Tychonoff's theorem without the axiom of choice

by

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Abstract. It is well known that the theorem of Tychonoff, that a product of compact spaces is compact, is equivalent in standard (ZF) set theory to the axiom of choice. The purpose of this paper is to point out that here, as elsewhere in general topology, the role of the axiom of choice is only to enable us to pick out the points of a given space, and that if we adopt the "locale-theoretic" view that what matters about a space is its lattice of open sets and not its points, then we can prove the theorem without any use of choice. We also give a choice-free construction of the Stone–Čech compactification of a locale, and discuss some topos-theoretic applications of these results.

0. Introduction. It is well known that the theorem of Tychonoff [25], that a product of compact spaces is compact, is equivalent in standard (ZF) set theory to the axiom of choice [17]. The purpose of this paper is to point out that here, as elsewhere in general topology, the role of the axiom of choice is only to enable us to pick out the points of a given space, and that if we adopt the "locale-theoretic" view that what matters about a space is its lattice of open sets and not its points, then we can prove the theorem without any use of choice. We also give a choice-free construction of the Stone–Čech compactification of a locale, and discuss some topos-theoretic applications of these results. Proofs of the Tychonoff theorem for locales already exist in the literature [20, 7], but both of these use the axiom of choice. Our approach is to be distinguished from that of Comfort [4], who eliminated the axiom of choice from the Tychonoff and Stone–Čech theorems by redefining compactness so that they both became trivial; with us it is the concept of space which is redefined, but that of compactness stays the same.

1. Locales and sublocales. The basic theory of locales or frames has been developed by Bénabou [3], Dowker and Strauss [5, 6, 7], Isbell [14] and Simmons [23]; but since there are considerable differences in terminology between these four, and since our own terminology differs in parts from all four of them, we begin with the definitions. A *frame* is a complete lattice A in which the infinite distributive law

$$a\wedge(\bigvee S) = \bigvee\{a\wedge s \mid s\in S\}$$

holds for all $a \in A$, $S \subseteq A$. A *frame homomorphism* $A \to B$ is a map preserving finite meets and arbitrary joins; thus we have a category **Frm** of frames. If X is a topological space, the lattice $\Omega(X)$ of open subsets of X is a frame; and if $f\colon X \to Y$ is a continuous map, then $f^{-1}\colon \Omega(Y) \to \Omega(X)$ is a frame homomorphism. Thus Ω is a contravariant functor from the category **Sp** of topological spaces to **Frm**.

Following Isbell [14], we shall write **Loc** for the opposite category $\mathbf{Frm}^{op}$, and call its objects *locales*. The reason for this dual terminology is that, by making $\Omega\colon \mathbf{Sp} \to \mathbf{Loc}$ into a covariant functor, we are entitled to use the names of familiar concepts in topology for their natural generalizations in **Loc**. For instance, we shall be able to talk about closed and dense sublocales of a given locale, whereas we should have had to refer (as in [5]) to quotient frames. We adopt the convention that if $f\colon A \to B$ is a morphism in **Loc**, $f^*\colon B \to A$ denotes the corresponding frame homomorphism, and f_* its right adjoint (called an *antimap* by Dowker and Strauss), defined by

$$f_*(a) = \bigvee \{b \in B \mid f^*(b) \leqslant a\}.$$

A *point* of a locale A is a locale morphism $2 \to A$, where 2 is the two-element locale $\{0, 1\}$. If $\mathrm{pt}(A)$ denotes the set of points of A, there is a natural map φ^* from A to the power-set of $\mathrm{pt}(A)$, which sends $a \in A$ to $\{p \in \mathrm{pt}(A) \mid p^*(a) = 1\}$. It is easily checked that φ^* is a frame homomorphism, and so its image is a topology on $\mathrm{pt}(A)$. Moreover, for every locale morphism $f\colon A \to B$, the map $\mathrm{pt}(A) \to \mathrm{pt}(B)$ induced by composition with f is continuous, and so pt is a functor $\mathbf{Loc} \to \mathbf{Sp}$. In fact pt is right adjoint to Ω; we call a space X *sober* (following Grothendieck [11]) if the unit map $X \to \mathrm{pt}(\Omega(X))$ is a homeomorphism, and we call a locale A *spatial* (Isbell uses *primal*) if the counit $\varphi\colon \Omega(\mathrm{pt}(A)) \to A$ is an isomorphism. The functors Ω and pt restrict to an equivalence of categories between sober spaces and spatial locales. Every Hausdorff space is sober and every sober space is T_0, but sobriety is incomparable with the T_1 axiom.

Monomorphisms in **Loc** (i.e. epimorphisms in **Frm**) are badly behaved — indeed, Isbell has shown that the locale $\Omega(R)$ has a proper class of non-isomorphic subobjects. However, regular monomorphisms (= equalizers), which correspond to surjective frame homomorphisms, are more manageable. If $f^*\colon A \to B$ is a surjective frame homomorphism, then the composite $j = f_* f^*\colon A \to A$ satisfies the conditions

$$j(a) \geqslant a,$$

$$j(j(a)) = j(a)$$

and

$$j(a \wedge b) = j(a) \wedge j(b)$$

for all $a, b \in A$; following Simmons [23], we call such a map a *nucleus* on A. Conversely, if j is a nucleus on A, we write A_j for the set $\{a \in A \mid j(a) = a\}$ with its induced order; then A_j is a locale, and j is a surjective frame homomorphism $A \to A_j$ (its right adjoint being the inclusion $A_j \to A$). Thus the regular subobjects of A in **Loc** may be identified with locales of the form A_j; we call these *sublocales* of A (Isbell uses the term "part", but we prefer to follow the tradition whereby in **Sp** the term "subspace" denotes a regular subobject and not an arbitrary subobject).

