# TYPE AND CONDUCTOR OF SIMPLICIAL AFFINE SEMIGROUPS 

RAHELEH JAFARI AND MARJAN YAGHMAEI


#### Abstract

We provide a generalization of pseudo-Frobenius numbers of numerical semigroups to the context of the simplicial affine semigroups. In this way, we characterize the Cohen-Macaulay type of the simplicial affine semigroup ring $\mathbb{K}[S]$. We define the type of $S$, type $(S)$, in terms of some Apéry sets of $S$ and show that it coincides with the Cohen-Macaulay type of the semigroup ring, when $\mathbb{K}[S]$ is Cohen-Macaulay. If $\mathbb{K}[S]$ is a $d$-dimensional CohenMacaulay ring of embedding dimension at most $d+2$, then type $(S) \leq 2$. Otherwise, type $(S)$ might be arbitrary large and it has no upper bound in terms of the embedding dimension. Finally, we present a generating set for the conductor of $S$ as an ideal of its normalization.


## 1. Introduction

Let $S$ be an affine semigroup in $\mathbb{N}^{d}$, where $\mathbb{N}$ denotes the set of nonnegative integers. The affine semigroup ring $\mathbb{K}[S]$, over a field $\mathbb{K}$, is defined as the subring $\left\{\oplus_{\mathbf{a} \in S} k_{\mathbf{a}} \mathbf{x}^{\mathrm{a}}: k_{\mathbf{a}} \in \mathbb{K}\right\}$ of the polynomial ring $\mathbb{K}[\mathbf{x}]=\mathbb{K}\left[x_{1}, \ldots, x_{d}\right]$. The structure of $\mathbb{K}[S]$ is intimately related to the structure of the affine semigroup $S$ and cone $(S)$, the rational polyhedral cone spanned by $S$. The Cohen-Macaulay, Gorenstein and Buchsbaum properties of $\mathbb{K}[S]$ have been characterized in terms of certain numerical and topological properties of $S$, see [14], [17], [18], [13], [23] and [10]. Our aim in this paper, is to characterize the conductor and Cohen-Macaulay type of $\mathbb{K}[S]$.

If $d=1$, then $S$ is a submonoid of $\mathbb{N}$. Let $h$ be the greatest common divisor of nonzero elements in $S$. Dividing all elements of $S$ by $h$, we obtain an isomorphic semigroup in $\mathbb{N}$. A submonoid $S$ of $\mathbb{N}$ such that $\operatorname{gcd}(s ; s \in S)=1$ is called a numerical semigroup. In other words, the study of affine semigroups in $\mathbb{N}$ reduces to the study of numerical semigroups. The condition $\operatorname{gcd}(s ; s \in S)=1$ is equivalent to say that $\mathbb{N} \backslash S$ is a finite set, [25, Lemma 2.1]. For an affine semigroup $S$, consider the natural partial ordering $\preceq_{S}$ on $\mathbb{N}^{d}$ where, for all elements $x, y \in \mathbb{N}^{d}, x \preceq_{S} y$ if $y-x \in S$. For a numerical semigroup $S \varsubsetneqq \mathbb{N}$, the maximal elements of $\mathbb{N} \backslash S$ with respect to $\preceq_{S}$ are called pseudo-Frobenius numbers. Fröberg, Gottlieb and Häggkvist [9], defined the type of the numerical semigroup $S$ as the cardinality of the set of its pseudo-Frobenius numbers. This notion of type coincides with the Cohen-Macaulay type of the numerical semigroup ring $\mathbb{K}[S]$, see [28] for a detailed proof.

By analogy, García-García, Ojeda, Rosales and Vingneron-Tenorio, define a pseudo-Frobenius element of $S$ to be an element $\mathbf{a} \in \mathbb{N}^{d} \backslash S$ such that $\mathbf{a}+S \backslash\{0\} \subseteq S$, in [12]. They show that the set of pseudo-Frobenius elements of $S, \operatorname{PF}(S)$, is not

[^0]empty, precisely when depth $\mathbb{K}[S]=1$. Thus, when $d>1$ and $\mathbb{K}[S]$ is a CohenMacaulay ring, the set of pseudo-Frobenius elements of $S$ is empty and express noting about the Cohen-Macaulay type of the semigroup ring.

In Section 3, we present another generalization of pseudo-Frobenius numbers, determining the Cohen-Macaulay type of the semigroup ring $\mathbb{K}[S]$, under the assumption that the affine semigroup $S \subset \mathbb{N}^{d}$ is also simplicial, i.e. cone $(S)$ has $d$ extremal rays. All fully embedded affine semigroups in $\mathbb{N}^{d}$, for $d=1,2$, are simplicial. Let $\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}$ be the componentwise smallest integer vectors in $S$, situated on each extremal ray of cone $(S)$, respectively. Then $\cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)$ is a finite set, where $\operatorname{Ap}\left(S, \mathbf{a}_{i}\right)=\left\{\mathbf{a} \in S ; \mathbf{a}-\mathbf{a}_{i} \notin S\right\}$ denotes the Apéry set of $S$ with respect to $\mathbf{a}_{i}$. In numerical case, where $d=1, \mathbf{a}_{1}$ is the smallest positive number in $S$ and the set of pseudo-Frobenius numbers is equal to the set $\left\{\mathbf{w}-\mathbf{a}_{1} ; \mathbf{w} \in \operatorname{Max}_{\preceq_{S}} \operatorname{Ap}\left(S, \mathbf{a}_{1}\right)\right\}$. In an analogue way, we consider the set

$$
\operatorname{QF}(S)=\left\{\mathbf{w}-\sum_{i=1}^{d} \mathbf{a}_{i} ; \mathbf{w} \in \max _{\preceq S} \cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)\right\} .
$$

We call the elements of $\mathrm{QF}(S)$, quasi-Frobenius elements, to distinguish from $\operatorname{PF}(S)$ introduced in [12]. In Proposition 3.3, we show that the cardinality of $\mathrm{QF}(S)$ which is called the type of $S$ and is denoted by type $(S)$, equals the Cohen-Macaulay type of $\mathbb{K}[S]$, when the semigroup ring is Cohen-Macaulay. Campillo and Gimenez in [5], study combinatorics of some simplicial complexes associated to elements of $S$. They compute the homologies of the simplicial complexes in terms of certain graph homologies, and show in [5, Theorem 4.2(ii)], that for Cohen-Macaulay simplicial affine semigroups, type $(S)$ equals the number of maximal elements of $\operatorname{Ap}(S, E)$ with respect to $\preceq_{S}$. Our result, Proposition 3.3, provides a different algebraic proof for this fact. Although $\operatorname{PF}(S)$ and $\mathrm{QF}(S)$ coincide in the numerical case where $d=1$, but for $d>1$ they have no common element, see Remark 3.2.

As an immediate consequence, we derive that the simplicial affine semigroup ring $\mathbb{K}[S]$ is Gorenstein if and only if it is Cohen-Macaulay and $\cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)$ has a single maximal element with respect to $\preceq_{S}$. This result is already proved in [5, 23], by different arguments. In the rest of Section 3, we investigate the above bound for type $(S)$. Generalizing [9, Theorem 11], we show in Theorem 3.5, that type $(S) \leq 2$, if $S$ is generated by $d+2$ elements and $\mathbb{K}[S]$ is Cohen-Macaulay. If either $\mathbb{K}[S]$ is not Cohen-Macaulay or if the minimal generating set of $S$ has more than $d+2$ elements, then type $(S)$ might be arbitrary large, see Example 3.7 and Example 3.8.

Recall that the normalization of an integral domain $R$ is the set of elements in its field of fractions satisfying a monic polynomial in $R[y]$. Then $R=\mathbb{K}[S]$ is an integral domain with normalization $\bar{R}=\mathbb{K}[\operatorname{group}(S) \cap \operatorname{cone}(S)]$, where $\operatorname{group}(S)$ denotes the group of differences of $S,[21$, Proposition 7.25$]$. The purpose of the last section, is to investigate the normality of $R$ and, to detect a generating set for the conductor of $R$, $C_{R}=\left(R:_{T} \bar{R}\right)$, where $T$ denotes the total ring of fractions of $R$. We will work with corresponding objects in $S$. In semigroup interpretation, $\bar{S}=\operatorname{group}(S) \cap \operatorname{cone}(S)$ and $\mathfrak{c}(S)=\{\mathbf{b} \in S ; \mathbf{b}+\bar{S} \subseteq S\}$, are called the normalization and the conductor of $S$, respectively, [3]. The ring $R$ is normal precisely when the semigroup $S$ is normal, i.e. $S=\bar{S},[4]$.

Quasi-Frobenius elements, are also profitable to recognize the normality of $S$. Note that quasi-Frobenius elements, might have negative components. Having more
negative components in elements of $\mathrm{QF}(S)$, makes the semigroup more close to being normal. More precisely, if $S$ is normal, then $-\mathrm{QF}(S) \subseteq$ cone $(S)$ and so $-\mathrm{QF}(S) \subseteq \mathbb{N}^{d}$. The converse holds, if $-\mathrm{QF}(S)$ is a subset of the relative interior of cone $(S)$. This is the subject of Theorem 4.6, which states also that in this case $-\mathrm{QF}(S) \subseteq S$.

If $S \subseteq \mathbb{N}$ is a numerical semigroup, then $\bar{S}=\mathbb{N}$ and $\mathfrak{c}(S)$ is a principal ideal of $\mathbb{N}$ generated by $F+1$, where $F=\max (\mathbb{Z} \backslash S)$. The rest of Section 4 , is devoted to find a generating set for $\mathfrak{c}(S)$, as an ideal of $\bar{S}$, where $S$ is an arbitrary simplicial affine semigroup.

Note that any element $\mathbf{c} \in \operatorname{cone}(S) \cap \mathbb{N}^{d}$, is uniquely presented as $\mathbf{c}=\sum_{i=1}^{d} n_{i} \mathbf{a}_{i}+$ $r(\mathbf{c})$ for some $n_{1}, \ldots, n_{d} \in \mathbb{N}$ and $r(\mathbf{c}) \in P_{S}$, where $P_{S}=\left\{\sum_{i=1}^{d} r_{i} \mathbf{a}_{i} ; 0 \leq\right.$ $\left.r_{1}, \ldots, r_{d}<1\right\}$ is the fundamental parallelogram of $S$. It is not difficult to observe that $P_{S} \cap \operatorname{group}(S)=\left\{r(\mathbf{w}) ; \mathbf{w} \in \cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)\right\}$, see Lemma 4.3. Let $P_{S} \cap \operatorname{group}(S)=\left\{0=\mathbf{b}_{0}, \mathbf{b}_{1}, \ldots, \mathbf{b}_{k}\right\}$. For $\mathbf{w}_{1}, \ldots, \mathbf{w}_{k} \in \cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)$ with $r\left(\mathbf{w}_{i}\right)=\mathbf{b}_{i}$, we consider the vector $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}=\sum_{i=1}^{d} f_{i} \mathbf{a}_{i}$, where $f_{i}$ is the maximum integer such that $\mathbf{w}_{j}-f_{i} \mathbf{a}_{i} \in P_{S}$ for some $j=1, \ldots, k$. We show in Theorem 4.9, that any element of the minimal generating set for $\mathfrak{c}(S)$, as an ideal of $\bar{S}$, is of the form $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}-\mathbf{b}_{j}+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i}$ for some $l_{i} \in\{0,1\}, j \in\{1, \ldots, k\}$ and $\mathbf{w}_{1}, \ldots, \mathbf{w}_{k} \in \cap \cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)$ such that $r\left(\mathbf{w}_{i}\right)=\mathbf{b}_{i}$ for $i=1, \ldots, k$. Moreover, at least for one $i$, we have $l_{i}=0$. As $\cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)$ is a finite set and $\mathfrak{c}(S)=\left\{\mathbf{c} \in S ; \mathbf{c}+\mathbf{b}_{i} \in S\right.$ for $\left.i=1, \ldots, k\right\}$, Theorem 4.9 provides an algorithmic way to find a generating set for $\mathfrak{c}(S)$. Let $\preceq_{c}$ denote the natural partial ordering with respect to cone $(S)$, that is $\mathbf{a} \preceq_{c} \mathbf{b}$ if $\mathbf{b}-\mathbf{a} \in \operatorname{cone}(S)$. We show in Corollary 4.17 that, if $\mathbb{K}[S]$ is Cohen-Macaulay and $\max _{\preceq_{c}} \cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)=\{\mathbf{w}\}$, then $\left\{\mathbf{w}-\mathbf{b} ; \mathbf{b} \in \max _{\preceq_{c}} r(\operatorname{Ap}(S, E))\right\}$, generates $\mathfrak{c}(S)$ as an ideal of $\bar{S}$. Several explicit examples are provided to illustrate the generating set of $\mathfrak{c}(S)$, as an ideal of $\bar{S}$.

## 2. Fundamentals

By an affine semigroup, we mean a finitely generated submonoid of $\mathbb{N}^{d}$, where $\mathbb{N}$ denotes the set of nonnegative integers and $d \in \mathbb{N} \backslash\{0\}$. Let $S$ be an affine semigroup minimally generated by $\operatorname{mgs}(S)=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{e}\right\}$. We write $S=\left\langle\mathbf{a}_{1}, \ldots, \mathbf{a}_{e}\right\rangle$, to indicate its generating set. The minimal generating set, $\operatorname{mgs}(S)$, is a unique finite set, see [24, Chapter 3]. The number of elements in $\operatorname{mgs}(S)$ is called the embedding dimension of $S$. For a field $\mathbb{K}$, the semigroup ring $\mathbb{K}[S]$ is the subalgebra of the polynomial ring $\mathbb{K}\left[x_{1}, \ldots, x_{d}\right]$ generated by the monomials with exponents in $S$. The ring $\mathbb{K}[S]=\mathbb{K}\left[\mathbf{x}^{\mathbf{a}_{1}}, \ldots, \mathbf{x}^{\mathbf{a}_{e}}\right]$ has a unique maximal monomial ideal $\mathfrak{m}=\left(\mathrm{x}^{\mathbf{a}_{1}}, \ldots, \mathrm{x}^{\mathbf{a}_{e}}\right)$.

