# TYPE AND CONDUCTOR OF SIMPLICIAL AFFINE SEMIGROUPS

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ABSTRACT. We provide a generalization of pseudo-Frobenius numbers of numerical semigroups to the context of the simplicial affine semigroups. In this way, we characterize the Cohen-Macaulay type of the simplicial affine semigroup ring  $\mathbb{K}[S]$ . We define the type of S, type(S), in terms of some Apéry sets of S and show that it coincides with the Cohen-Macaulay type of the semigroup ring, when  $\mathbb{K}[S]$  is Cohen-Macaulay. If  $\mathbb{K}[S]$  is a d-dimensional Cohen-Macaulay ring of embedding dimension at most d + 2, then type $(S) \leq 2$ . Otherwise, type(S) might be arbitrary large and it has no upper bound in terms of the embedding dimension. Finally, we present a generating set for the conductor of S as an ideal of its normalization.

## 1. INTRODUCTION

Let S be an affine semigroup in  $\mathbb{N}^d$ , where  $\mathbb{N}$  denotes the set of nonnegative integers. The affine semigroup ring  $\mathbb{K}[S]$ , over a field  $\mathbb{K}$ , is defined as the subring  $\{\bigoplus_{\mathbf{a}\in S} k_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} : k_{\mathbf{a}} \in \mathbb{K}\}$  of the polynomial ring  $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_d]$ . The structure of  $\mathbb{K}[S]$  is intimately related to the structure of the affine semigroup S and cone(S), the rational polyhedral cone spanned by S. The Cohen-Macaulay, Gorenstein and Buchsbaum properties of  $\mathbb{K}[S]$  have been characterized in terms of certain numerical and topological properties of S, see [14], [17], [18], [13], [23] and [10]. Our aim in this paper, is to characterize the conductor and Cohen-Macaulay type of  $\mathbb{K}[S]$ .

If d = 1, then S is a submonoid of N. Let h be the greatest common divisor of nonzero elements in S. Dividing all elements of S by h, we obtain an isomorphic semigroup in N. A submonoid S of N such that  $gcd(s; s \in S) = 1$  is called a *numerical semigroup*. In other words, the study of affine semigroups in N reduces to the study of numerical semigroups. The condition  $gcd(s; s \in S) = 1$  is equivalent to say that  $\mathbb{N} \setminus S$  is a finite set, [25, Lemma 2.1]. For an affine semigroup S, consider the natural partial ordering  $\leq_S$  on  $\mathbb{N}^d$  where, for all elements  $x, y \in \mathbb{N}^d$ ,  $x \leq_S y$ if  $y - x \in S$ . For a numerical semigroup  $S \subsetneqq \mathbb{N}$ , the maximal elements of  $\mathbb{N} \setminus S$ with respect to  $\leq_S$  are called *pseudo-Frobenius numbers*. Fröberg, Gottlieb and Häggkvist [9], defined the type of the numerical semigroup S as the cardinality of the set of its pseudo-Frobenius numbers. This notion of type coincides with the Cohen-Macaulay type of the numerical semigroup ring  $\mathbb{K}[S]$ , see [28] for a detailed proof.

By analogy, García-García, Ojeda, Rosales and Vingneron-Tenorio, define a pseudo-Frobenius element of S to be an element  $\mathbf{a} \in \mathbb{N}^d \setminus S$  such that  $\mathbf{a}+S \setminus \{0\} \subseteq S$ , in [12]. They show that the set of pseudo-Frobenius elements of S, PF(S), is not

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empty, precisely when depth  $\mathbb{K}[S] = 1$ . Thus, when d > 1 and  $\mathbb{K}[S]$  is a Cohen-Macaulay ring, the set of pseudo-Frobenius elements of S is empty and express noting about the Cohen-Macaulay type of the semigroup ring.

In Section 3, we present another generalization of pseudo-Frobenius numbers, determining the Cohen-Macaulay type of the semigroup ring  $\mathbb{K}[S]$ , under the assumption that the affine semigroup  $S \subset \mathbb{N}^d$  is also simplicial, i.e.  $\operatorname{cone}(S)$  has d extremal rays. All fully embedded affine semigroups in  $\mathbb{N}^d$ , for d = 1, 2, are simplicial. Let  $\mathbf{a}_1, \ldots, \mathbf{a}_d$  be the componentwise smallest integer vectors in S, situated on each extremal ray of  $\operatorname{cone}(S)$ , respectively. Then  $\bigcap_{i=1}^d \operatorname{Ap}(S, \mathbf{a}_i)$  is a finite set, where  $\operatorname{Ap}(S, \mathbf{a}_i) = \{\mathbf{a} \in S; \mathbf{a} - \mathbf{a}_i \notin S\}$  denotes the Apéry set of S with respect to  $\mathbf{a}_i$ . In numerical case, where  $d = 1, \mathbf{a}_1$  is the smallest positive number in S and the set of pseudo-Frobenius numbers is equal to the set  $\{\mathbf{w} - \mathbf{a}_1; \mathbf{w} \in \operatorname{Max}_{\leq S} \operatorname{Ap}(S, \mathbf{a}_1)\}$ . In an analogue way, we consider the set

$$QF(S) = \{ \mathbf{w} - \sum_{i=1}^{d} \mathbf{a}_i ; \mathbf{w} \in \max_{\leq S} \cap_{i=1}^{d} Ap(S, \mathbf{a}_i) \}.$$