For any element a of a locale A, the map $c(a) = a \vee (-)\colon A \to A$ is a nucleus; the corresponding sublocale is the set $\uparrow(a) = \{b \in A \mid b \geqslant a\}$. Such a sublocale is called *closed*. A nucleus j (or the corresponding sublocale A_j) is called *dense* if $j(0) = 0$. It is easily verified that for a T_0 topological space X, the continuous maps $Y \to X$ which induce regular monomorphisms $\Omega(Y) \to \Omega(X)$ are precisely the inclusions of subspaces of X, and that in this context the words "closed" and "dense" restrict to their usual meanings in topology. However, one respect in which locales differ sharply from spaces is that every locale A has a smallest dense sublocale; the corresponding nucleus is the "double negation" map which sends $a \in A$ to the largest element disjoint from all elements disjoint from a.

Bénabou [3] showed that the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$ has a left adjoint. We shall make use of the following more general method of constructing frames from "generators and relations". Let A be a meet-semilattice; by a *coverage* on A we mean a function C assigning to every $a \in A$ a set $C(a)$ of subsets of $\downarrow(a)$, called "covers of a", such that if $S \in C(a)$ for some a, then $\{s \wedge b \mid s \in S\} \in C(b)$ for every $b \leqslant a$. (Cognoscenti will recognize here the essential part of the definition of a Grothendieck pretopology [12].) Given a coverage C, we define a *C-ideal* in A to be a subset $I \subseteq A$ which is closed downwards (i.e. $a \in I$ and $b \leqslant a$ implies $b \in I$) and closed under covers in C (i.e. $S \subseteq I$ for some $S \in C(a)$ implies $a \in I$).

PROPOSITION 1.1. *Let (A, C) be a meet-semilattice with a coverage. Then the set $C\text{-Idl}(A)$ of C-ideals of A, ordered by inclusion, is a frame; and there is a function $A \to C\text{-Idl}(A)$ which is universal among meet-semilattice homomorphisms f from A to a frame B which transform C-covers to joins (i.e. satisfy $f(a) = \bigvee \{f(s) \mid s \in S\}$ for every $S \in C(a)$), in the sense that every such f factors uniquely through $A \to C\text{-Idl}(A)$ by a frame homomorphism.*

Proof. Consider first the case when C is trivial, so that a C-ideal is just a downward-closed subset of A. In this case the set $C\text{-Idl}(A)$ (which we shall denote by $\downarrow\mathrm{Cl}(A)$) is a sub-complete-lattice of the power-set of A, and hence a frame; we may embed A in $\downarrow\mathrm{Cl}(A)$ via the map which sends a to $\downarrow(a)$, and then any semilattice homomorphism $f\colon A \to B$, where B is a frame, extends uniquely to $\bar{f}\colon \downarrow\mathrm{Cl}(A) \to B$, where $\bar{f}(S) = \bigvee \{f(s) \mid s \in S\}$.

In the general case, consider the map $j_C\colon \downarrow\mathrm{Cl}(A) \to \downarrow\mathrm{Cl}(A)$ which sends a downward-closed subset to the intersection of all C-ideals containing it. Since an intersection of C-ideals is a C-ideal, it is clear that we have $S \subseteq j_C(S) = j_C(j_C(S))$, and that the image of j_C is precisely $C\text{-Idl}(A)$. So to prove that $C\text{-Idl}(A)$ is a sublocale of $\downarrow\mathrm{Cl}(A)$, it suffices to prove that j_C commutes with finite meets (intersections). Clearly $j_C(S \cap T) \subseteq j_C(S) \cap j_C(T)$, since the right-hand side is a C-ideal containing $S \cap T$. Conversely, let I be a C-ideal containing $S \cap T$, and consider the set

$$J = \{a \in A \mid a \wedge s \in I \text{ for all } s \in S\}.$$

J is a C-ideal, for if $U \in C(b)$ and $U \subseteq J$, then $\{u \wedge s \mid u \in U\} \in C(b \wedge s)$ for all $s \in S$, $\{u \wedge s \mid u \in U\} \subseteq I$ and so $b \wedge s \in I$ for all $s \in S$, i.e. $b \in J$. And $T \subseteq J$, since $S \cap T \subseteq I$. Similarly, we have a C-ideal

$$K = \{a \in A \mid a \wedge j \in I \text{ for all } j \in J\}$$

which contains S. But by construction it is clear that $J \cap K \subseteq I$, so we deduce $j_C(S) \cap j_C(T) \subseteq I$ and hence $j_C(S) \cap j_C(T) \subseteq j_C(S \cap T)$.

Now it is clear that is $S \in C(a)$, then the join in C-Idl(A) of the C-ideals $j_C(\downarrow(s))$, $s \in S$, must contain a; hence the composite map

$$\eta_C\colon A \xrightarrow{\downarrow(-)} \downarrow\mathrm{Cl}(A) \xrightarrow{j_C} C\text{-Idl}(A)$$

has the properties described in the statement of the proposition. But if $f\colon A \to B$ is any map with the same properties, we may first extend it to $\bar{f}\colon \downarrow\mathrm{Cl}(A) \to B$ as above, and then observe that the surjective part of $\bar{f}$ gives rise to a larger nucleus on $\downarrow\mathrm{Cl}(A)$ than j_C (equivalently, a smaller sublocale of $\downarrow\mathrm{Cl}(A)$ than C-Idl(A)). So $\bar{f}$ factors uniquely through j_C, as required. ■

As an application of Proposition 1.1, we give the construction of products in **Loc** (i.e. of coproducts in **Frm**). This construction has been given before by Dowker and Strauss [7] (though there is an easily rectifiable error in their description, concerning the least element of the product locale), but the use of Proposition 1.1 helps to make the reasons for the construction more transparent.