Given two sets $A, B \subseteq \mathbb{N}^{d}$, we write $A+B$ for the set $\{\mathbf{a}+\mathbf{b} ; \mathbf{a} \in A, \mathbf{b} \in B\}$. If $A=\{\mathbf{a}\}$, we simply write $\mathbf{a}+B$, instead of $\{\mathbf{a}\}+B$. Recall that an ideal of $S$ is a nonempty set $H \subseteq S$, such that $H+S \subseteq S$. For any ideal $H$ of $S$, there exists a set of vectors $B=\left\{\mathbf{b}_{1}, \ldots, \mathbf{b}_{l}\right\}$ such that $H=B+S=\cup_{i=1}^{l} \mathbf{b}_{i}+S$. In this case, $B$ is called a generating set of $H$. If no proper subset of $B$ generates $H$, we refer to $B$ as the minimal generating set of $H$. Any ideal of an affine semigroup has a unique minimal generating set. Note that for ideals $H_{1}$ and $H_{2}$ of $S, H_{1}+H_{2}$ is also an ideal of $S$. In particular, for an ideal $H$ of $S, n$ times summation of $H$ which is denoted by $n H$, is again an ideal of $S$. Let $M=S \backslash\{0\}$ be the maximal ideal of $S$.

A monomial in the semigroup ring $\mathbb{K}[S]$ is an element of the form $\mathbf{x}^{\mathbf{a}}$ for $\mathbf{a} \in S$. An ideal $I \subseteq \mathbb{K}[S]$ is a monomial ideal if it is generated by monomials. For any
subset $H$ of $S$, let $\mathbb{K}[H]$ denote the $\mathbb{K}$-linear span of the monomials $\mathbf{x}^{\mathbf{a}}$ with $\mathbf{a} \in H$. Then $I$ is a monomial ideal if and only if $I=\mathbb{K}[H]$ for some ideal $H$ of $S$, or equivalently, if $I$ is homogeneous with respect to the tautological grading on $\mathbb{K}[S]$, which is defined by $\operatorname{deg}\left(\mathbf{x}^{\mathbf{a}}\right)=\mathbf{a}$. Note that $\mathfrak{m}=\mathbb{K}[M]$.

The affine semigroup $S \subseteq \mathbb{N}^{d}$ is called simplicial if there exist $d$ elements $\mathbf{a}_{i_{1}}, \ldots, \mathbf{a}_{i_{d}} \in \operatorname{mgs}(S)$ such that they are linearly independent over the field of rational numbers $\mathbb{Q}$ (equivalently, over the field of real numbers $\mathbb{R}$ ), and for each element $\mathbf{a} \in S$, we have $n \mathbf{a} \in \mathbb{N} \mathbf{a}_{i_{1}}+\cdots+\mathbb{N} \mathbf{a}_{i_{d}}$, for some positive integer $n$. There is a geometrical interpretation for the simplicial property. Let

$$
\operatorname{cone}(S)=\left\{\sum_{i=1}^{e} \lambda_{i} \mathbf{a}_{i} ; \lambda_{i} \in \mathbb{R}_{\geq 0}, \text { for } i=1, \ldots, e\right\}
$$

denote the rational polyhedral cone generated by $S$. The dimension (or rank) of $S$ is defined as the dimension of the affine subspace it generates, which is the same as the dimension of the subspace generated by cone $(S)$. The cone $(S)$ is polyhedral i.e. it is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^{d}$, each of whose bounding hyperplanes contains the origin, [27, Corollary 7.1(a)]. These half-spaces are called support hyperplanes. The integral vectors in each support hyperplane, is a face of $S$, and all maximal faces (called facets) are in this form. The intersection of any two adjacent support hyperplanes is a one-dimensional vector space, which is called an extremal ray. The cone $(S)$ has at least $d$ facets and at least $d$ extremal rays. It has $d$ facets (equivalently, it has $d$ extremal rays), precisely when $S$ is simplicial.

Throughout this paper, we assume that $S$ is a simplicial affine semigroup. On each extremal ray of cone $(S)$, the componentwise smallest element from $S$, is called an extremal ray of $S$. Denote by $\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}$ the extremal rays of $S$. These form a basis for cone $(S)$. For $\mathbf{z} \in \mathbb{R}^{d}$ such that $\mathbf{z}=\sum_{i=1}^{d} \lambda_{i} \mathbf{a}_{i}$ with $\lambda_{i} \in \mathbb{Q}, i=1, \ldots, d$, we set $[\mathbf{z}]_{i}=\lambda_{i}$ for $i=1, \ldots, d$. For each element $\mathbf{a} \in S$, we have $n \mathbf{a} \in \mathbb{N} \mathbf{a}_{1}+\cdots+\mathbb{N} \mathbf{a}_{d}$, for some positive integer $n$. In other words, $\left\{\mathbf{x}^{\mathbf{a}_{1}}, \ldots, \mathbf{x}^{\mathbf{a}_{d}}\right\}$ provides a monomial system of parameters for $\mathbb{K}[S]$. The fundamental (semi-open) parallelotope of $S$ is the set

$$
\begin{aligned}
P_{S} & =\left\{\mathbf{z} \in \mathbb{R}^{d} ; 0 \leq[\mathbf{z}]_{i}<1 \text { for } i=1, \ldots, d\right\} \\
& =\left\{\sum_{i=1}^{d} \lambda_{i} \mathbf{a}_{i} ; \lambda_{i} \in \mathbb{Q}, 0 \leq \lambda_{i}<1 \text { for } i=1, \ldots, d\right\}
\end{aligned}
$$

Its closure in $\mathbb{R}^{d}$ is the set $\bar{P}_{S}=\left\{\mathbf{z} \in \mathbb{R}^{d} ; 0 \leq[\mathbf{z}]_{i} \leq 1\right.$ for $\left.i=1, \ldots, d\right\}$. It is well known, and easy to see, that any a in $\operatorname{cone}(S) \cap \mathbb{N}^{d}$ decomposes uniquely as $\mathbf{a}=\sum_{i=1}^{d} n_{i} \mathbf{a}_{i}+r(\mathbf{a})$, with $r(\mathbf{a}) \in P_{S} \cap \mathbb{N}^{d}$ and nonnegative integers $n_{1}, \ldots, n_{d}$. We will call $r(\mathbf{a})$ the remainder of $\mathbf{a}$ in $P_{S}$.

Remark 2.1. Let $\mathbf{a} \in \operatorname{cone}(S) \cap \mathbb{N}^{d}$ and let $n_{i}=\left\lfloor[\mathbf{a}]_{i}\right\rfloor$ be the unique integer such that $n_{i} \leq[\mathbf{a}]_{i}<n_{i}+1$, for $i=1, \ldots, d$. Then $r(\mathbf{a})=\sum_{i=1}^{d}\left([\mathbf{a}]_{i}-n_{i}\right) \mathbf{a}_{i}$.

We consider the natural partial orderings $\preceq_{S}$ and $\preceq_{c}$ on $\mathbb{N}^{d}$ where, for all elements $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{N}^{d}, \mathbf{b} \preceq_{S} \mathbf{a}\left(\mathbf{b} \preceq_{c} \mathbf{a}\right)$, if there is an element $\mathbf{c} \in S(\mathbf{c} \in \operatorname{cone}(S))$ such that $\mathbf{a}=\mathbf{b}+\mathbf{c}$. The partial order $\preceq_{c}$ is indeed the coordinatewise order on cone $(S) \cap \mathbb{N}^{d}$. More precisely, for $\mathbf{a}, \mathbf{b} \in \operatorname{cone}(S) \cap \mathbb{N}^{d}$, $\mathbf{a} \preceq_{c} \mathbf{b}$ if and only if $[\mathbf{a}]_{i} \leq[\mathbf{b}]_{i}$ for $i=1, \ldots, d$.

An element $\mathbf{a} \in S$, may be written as $\mathbf{a}=\sum_{i=1}^{d+r} l_{i} \mathbf{a}_{i}$ for some nonnegative integers $l_{1}, \ldots, l_{d+r}$. The value $\sum_{i=1}^{d+r} l_{i}$ is called the length of the expression $\sum_{i=1}^{d+r} l_{i} \mathbf{a}_{i}$. The maximum integer $n$ such that $n M$ contains $\mathbf{a}$, is called the order of $\mathbf{a}$ and it is denoted by ord( $\mathbf{a}$ ). In other words, $\mathbf{a} \in n M \backslash(n+1) M$ if and only if $n=\operatorname{ord}(\mathbf{a})$. The expression of length $\operatorname{ord}(\mathbf{a})$ of $\mathbf{a}$, is called a maximal expression of $\mathbf{a}$.

The Apéry set of an element $\mathbf{b} \in S$ is defined as $\operatorname{Ap}(S, \mathbf{b})=\{\mathbf{a} \in S ; \mathbf{a}-\mathbf{b} \notin S\}$. We will denote the zero vector of $\mathbb{N}^{d}$ by 0 . Since $S \subseteq \mathbb{N}^{d}$, for $\mathbf{b} \neq 0$ we have $0 \in \operatorname{Ap}(S, \mathbf{b})$. Note that if $\mathbf{a} \in \operatorname{Ap}(S, \mathbf{b})$ and $\mathbf{z} \in S$ such that $\mathbf{z} \preceq_{S} \mathbf{a}$, then $\mathbf{z} \in \operatorname{Ap}(S, \mathbf{b})$. For a subset $E$, we set

$$
\operatorname{Ap}(S, E)=\{\mathbf{a} \in S ; \mathbf{a}-\mathbf{b} \notin S, \text { for all } \mathbf{b} \in E\}
$$

Throughout the paper, $E=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right\}$ will denote the set of extremal rays of $S$. Then $\operatorname{Ap}(S, E)=\cap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right)$.

Let $I_{S}$ denote the kernel of the $\mathbb{K}$-algebra homomorphism $\varphi: \mathbb{K}\left[z_{1}, \ldots, z_{d+r}\right] \longrightarrow$ $\mathbb{K}[S]$, defined by $z_{i} \mapsto \mathbf{x}^{\mathbf{a}_{i}}$, for $i=1, \ldots, d+r$. Then $I_{S}$ is a binomial prime ideal, [16, Proposition 1.4]. Note that $\mathbb{K}[S] \cong \mathbb{K}\left[z_{1}, \ldots, z_{d+r}\right] / I_{S}$ has $S$-graded structure defined by $\operatorname{deg}_{S}\left(z_{1}^{n_{1}} \ldots z_{d+r}^{n_{d+r}}\right)=\sum_{i=1}^{d+r} n_{i} \mathbf{a}_{i}$. Let $R^{\prime}=\frac{\mathbb{K}\left[z_{1}, \ldots, z_{d+r}\right]}{I_{S}+\left(z_{1}, \ldots, z_{d}\right)}$. Then, as a $\mathbb{K}$-vector space, $R^{\prime}$ is generated by the set of monomials $\mathbf{z}^{\mathbf{a}}$ such that $\mathbf{a} \notin$ $I_{S}+\left(z_{1}, \ldots, z_{d}\right)$. Let $B$ denote the monomial $\mathbb{K}$-basis of $R^{\prime}$. From [22, Theorem 3.3], we have

$$
\operatorname{Ap}(S, E)=\left\{\operatorname{deg}_{S}(u) ; u \in B\right\}
$$

Therefore, as an algorithm to find $\operatorname{Ap}(S, E)$, one may first compute $I_{S}$, using any of computer algebra systems GAP [8], Singular [7], CoCoA [1] or Macaulay2 [15], and then, find the monomial basis of $\frac{\mathbb{K}\left[z_{1}, \ldots, z_{d+r}\right]}{I_{S}+\left(z_{1}, \ldots, z_{d}\right)}$.
Example 2.2. Let $\mathbf{a}_{1}=(5,3,1), \mathbf{a}_{2}=(1,5,2), \mathbf{a}_{3}=(8,3,5), \mathbf{a}_{4}=(2,1,1), \mathbf{a}_{5}=$ $(2,2,1)$. A computation by Macaulay2 [15], shows that

$$
I_{S}=\left(z_{5}^{5}-z_{1} z_{2} z_{4}^{2}, z_{4}^{19}-z_{1}^{2} z_{3}^{3} z_{5}^{2}, z_{4}^{17} z_{5}^{3}-z_{1}^{3} z_{2} z_{3}^{3}\right)
$$

Consequently, $I_{S}+\left(z_{1}, z_{2}, z_{3}\right)=\left(z_{1}, z_{2}, z_{3}, z_{4}^{19}, z_{5}^{5}, z_{4}^{17} z_{5}^{3}\right)$. The image of

$$
\left\{1, z_{4}^{r} z_{5}, z_{4}^{r} z_{5}^{2}, z_{4}^{s} z_{5}^{3}, z_{4}^{s} z_{5}^{4}, z_{4}^{t} ; 0 \leq r \leq 18,0 \leq s \leq 16,1 \leq t \leq 18\right\}
$$

in $\frac{\mathbb{K}\left[z_{1}, \ldots, z_{5}\right]}{I_{S}+\left(z_{1}, z_{2}, z_{3}\right)}$, provides a $\mathbb{K}$-basis. Therefore, $\operatorname{Ap}(S, E)$ is equal to the set
$\left\{0, r \mathbf{a}_{4}+\mathbf{a}_{5}, r \mathbf{a}_{4}+2 \mathbf{a}_{5}, s \mathbf{a}_{4}+3 \mathbf{a}_{5}, s \mathbf{a}_{4}+4 \mathbf{a}_{5}, t \mathbf{a}_{4} ; 0 \leq r \leq 18,0 \leq s \leq 16,1 \leq t \leq 18\right\}$.
We write $\operatorname{group}(S)$ for the group of differences of $S$, i.e. $\operatorname{group}(S)$ is the smallest group (up to isomorphism) that contains $S$.

$$
\operatorname{group}(S)=\{\mathbf{a}-\mathbf{b} \mid \mathbf{a}, \mathbf{b} \in S\}
$$

By $\operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)$, we mean the smallest group that contains $\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right\}$, equivalently $\operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)=\left\{\sum_{i=1}^{d} z_{i} \mathbf{a}_{i} ; z_{i} \in \mathbb{Z}\right\}$.
Remark 2.3. For $\mathbf{c} \in \operatorname{group}(S)$, there exists $\mathbf{b} \in S$ such that $\mathbf{c}+\mathbf{b} \in S$. As $S$ is simplicial, $n \mathbf{b} \in \sum_{i=1}^{d} \mathbb{N} \mathbf{a}_{i}$ for some positive integer $n$. Therefore, $\mathbf{c}+n \mathbf{b}=$ $\mathbf{c}+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i} \in S$, for some $r_{i} \in \mathbb{N}$. Consider $r_{1}, \ldots, r_{d}$, as small as possible with this property, i.e.

$$
\begin{equation*}
\mathbf{c}+\left(r_{j}-1\right) \mathbf{a}_{j}+\sum_{i=1, i \neq j}^{d} r_{i} \mathbf{a}_{i} \notin S \tag{2.1}
\end{equation*}
$$

for $j=1, \ldots, d$. Let $\mathbf{c}+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i}=\mathbf{w}+\sum_{i=1}^{d} s_{i} \mathbf{a}_{i}$ for some $\mathbf{w} \in \operatorname{Ap}(S, E)$ and $s_{1}, \ldots, s_{d} \in \mathbb{N}$. If $r_{j}>0$, for some $1 \leq j \leq d$, then

$$
\mathbf{c}+\left(r_{j}-1\right) \mathbf{a}_{j}+\sum_{i=1, i \neq j}^{d} r_{i} \mathbf{a}_{i}=\mathbf{w}+\left(s_{j}-1\right) \mathbf{a}_{j}+\sum_{i=1, i \neq j}^{d} r_{i} \mathbf{a}_{i}
$$

By (2.1), we get $s_{j}=0$.
As an advantage of considering $\operatorname{Ap}(S, E)$, we recall the following criteria for the Cohen-Macaulay property of $\mathbb{K}[S]$.

