}\sup_{x\in Q}\|(I-P_n)x\| = \lim_{n\to\infty}\sup_i\|(I-P_n)Ae_i\|\\ &= \lim_{n\to\infty}\sup_i\{\sum_{j=n+1}^{\infty}|a_{ji}|^p\}^{\frac{1}{p}}.\end{aligned}$$

The inequality $\chi(Q) \le \chi(AS)$ yields

$$\chi(AS) = \|A\|_\chi \ge \lim_{n\to\infty}\sup_k\{\sum_{j=n+1}^{\infty}|a_{jk}|^p\}^{\frac{1}{p}}.$$

Finally, we can conclude

$$\|A\|_\chi = \lim_{n\to\infty}\sup_k\{\sum_{j=n+1}^{\infty}|a_{jk}|^p\}^{\frac{1}{p}}.$$

□

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