Let $(A_\gamma \mid \gamma \in \Gamma)$ be a family of locales. Write B for the Cartesian product of the A_γ (which is of course their product in **Frm**), and $p_\gamma\colon B \to A_\gamma$ for the γth product projection. p_γ has a right adjoint q_γ, which sends $a \in A_\gamma$ to the Γ-tuple with a in the γth place and 1's elsewhere. Let A denote the sub-meet-semilattice of B generated by the union of the images of the q_γ, i.e. the set of all $(a_\gamma \mid \gamma \in \Gamma)$ in B such that $a_\gamma = 1$ for all but finitely many $\gamma \in \Gamma$; then A is easily seen to be the coproduct of the A_γ in the category of meet-semilattices. Now if F is the coproduct of the A_γ in **Frm**, the universal property of coproducts yields a semilattice homomorphism $f\colon A \to F$, which is clearly universal among homomorphisms with the property that $f(q_\gamma(a)) = \bigvee\{f(q_\gamma(s)) \mid s \in S\}$ whenever S is a subset of A_γ with join a.

The covering families in A which appear in the above equation do not themselves form a coverage, since they fail to satisfy the stability condition; but if f preserves these covers, it will clearly also preserve any cover obtained from one of them by taking meets with a fixed element of A. We therefore define a coverage C on A by letting $C(a)$ consist of all sets of the form

$$S_\gamma[a] = \{a \wedge q_\gamma(s) \mid s \in S\}$$

where $a = (a_\gamma \mid \gamma \in \Gamma)$ and S is a set in A_γ with join a_γ. Then it is clear from the above argument that we have

PROPOSITION 1.2. *Let $(A_\gamma \mid \gamma \in \Gamma)$ be a family of locales, and let C be the coverage defined above on the semilattice-coproduct A of the A_γ. Then C-Idl(A) is the product of the A_γ in* **Loc**. ■

2. Tychonoff's theorem. A locale A is said to be *compact* if, given any $S \subseteq A$ with $\bigvee S = 1$, there is a finite $F \subseteq S$ with $\bigvee F = 1$. In this section we shall give a choice-free proof of the theorem that a product of compact locales is compact. As we have already remarked, proofs of this theorem exist in the literature [20, 7]; but both of these use the result that a locale is compact if and only if all its maximal ideals are principal, whose proof requires a fairly obvious application of Zorn's lemma. To give a proof which proceeds directly from the definition of compactness, we shall find it convenient to deal separately with the finite and directed covers in the coverage C of Proposition 1.2; so, in the notation of that proposition, let C_f be the sub-coverage of C such that $C_f(a)$ consists of all finite sets in $C(a)$.

LEMMA 2.1. *Let S be a downward-closed subset of A, and let $P(S)$ be the set of all those $(a_\gamma \mid \gamma \in \Gamma)$ in A for which there is a finite set $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ and a finite cover S_{γ_i} of a_{γ_i} for each i, such that every element of the form $(s_\gamma \mid \gamma \in \Gamma)$, where each s_{γ_i} is in S_{γ_i} and $s_\gamma = a_\gamma$ if $\gamma \notin \{\gamma_1, \ldots, \gamma_n\}$, is in S. Then $P(S)$ is the C_f-ideal generated by S.*

Proof. It is clear that every member of $P(S)$ must be in every C_f-ideal which contains S; so it suffices to prove that $P(S)$ is a C_f-ideal. But this follows easily from the fact that any finite set of finite covers of a given element in a locale has a common finite refinement. ■

COROLLARY 2.2. *For any family of locales $(A_\gamma \mid \gamma \in \Gamma)$, the locale C_f-Idl(A) is compact.*

Proof. Let X be a set of C_f-ideals. The join of X in C_f-Idl(A) is the C_f-ideal $P(\bigcup X)$ generated by the union of the members of X. But if 1_A is in $P(\bigcup X)$, then only finitely many members of $\bigcup X$ are involved in the proof that it is, and so $1_A \in P(\bigcup Y)$ for some finite $Y \subseteq X$. ■

Next we consider directed joins. If S is a downward-closed subset of A, let $D(S)$ denote the set of all joins (in A) of (upwards) directed subsets of S.

LEMMA 2.3. *$D(S)$ is contained in every C-ideal which contains S.*

Proof. Let $a = (a_\gamma) \in D(S)$ and suppose $a = \bigvee T$ where $T \subseteq S$ is directed. Pick $t \in T$; since $t_\gamma = 1$ for all but finitely many γ, we have $t_\gamma = a_\gamma$ for all but a finite set $F = \{\gamma_1, \ldots, \gamma_n\}$ of indices. Moreover, directedness of T ensures that $a = \bigvee\{t' \in T \mid t' \geqslant t\}$, so by cutting down to this subset we may assume that every member of T differs from a in at most this set of factors.