Proposition 2.4. [23, Corollary 1.6] The following statements are equivalent.
(1) $\mathbb{K}[S]$ is Cohen-Macaulay.
(2) For all $\mathbf{w}_{1}, \mathbf{w}_{2} \in \operatorname{Ap}(S, E)$, if $\mathbf{w}_{1}-\mathbf{w}_{2} \in \operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)$, then $\mathbf{w}_{1}=\mathbf{w}_{2}$.

The Cohen-Macaulay property is indeed equivalent to have a one to one correspondence between elements in $\operatorname{Ap}(S, E)$ and their remainders. More precisely, let

$$
r(\operatorname{Ap}(S, E))=\{r(\mathbf{w}) ; \mathbf{w} \in \operatorname{Ap}(S, E)\}=\left\{0=\mathbf{b}_{0}, \mathbf{b}_{1}, \ldots, \mathbf{b}_{k}\right\}
$$

Then the family of subsets $C_{j}=\left\{\mathbf{w} \in \operatorname{Ap}(S, E) ; r(\mathbf{w})=\mathbf{b}_{j}\right\}$, for $j=0, \ldots, k$, defines a partition of $\operatorname{Ap}(S, E)$.

Lemma 2.5. The following statements hold.
(1) $\mathbb{K}[S]$ is Cohen-Macaulay if and only if $C_{i}$ is a singleton for $i=1, \ldots, k$.
(2) $\mathbb{K}[S]$ is Buchsbaum if and only if, either $C_{i}$ is a singleton or $C_{i}=\{\mathbf{c}+$ $\left.\mathbf{a}_{1}, \ldots, \mathbf{c}+\mathbf{a}_{d}\right\}$ for some $\mathbf{c} \in \mathbb{N}^{d}$ such that $\mathbf{c}+(S \backslash\{0\}) \subset S$, for $i=1, \ldots, d$.

Proof. Let $\mathbf{v}, \mathbf{w} \in \operatorname{Ap}(S, E)$. Then $\mathbf{v}-\mathbf{w}=r(\mathbf{v})+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i}-r(\mathbf{w})-\sum_{i=1}^{d} s_{i} \mathbf{a}_{i}$, for some $r_{i}, s_{i} \in \mathbb{N}$. In particular, $\mathbf{v}-\mathbf{w}=r(\mathbf{v})-r(\mathbf{w})+\mathbf{b}$, where $\mathbf{b}=\sum_{i=1}^{d}\left(r_{i}-\right.$ $\left.s_{i}\right) \mathbf{a}_{i} \in \operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)$. Therefore, $\mathbf{v}-\mathbf{w} \in \operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)$ if and only if $r(\mathbf{v})=r(\mathbf{w})$, equivalently $\mathbf{v}, \mathbf{w} \in C_{i}$, for some $0 \leq i \leq k$. Thus, $C_{i} \mathrm{~s}$ are precisely the equivalence classes under the equivalence relation $\sim$ on $\operatorname{Ap}(S, E)$, where $\mathbf{w}_{i} \sim \mathbf{w}_{j}$ if $\mathbf{w}_{i}-\mathbf{w}_{j} \in \operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)$. Now, the statement (1) follows by Proposition 2.4, and the statement (2) is a consequence of [10, Theorem 11].

When $d=1$, equivalently $S$ is isomorphic to a numerical semigroup, then $\mathbb{K}[S]$ is a one-dimesional domain and consequently it is Cohen-Macaulay. From another point of view, since all elements of $S$ belong to the real line, one may easily use Proposition 2.4 or Lemma 2.5 to check that $\mathbb{K}[S]$ is Cohen-Macaulay. The following lemma, extends this property to simplicial affine semigroups in $\mathbb{N}^{d}$.

Lemma 2.6. Let $S$ be of embedding dimension $d+r$. If the vectors $\mathbf{a}_{d+1}, \ldots, \mathbf{a}_{d+r}$ belong to the same line passing through the origin of coordinates, then $\mathbb{K}[S]$ is Cohen-Macaulay.

Proof. As $\mathbf{a}_{d+1}, \ldots, \mathbf{a}_{d+r}$ belong to the same line passing through the origin of coordinates, there exist nonnegative rational numbers $l_{i} \in \mathbb{Q}$ such that $\mathbf{a}_{d+i}=l_{i} \mathbf{a}_{d+1}$, for $i=1, \ldots, r$. Let $\mathbf{w}_{1}, \mathbf{w}_{2} \in \operatorname{Ap}(S, E)$, then $\mathbf{w}_{j}=\lambda_{j} \mathbf{a}_{d+1}=\lambda_{j}\left(\sum_{i=1}^{d} \mu_{i} \mathbf{a}_{i}\right)$, for some nonnegative rational numbers $\lambda_{1}, \lambda_{2}, \mu_{1}, \ldots, \mu_{d}$. If $\mathbf{w}_{1}-\mathbf{w}_{2} \in \operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)$, then

$$
\left(\lambda_{1}-\lambda_{2}\right)\left(\sum_{i=1}^{d} \mu_{i} \mathbf{a}_{i}\right)=\mathbf{w}_{1}-\mathbf{w}_{2}=\sum_{i=1}^{d} z_{i} \mathbf{a}_{i}
$$

for some integers $z_{1}, \ldots, z_{d}$. Hence, for all $i, z_{i}=\left(\lambda_{1}-\lambda_{2}\right) \mu_{i}$, which forces the signs of $z_{i}$ to be the same for all $i$. If they are all nonnegative (nonpositive), the equation $\mathbf{w}_{1}=\mathbf{w}_{2}+\sum_{i=1}^{d} z_{i} \mathbf{a}_{i}\left(\mathbf{w}_{2}=\mathbf{w}_{1}+\sum_{i=1}^{d}\left(-z_{i}\right) \mathbf{a}_{i}\right)$, implies $z_{i}=0$, since $\mathbf{w}_{1}, \mathbf{w}_{2} \in \operatorname{Ap}(S, E)$. Therefore, $\mathbb{K}[S]$ is Cohen-Macaulay by Proposition 2.4.

The following lemma states an easy but useful property about maximal expressions of Apéry elements in $\operatorname{Ap}(S, E)$, when $S$ has only two nonextremal generators that belong to the same line passing through the origin of coordinates.
Lemma 2.7. Let $S$ be of embedding dimension $d+2$. Then
(1) There are no elements in $\operatorname{Ap}(S, E)$ having two different expressions with the same length. In particular, each element in $\operatorname{Ap}(S, E)$ has a unique maximal expression.
(2) Assume that $r \mathbf{a}_{d+1}=s \mathbf{a}_{d+2}$, where $r$ and $s$ are relatively prime positive integers with $r>s$. For $\mathbf{b} \in \operatorname{Ap}(S, E)$, an expression $\mathbf{b}=n_{1} \mathbf{a}_{d+1}+n_{2} \mathbf{a}_{d+2}$ is maximal if and only if $n_{2}<s$.
Proof. (1). Let $\operatorname{mgs}(S)=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}, \mathbf{a}_{d+1}, \mathbf{a}_{d+2}\right\}$. Assume on the contrary that, an element $\mathbf{b} \in \operatorname{Ap}(S, E)$ has two expressions of the same length

$$
\mathbf{b}=n_{1} \mathbf{a}_{d+1}+n_{2} \mathbf{a}_{d+2}=n_{1}^{\prime} \mathbf{a}_{d+1}+n_{2}^{\prime} \mathbf{a}_{d+2}
$$

Then $\left(n_{1}-n_{1}^{\prime}\right) \mathbf{a}_{d+1}+\left(n_{2}-n_{2}^{\prime}\right) \mathbf{a}_{d+2}=0$. Since $n_{1}+n_{2}=n_{1}^{\prime}+n_{2}^{\prime}, n_{1}-n_{1}^{\prime}=$ $n_{2}^{\prime}-n_{2} \neq 0$. Therefore $\mathbf{a}_{d+1}=-\mathbf{a}_{d+2}$, a contradiction.
(2). If $n_{2} \geq s$, then $\mathbf{b}=\left(n_{1}+r\right) \mathbf{a}_{d+1}+\left(n_{2}-s\right) a_{d+2}$. Since $\left(n_{1}+r+n_{2}-s\right)>$ $n_{1}+n_{2}$, the expression $n_{1} \mathbf{a}_{d+1}+n_{2} \mathbf{a}_{d+2}$ is not maximal.

Now assume that $n_{2}<s$. Let $\mathbf{b}=n_{1}^{\prime} \mathbf{a}_{d+1}+n_{2}^{\prime} \mathbf{a}_{d+2}$ be a maximal expression. Then $n_{2}^{\prime}$ is also smaller than $s$, by our first argument. Comparing the two expressions of $\mathbf{b}$, we derive

$$
\left(n_{1}-n_{1}^{\prime}\right) \mathbf{a}_{d+1}=\left(n_{2}^{\prime}-n_{2}\right) \mathbf{a}_{d+2}=\frac{r\left(n_{2}^{\prime}-n_{2}\right)}{s} \mathbf{a}_{d+1}
$$

Since $r$ and $s$ are relatively prime, this follows $\left(n_{2}^{\prime}-n_{2}\right)$ is a multiple of $s$, which contradicts $\left|n_{2}^{\prime}-n_{2}\right|<s$, unless $n_{2}^{\prime}=n_{2}$.

## 3. The type of simplicial affine semigroups

In this section, we are looking for a characterization of the Cohen-Macaulay type of the affine semigroup ring $\mathbb{K}[S]$, in terms of some numerical invariants of $S$. All over the section, as thorough the paper, $S$ is a $d$-dimensional simplicial affine semigroup with $\operatorname{mgs}(S)=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d+r}\right\}$, where $\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}$ are the extremal rays of $S$.

If $d=1$, then dividing elements of $S$ by the greatest common divisor of $\mathbf{a}_{1}, \ldots$, $\mathbf{a}_{d+r}$, we obtain an isomorphic semigroup. So, we may assume that $S$ is a numerical semigroup, equivalently $S$ is a submonid of $\mathbb{N}$ such that $\mathbb{N} \backslash S$ is a finite set. If $S \neq \mathbb{N}$, the elements of $\operatorname{PF}(S):=\max _{\preceq_{s}} \mathbb{N} \backslash S=\{f \in \mathbb{N} \backslash S ; f+s \in S$, for all $s \in S \backslash\{0\}\}$ are called pseudo-Frobenius numbers of $S$. The cardinality of $\operatorname{PF}(S)$ is called the type of $S$, by Fröberg, Gottlieb and Häggkvist, in [9]. This notion of type coincides with the Cohen-Macaulay type of the numerical semigroup ring $\mathbb{K}[S]$, see [28] for a detailed proof.

In general case that $d \geq 1$, the pseudo-Frobenius elements of $S$ are defined by analogy, in [12], to be $\operatorname{PF}(S)=\left\{\mathbf{a} \in \mathbb{N}^{d} \backslash S ; \mathbf{a}+S \backslash\{0\} \subseteq S\right\}$. As $\mathbb{N}^{d} \backslash S$ is not
necessarily a finite set, $\mathrm{PF}(S)$ might be an empty set. Indeed $\mathrm{PF}(S) \neq \emptyset$ if and only if depth $\mathbb{K}[S]=1,[12$, Theorem 6]. The pseudo-Frobenius numbers of a numerical semigroup, can be described in terms of Apéry sets. Let a be a nonzero element of a numerical semigroup $S$. Then $\operatorname{PF}(S)=\left\{\mathbf{w}-\mathbf{a} ; \mathbf{w} \in \operatorname{Max}_{\preceq_{S}} \operatorname{Ap}(S, \mathbf{a})\right\}$, see $[25$, 2.20]. First, we present a generalization of $\operatorname{PF}(S)$, for simplicial affine semigroups $S \subseteq \mathbb{N}^{d}$, in terms of some Apéry sets. Let $E=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right\}$ and let $l_{i}$ be the smallest positive integer such that $l_{i} \mathbf{a}_{d+i} \in \sum_{j=1}^{d} \mathbb{N a}_{j}$, for $i=1, \ldots, r$. Then

$$
\operatorname{Ap}(S, E)=\bigcap_{i=1}^{d} \operatorname{Ap}\left(S, \mathbf{a}_{i}\right) \subseteq\left\{\sum_{i=1}^{r} n_{i} \mathbf{a}_{d+i} ; 0 \leq n_{i}<l_{i}\right\}
$$

is a finite set. The last set in the above equation is called $\Gamma$ in [23].
Definition 3.1. The element $\mathbf{b}-\sum_{i=1}^{d} \mathbf{a}_{i}$, where $\mathbf{b} \in \operatorname{Max}_{\preceq_{S}} \operatorname{Ap}(S, E)$, is called a quasi-Frobenius element. The set of quasi-Frobenius elements of $S$ is denoted by $\mathrm{QF}(S)$. The number of quasi-Frobenius elements, is called the type of $S$ and is denoted by type $(S)$.

Remark 3.2. Let $d>1$. If $\mathbf{f} \in \mathrm{QF}(S) \cap \operatorname{PF}(S)$, then $\mathbf{f}+\mathbf{a}_{1}=\mathbf{m}-\sum_{i=2}^{d} \mathbf{a}_{i}$, where $\mathbf{m} \in \operatorname{Max}_{\preceq_{S}} \operatorname{Ap}(S, E)$. Since $\mathbf{f} \in \operatorname{PF}(S)$, this follows $\mathbf{f}+\mathbf{a}_{1} \in S$, which contradicts $\mathbf{m}-\sum_{i=2}^{d} \mathbf{a}_{i} \notin S$.