We call the elements of QF(S), quasi-Frobenius elements, to distinguish from PF(S)introduced in [12]. In Proposition 3.3, we show that the cardinality of QF(S) which is called the *type* of S and is denoted by type(S), equals the Cohen-Macaulay type of  $\mathbb{K}[S]$ , when the semigroup ring is Cohen-Macaulay. Campillo and Gimenez in [5], study combinatorics of some simplicial complexes associated to elements of S. They compute the homologies of the simplicial complexes in terms of certain graph homologies, and show in [5, Theorem 4.2(ii)], that for Cohen-Macaulay simplicial affine semigroups, type(S) equals the number of maximal elements of Ap(S, E) with respect to  $\preceq_S$ . Our result, Proposition 3.3, provides a different algebraic proof for this fact. Although PF(S) and QF(S) coincide in the numerical case where d = 1, but for d > 1 they have no common element, see Remark 3.2.

As an immediate consequence, we derive that the simplicial affine semigroup ring  $\mathbb{K}[S]$  is Gorenstein if and only if it is Cohen-Macaulay and  $\cap_{i=1}^{d} \operatorname{Ap}(S, \mathbf{a}_i)$  has a single maximal element with respect to  $\leq_S$ . This result is already proved in [5, 23], by different arguments. In the rest of Section 3, we investigate the above bound for type(S). Generalizing [9, Theorem 11], we show in Theorem 3.5, that type(S)  $\leq 2$ , if S is generated by d + 2 elements and  $\mathbb{K}[S]$  is Cohen-Macaulay. If either  $\mathbb{K}[S]$  is not Cohen-Macaulay or if the minimal generating set of S has more than d + 2 elements, then type(S) might be arbitrary large, see Example 3.7 and Example 3.8.

Recall that the normalization of an integral domain R is the set of elements in its field of fractions satisfying a monic polynomial in R[y]. Then  $R = \mathbb{K}[S]$  is an integral domain with normalization  $\overline{R} = \mathbb{K}[\operatorname{group}(S) \cap \operatorname{cone}(S)]$ , where  $\operatorname{group}(S)$  denotes the group of differences of S, [21, Proposition 7.25]. The purpose of the last section, is to investigate the normality of R and, to detect a generating set for the conductor of R,  $C_R = (R :_T \overline{R})$ , where T denotes the total ring of fractions of R. We will work with corresponding objects in S. In semigroup interpretation,  $\overline{S} = \operatorname{group}(S) \cap \operatorname{cone}(S)$ and  $\mathfrak{c}(S) = \{\mathbf{b} \in S ; \mathbf{b} + \overline{S} \subseteq S\}$ , are called the *normalization* and the *conductor* of S, respectively, [3]. The ring R is normal precisely when the semigroup S is normal, i.e.  $S = \overline{S}$ , [4].

Quasi-Frobenius elements, are also profitable to recognize the normality of S. Note that quasi-Frobenius elements, might have negative components. Having more negative components in elements of QF(S), makes the semigroup more close to being normal. More precisely, if S is normal, then  $-QF(S) \subseteq \operatorname{cone}(S)$  and so  $-QF(S) \subseteq \mathbb{N}^d$ . The converse holds, if -QF(S) is a subset of the relative interior of  $\operatorname{cone}(S)$ . This is the subject of Theorem 4.6, which states also that in this case  $-QF(S) \subseteq S$ .

If  $S \subseteq \mathbb{N}$  is a numerical semigroup, then  $\overline{S} = \mathbb{N}$  and  $\mathfrak{c}(S)$  is a principal ideal of  $\mathbb{N}$  generated by F + 1, where  $F = \max(\mathbb{Z} \setminus S)$ . The rest of Section 4, is devoted to find a generating set for  $\mathfrak{c}(S)$ , as an ideal of  $\overline{S}$ , where S is an arbitrary simplicial affine semigroup.