Now for $t' \in T$ and $0 \leqslant j \leqslant n$, define $t'[j]$ to be the result of substituting a_{γ_i} for t'_{γ_i} in t', for $1 \leqslant i \leqslant j$. (Thus $t'[0] = t'$ and $t'[n] = a$.) Now since $a_{\gamma_j} = \bigvee\{t'_{\gamma_j} \mid t' \in T\}$, it is clear that every $t'[j]$ is C-covered by elements which lie below some $t''[j-1]$. So by induction on j we deduce that each $t'[j]$ is in every C-ideal containing S, and hence $t[n] = a$ is in every such C-ideal. ■

LEMMA 2.4. *If I is a C_f-ideal, so is $D(I)$.*

Proof. It is sufficient to consider C_f-covers with just two elements. So let $a = \bigvee S$, $b = \bigvee T$ be elements of $D(I)$ (where S, T are directed subsets of I) which differ only in the γth entry. For $s \in S$, $t \in T$, define $f_\gamma(s, t) \in A$ by

$$(f_\gamma(s, t))_\delta = \begin{cases} s_\delta \vee t_\delta & \text{for } \delta = \gamma, \\ s_\delta \wedge t_\delta & \text{otherwise.} \end{cases}$$

Since I is a C_f-ideal, each $f_\gamma(s, t)$ is in I; and since f_γ is an order-preserving function of each variable, it is easy to see that $\{f_\gamma(s, t) \mid s \in S, t \in T\}$ is directed. But finite meets in A_δ ($\delta \neq \gamma$) distribute over arbitrary joins, so that

$$\bigvee \{f_\gamma(s, t) \mid s \in S, t \in T\} = a \vee b .$$

Hence $a \vee b \in D(I)$. ∎

LEMMA 2.5. *A C_f-ideal I is a C-ideal if and only if $I = D(I)$.*

Proof. This is trivial from Lemma 2.3 and the fact that arbitrary joins can be constructed from finite joins and directed joins. ∎

LEMMA 2.6. *Suppose the locales A_γ are all compact. Then a downward-closed subset S of A generates the whole of A as a C-ideal if and only if it generates A as a C_f-ideal.*

Proof. Given S, we define an ordinal sequence $(I_\alpha \mid \alpha \in On)$ of C_f-ideals by

$$\begin{aligned} I_0 &= P(S), \\ I_{\alpha+1} &= D(I_\alpha), \\ I_\alpha &= \bigcup \{I_\beta \mid \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The sequence (I_α) is clearly increasing, but since there is only a set of distinct C_f-ideals it cannot be strictly increasing; i.e. there must exist an α with $I_\alpha = I_{\alpha+1}$. Then I_α is a C-ideal by Lemma 2.5; but it is contained in every C-ideal which contains S, by Lemmas 2.1 and 2.3, and so it is the C-ideal $j_C(S)$ generated by S.

Now suppose $j_C(S) = A$, and consider the least α for which $1_A \in I_\alpha$. Clearly, α cannot be a limit ordinal. Suppose $\alpha = \beta + 1$; then we have a directed $T \subseteq I_\beta$ with $\bigvee T = 1$. Going through the argument of Lemma 2.3 and using the compactness of each A_{γ_i}, we may now deduce inductively that $t[j] \in I_\beta$ for every $t \in T$ and every j (where we take $a = 1$). Hence $1 \in I_\beta$; so the least α is not a successor. So we must have $\alpha = 0$; i.e. $1_A \in P(S)$ and so S generates A as a C_f-ideal. ∎

Combining Lemmas 2.2 and 2.6, we at once obtain

THEOREM 2.7 (Tychonoff's theorem for locales). *A product of compact locales is compact.*

Proof. Let X be a set of C-ideals whose join in $C\text{-Idl}(A)$ is A. Then the set $S = \bigcup X$ generates A as a C-ideal, so by Lemma 2.6 it generates A as a C_f-ideal. Then by Lemma 2.2 there is a finite $Y \subseteq X$ such that $\bigcup Y$ generates A as a C_f-ideal and hence as a C-ideal. ∎

3. Local compactness. The proof of the Tychonoff theorem given above, whilst it does not use the axiom of choice, does use just about the full strength of ZF set theory: in particular, the axiom of replacement is used in the assertion that the increasing sequence (I_α) of Lemma 2.6 is eventually stationary. The proof would clearly be simpler if we could be assured that the sequence becomes stationary after some definite number of steps; but in general there seems no reason to suppose that this should be so. However, if we add the further hypothesis that the locales A_γ are locally compact, then the sequence becomes stationary after a single step.

Let a and b be elements of a complete lattice A. We write $a \ll b$ (read "*a is way below b*") if, whenever we have a directed $S \subseteq A$ with $\bigvee S \geqslant b$, there exists $s \in S$ with $s \geqslant a$. We say a locale A is *locally compact* if every $a \in A$ satisfies

$$a = \bigvee \{b \in A \mid b \ll a\} .$$

Locally compact locales (under the name "distributive continuous lattices") have been studied by Hofmann and Lawson [13] and Banaschewski [2]. They have shown (using the axiom of choice) that every locally compact locale is spatial, and in fact the category of locally compact locales is equivalent to that of locally compact sober spaces.

It is easy to show that for any $a \in A$ the set

$$\twoheaddownarrow(a) = \{b \in A \mid b \ll a\}$$

is actually an ideal of A, so that the join appearing in the definition of local compactness is directed. Moreover, if a is the join of a directed set S, then every element way below a is below some member of S. Thus if I is a downward-closed set and we know $a = \bigvee S$ for some directed $S \subseteq I$, there is a canonical way of choosing such an S, namely $S = \twoheaddownarrow(a)$. This idea is used to prove

LEMMA 3.1. *With the notation of Section 2, suppose that all the locales A_γ are locally compact, and all but a finite number are compact. Then $D(I)$ is a C-ideal for every C_f-ideal $I \subseteq A$.*

Proof. The semilattice A is not in general a locale (it has no least element if Γ is infinite); but it does have joins for all nonempty subsets (in particular for all directed subsets) and so we can interpret the relation $\ll$ in it. Now it is clear that $a_\gamma \ll b_\gamma$ for all $\gamma \in \Gamma$ implies $(a_\gamma \mid \gamma \in \Gamma) \ll (b_\gamma \mid \gamma \in \Gamma)$, and compactness of A_γ is equivalent to the assertion that $1 \ll 1$ in A_γ; so it is clear from the hypotheses of the lemma that A itself is "locally compact".