The type of a $d$-dimensional Cohen-Macaulay local $\operatorname{ring}(R, \mathfrak{m})$ is type $(R)=$ $\operatorname{dim}_{R / \mathfrak{m}} \operatorname{Ext}_{R}^{d}(R / \mathfrak{m}, R)$. For a Cohen-Macaulay ring $R$, the type is defined as the maximum of type $\left(R_{\mathfrak{p}}\right)$, where $\mathfrak{p}$ ranges in the set of maximal ideals of $R$.

The ring $\mathbb{K}[S]$ is $\mathbb{N}$-graded by setting $\operatorname{deg}\left(\mathbf{x}^{\mathbf{a}}\right)=|\mathbf{a}|$, for all $\mathbf{a} \in S$, where $\left|\left(a_{1}, \ldots, a_{d}\right)\right|=\sum_{i=1}^{d} a_{i}$, denotes the total degree. Therefore,

$$
\operatorname{type}(\mathbb{K}[S])=\operatorname{type}\left(\mathbb{K}[S]_{\mathfrak{m}}\right)
$$

by [2, Theorem]. The following result can be also deduced from [5, Theorem 4.2(ii)], where the authors provide a combinatorial method to study some simplicial complexes associated to the elements of $S$. We bring a different algebraic proof here.

Proposition 3.3. If $\mathbb{K}[S]$ is a Cohen-Macaulay ring, then type $(S)=\operatorname{type}\left(\mathbb{K}[S]_{\mathfrak{m}}\right)=$ type( $\mathbb{K} \llbracket S \rrbracket)$.

Proof. The ring map $\mathbb{K}[S]_{\mathfrak{m}} \longrightarrow \mathbb{K} \llbracket S \rrbracket$ is flat and has only one trivial fiber which is the field $\mathbb{K}$. Since $\mathbb{K} \llbracket S \rrbracket / \mathfrak{m} \mathbb{K} \llbracket S \rrbracket$, as an $\mathbb{K} \llbracket S \rrbracket$-module, has type 1 and depth 0 , we have

$$
\begin{aligned}
& \operatorname{type}\left(\mathbb{K} \llbracket S \rrbracket_{\mathbb{K}[S]_{\mathfrak{m}}}^{\otimes} \mathbb{K}[S]_{\mathfrak{m}}\right)=\operatorname{type}\left(\mathbb{K}[S]_{\mathfrak{m}}\right), \\
& \operatorname{depth}\left(\mathbb{K} \llbracket S \rrbracket_{\mathbb{K}[S]_{\mathfrak{m}}}^{\otimes} \mathbb{K}[S]_{\mathfrak{m}}\right)=\operatorname{depth}\left(\mathbb{K}[S]_{\mathfrak{m}}\right),
\end{aligned}
$$

by [4, Proposition 1.2.16]. Thus, $\mathbb{K} \llbracket S \rrbracket$ is Cohen-Macaulay and type $\left(\mathbb{K}[S]_{\mathfrak{m}}\right)=$ type $(\mathbb{K} \llbracket S \rrbracket)$. Let $R=\mathbb{K} \llbracket S \rrbracket$. Then $R$ is a local ring with maximal ideal $\mathfrak{m}=$ $\left(\mathbf{x}^{\mathbf{a}_{1}}, \ldots, \mathbf{x}^{\mathbf{a}_{d+r}}\right)$. Note that $\mathfrak{q}=\left(\mathbf{x}^{\mathbf{a}_{1}}, \ldots, \mathbf{x}^{\mathbf{a}_{d}}\right)$ is a parameter ideal of $R$, since $S$ is simplicial. As $R$ is Cohen-Macaulay, $\mathbf{x}^{\mathbf{a}_{1}}, \ldots, \mathbf{x}^{\mathbf{a}_{d}}$ provide a maximal $R$-regular sequence. By [4, Lemma 1.2.19],

$$
\operatorname{type}(R)=\operatorname{dim}_{R / \mathfrak{m}}\left(\operatorname{Hom}_{R}(R / \mathfrak{m}, R / \mathfrak{q})\right)
$$

Since $\operatorname{Hom}_{R}(R / \mathfrak{m}, R / \mathfrak{q}) \cong\left(0:_{R / \mathfrak{q}} \mathfrak{m}\right)=\{r \in R / \mathfrak{q} ; r \mathfrak{m}=0\}$, it is enough to show that $\left(0:_{R / \mathfrak{q}} \mathfrak{m}\right)$ is the $R / \mathfrak{m}$-vector space generated by residue classes of $\mathbf{x}^{\mathbf{s}}$, where
$\mathbf{s} \in \max _{\preceq_{S}} \operatorname{Ap}(S, E)$. For an element $\mathbf{f} \in R$, the residue of $\mathbf{f}$ in $R / \mathfrak{q}$ is equal to the residue of $\sum_{i \geq 1} r_{i} \mathbf{x}^{\mathbf{s}_{i}}$, for some $r_{i} \in \mathbb{K}$ and $\mathbf{s}_{i} \in \operatorname{Ap}(S, E)$. If the residue of $\mathbf{f}$ in $R / \mathfrak{q}$, belongs to $\left(0:_{R / \mathfrak{q}} \mathfrak{m}\right)$, then we derive $\mathbf{x}^{\mathbf{s}_{i}+\mathbf{a}_{j}} \in \mathfrak{q}$, for $i \geq 1$ and $1 \leq j \leq d+r$ which implies $\mathbf{s}_{i} \in \max _{\preceq_{S}} \operatorname{Ap}(S, E)$. Conversely, let $\mathbf{s} \in \max _{\preceq_{S}} \operatorname{Ap}(S, E)$. Since $\mathbf{s}+\mathbf{a}_{i} \notin \operatorname{Ap}(S, E)$, for $i=d+1, \ldots, d+r$, we get $\mathbf{x}^{\mathbf{s}+\mathbf{a}_{i}} \in \mathfrak{q} R$.

Recall that a Cohen-Macaulay ring is Gorenstein precisely when its CohenMacaulay type is one. As an immediate consequence of Proposition 3.3, we derive the following

Corollary 3.4. [23, 4.6 and 4.8$][5,4.2] \mathbb{K}[S]$ is a Gorenstein ring if and only if it is Cohen-Macaulay and $\operatorname{Ap}(S, E)$ has a single maximal element with respect to $\preceq_{S}$.

If $S$ is a numerical semigroup of embedding dimension three, then type $(S) \leq 2$, [9, Theorem 11]. The following result is a generalization of this fact, to simplicial affine semigroups of embedding dimension $d+2$.

Theorem 3.5. Let $S$ be of embedding dimension $d+2$. If $\mathbb{K}[S]$ is Cohen-Macaulay, then type $(S) \leq 2$.
Proof. Let $\operatorname{Max}_{\preceq_{S}} \operatorname{Ap}(S, E)=\left\{\mathbf{m}_{1}, \ldots, \mathbf{m}_{t}\right\}$. By Lemma 2.7, $\mathbf{m}_{i}$ has a unique maximal expression

$$
\mathbf{m}_{i}=r_{i_{1}} \mathbf{a}_{d+1}+r_{i_{2}} \mathbf{a}_{d+2}
$$

for $i=1, \ldots, t$. Let $s, k$ be such that $r_{s_{1}}=\max \left\{r_{i_{1}} ; i=1, \ldots, t\right\}$ and $r_{k_{2}}=$ $\max \left\{r_{i_{2}} ; i=1, \ldots, t\right\}$. Assume on the contrary, that $t \geq 3$ and let $l \in\{1, \ldots, t\} \backslash$ $\{s, k\}$. Since $\mathbf{m}_{l} \not \varliminf_{S} \mathbf{m}_{s}$ and $\mathbf{m}_{l} \not \varliminf_{S} \mathbf{m}_{k}$, we have

$$
r_{k_{1}}<r_{l_{1}}<r_{s_{1}}, r_{s_{2}}<r_{l_{2}}<r_{k_{2}} .
$$

Moreover,

$$
\begin{align*}
& \mathbf{m}_{l}+\mathbf{a}_{d+1}=\left(1+r_{l_{1}}\right) \mathbf{a}_{d+1}+r_{l_{2}} \mathbf{a}_{d+2}=\mathbf{w}_{1}+\sum_{i=1}^{d} n_{i} \mathbf{a}_{i}  \tag{3.1}\\
& \mathbf{m}_{l}+\mathbf{a}_{d+2}=r_{l_{1}} \mathbf{a}_{d+1}+\left(1+r_{l_{2}}\right) \mathbf{a}_{d+2}=\mathbf{w}_{2}+\sum_{i=1}^{d} m_{i} \mathbf{a}_{i} \tag{3.2}
\end{align*}
$$

where $\mathbf{w}_{1}, \mathbf{w}_{2} \in \operatorname{Ap}(S, E)$ and $n_{i}, m_{i} \in \mathbb{N}$ for $i=1, \ldots, d$. Note that

$$
\mathbf{w}_{1}=c_{1} \mathbf{a}_{d+1}+c_{2} \mathbf{a}_{d+2}, \quad \mathbf{w}_{2}=e_{1} \mathbf{a}_{d+1}+e_{2} \mathbf{a}_{d+2}
$$

for some $c_{1}, c_{2}, e_{1}, e_{2} \in \mathbb{N}$. Since $\mathbf{m}_{l}+\mathbf{a}_{d+i} \notin \operatorname{Ap}(S, E)$, for $i=1,2$, both $\sum_{i=1}^{d} n_{i} \mathbf{a}_{i}$ and $\sum_{i=1}^{d} m_{i} \mathbf{a}_{i}$ are nonzero. As $r_{l_{1}} \mathbf{a}_{d+1}+r_{l_{2}} \mathbf{a}_{d+2} \in \operatorname{Ap}(S, E)$, we get $c_{1}=e_{2}=0$. Note that, $1+r_{l_{1}} \leq r_{s_{1}}$ and $1+r_{l_{2}} \leq r_{k_{2}}$. Therefore, $\left(1+r_{l_{1}}\right) \mathbf{a}_{d+1}$ and $\left(1+r_{l_{2}}\right) \mathbf{a}_{d+2}$ belong to $\operatorname{Ap}(S, E)$. Consequently,

$$
\begin{equation*}
r_{l_{1}}>e_{1} \text { and } r_{l_{2}}>c_{2} \tag{3.3}
\end{equation*}
$$

which implies $\left(1+e_{1}\right) \mathbf{a}_{d+1}$ and $\left(1+c_{2}\right) \mathbf{a}_{d+2}$ are also in $\operatorname{Ap}(S, E)$.
Now, subtracting (3.2) from (3.1), we derive

$$
\left(1+e_{1}\right) \mathbf{a}_{d+1}-\left(1+c_{2}\right) \mathbf{a}_{d+2}=\sum_{i=1}^{d}\left(n_{i}-m_{i}\right) \mathbf{a}_{i} .
$$

Since $\mathbb{K}[S]$ is Cohen-Macaulay, Proposition 2.4 implies that

$$
\left(1+e_{1}\right) \mathbf{a}_{d+1}=\left(1+c_{2}\right) \mathbf{a}_{d+2}
$$

Without loss of generality we may assume that $e_{1}>c_{2}$. Then

$$
r_{l_{2}} \mathbf{a}_{d+2}=\left(r_{l_{2}}-c_{2}-1\right) \mathbf{a}_{d+2}+\left(1+e_{1}\right) \mathbf{a}_{d+1}
$$

Consequently, ord $\left(r_{l_{2}} \mathbf{a}_{d+2}\right) \geq r_{l_{2}}-c_{2}+e_{1}>r_{l_{2}}$, in contradiction to our choice of maximal expression of $\mathbf{m}_{l}$.

The converse of Theorem 3.5 is not true.
Example 3.6. Let $\mathbf{a}_{1}=(2,0), \mathbf{a}_{2}=(0,2), \mathbf{a}_{3}=(4,1), \mathbf{a}_{4}=(2,3)$. Since $\mathbf{a}_{1}-\mathbf{a}_{2}=$ $\mathbf{a}_{3}-\mathbf{a}_{4}, \mathbb{K}[S]$ is not Cohen-Macaulay, by Proposition 2.4. However,

$$
\operatorname{Ap}(S, E) \backslash\{0\}=\max _{\preceq S} \operatorname{Ap}(S, E)=\{(4,1),(2,3)\}
$$

The following two examples show that either if $\mathbb{K}[S]$ is not Cohen-Macaulay or if $r \geq 3$, then type $(S)$ does not have any upper bound in terms of its embedding dimension.