Now let S be a directed subset of $D(I)$, and let $a = \bigvee S$. By the remarks above, we have $\twoheaddownarrow(s) \subseteq I$ for each $s \in S$, and

$$a = \bigvee \{\bigvee \twoheaddownarrow(s) \mid s \in S\} = \bigvee (\bigcup \{\twoheaddownarrow(s) \mid s \in S\}) .$$

So it suffices to show that the set $T = \bigcup \{\twoheaddownarrow(s) \mid s \in S\}$ is directed. Let $t_1, t_2 \in T$, and suppose $t_i \ll s_i \in S$. Then there is an upper bound s_3 for s_1 and s_2 in S, and we have $t_1 \ll s_3$, $t_2 \ll s_3$ and hence $(t_1 \vee t_2) \ll s_3$. So $t_1 \vee t_2 \in T$.

Thus $D(I) = D(D(I))$; hence by Lemmas 2.4 and 2.5 it is a C-ideal. ∎

THEOREM 3.2. *For compact and locally compact locales, the Tychonoff theorem holds in Zermelo set theory (i.e. without the axiom of replacement).* ∎

In fact for locally compact locales we can say rather more than the Tychonoff theorem:

PROPOSITION 3.3. *Let $(A_\gamma|\ \gamma\in\Gamma)$ be a family of locally compact locales, of which all but a finite number are compact. Then the product of the A_γ in* **Loc** *is locally compact.*

Proof. With the same notation as before, let $a\in A$ and let $I(a)$ denote the principal C-ideal generated by a (i.e. the smallest C-ideal containing a). We shall show that $b\ll a$ in A implies $I(b)\ll I(a)$ in $C\text{-Idl}(A)$, and that

$$I(a)=\bigvee\{I(b)|\ b\ll a \text{ in } A\}\,;$$

since every C-ideal can be represented as a union (and hence a join) of principal C-ideals, this will suffice.

For the first assertion, suppose we have a directed family S of C-ideals with $I(a)\subseteq\bigvee S$. Now directedness of S clearly implies that the union of the members of S is a C_f-ideal; and so $\bigvee S$, being the C-ideal generated by $\bigcup S$, is equal to $D(\bigcup S)$ by Lemma 3.1. So there is a directed family $T\subseteq\bigcup S$ with $a=\bigvee T$. Now $b\ll a$ implies $b\leqslant t$ for some $t\in T$, and hence $b\in I$ for some $I\in S$.

For the second assertion, observe that we have

$$a=\bigvee\{b\in A|\ b\ll a\}$$

and that this join is directed; so $a\in D(\bigcup\{I(b)|\ b\ll a\})$. So by Lemma 2.3 the join of the $I(b)$ in $C\text{-Idl}(A)$ contains a, and hence equals $I(a)$. ∎

Now a product of spatial locales need not coincide with the open-set locale of the corresponding product of spaces (for counter-examples, see [14] and [7]); but the adjunction $(\Omega\dashv \mathrm{pt})$ ensures (for sober spaces X_γ) that $\prod X_\gamma$ is the space of points of the locale product of the $\Omega(X_\gamma)$, and hence the two do coincide whenever the locale product is itself spatial. So recalling the theorem of Hofmann–Lawson and Banaschewski that locally compact locales are spatial, we recover a special case of the classical Tychonoff theorem:

THEOREM 3.4. *Assume the axiom of choice. Then a product of compact, locally compact sober spaces is compact.* ∎

Even with the axiom of choice, it does not seem possible to recover the full Tychonoff theorem directly from Theorem 2.7 in this way, since a compact locale need not be spatial. For example, Isbell [14] has observed that if αQ denotes the one-point compactification of the space of rationals, then the locale product of $\Omega(\alpha Q)$ with itself is not spatial, and so cannot be isomorphic to $\Omega(\alpha Q\times\alpha Q)$. (However, since the first draft of this section was written, D. Strauss has shown the author a proof — using the axiom of choice — that if $(X_\gamma|\ \gamma\in\Gamma)$ is any family of compact spaces with open-set locales $A_\gamma=\Omega(X_\gamma)$, then the canonical map

$$\Omega(\textstyle\prod X_\gamma)\to C\text{-Idl}(A)\,,$$

which sends an open set to the C-ideal of open rectangles which it contains, preserves families with join 1. Using this, one may immediately deduce the full Tychonoff theorem from Theorem 2.7.)

It is not clear whether the special case of Tychonoff's theorem given above is already sufficient to imply the axiom of choice: one certainly cannot prove it by the method of Kelley [17], since the latter makes essential use of non-sober topologies. It is known that the Tychonoff theorem for compact Hausdorff spaces is equivalent to the Boolean prime ideal theorem [18]; and it seems quite likely that the same is true of the slightly more general result above. However, I do not know how to prove Theorem 3.4 using only the Boolean prime ideal theorem.

4. The Stone–Čech compactification. We wish to construct a left adjoint for the inclusion in **Loc** of the category of compact "Hausdorff" locales. Unfortunately, there are difficulties in saying "Hausdorff" without mentioning points (for a discussion of these difficulties, see Isbell [14]); and although Simmons [24] has succeeded in giving a lattice-theoretic condition which is equivalent in spatial locales to the Hausdorff axiom, his condition seems rather unwieldy for use in the non-spatial case. We therefore follow the approach of Dowker and Strauss in taking regularity as our basic separation property; for compact locales, this is justified by the well-known facts that a compact Hausdorff space is regular, and that a regular T_0-space (in particular a regular sober space) is Hausdorff.

We say a locale A is *regular* if every $a\in A$ satisfies

$$a=\bigvee\{b\in A|\ \exists c\in A \text{ with } b\wedge c=0 \text{ and } a\vee c=1\}\,.$$

We write $b\lessdot a$ as an abbreviation for "there exists c with $b\wedge c=0$ and $a\vee c=1$"; in a spatial locale this relation holds iff the closure of the open set b is contained in a, from which it is easy to show that the above definition is equivalent to the usual notion of regularity.

LEMMA 4.1. *In a compact locale, $b\lessdot a$ implies $b\ll a$. Hence any compact regular locale is locally compact.*

Proof. Suppose $b\lessdot a$, and let S be directed with $\bigvee S\geqslant a$. Then (if c satisfies $b\wedge c=0$ and $a\vee c=1$) the set $\{s\vee c|\ s\in S\}$ is directed and has join 1; so by compactness there exists $s\in S$ with $s\vee c=1$. Then

$$b=(s\vee c)\wedge b=(s\wedge b)\vee(c\wedge b)=s\wedge b\,,$$

i.e. $b\leqslant s$. So $b\ll a$; and the second statement is immediate from the first. ∎

LEMMA 4.2. (i) *A product of regular locales is regular.* (ii) *A sublocale of a regular locale is regular.*

Proof. (i) is proved by Dowker and Strauss [7, Proposition 6]. (ii) Let A be a regular locale, j a nucleus on A, $a\in A_j$. Since $j\colon A\to A_j$ is a lattice homomorphism, $b\lessdot a$ in A implies $j(b)\lessdot a$ in A_j. So

$$a=\bigvee_A\{b\in A|\ b\lessdot a\}=\bigvee_{A_j}\{j(b)|\ j(b)\lessdot a \text{ in } A_j\}\,. \ ∎$$

Unlike regularity, compactness is not inherited by arbitrary sublocales; but it is inherited by closed sublocales, since the join of any nonempty set in $\uparrow(a)\subseteq A$ is just its join in A. Moreover, we have

LEMMA 4.3. *If A is regular, then the equalizer of a pair of maps*

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

in **Loc** *is a closed sublocale of B.*

Proof. Isbell [14, 2.3(1)] showed that a regular locale is "strongly Hausdorff" in the sense that the diagonal $A \to A \times A$ is a closed sublocale of $A \times A$. Since the equalizer of f and g may be obtained by pulling back the diagonal along (f, g): $B \to A \times A$, it suffices to prove that a pullback of a closed sublocale is a closed sublocale. But the latter fact follows easily from the observation that, given $h: C \to D$ and $d \in D$, h factors (uniquely) through the closed sublocale $\uparrow(d)$ iff the nucleus $h_* h^*$ fixes fewer elements than $(-) \vee d$, iff $d \leqslant h_* h^*(0)$, iff $h^*(d) = 0$. ■

Combining all the above results with Theorem 3.2, we deduce

THEOREM 4.4. *The full subcategory* **KRegLoc** *of compact regular locales is closed under the formation of arbitrary limits in* **Loc**.

Proof. It is sufficient to check products and equalizers [19, p. 109]; but products follow from 3.2, 4.1 and 4.2(i), and equalizers from 4.2(ii), 4.3 and the remark between them. ■

Since the category **Loc** is complete and has small hom-sets, all that remains is to verify the "solution-set condition" [19, p. 117] for an application of the Adjoint Functor Theorem. This turns out to be straightforward: recall that an element of a locale A is said to be *regular* if it is fixed by the double-negation nucleus (equivalently, if it is a member of every dense sublocale of A). Then we have

LEMMA 4.5. *A regular locale is "semi-regular", i.e. every element is a join of regular elements.*

Proof. It is easily verified that $b \lessdot a$ implies $\neg\neg b \leqslant a$; so

$$a = \bigvee \{b \mid b \lessdot a\} = \bigvee \{\neg\neg b \mid \neg\neg b \leqslant a\}.$$

But elements of the form $\neg\neg b$ are regular. ■

COROLLARY 4.6. *Let f: $A \to B$ be a morphism in* **Loc**, *where B is regular and the image of f is a dense sublocale of B. Then B is isomorphic to a sub-poset of the power-set of A.*

Proof. Since the image of f is dense, every regular element of B is fixed by the nucleus $f_* f^*$. So the (order-preserving) map which sends $b \in B$ to

$$\{f^*(b') \mid b' \text{ regular, } b' \lessdot b\} \subseteq A$$

has a one-sided inverse, namely

$$S \mapsto \bigvee_B \{f_*(s) \mid s \in S\};$$

in particular, it is one-to-one. ■

THEOREM 4.7 (Stone–Čech compactification for locales). *The inclusion functor* **KRegLoc** $\to$ **Loc** *has a left adjoint β.*

Proof. Given a locale A, Corollary 4.6 enables us to construct a set Γ (in fact a subset of the power-set of A) indexing the isomorphism classes of morphisms f_γ: $A \to B_\gamma$, where B_γ is compact regular and the image of f_γ is dense in B_γ. Let B denote the product of the B_γ in **Loc**, and f: $A \to B$ the morphism induced by the f_γ. There is a smallest closed sublocale of B through which f factors, namely $\uparrow(f_*(0))$; we define this to be βA.