Example 3.7. Let $\mathbf{a}_{1}=(3,0), \mathbf{a}_{2}=\left(0,3^{n}\right), \mathbf{a}_{3}=(5,2), \mathbf{a}_{4}=\left(2,2+3^{n}\right)$, where $n$ is a positive integer. Since $\mathbf{a}_{4}-\mathbf{a}_{3}=\mathbf{a}_{2}-\mathbf{a}_{1}, \mathbb{K}[S]$ is not Cohen-Macaulay, by Proposition 2.4. First, we show that

$$
\begin{equation*}
\operatorname{Ap}(S, E)=\left\{r \mathbf{a}_{3}+s \mathbf{a}_{4} ; \text { for all } r, s \in \mathbb{N} \text { such that } r+s<3^{n}\right\} \tag{3.4}
\end{equation*}
$$

Assume on the contrary that, $r \mathbf{a}_{3}+s \mathbf{a}_{4} \notin \operatorname{Ap}(S, E)$ for some $r, s \in \mathbb{N}$ such that $r+s<3^{n}$. Then $r \mathbf{a}_{3}+s \mathbf{a}_{4}=\sum_{i=1}^{4} n_{i} \mathbf{a}_{i}$ for some integers $n_{i} \geq 0$ such that either $n_{1}>0$ or $n_{2}>0$. Consequently

$$
\left(r-n_{3}\right)(5,2)+\left(s-n_{4}\right)\left(2,2+3^{n}\right)=n_{1}(3,0)+n_{2}\left(0,3^{n}\right)
$$

which implies

$$
\begin{array}{r}
2\left(r+s-n_{3}-n_{4}\right)+3^{n}\left(s-n_{4}\right)=3^{n} n_{2} \\
5\left(r-n_{3}\right)+2\left(s-n_{4}\right)=3 n_{1} . \tag{3.6}
\end{array}
$$

Therefore, $r+s=3^{n} k+n_{3}+n_{4}$ for some integer $k$. Since $r+s<3^{n}$, we get $k \leq 0$. Then $r+s-n_{3}-n_{4} \leq 0$. Since $n_{2} \geq 0$, we get by (3.5) that $s-n_{4} \geq 0$ and $r-n_{3} \leq 0$. Note that

$$
3\left(r-n_{3}\right)+2\left(r+s-n_{3}-n_{4}\right)=3 n_{1},
$$

from (3.6). Since $r-n_{3} \leq 0, r+s-n_{3}-n_{4} \leq 0$ and $n_{1} \geq 0$, we get

$$
r-n_{3}=r+s-n_{3}-n_{4}=n_{1}=0
$$

Then $n_{2}=0$, from (3.5), a contradiction. We have shown that, the set on the right hand side of (3.4) is a subset of $\operatorname{Ap}(S, E)$. Now, let $r, s \in \mathbb{N}$ such that $r+s=3^{n}$. So

$$
\begin{aligned}
r \mathbf{a}_{3}+s \mathbf{a}_{4}-\mathbf{a}_{1} & =r(5,2)+s\left(2,2+3^{n}\right)-(3,0) \\
& =\left(2(r+s)+3(r-1), 2(r+s)+3^{n} s\right) \\
& =\left(2 \times 3^{n-1}+(r-1)\right)(3,0)+(2+s)\left(0,3^{n}\right)
\end{aligned}
$$

Therefore $r \mathbf{a}_{3}+s \mathbf{a}_{4} \notin \operatorname{Ap}(S, E)$. If $r+s>3^{n}$, then

$$
r \mathbf{a}_{3}+s \mathbf{a}_{4}=r^{\prime} \mathbf{a}_{3}+s^{\prime} \mathbf{a}_{4}+\left(r-r^{\prime}\right) \mathbf{a}_{3}+\left(s-s^{\prime}\right) \mathbf{a}_{4}
$$

for some nonnegative integers $r^{\prime}, s^{\prime}$ such that $r>r^{\prime}, s>s^{\prime}$ and $r^{\prime}+s^{\prime}=3^{n}$. Consequently, $r \mathbf{a}_{3}+s \mathbf{a}_{4} \notin \operatorname{Ap}(S, E)$. Therefore, (3.4) is proved. Now, one can easily see that

$$
\max _{\preceq} \operatorname{Ap}(S, E)=\left\{r \mathbf{a}_{3}+s \mathbf{a}_{4} ; \text { for all } r, s \in \mathbb{N} \text { such that } r+s=3^{n}-1\right\}
$$

and so type $(S)=\left|\left\{(r, s) \in \mathbb{N}^{2} ; r+s=3^{n}-1\right\}\right|=3^{n}$.
Example 3.8. For an integer $a \geq 3$, let $S$ be the affine semigroup generated by $\mathbf{a}_{1}=\left(a^{2}, 0\right), \mathbf{a}_{2}=\left(0, a^{2}\right), \mathbf{a}_{3}=\left(a^{2}-a, a^{2}-a\right), \mathbf{a}_{4}=\left(a^{2}-a+1, a^{2}-a+1\right), \mathbf{a}_{5}=$ $\left(a^{2}-1, a^{2}-1\right)$. Then $S$ is simplicial with extremal rays $\mathbf{a}_{1}, \mathbf{a}_{2}$. Let $T$ be the numerical semigroup generated by $\left\{a^{2}-a, a^{2}-a+1, a^{2}-1, a^{2}\right\}$. Then

$$
\begin{equation*}
\operatorname{Ap}(S, E)=\operatorname{Ap}\left(S, \mathbf{a}_{1}+\mathbf{a}_{2}\right)=\left\{(s, s) ; s \in \operatorname{Ap}\left(T, a^{2}\right)\right\} \tag{3.7}
\end{equation*}
$$

Therefore, $\operatorname{type}(S)=\operatorname{type}(T)=2 a-4$, by $[6,(3.4)$ Proposition $]$. Note that $\mathbb{K}[S]$ is Cohen-Macaulay, by Lemma 2.6.

## 4. The conductor ideal of simplicial affine semigroups

Thorough this section, $S \subseteq \mathbb{N}^{d}$ is a simplicial affine semigroup with $\operatorname{mgs}(S)=$ $\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d+r}\right\}$, where $\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}$ are the extremal rays of $S$. Let $R=\mathbb{K}[S]$ be the affine semigroup ring. Recall that the normalization of an integral domain $R$ is the set of elements in its field of fractions satisfying a monic polynomial in $R[y]$. Then $R=\mathbb{K}[S]$ is an integral domain with normalization $\bar{R}=\mathbb{K}[\operatorname{group}(S) \cap \operatorname{cone}(S)]$ [21, Proposition 7.25]. Recall that the conductor of $R, C_{R}=\left(R:_{T} \bar{R}\right)$, where $T$ denotes the total ring of fractions of $R$, is the largest common ideal of $R$ and $\bar{R}$, [19, Exercise 2.11]. The purpose of this section, is to investigate the normality of $R$ and the generating set of $C_{R}$ as an ideal of $\bar{R}$.

The integral closure of $S$ in $\operatorname{group}(S), \bar{S}=\{\mathbf{a} \in \operatorname{group}(S) ; n \mathbf{a} \in S$ for some $n \in$ $\mathbb{N}\}$, is called the normalization of $S$. As a geometrical interpretation, one can see that $\bar{S}=\operatorname{cone}(S) \cap \operatorname{group}(S)$. The semigroup $S$ is normal when $S=\bar{S}$, equivalently $\mathbb{K}[S]$ is a normal ring, $[3,4]$. Since $S$ is finitely generated, cone $(S)$ is generated by finitely many rational vectors, i.e. it is the intersection of finitely many rational vector halfspaces, [27, Corollary 7.1(a)]. By Gordan's lemma, $\bar{S}$ is also finitely generated.

The conductor of $S$ is defined as $\mathfrak{c}(S)=\{\mathbf{b} \in S ; \mathbf{b}+\bar{S} \subseteq S\}$. The conductor, $\mathfrak{c}(S)$, is the largest ideal of $S$ that is also an ideal of $\bar{S}$, [3, Exercise 2.9].

Remark 4.1. As $\mathfrak{c}(S)$ is an ideal of $S$, we have $S=\bar{S}$ precisely when $0 \in \mathfrak{c}(S)$. In other words, $S$ is normal if and only if $\mathfrak{c}(S)=S$.

When $S$ is fully embedded in $\mathbb{N}^{d}$, that is the affine subspace it generates coincides with $\mathbb{R}^{d}$, we have $\operatorname{group}(S) \cong \mathbb{Z}^{d}$. In the case that $\operatorname{group}(S)=\mathbb{Z}^{d}$, we have $\bar{S}=\mathbb{N}^{d} \cap \operatorname{cone}(S)$. The later property may happen also for affine semigroups that $\operatorname{group}(S) \neq \mathbb{Z}^{d}$. For instance, it holds when $(\operatorname{cone}(S) \backslash S) \bigcap \mathbb{N}^{d}$ is finite, such semigroups are considered in [11].
Lemma 4.2. If $(\operatorname{cone}(S) \backslash S) \bigcap \mathbb{N}^{d}$ is finite, then $\operatorname{group}(S) \cap \operatorname{cone}(S)=\mathbb{N}^{d} \cap \operatorname{cone}(S)$. In particular, $S$ is normal if and only if $S=\mathbb{N}^{d} \cap \operatorname{cone}(S)$.

Proof. All vectors in cone $(S)$ have nonnegative components. Thus group $(S) \cap$ $\operatorname{cone}(S) \subseteq \mathbb{N}^{d} \cap \operatorname{cone}(S)$. Let $\mathbf{a} \in \mathbb{N}^{d} \cap \operatorname{cone}(S)$. Then $\mathbf{a}+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i} \in \operatorname{cone}(S) \cap \mathbb{N}^{d}$, for all $l_{1}, \ldots, l_{d} \in \mathbb{N}$. Since $\left(\operatorname{cone}(S) \cap \mathbb{N}^{d}\right) \backslash S$ is a finite set, we have $\mathbf{a}+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i} \in S$,
for some $l_{1}, \ldots, l_{d} \in \mathbb{N}$, which implies $\mathbf{a} \in \operatorname{group}(S)$. Therefore, $\mathbb{N}^{d} \cap \operatorname{cone}(S)=$ $\operatorname{group}(S) \cap \operatorname{cone}(S)$.
Lemma 4.3. As an affine semigroup, $\bar{S}$ is generated by $\left(P_{S} \cap \operatorname{group}(S)\right) \cup\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right\}$, and $P_{S} \cap \operatorname{group}(S)=\{r(\mathbf{w}) ; \mathbf{w} \in \operatorname{Ap}(S, E)\}$.
Proof. An element $\mathbf{a} \in \bar{S}$ can be written as $\mathbf{a}=r(\mathbf{a})+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i}$ for some $r_{1}, \ldots, r_{d} \in \mathbb{N}$. Therefore, $\left(P_{S} \cap \operatorname{group}(S)\right) \cup\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right\}$ provides a generating set for $\bar{S}$. For the last statement, let $\mathbf{b} \in P_{S} \cap \operatorname{group}(S)$. By Remark 2.3 , there exist $r_{1}, \ldots, r_{d}, s_{1}, \ldots, s_{d} \in$ $\mathbb{N}$ and $\mathbf{w} \in \operatorname{Ap}(S, E)$ such that $\mathbf{b}+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i}=\mathbf{w}+\sum_{i=1}^{d} s_{i} \mathbf{a}_{i}$. Since $\mathbf{b} \in P_{S}$, it follows that $\mathbf{b}=r(\mathbf{w})$.

As an immediate consequence of Lemma 4.3,

$$
\begin{equation*}
\mathfrak{c}(S)=\{\mathbf{a} \in S ; \mathbf{a}+r(\mathbf{w}) \in S \text { for all } \mathbf{w} \in \operatorname{Ap}(S, E)\} \tag{4.1}
\end{equation*}
$$

Example 4.4. Let $S$ be a numerical semigroup, that is $d=1$. Then $\bar{S}=\mathbb{N}$ and $\mathfrak{c}(S)=\{n ; n \geq F+1\}$, where $F$ is the maximal integer in $\mathbb{Z} \backslash S$. Therefore, $\mathfrak{c}(S)$ is generated by $\{F+1\}$ as an ideal of $\mathbb{N}$. In this regard, $F+1$ is called the conductor of $S$.

Lemma 4.5. The following statements are equivalent.
(1) $-\mathrm{QF}(S) \subseteq$ cone $(S)$;
(2) $\operatorname{Ap}(S, E) \subseteq \bar{P}_{S}$.

Proof. (1) $\Longrightarrow(2):$ Let $\mathbf{w} \in \operatorname{Ap}(S, E)$. Then $\mathbf{m}-\mathbf{w} \in S$, for some $\mathbf{m} \in \operatorname{Max} \preceq_{S} \operatorname{Ap}(S, E)$. As $\mathbf{m}-\sum_{i=1}^{d} \mathbf{a}_{i} \in \mathrm{QF}(S)$, we get $\sum_{i=1}^{d}\left(1-[\mathbf{m}]_{i}\right) \mathbf{a}_{i} \in \operatorname{cone}(S)$ precisely when $[\mathbf{w}]_{i} \leq[\mathbf{m}]_{i} \leq 1$.
$(2) \Longrightarrow(1):$ Let $\mathbf{f}=\mathbf{m}-\sum_{i=1}^{d} \mathbf{a}_{i} \in \operatorname{QF}(S)$ for some $\mathbf{m} \in \operatorname{Max}_{\preceq_{S}} \operatorname{Ap}(S, E)$. As $[\mathbf{m}]_{i} \leq 1$ for $i=1, \ldots, d$, we get $-\mathbf{f}=\sum_{i=1}^{d}\left(1-[\mathbf{m}]_{i}\right) \mathbf{a}_{i} \in \operatorname{cone}(S)$.

If we replace $\bar{P}_{S}$ in the above lemma, with $P_{S}$, then we derive an equivalent condition for $S$ to be normal. Roughly speaking, having more negative coefficients $[\mathbf{f}]_{i}$ for $\mathbf{f} \in \mathrm{QF}(S)$, makes the semigroup more close to being normal. In this order, we need to consider the relative interior of cone $(S)$. Let relint $(S)$ denote the elements of $\mathbb{R}^{d}$ that belong to the relative interior of cone $(S)$,

$$
\operatorname{relint}(S)=\left\{\mathbf{b} \in \operatorname{cone}(S) ; \mathbf{b}=\sum_{i=1}^{d} \lambda_{i} \mathbf{a}_{i} \text { with } \lambda_{i} \in \mathbb{R}_{>0} \text { for all } i=1, \ldots, d\right\}
$$

Theorem 4.6. The following statements are equivalent.
(1) $S$ is normal;
(2) $-\mathrm{QF}(S) \subseteq S \cap \operatorname{relint}(S)$;
(3) $-\mathrm{QF}(S) \subseteq \operatorname{relint}(S)$;
(4) $\operatorname{Ap}(S, E) \subseteq P_{S}$.

Proof. (1) $\Longrightarrow(2)$ : Let $\mathbf{f} \in \mathrm{QF}(S)$. Then $\mathbf{f}=\mathbf{m}-\sum_{i=1}^{d} \mathbf{a}_{i}$, for some $\mathbf{m} \in$ $\operatorname{Max}_{\preceq_{S}} \operatorname{Ap}(S, E)$. If $[\mathbf{m}]_{j} \geq 1$, for some $1 \leq j \leq d$, then $\mathbf{m}=\mathbf{a}_{j}+\left([\mathbf{m}]_{j}-1\right) \mathbf{a}_{j}+$ $\sum_{i=1, i \neq j}^{d}[\mathbf{m}]_{i} \mathbf{a}_{i}$, which implies $\mathbf{m}-\mathbf{a}_{j} \in \bar{S}=S$, which contradicts $\mathbf{m} \in \operatorname{Ap}(S, E)$. Therefore $[\mathbf{m}]_{i}<1$, for $i=1, \ldots, d$. Now, $-\mathbf{f}=\sum_{i=1}^{d}\left(1-[\mathbf{m}]_{i}\right) \mathbf{a}_{i} \in \operatorname{relint}(S) \cap$ $\operatorname{group}(S) \subseteq S$.
$(2) \Longrightarrow(3)$ is clear.
$(3) \Longrightarrow(4):$ Let $\mathbf{w} \in \operatorname{Ap}(S, E)$. Then $\mathbf{m}-\mathbf{w} \in S$, for some $\mathbf{m} \in \operatorname{Max}_{\preceq_{S}} \operatorname{Ap}(S, E)$. As $\mathbf{m}-\sum_{i=1}^{d} \mathbf{a}_{i} \in \mathrm{QF}(S)$, we get $\sum_{i=1}^{d}\left(1-[\mathbf{m}]_{i}\right) \mathbf{a}_{i} \in \operatorname{relint}(S)$. Thus, $1-[\mathbf{m}]_{i}>0$ which implies $[\mathbf{w}]_{i} \leq[\mathbf{m}]_{i}<1$.
$(4) \Longrightarrow(1): \operatorname{Ap}(S, E) \subseteq P_{S}$ is equivalent to $\operatorname{Ap}(S, E)=r(\operatorname{Ap}(S, E))$. As the later equality holds precisely when $r(\operatorname{Ap}(S, E)) \subseteq S$, the result follows by Lemma 4.3.

Our next aim in this section is to find a generating set for $\mathfrak{c}(S)$ as an ideal of $\bar{S}$. Recall from Section 2, that $C_{j}=\left\{\mathbf{w} \in \operatorname{Ap}(S, E) ; r(\mathbf{w})=\mathbf{b}_{j}\right\}$, for $j=0, \ldots, k$, where $r(\operatorname{Ap}(S, E))=\{r(\mathbf{w}) ; \mathbf{w} \in \operatorname{Ap}(S, E)\}=\left\{0=\mathbf{b}_{0}, \mathbf{b}_{1}, \ldots, \mathbf{b}_{k}\right\}$. For any $\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right) \in C_{1} \times \cdots \times C_{k}$, we consider the vector

$$
\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}=\sum_{i=1}^{d} f_{i} \mathbf{a}_{i}
$$

where $f_{i}=\max \left\{\left[\mathbf{w}_{j}-r\left(\mathbf{w}_{j}\right)\right]_{i} ; j=1, \ldots, k\right\}$, for $i=1, \ldots, d$. Note that

$$
f_{i}=\max \left\{\left\lfloor\left[\mathbf{w}_{j}\right]_{i}\right\rfloor ; j=1, \ldots, k\right\}
$$

for $i=1, \ldots, d$, where $\left\lfloor\left[\mathbf{w}_{j}\right]_{i}\right\rfloor$ denotes the greatest integer less than or equal to $\left[\mathbf{w}_{j}\right]_{i}$.

Lemma 4.7. Given $\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right) \in C_{1} \times \cdots \times C_{k}$, the vector $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}$ belongs to $\mathfrak{c}(S)$.

Proof. Let $\mathbf{f}=\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}$. By (4.1), it is enough to show that $\mathbf{f}+\mathbf{b}_{j} \in S$ for $j=1, \ldots, k$. Let $\mathbf{b}_{j}=r\left(\mathbf{w}_{j}\right)$. Note that $\mathbf{f}=\sum_{i=1}^{d}\left(\left[\mathbf{w}_{j}-\mathbf{b}_{j}\right]_{i}+r_{i}\right) \mathbf{a}_{i}$, for some $r_{i} \in \mathbb{N}$. Therefore, $\mathbf{f}+\mathbf{b}_{j}=\mathbf{w}_{j}+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i} \in S$, for $j=1, \ldots, k$.

Corollary 4.8. The following statements are equivalent.
(1) $S$ is normal;
(2) $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}=0$, for all $\mathbf{w}_{i} \in C_{i}$ and $i=1, \ldots, k$;
(3) $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}=0$, for some $\mathbf{w}_{i} \in C_{i}$ and $i=1, \ldots, k$.

Proof. (1) $\Longrightarrow(2)$ follows from Theorem 4.6.
$(2) \Longrightarrow(3)$ is clear.
$(3) \Longrightarrow(1)$ : By Lemma $4.7,0=\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}$ belongs to $\mathfrak{c}(S)$ which is an ideal of
$S$. Therefore, $S$ is normal, see Remark 4.1.
Theorem 4.9. Let $\mathbf{c}$ be a minimal generator of $\mathfrak{c}(S)$. Then there exist $\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right) \in$ $C_{1} \times \cdots \times C_{k}$ such that $\mathbf{c}=\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}-\mathbf{b}_{j}+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i}$ for some $l_{i} \in\{0,1\}$ and $j \in\{0, \ldots, k\}$. Moreover, at least for one $i$, we have $l_{i}=0$.

Proof. Since $\mathbf{c}+\mathbf{b}_{j} \in S$, for $j=0, \ldots, k$,

$$
\begin{equation*}
\mathbf{c}+\mathbf{b}_{j}=\mathbf{w}_{t_{j}}+\sum_{i=1}^{d} r_{j_{i}} \mathbf{a}_{i} \tag{4.2}
\end{equation*}
$$

for some $\mathbf{w}_{t_{j}} \in \operatorname{Ap}(S, E)$ and $r_{j_{i}} \in \mathbb{N}$. Note that $r\left(\mathbf{w}_{t_{j}}\right) \neq r\left(\mathbf{w}_{t_{i}}\right)$, for $0 \leq i \neq j \leq k$, since otherwise $\mathbf{b}_{j}-\mathbf{b}_{i} \in \operatorname{group}\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}\right)$ which is not possible. Let $\left\{t_{1}, \ldots, t_{k}\right\}=$ $\{1, \ldots, k\}$ such that $\mathbf{w}_{i} \in C_{i}$, and let $\mathbf{f}=\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}$. Then

$$
[\mathbf{f}]_{j}=\max \left\{\left\lfloor\left[\mathbf{w}_{i}\right]_{j}\right\rfloor ; i=1, \ldots, k\right\}
$$

for $j=1, \ldots, d$. Let $1 \leq s \leq d$. If $r_{j_{s}} \geq 1$ for $j=0, \ldots, k$, then

$$
\mathbf{c}-\mathbf{a}_{s}+\mathbf{b}_{i}=\mathbf{w}_{t_{i}}+\sum_{j=1, j \neq s}^{d} r_{i_{j}} \mathbf{a}_{j}+\left(r_{i_{s}}-1\right) \mathbf{a}_{s} \in S
$$

for $i=0, \ldots, k$, which implies $\mathbf{c}-\mathbf{a}_{s} \in \mathfrak{c}(S)$, a contradiction. Thus, $r_{j_{s}}=0$ for some $0 \leq j \leq k$. Consider $0 \leq h \leq k$ such that

$$
\left[\mathbf{b}_{h}\right]_{s}=\max \left\{\left[\mathbf{b}_{j}\right]_{s} ; r_{j_{s}}=0,0 \leq j \leq k\right\}
$$

Then $\left[\mathbf{w}_{t_{h}}\right]_{s}=\left[\mathbf{c}+\mathbf{b}_{h}\right]_{s}=\max \left\{\left[\mathbf{w}_{t_{j}}\right]_{s} ; r_{j_{s}}=0,0 \leq j \leq k\right\}$. If $r_{i_{s}}>0$ for some $0 \leq i \leq k$, then

$$
\left[\mathbf{w}_{t_{h}}\right]_{s}-\left[\mathbf{b}_{h}\right]_{s}=[\mathbf{c}]_{s}=\left[\mathbf{w}_{t_{i}}\right]_{s}+r_{i_{s}}-\left[\mathbf{b}_{i}\right]_{s}>\left[\mathbf{w}_{t_{i}}\right]_{s}
$$

Consequently,

$$
\left[\mathbf{w}_{t_{h}}\right]_{s}=\max \left\{\left[\mathbf{w}_{i}\right]_{s} ; i=1, \ldots, k\right\}
$$

and thus

$$
\left\lfloor\left[\mathbf{w}_{t_{h}}\right]_{s}\right\rfloor=[\mathbf{f}]_{s} .
$$

Let $c_{s}=\left\lceil[\mathbf{c}]_{s}\right\rceil$, where $\left\lceil[\mathbf{c}]_{s}\right\rceil=\operatorname{ceil}\left([\mathbf{c}]_{s}\right)$ denotes the least integer greater than or equal to $[\mathbf{c}]_{s}$, for $s=1, \ldots, d$. Now, we distinguish the following two possibilities:
(1) If $\left[\mathbf{b}_{h}\right]_{s} \geq c_{s}-[\mathbf{c}]_{s}$, then $\left[\mathbf{w}_{t_{h}}\right]_{s}=\left[\mathbf{b}_{h}+\mathbf{c}\right]_{s} \geq c_{s}$. As $c_{s}$ is an integer, it follows that $\left\lfloor\left[\mathbf{w}_{t_{h}}\right]_{s}\right\rfloor \geq c_{s}$. If $\left\lfloor\left[\mathbf{w}_{t_{h}}\right]_{s}\right\rfloor>c_{s}$, then $\left\lfloor\left[\mathbf{w}_{t_{h}}\right]_{s}\right\rfloor \geq 1+c_{s}=1+\left\lceil[\mathbf{c}]_{s}\right\rceil$, which implies $\left[\mathbf{b}_{h}\right]_{s}=\left[\mathbf{w}_{t_{h}}\right]_{s}-[\mathbf{c}]_{s} \geq 1$, a contradiction. Therefore,

$$
c_{s}=\left\lfloor\left[\mathbf{c}+\mathbf{b}_{h}\right]_{s}\right\rfloor=\left\lfloor\left[\mathbf{w}_{t_{h}}\right]_{s}\right\rfloor=[\mathbf{f}\rfloor_{s}
$$

(2) If $\left[\mathbf{b}_{h}\right]_{s}<c_{s}-[\mathbf{c}]_{s}$, then $\left\lfloor\left[\mathbf{b}_{h}+\mathbf{c}\right]_{s}\right\rfloor=\left\lfloor[\mathbf{c}]_{s}\right\rfloor$. Note that, $c_{s}-[\mathbf{c}]_{s}>0$ which means $[\mathbf{c}]_{s}$ is not an integer. Thus, $\left\lfloor[\mathbf{c}]_{s}\right\rfloor=c_{s}-1$. Therefore,

$$
c_{s}=\left\lfloor[\mathbf{c}]_{s}\right\rfloor+1=\left\lfloor\left[\mathbf{b}_{h}+\mathbf{c}\right]_{s}\right\rfloor+1=\left\lfloor\left[\mathbf{w}_{t_{h}}\right]_{s}\right\rfloor+1=[\mathbf{f}]_{s}+1
$$

If $c_{s}$ satisfies (1), then let $l_{s}=0$, and otherwise let $l_{s}=1$. Then

$$
\mathbf{c}=\mathbf{f}-\sum_{s=1}^{d}\left(c_{s}-[\mathbf{c}]_{s}\right) \mathbf{a}_{s}+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i}
$$

Note that $\sum_{s=1}^{d}\left(c_{s}-[\mathbf{c}]_{s}\right) \mathbf{a}_{s}=\sum_{s=1}^{d} c_{s} \mathbf{a}_{s}-\mathbf{c}$ belongs to group $(S) \cap P_{S}=r(\operatorname{Ap}(S, E))$, see Lemma 4.3.

For the last statement, let $1 \leq j \leq k$ and let

$$
\left\{i_{1}, \ldots, i_{t}\right\}=\left\{i ; 1 \leq i \leq d,\left[\mathbf{b}_{j}\right]_{i}>0\right\}
$$

Then $\sum_{s=1}^{t} \mathbf{a}_{i_{s}}-\mathbf{b}_{j} \in \operatorname{group}(S) \cap P_{S}=r(\operatorname{Ap}(S, E))$. In particular, $\mathbf{f}+\sum_{i=1}^{d} \mathbf{a}_{i}-\mathbf{b}_{j} \in$ $\{\mathbf{f}\}+\bar{S}$. As $\mathbf{f} \in \mathfrak{c}(S)$ by Lemma 4.7, it means that $\mathbf{f}+\sum_{i=1}^{d} \mathbf{a}_{i}-\mathbf{b}_{j}$ is not in the minimal generating set of $\mathfrak{c}(S)$.

Corollary 4.10. If $\mathbf{c} \in \mathfrak{c}(S)$ such that $[\mathbf{c}]_{i} \in \mathbb{N}$ for $i=1, \ldots, d$, then $[\mathbf{c}]_{i} \geq$ $\left[\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}\right]_{i}$ for some $\mathbf{w}_{1}, \ldots, \mathbf{w}_{k} \in C_{1} \times \cdots \times C_{k}$.
Proof. Note that $\mathbf{c}=\mathbf{c}^{\prime}+\mathbf{b}$ for a minimal generator $\mathbf{c}^{\prime}$ of $\mathbf{c}(S)$ and some $\mathbf{b} \in \bar{S}$. By Theorem 4.9, there exist $\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right) \in C_{1} \times \cdots \times C_{k}$ and $l_{1}, \ldots, l_{d} \in\{0,1\}$ such that $\mathbf{c}^{\prime}=\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}-\mathbf{b}_{j}+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i}$ for some $j \in\{0, \ldots, k\}$. Thus

$$
[\mathbf{c}]_{i} \geq\left[\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}\right]_{i}-\left[\mathbf{b}_{j}\right]_{i} .
$$

As $[\mathbf{c}]_{i}$ is an integer, and $0 \leq\left[\mathbf{b}_{j}\right]_{i}<1$, we have $[\mathbf{c}]_{i} \geq\left[\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}\right]_{i}$.