Now if h: $A \to C$ is any map from A to a compact regular locale, we may factor it through the closed sublocale $\uparrow(h_*(0))$, and this factorization has dense image; so $\uparrow(h_*(0))$ is isomorphic to B_γ for some $\gamma \in \Gamma$, and we can construct a commutative diagram

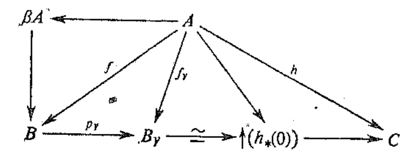

Fig. 1

So there exists a factorization of h through $A \to \beta A$. Uniqueness follows from the fact that the equalizer of two such factorizations would be a closed sublocale of βA (and hence of B) containing the image of f, and hence would be the whole of βA. ■

The reader may wonder why we have chosen to follow what is surely the least explicit and most "abstractly nonsensical" of all the many constructions of the Stone–Čech compactification available in the literature. (Its spatial equivalent may be found on p. 121 of [19], or as Exercise 1E in [27].) The reason is that all the other constructions rely upon the fact (made explicit on p. 127 of [19]) that the unit interval $[0, 1]$ is a cogenerator in the category of compact Hausdorff spaces, i.e. that compact Hausdorff spaces are completely regular. This in turn rests on Urysohn's Lemma [26]. Now Urysohn's Lemma may easily be proved for locales [6] (see also [10] for a purely localic definition of the real numbers), but its proof does involve a fairly obvious use of (countable) dependent choice, in that we have to construct an infinite sequence of elements of the locale in which there is an arbitrary choice at each step.

One way to sidestep this problem would be to work with the category **KCRegLoc** of compact completely regular locales, instead of **KRegLoc**. (It is not hard to define complete regularity for locales; one simply takes the definition of regularity and replaces the relation "$b \lessdot a$" with "there exists a family of elements $(c_q \mid q \in \mathbb{Q})$ with $b \leqslant c_p \lessdot c_q \leqslant a$ whenever $p < q$". Alternatively, one may follow the approach of Reynolds [21], who axiomatizes the properties of cozero-set lattices under the name "Alexandroff algebra", and defines a locale to be completely regular if it is generated by an Alexandroff algebra. It is not hard to prove that the two approaches are equivalent.) Having done this, it is then possible to follow one of the classical constructions of βX (for example, Čech's original construction by embedding X in a product of copies of $[0, 1]$) to obtain a left adjoint for the inclusion **KCRegLoc** $\to$ **Loc**. (See, for example, Theorem 12 of [7].)

Thus we are faced with the possibility that, in the absence of the axiom of choice, there may be two different functors on the category of locales, both of which generalize the usual Stone–Čech compactification for spaces. (I have to admit that I do not know any example of a compact regular locale which is not completely regular; but it should not be impossibly difficult to construct one in some Fraenkel–Mostowski model of set theory.) It is not at all clear which of the two constructions actually deserves the name "Stone–Čech compactification". (A frivolous suggestion: perhaps one could be called "Stone–Čech compactification" and the other "Čech–Stone compactification"!)

5. Topos-theoretic applications. The author's interest in the subject of this paper was originally aroused by the study of internal locales in a topos, and it may be helpful to give a brief account of the reasons for this study, and the way in which the results of this paper may be applied to it. For topos-theoretic definitions and notation, we refer the reader to [15].

We shall say a geometric morphism $f\colon \underset{\sim}{F} \to \underset{\sim}{E}$ is *localic* if 1 is an object of generators [15, 4.43] for $\underset{\sim}{F}$ over $\underset{\sim}{E}$. In this case the relative Giraud theorem [15, 4.46] may be stated in a more explicit form: $\underset{\sim}{F}$ is equivalent to the topos of $\underset{\sim}{E}$-valued sheaves (for the internal "canonical topology") on an internal locale A in $\underset{\sim}{E}$; moreover, A is determined up to isomorphism by f, being $f_*(\Omega_F)$ [15, 5.38]. Inspection of the proof of Lemma 4.44 in [15] shows that both halves of the lemma remain true with "bounded" replaced by "localic"; in particular, if $\underset{\sim}{F}$ is a Grothendieck topos satisfying (SG) [15, 5.31] (and hence localic over **Set**), then any geometric morphism from $\underset{\sim}{F}$ to another **Set**-topos $\underset{\sim}{E}$ is localic.

Let $f\colon Y \to X$ be a continuous map of topological spaces, and let f also denote the geometric morphism $\mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ which it induces. By the remarks above, this geometric morphism is localic, and so is entirely determined by the internal locale $f_*(\Omega_Y)$ in $\mathrm{Shv}(X)$. (The underlying sheaf of sets of $f_*(\Omega_Y)$ is the assignment

$$U \mapsto \{V \in \Omega(Y) \mid V \subseteq f^{-1}(U)\}.$$

Now, just as many interesting topological properties of X may be viewed as properties of the topos $\mathrm{Shv}(X)$ (or equivalently of the locale $\Omega(X)$), so it seems plausible that many interesting properties of the map f (at least those which are "local on the base") may be viewed as properties of $\mathrm{Shv}(Y)$ as a $\mathrm{Shv}(X)$-topos, i.e. of the internal locale $f_*(\Omega_Y)$.

(Of course, the same techniques can be used when X and Y are replaced by locales; the point is, however, that even if we start with "honest" spaces X and Y, the internal locale $f_*(\Omega_Y)$ will not in general be spatial. In fact points of $f_*(\Omega_Y)$ correspond to sections of f, so — under mild separation assumptions on Y — $f_*(\Omega_Y)$

is spatial iff f has "enough local sections", in the sense that every $y \in Y$ is in the image of a section of f over an open neighbourhood of $f(y)$.)

In particular, the techniques described above can be applied to the notion of propriety. Recall that a map $f\colon Y \to X$ is said to be *proper* if

(i) For every $x \in X$, the fibre $f^{-1}(x)$ is compact.