Example 4.11. Let $\mathbf{a}_{1}=(3,0), \mathbf{a}_{2}=(0,3), \mathbf{a}_{3}=(5,2), \mathbf{a}_{4}=(2,5)$. As we have seen in Example 3.7,

$$
\begin{aligned}
& \begin{array}{r}
\operatorname{Ap}(S, E)=\left\{0, \mathbf{a}_{3}, \mathbf{a}_{4}, \mathbf{a}_{3}+\mathbf{a}_{4}, 2 \mathbf{a}_{3}, 2 \mathbf{a}_{4}\right\} \\
\\
=\left\{0, \mathbf{w}_{1}=(5,2), \mathbf{w}_{2}=(2,5), \mathbf{w}_{3}=(7,7), \mathbf{w}_{4}=(10,4), \mathbf{w}_{5}=(4,10)\right\} \\
\text { and } r(\operatorname{Ap}(S, E))=\left\{0, \mathbf{b}_{1}=(1,1), \mathbf{b}_{2}=(2,2)\right\} . \text { Note that } C_{1}=\left\{\mathbf{w}_{3}, \mathbf{w}_{4}, \mathbf{w}_{5}\right\}, \\
C_{2}=\left\{\mathbf{w}_{1}, \mathbf{w}_{2}\right\} \text { and } \mathbf{f}_{\left(\mathbf{w}_{3}, \mathbf{w}_{i}\right)}=2 \mathbf{a}_{1}+2 \mathbf{a}_{2}=(6,6), \mathbf{f}_{\left(\mathbf{w}_{4}, \mathbf{w}_{i}\right)}=3 \mathbf{a}_{1}+\mathbf{a}_{2}=(9,3), \\
\mathbf{f}_{\left(\mathbf{w}_{5}, \mathbf{w}_{i}\right)}=\mathbf{a}_{1}+3 \mathbf{a}_{2}=(3,9), \text { for } i=1,2 . \\
\quad \operatorname{As}\{(6,6)-(1,1),(9,3)-(1,1),(3,9)-(1,1)\}+r(\operatorname{Ap}(S, E)) \subset S, \text { we have } \\
\qquad\{(5,5),(8,2),(2,8)\} \subset \mathfrak{c}(S) .
\end{array}
\end{aligned}
$$

If $\mathfrak{c}(S) \neq\{(5,5),(8,2),(2,8)\}+\bar{S}$, the other generators of $\mathfrak{c}(S)$ are among

$$
\{(9,3),(3,9),(6,6)\}+\left\{l_{i} \mathbf{a}_{i}-(2,2) ; l_{i} \in\{0,1\}, i=1,2\right\},
$$

by Theorem 4.9. Since the above set which equals

$$
\{(7,1),(1,7),(4,4),(10,1),(1,10),(4,7),(7,4)\}
$$

has no element in $S, \mathfrak{c}(S)$ is generated by $\{(5,5),(8,2),(2,8)\}=\left\{\mathbf{f}_{\left(\mathbf{w}_{3}, \mathbf{w}_{1}\right)}-\mathbf{b}_{1}, \mathbf{f}_{\left(\mathbf{w}_{4}, \mathbf{w}_{1}\right)}-\right.$ $\left.\mathbf{b}_{1}, \mathbf{f}_{\left(\mathbf{w}_{5}, \mathbf{w}_{1}\right)}-\mathbf{b}_{1}\right\}$, as an ideal of $\bar{S}=\langle(3,0),(0,3),(1,1)\rangle$.

The following example shows that the summand $\sum_{i=1}^{d} l_{i} \mathbf{a}_{i}$ in the statement of Theorem 4.9 can not be removed.

Example 4.12. Let $\mathbf{a}_{1}=(5,2), \mathbf{a}_{2}=(2,2), \mathbf{a}_{3}=(2,1), \mathbf{a}_{4}=(5,3)$. Then $\operatorname{Ap}(S, E)=\left\{0, \mathbf{w}_{1}=(2,1), \mathbf{w}_{2}=(4,2), \mathbf{w}_{3}=(6,3), \mathbf{w}_{4}=(8,4), \mathbf{w}_{5}=(5,3)\right\}$ and $r(\operatorname{Ap}(S, E))=\left\{0, \mathbf{b}_{1}=(2,1), \mathbf{b}_{2}=(4,2), \mathbf{b}_{3}=(1,1), \mathbf{b}_{4}=(3,2), \mathbf{b}_{5}=(5,3)\right\}$. Note that $C_{i}=\left\{\mathbf{w}_{i}\right\}$ for $i=1, \ldots, 5$ and $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{5}\right)}=\mathbf{a}_{1}$. By Theorem 4.9, the generators of $\mathfrak{c}(S)$ are among

$$
\left\{\mathbf{a}_{1}-\mathbf{b}_{i}, 2 \mathbf{a}_{1}-\mathbf{b}_{i}, \mathbf{a}_{1}+\mathbf{a}_{2}-\mathbf{b}_{i} ; i=0, \ldots, 5\right\}
$$

The only elements of the above set, that belong also to $S$ are

$$
\{(5,2),(10,4),(5,3),(2,1),(7,4),(4,2),(6,3)\} .
$$

Note that $(2,1)+(1,1) \notin S,\{(5,2),(4,2)\}+r(\operatorname{Ap}(S, E)) \subseteq S,\{(10,4),(7,4),(6,3)\} \subset$ $(5,2)+\bar{S}$ and $(5,3)=(4,2)+\mathbf{b}_{3}$. Therefore, $\mathfrak{c}(S)$ is generated by $\{(5,2),(4,2)\}=$ $\left\{\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{5}\right)}, \mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{5}\right)}+\mathbf{a}_{2}-\mathbf{b}_{4}\right\}$, as an ideal of $\bar{S}=\langle(1,1),(2,1),(5,2)\rangle$.
Proposition 4.13. Assume that there is a fixed class $C_{j}$ such that for any $\mathbf{w} \in C_{j}$ and $\mathbf{w}^{\prime} \in \operatorname{Ap}(S, E) \backslash C_{j}$, one has $\max _{\preceq_{c}}\left(\mathbf{w}, \mathbf{w}^{\prime}\right)=\mathbf{w}$. If either $C_{j}$ is a singleton or $\mathbf{b}_{j}=\min _{\preceq_{c}}(r(\operatorname{Ap}(S, E)) \backslash\{0\})$, then $\mathfrak{c}(S)$ is generated by

$$
\left\{\mathbf{w}-\mathbf{b} ; \mathbf{w} \in C_{j}, \mathbf{b} \in \max _{\preceq_{c}}\left\{\mathbf{b}_{1}, \ldots, \mathbf{b}_{k}\right\}\right\}
$$

as an ideal of $\bar{S}$.
Proof. First we show that

$$
\begin{equation*}
\mathbf{w}-\mathbf{b}_{l}+\mathbf{b}_{s} \in S \tag{4.3}
\end{equation*}
$$

for any $\mathbf{w} \in C_{j}$ and $0 \leq l, s \leq k$.
As $\mathbf{w}-\mathbf{b}_{l}+\mathbf{b}_{s} \in \operatorname{group}(S)$, we have by Remark 2.3 , that $\mathbf{w}-\mathbf{b}_{l}+\mathbf{b}_{s}+$ $\sum_{i=1}^{d} r_{i} \mathbf{a}_{i}=\mathbf{w}^{\prime}+\sum_{i=1}^{d} s_{i} \mathbf{a}_{i}$ for some $\mathbf{w}^{\prime} \in \operatorname{Ap}(S, E)$ and nonnegative integers $r_{1}, \ldots, r_{d}, s_{1}, \ldots, s_{d}$. If $r_{1}=\cdots=r_{d}=0$, then (4.3) is clear. Assume that $r_{h}>0$, for some $1 \leq h \leq d$. Then $s_{h}=0$ by our choice in Remark 2.3, and $[\mathbf{w}]_{h}<\left[\mathbf{w}^{\prime}\right]_{h}$.

If $\mathbf{w}^{\prime} \notin C_{j}$, then $\max _{\preceq_{c}}\left(\mathbf{w}, \mathbf{w}^{\prime}\right)=\mathbf{w}$ which implies $[\mathbf{w}]_{h} \geq\left[\mathbf{w}^{\prime}\right]_{h}$, a contradiction. Thus $\mathbf{w}^{\prime} \in C_{j}$. In other words, $\mathbf{w}$ and $\mathbf{w}^{\prime}$ have the same remainder $\mathbf{b}_{j}$. Therefore, $\mathbf{b}_{s}=\mathbf{b}_{l}$, and (4.3) holds. Consequently, $\mathbf{w}-\mathbf{b}_{r} \in \mathfrak{c}(S)$ for $r=0, \ldots, k$.

Let $\mathbf{c}$ be a minimal generator of $\mathfrak{c}(S)$. By Theorem 4.9, $\mathbf{c}$ can be written as

$$
\begin{equation*}
\mathbf{c}=\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}-\mathbf{b}_{t}+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i} \tag{4.4}
\end{equation*}
$$

for some $\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right) \in C_{1} \times \cdots \times C_{k}, 0 \leq t \leq k$ and some $l_{1}, \ldots, l_{d} \in\{0,1\}$. Note that $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}=\mathbf{w}_{j}-\mathbf{b}_{j}$ and $\mathbf{b}_{j}+\mathbf{b}_{t}=\mathbf{b}_{s}+\sum_{i=1}^{d} l_{i}^{\prime} \mathbf{a}_{i}$ for some $l_{1}^{\prime}, \ldots, l_{d}^{\prime} \in\{0,1\}$ and $0 \leq s \leq k$. Therefore, we get

$$
\begin{equation*}
\mathbf{c}=\mathbf{w}_{j}-\left(\mathbf{b}_{j}+\mathbf{b}_{t}\right)+\sum_{i=1}^{d} l_{i} \mathbf{a}_{i}=\mathbf{w}_{j}-\mathbf{b}_{s}+\sum_{i=1}^{d}\left(l_{i}-l_{i}^{\prime}\right) \mathbf{a}_{i} \tag{4.5}
\end{equation*}
$$

from (4.4). If $\mathbf{b}_{s}=0$, then $\mathbf{c}$ and $\mathbf{w}_{j}$ are in the same congruence class modulo the group spanned by the extremal rays. In other words, $r(\mathbf{c})=r\left(\mathbf{w}_{j}\right)=\mathbf{b}_{j}$. Thus, $\mathbf{c}=\mathbf{w}+\sum_{i=1}^{d} h_{i} \mathbf{a}_{i}$, for some $\mathbf{w} \in C_{j}$ and $h_{1}, \ldots, h_{d} \in \mathbb{N}$. This provides a contradiction with minimality of $\mathbf{c}$, as $\mathbf{w}-\mathbf{b}_{i} \in \mathfrak{c}(S)$, for $1 \leq i \leq k$. Therefore, $\mathbf{b}_{s} \neq 0$. Now, we distinguish the following two cases:

Case 1, $\mathbf{b}_{j}=\min _{\preceq_{c}}(r(\operatorname{Ap}(S, E)) \backslash\{0\})$. Then $\left[\mathbf{b}_{j}\right]_{i} \leq\left[\mathbf{b}_{s}\right]_{i}=\left[\mathbf{b}_{j}\right]_{i}+\left[\mathbf{b}_{t}\right]_{i}-l_{i}^{\prime}$, which implies $l_{i}^{\prime}=0$, for $i=1, \ldots, d$.

Case 2: $C_{j}$ is a singleton. From equation (4.5), we have $r\left(\mathbf{c}+\mathbf{b}_{s}\right)=r\left(\mathbf{w}_{j}\right)=\mathbf{b}_{j}$ which implies $\mathbf{c}+\mathbf{b}_{s}=\mathbf{w}_{j}+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i}$ for some $r_{1}, \ldots, r_{d} \in \mathbb{N}$. Now, looking again at (4.5), we derive

$$
\mathbf{w}_{j}+\sum_{i=1}^{d} r_{i} \mathbf{a}_{i}=\mathbf{w}_{j}+\sum_{i=1}^{d}\left(l_{i}-l_{i}^{\prime}\right) \mathbf{a}_{i} .
$$

As $\mathbf{a}_{1}, \ldots, \mathbf{a}_{d}$ are linearly independent, we get $l_{i}-l_{i}^{\prime}=r_{i} \geq 0$, for $i=1, \ldots, d$.
Thus in both cases, $l_{i}-l_{i}^{\prime} \geq 0$, for $i=1, \ldots, d$. Since $\mathbf{w}_{j}-\mathbf{b}_{s} \in \mathfrak{c}(S)$ and $\mathbf{c}$ is a minimal generator of $\mathfrak{c}(S)$, we derive from (4.5), that $l_{i}-l_{i}^{\prime}=0$ for $i=1, \ldots, d$ and $\mathbf{c}=\mathbf{w}_{j}-\mathbf{b}_{s}$.

If $\mathbf{b}_{s} \preceq_{c} \mathbf{b}_{r}$ for some $1 \leq r \leq k$, then $\mathbf{b}_{r}-\mathbf{b}_{s} \in \operatorname{cone}(S) \cap \operatorname{group}(S)=\bar{S}$ and $\mathbf{w}_{j}-\mathbf{b}_{s}=\mathbf{w}_{j}-\mathbf{b}_{r}+\mathbf{b}_{r}-\mathbf{b}_{s}$. As $\mathbf{c}$ is a minimal generator, we get $\mathbf{b}_{s}=\mathbf{b}_{r}$. Thus, $\mathbf{b}_{s} \in \max _{\preceq_{c}}\left\{\mathbf{b}_{1}, \ldots, \mathbf{b}_{k}\right\}$.

Applying the above proposition to the semigroup in Example 4.11, provides an easier argument to find the minimal generating set of $\mathfrak{c}(S)$.

Example 4.14. Let $\mathbf{a}_{1}=(3,0), \mathbf{a}_{2}=(0,3), \mathbf{a}_{3}=(5,2), \mathbf{a}_{4}=(2,5)$. As we have seen in Example 4.11, $\operatorname{Ap}(S, E)=\left\{0, \mathbf{w}_{1}=(5,2), \mathbf{w}_{2}=(2,5), \mathbf{w}_{3}=(7,7), \mathbf{w}_{4}=\right.$ $\left.(10,4), \mathbf{w}_{5}=(4,10)\right\}, r(\operatorname{Ap}(S, E))=\left\{0, \mathbf{b}_{1}=(1,1), \mathbf{b}_{2}=(2,2)\right\}, C_{1}=\left\{\mathbf{w}_{3}, \mathbf{w}_{4}, \mathbf{w}_{5}\right\}$ and $C_{2}=\left\{\mathbf{w}_{1}, \mathbf{w}_{2}\right\}$. Note that $\max _{\preceq_{c}}\left\{\mathbf{w}_{i}, \mathbf{w}_{j}\right\}=\mathbf{w}_{j}$ for $i=1,2$ and $j=$ $3,4,5$. Therefore, $\mathfrak{c}(S)$ is generated by $\left\{\mathbf{w}_{3}-(2,2), \mathbf{w}_{4}-(2,2), \mathbf{w}_{5}-(2,2)\right\}=$ $\{(5,5),(8,2),(2,8)\}$, as an ideal of $\bar{S}=\langle(3,0),(0,3),(1,1)\rangle$.
Example 4.15. Let $\mathbf{a}_{1}=(2,0), \mathbf{a}_{2}=(0,2), \mathbf{a}_{3}=(4,1), \mathbf{a}_{4}=(2,3)$. We have $\operatorname{Ap}(S, E)=\left\{0, \mathbf{w}_{1}=(4,1), \mathbf{w}_{2}=(2,3)\right\}$ and $r(\operatorname{Ap}(S, E))=\{0,(0,1)\}$. More precisely, $\mathbf{w}_{1}-2 \mathbf{a}_{1}=\mathbf{w}_{2}-\mathbf{a}_{1}-\mathbf{a}_{2}=1 / 2 \mathbf{a}_{2}$. Thus $k=1$ and $C_{1}=\{(4,1),(2,3)\}$. By Proposition 4.13, $\mathfrak{c}(S)$ is generated by $\{(4,1)-(0,1),(2,3)-(0,1)\}=\{(4,0),(2,2)\}$ as an ideal of $\bar{S}=\langle(2,0),(0,1)\rangle$.