(ii) f is closed, i.e. the image of a closed subset of Y is closed in X.

(iii) f is separated, i.e. the diagonal map $\Delta\colon Y \to Y \times_X Y$ is closed.

It can be shown that under mild separation conditions on X (the T_D-axiom of Aull and Thron [1] is sufficient) f is proper iff the locale $f_*(\Omega_Y)$ is compact regular. (Details of the proof will appear elsewhere [16].) It therefore seems reasonable to *define* a map of locales $f\colon B \to A$ (or more generally a localic geometric morphism $f\colon \underset{\sim}{F} \to \underset{\sim}{E}$) to be proper iff the internal locale $f_*(\Omega)$ is compact regular in $\mathrm{Shv}(A)$ (respectively $\underset{\sim}{E}$).

To apply the Tychonoff and Stone–Čech theorems in a topos, we need to know that their proofs are not only choice-free but also constructively valid, i.e. that they do not make unjustified use of the law of excluded middle. Now the proof of the Tychonoff theorem given in Section 2 is not constructive, because of the transfinite induction involved in Lemma 2.6; but that in Section 3, by reducing the induction to a single step, is constructive at least as far as the locales A_γ are concerned. As presented, however, the proof did make use of the law of excluded middle for the index set Γ, in definitions like that of $f_\gamma(s, t)$ in Lemma 2.4.

In fact what is wrong here is our definition of the semilattice coproduct A; if the index set Γ does not obey the law of excluded middle, we cannot identify A with a subset of the semilattice product B. So we have to regard its elements as *formal* finite meets of elements of the form $q_\gamma(a_\gamma)$, subject to the obvious identifications; and then it will be found that all the relevant arguments in Sections 2 and 3 can be carried out constructively (though at some cost in added notational complexity), with the exception of the proof of Lemma 2.4. For arbitrary locales A_γ, the latter appears to be illegitimate if Γ does not satisfy the law of excluded middle; but in the case which interests us, when the A_γ are compact and locally compact, we can repair the damage by arguments like those of Lemma 3.1. As an example of the methods employed, we shall give a constructive proof of a lemma which covers both 2.4 and 3.1. First, however, let us remark that the word "finite", as applied to subsets of Γ, must be interpreted in a topos as "Kuratowski-finite" [15, 9.11]; and Theorem 9.20 of [15], which asserts that Kuratowski-finite objects are "finitely enumerable", is required to justify both the finite induction used in the proof of Lemma 2.3 and the notation used in what follows.

LEMMA 5.1. *With the notation of Section 2, suppose the locales A_γ are compact and locally compact. Then for any C_f-ideal I in A, $D(I)$ is a C-ideal.*

Proof. Let $a = \bigwedge_{i=1}^{n} q_{\gamma_i}(a_{\gamma_i})$ be a typical element of A, and suppose we have an expression $a_{\gamma_1} = \bigvee S$ in A_{γ_1} such that each $q_{\gamma_1}(s) \wedge a$ is in $D(I)$. Then for every choice of $b_{\gamma_i} \ll a_{\gamma_i}$ for each i and $t \ll s$ for some $s \in S$, we have $q_{\gamma_i}(t) \wedge \bigwedge_{i=1}^{n} q_{\gamma_i}(b_{\gamma_i}) \in I$. And since I is a C_f-ideal, we deduce that every element of the form

$$q_{\gamma_i}(t_1 \vee \ldots \vee t_m) \wedge \bigwedge_{i=1}^{n} q_{\gamma_i}(b_{\gamma_i})$$

is in I. But the family of all such elements is directed, and their join in A is $q_{\gamma_i}(\bigvee S) \wedge \bigwedge_{i=1}^{n} q_{\gamma_i}(a_{\gamma_i}) = a$. So $a \in D(I)$. ■

Thus we may prove the Tychonoff theorem for arbitrary ($\underset{\sim}{E}$-indexed) families of compact and locally compact locales in an arbitrary topos $\underset{\sim}{E}$. The arguments of Section 4 are all constructive, so we deduce that the Stone–Čech theorem holds too. Applying this to the concept of proper map which we discussed earlier, we deduce

Theorem 5.2. *Let $f\colon \underset{\sim}{F} \to \underset{\sim}{E}$ be a localic geometric morphism. Then there exists a (unique) best possible factorization of f through a proper map $f'\colon \underset{\sim}{F}' \to \underset{\sim}{E}$.*

Proof. Let $\underset{\sim}{F}'$ be the topos of $\underset{\sim}{E}$-valued sheaves on the Stone–Čech compactification of $f_*(\Omega_F)$. The unit map $f_*(\Omega_F) \to \beta(f_*(\Omega_F))$ induces a geometric morphism $\underset{\sim}{F} \to \underset{\sim}{F}'$ over $\underset{\sim}{E}$ with the required universal property. ■

In fact the word "localic" could be dropped from the hypotheses of the theorem, since it is well known that any geometric morphism f has a best possible factorization through a localic one, namely that which corresponds to $f_*(\Omega)$ (cf. [12], IV 7.8(d)). For spatial toposes, the factorization of Theorem 5.2 is that constructed by Dyckhoff [8, 9], and it seems likely that many of the properties of Dyckhoff's factorization can be obtained directly from known properties of Stone–Čech compactifications. Further applications of this idea will appear in [16].

Added in proof. Since this paper was submitted, I have learned that the Stone-Čech compactification of locales in a topos has been considered independently by B. Banaschewski and C. J. Mulvey [28]. Their methods are different from those of the present paper, and do not involve the Tychonoff theorem (although, of course, the Tychonoff theorem for compact regular locales may be deduced from their work).

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