Remark 4.16. If $C_{i}$ is a singleton for $i=1, \ldots, k$, then the hypothesis on the existence of $j$ in Proposition 4.13 is equivalent to existence of a single maximal element of $\operatorname{Ap}(S, E)$ with respect to $\preceq_{c}$. This condition can not be removed. For instance, let $S$ be the semigroup defined in Example 4.12. Then $\max _{\varliminf_{c}} \operatorname{Ap}(S, E)=$ $\left\{\mathbf{w}_{4}, \mathbf{w}_{5}\right\}=\{(8,4),(5,3)\}$ and $\max _{\preceq_{c}}(r(\operatorname{Ap}(S, E)))=\left\{\mathbf{b}_{5}=(5,3)\right\}$. But, $(8,4)-$ $(5,3)$ is not in $S$ and in particular, it is not in $\mathfrak{c}(S)$.

Corollary 4.17. If $\mathbb{K}[S]$ is a Cohen-Macaulay ring, and $\max _{\preceq_{c}} \operatorname{Ap}(S, E)=\{\mathbf{w}\}$, then $\mathfrak{c}(S)$ is generated by $\left\{\mathbf{w}-\mathbf{b} ; \mathbf{b} \in \max _{\preceq_{c}} r(\operatorname{Ap}(S, E))\right\}$, as an ideal of $\bar{S}$. In particular, if $\max _{\preceq_{c}}(r(\operatorname{Ap}(S, E)))=\{\mathbf{b}\}$, then $\mathfrak{c}(S)$ is a principal ideal of $\bar{S}$ generated by $\mathbf{w}-\mathbf{b}$.

As Example 4.12 shows, for an affine semigroup with Cohen-Macaulay semigroup ring, $\max _{\preceq_{c}} \operatorname{Ap}(S, E)$ is not necessarily a singleton. Here, we have an example of an affine semigroup satisfying the conditions of Corollary 4.17.

Example 4.18. Let $\mathbf{a}_{1}=(1,5), \mathbf{a}_{2}=(5,1), \mathbf{a}_{3}=(2,2), \mathbf{a}_{4}=(3,3)$. Then

$$
\begin{aligned}
\operatorname{Ap}(S, E) & =\left\{0, \mathbf{a}_{3}, \mathbf{a}_{4}, 2 \mathbf{a}_{3}, \mathbf{a}_{3}+\mathbf{a}_{4}, 2 \mathbf{a}_{3}+\mathbf{a}_{4}\right\} \\
& =\left\{0, \mathbf{w}_{1}=(2,2), \mathbf{w}_{2}=(3,3), \mathbf{w}_{3}=(4,4), \mathbf{w}_{4}=(5,5), \mathbf{w}_{5}=(7,7)\right\}
\end{aligned}
$$

and $r\left(\operatorname{Ap}(S, E)=\left\{0, \mathbf{b}_{1}=(2,2), \mathbf{b}_{2}=(3,3), \mathbf{b}_{3}=(4,4), \mathbf{b}_{4}=(5,5), \mathbf{b}_{5}=\right.\right.$ $(1,1)\}$. As $C_{i}=\left\{\mathbf{w}_{i}\right\}$, for $i=1, \ldots, 5, \mathbb{K}[S]$ is Cohen-Macaulay, by Lemma 2.5. Moreover, $\max _{\preceq_{c}} \operatorname{Ap}(S, E)=\left\{\mathbf{w}_{5}\right\}$ and $\max _{\preceq_{c}} r(\operatorname{Ap}(S, E))=\left\{\mathbf{b}_{4}\right\}$. Therefore, by Corollary $4.17, \mathfrak{c}(S)$ is generated by $\mathbf{w}_{5}-\mathbf{b}_{4}=(2,2)$, as an ideal of $\bar{S}=$ $\langle(1,1),(1,5),(5,1)\rangle$.

For an affine semigroup $S$ with Cohen-Macaulay semigroup ring, if $\max _{\preceq_{c}} \operatorname{Ap}(S, E)$ is a singleton, it does not imply that $\max _{\preceq_{c}} r(\operatorname{Ap}(S, E))$ is also a singleton.

Example 4.19. Let $\mathbf{a}_{1}=(2,1), \mathbf{a}_{2}=(1,5), \mathbf{a}_{3}=(1,1), \mathbf{a}_{4}=(4,5)$. As the computation in [20, Example 2] shows, $\max _{\preceq_{c}} \operatorname{Ap}(S, E)=(6,7), \max _{\preceq_{c}} r(\operatorname{Ap}(S, E))=$ $\{(2,2),(2,3),(2,4),(2,5)\}$ and $\mathbb{K}[S]$ is Cohen-Macaulay. Therefore, $\mathfrak{c}(S)$ is generated by $\{(4,5),(4,4),(4,3),(4,2)\}$ as an ideal of $\bar{S}=\langle(1,1),(1,2),(1,3),(1,4),(1,5)$, $(2,1)\rangle$, by Corollary 4.17.

Note that $\max _{\preceq_{c}} \operatorname{Ap}(S, E) \subseteq \max _{\preceq_{S}} \operatorname{Ap}(S, E)$. In particular, if $\mathbb{K}[S]$ is Gorenstein, then $\max \preceq_{c} \operatorname{Ap}(S, E)$ has a single element. The converse is not true, for instance $\max \preceq_{S} \operatorname{Ap}(S, E)=\{(6,7),(5,5)\}$ in Example 4.19, while $\max _{\preceq_{c}} \operatorname{Ap}(S, E)$ has a single element.
Corollary 4.20. If $\mathbb{K}[S]$ is a Gorenstein ring and $\max _{\preceq_{c}}(r(\operatorname{Ap}(S, E)))$ has a single element, then $\mathfrak{c}(S)$ is a principal ideal of $\bar{S}$.

Remark 4.21. Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity $e$, that is $e=\min (S \backslash\{0\})$. As we mentioned in Example 4.4, $\mathfrak{c}(S)$ is generated by $F+1$ as an ideal of $\bar{S}=\mathbb{N}$, where $F=\max (\mathbb{Z} \backslash S)$. Note that $r(\operatorname{Ap}(S, e))=\{0,1, \ldots, e-1\}$. As an immediate consequence of Corollary 4.17, we derive that $F=w-e$, where $w$ is the maximal number in $\operatorname{Ap}(S, e)$. This is a fact already proved differently in [26], see also [25, Theorem 2.12]. As Example 4.12 shows, the conductor of a Cohen-Macaulay affine semigroup $S$ is not necessarily a principal ideal of $\bar{S}$.

The following is an example of a Cohen-Macaulay simplicial affine semigroup, for which $\max _{\preceq_{c}} \operatorname{Ap}(S, E)$ is a singleton but $\mathfrak{c}(S)$ is not principal.

Example 4.22. Let $\mathbf{a}_{1}=(3,0), \mathbf{a}_{2}=(0,3), \mathbf{a}_{3}=(2,1)$. Then $\operatorname{Ap}(S, E)=\left\{0, \mathbf{w}_{1}=\right.$ $\left.(2,1), \mathbf{w}_{2}=(4,2)\right\}$ and $r(\operatorname{Ap}(S, E))=\left\{0, \mathbf{b}_{1}=(2,1), \mathbf{b}_{2}=(1,2)\right\}$. Since $C_{i}=\left\{\mathbf{w}_{i}\right\}$ for $i=1,2, \mathbb{K}[S]$ is Cohen-Macaulay by Lemma 2.5. Moreover, $\max _{\preceq_{c}}\left\{\mathbf{w}_{1}, \mathbf{w}_{2}\right\}=$ $\left\{\mathbf{w}_{2}\right\}$ and $\max _{\preceq_{c}} r(\operatorname{Ap}(S, E))=\left\{\mathbf{b}_{1}, \mathbf{b}_{2}\right\}$. By Proposition 4.13, $\mathfrak{c}(S)$ is generated by $\left\{\mathbf{w}_{2}-\mathbf{b}_{1}, \mathbf{w}_{2}-\mathbf{b}_{2}\right\}=\{(2,1),(3,0)\}$ as an ideal of $\bar{S}=\langle(3,0),(0,3),(1,2),(2,1)\rangle$.

Example 4.23. Let $S$ be the affine semigroup presented in Example 2.2. Based on a computation by Macaulay $2[15], \mathbb{K}[S]$ is Cohen-Macaulay. Note that $\operatorname{Ap}(S, E)$ has two maximal elements $18 \mathbf{a}_{4}+2 \mathbf{a}_{5}=(40,22,20)$ and $16 \mathbf{a}_{4}+4 \mathbf{a}_{5}=(40,24,20)$. Computing the coordinates of these two elements, we find that $\mathbf{f}_{\left(\mathbf{w}_{1}, \ldots, \mathbf{w}_{k}\right)}=3 \mathbf{a}_{1}+$ $\mathbf{a}_{2}+3 \mathbf{a}_{3}=(40,23,20)$. One can use GAP [8] with an implementation based on Theorem 4.9 (provided to us by the referee), to find that $\mathfrak{c}(S)$ is minimally generated by $\{(29,15,14),(29,16,14),(29,18,15),(30,15,14),(30,16,14),(31,15,15)$, $(31,16,14),(31,18,16),(33,18,17),(33,19,15),(35,18,18)\}$.

We close this section by an example in dimension three.
Example 4.24. Let $\mathbf{a}_{1}=(1,2,1), \mathbf{a}_{2}=(2,3,1), \mathbf{a}_{3}=(2,1,3), \mathbf{a}_{4}=(2,3,2), \mathbf{a}_{5}=$ $(2,2,2), \mathbf{a}_{6}=(3,3,3)$. Then $\operatorname{Ap}(S, E)=\left\{0, \mathbf{w}_{1}=\mathbf{a}_{4}, \mathbf{w}_{2}=\mathbf{a}_{5}, \mathbf{w}_{3}=\mathbf{a}_{6}, \mathbf{w}_{4}=\mathbf{a}_{5}+\right.$ $\left.\mathbf{a}_{6}\right\}$ and $r(\operatorname{Ap}(S, E))=\left\{0, \mathbf{b}_{1}=(1,1,1), \mathbf{b}_{2}=(2,2,2), \mathbf{b}_{3}=(3,3,3)\right\}$. Moreover, $C_{1}=\left\{\mathbf{w}_{1}=\mathbf{a}_{1}+\mathbf{b}_{1}, \mathbf{w}_{4}=\mathbf{a}_{2}+\mathbf{a}_{3}+\mathbf{b}_{1}=2 \mathbf{a}_{5}+\mathbf{b}_{1}\right\}, C_{2}=\left\{\mathbf{w}_{2}=\mathbf{b}_{2}\right\}$ and $C_{3}=\left\{\mathbf{w}_{3}=\mathbf{b}_{3}\right\}$. As $\mathbf{b}_{2}, \mathbf{b}_{3} \in S$, we have $\mathfrak{c}(S)=\left\{\mathbf{a} \in S ; \mathbf{a}+\mathbf{b}_{1} \in S\right\}$. Since $\mathbf{a}_{i}+\mathbf{b}_{1} \in S$ for $i \in\{1,4,5,6\}$, we have $\mathbf{a}_{1}, \mathbf{a}_{4}=\mathbf{a}_{1}+\mathbf{b}_{1}, \mathbf{a}_{5}, \mathbf{a}_{6}=\mathbf{a}_{5}+\mathbf{b}_{1}$ are in $\mathfrak{c}(S)$. If $\mathbf{c}$ is a minimal generator of $\mathfrak{c}(S)$ that is not in $\left\{\mathbf{a}_{1}, \mathbf{a}_{5}\right\}$, then $\mathbf{c}=t \mathbf{a}_{2}+s \mathbf{a}_{3}$ for some $t, s \in \mathbb{N}$. Thus, $r\left(\mathbf{c}+\mathbf{b}_{1}\right)=\mathbf{b}_{1}$, and we get $\mathbf{c}+\mathbf{b}_{1}=\mathbf{w}+\sum_{i=1}^{3} l_{i} \mathbf{a}_{i}$ for some $\mathbf{w} \in C_{1}$ and $l_{1}, l_{2}, l_{3} \in \mathbb{N}$.

If $\mathbf{w}=\mathbf{w}_{1}$, then $\mathbf{c}+\mathbf{b}_{1}=\mathbf{a}_{1}+\mathbf{b}_{1}+\sum_{i=1}^{3} l_{i} \mathbf{a}_{i}$, consequently $\mathbf{c} \in \mathbf{a}_{1}+\bar{S}$, which is a contradiction.

If $\mathbf{w}=\mathbf{w}_{4}$, then $\mathbf{c}+\mathbf{b}_{1}=2 \mathbf{a}_{5}+\mathbf{b}_{1}+\sum_{i=1}^{3} l_{i} \mathbf{a}_{i}$, which implies $\mathbf{c} \in \mathbf{a}_{5}+\bar{S}$, a contradiction.

Thus, as an ideal of $\bar{S}, \mathfrak{c}(S)$ is minimally generated by $\left\{\mathbf{a}_{1}, \mathbf{a}_{5}\right\}$. Note that $\mathbf{f}_{\left(\mathbf{w}_{1}, \mathbf{w}_{2}, \mathbf{w}_{3}\right)}=\mathbf{a}_{1}, \mathbf{f}_{\left(\mathbf{w}_{4}, \mathbf{w}_{2}, \mathbf{w}_{3}\right)}=\mathbf{a}_{2}+\mathbf{a}_{3}$ and $\mathbf{a}_{5}=\mathbf{a}_{2}+\mathbf{a}_{3}-\mathbf{b}_{2}$. In particular, $\mathfrak{c}(S)$ is generated by $\left\{\mathbf{f}_{\left(\mathbf{w}_{1}, \mathbf{w}_{2}, \mathbf{w}_{3}\right)}, \mathbf{f}_{\left(\mathbf{w}_{4}, \mathbf{w}_{2}, \mathbf{w}_{3}\right)}-\mathbf{b}_{2}\right\}$ as an ideal of $\bar{S}=\left\langle\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \mathbf{b}_{1}\right\rangle$.

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Mosaheb Institute of Mathematics, Kharazmi University, Tehran, Iran
Email address: rjafari@ipm.ir
Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran Email address: hasti.tmu83@yahoo.com


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