Note that any element  $\mathbf{c} \in \operatorname{cone}(S) \cap \mathbb{N}^d$ , is uniquely presented as  $\mathbf{c} = \sum_{i=1}^d n_i \mathbf{a}_i + r(\mathbf{c})$  for some  $n_1, \ldots, n_d \in \mathbb{N}$  and  $r(\mathbf{c}) \in P_S$ , where  $P_S = \{\sum_{i=1}^d r_i \mathbf{a}_i ; 0 \leq r_1, \ldots, r_d < 1\}$  is the fundamental parallelogram of S. It is not difficult to observe that  $P_S \cap \operatorname{group}(S) = \{r(\mathbf{w}) ; \mathbf{w} \in \bigcap_{i=1}^d \operatorname{Ap}(S, \mathbf{a}_i)\}$ , see Lemma 4.3. Let  $P_S \cap \operatorname{group}(S) = \{0 = \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_k\}$ . For  $\mathbf{w}_1, \ldots, \mathbf{w}_k \in \bigcap_{i=1}^d \operatorname{Ap}(S, \mathbf{a}_i)$  with  $r(\mathbf{w}_i) = \mathbf{b}_i$ , we consider the vector  $\mathbf{f}_{(\mathbf{w}_1,\ldots,\mathbf{w}_k)} = \sum_{i=1}^d f_i \mathbf{a}_i$ , where  $f_i$  is the maximum integer such that  $\mathbf{w}_j - f_i \mathbf{a}_i \in P_S$  for some  $j = 1, \ldots, k$ . We show in Theorem 4.9, that any element of the minimal generating set for  $\mathbf{c}(S)$ , as an ideal of  $\bar{S}$ , is of the form  $\mathbf{f}_{(\mathbf{w}_1,\ldots,\mathbf{w}_k)} - \mathbf{b}_j + \sum_{i=1}^d l_i \mathbf{a}_i$  for some  $l_i \in \{0,1\}$ ,  $j \in \{1,\ldots,k\}$  and  $\mathbf{w}_1, \ldots, \mathbf{w}_k \in \bigcap_{i=1}^d \operatorname{Ap}(S, \mathbf{a}_i)$  such that  $r(\mathbf{w}_i) = \mathbf{b}_i$  for  $i = 1, \ldots, k$ . Moreover, at least for one i, we have  $l_i = 0$ . As  $\bigcap_{i=1}^d \operatorname{Ap}(S, \mathbf{a}_i)$  is a finite set and  $\mathbf{c}(S) = \{\mathbf{c} \in S ; \mathbf{c} + \mathbf{b}_i \in S \text{ for } i = 1, \ldots, k\}$ , Theorem 4.9 provides an algorithmic way to find a generating set for  $\mathbf{c}(S)$ . Let  $\preceq_c$  denote the natural partial ordering with respect to  $\operatorname{cone}(S)$ , that is  $\mathbf{a} \preceq_c \mathbf{b}$  if  $\mathbf{b} - \mathbf{a} \in \operatorname{cone}(S)$ . We show in Corollary 4.17 that, if  $\mathbb{K}[S]$  is Cohen-Macaulay and  $\max_{\preceq_c} \bigcap_{i=1}^d \operatorname{Ap}(S, \mathbf{a}_i) = \{\mathbf{w}\}$ , then  $\{\mathbf{w} - \mathbf{b}; \mathbf{b} \in \max_{\preceq_c} r(\operatorname{Ap}(S, E))\}$ , generates  $\mathbf{c}(S)$  as an ideal of  $\bar{S}$ . Several explicit examples are provided to illustrate the generating set of  $\mathbf{c}(S)$ , as an ideal of  $\bar{S}$ .

#### 2. Fundamentals

By an affine semigroup, we mean a finitely generated submonoid of  $\mathbb{N}^d$ , where  $\mathbb{N}$  denotes the set of nonnegative integers and  $d \in \mathbb{N} \setminus \{0\}$ . Let S be an affine semigroup minimally generated by  $\operatorname{mgs}(S) = \{\mathbf{a}_1, \ldots, \mathbf{a}_e\}$ . We write  $S = \langle \mathbf{a}_1, \ldots, \mathbf{a}_e \rangle$ , to indicate its generating set. The minimal generating set,  $\operatorname{mgs}(S)$ , is a unique finite set, see [24, Chapter 3]. The number of elements in  $\operatorname{mgs}(S)$  is called the *embedding dimension* of S. For a field  $\mathbb{K}$ , the semigroup ring  $\mathbb{K}[S]$  is the subalgebra of the polynomial ring  $\mathbb{K}[x_1, \ldots, x_d]$  generated by the monomials with exponents in S. The ring  $\mathbb{K}[S] = \mathbb{K}[\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_e}]$  has a unique maximal monomial ideal  $\mathfrak{m} = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_e})$ .

Given two sets  $A, B \subseteq \mathbb{N}^d$ , we write A + B for the set  $\{\mathbf{a} + \mathbf{b} ; \mathbf{a} \in A, \mathbf{b} \in B\}$ . If  $A = \{\mathbf{a}\}$ , we simply write  $\mathbf{a} + B$ , instead of  $\{\mathbf{a}\} + B$ . Recall that an *ideal* of S is a nonempty set  $H \subseteq S$ , such that  $H + S \subseteq S$ . For any ideal H of S, there exists a set of vectors  $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_l\}$  such that  $H = B + S = \bigcup_{i=1}^l \mathbf{b}_i + S$ . In this case, B is called a *generating set* of H. If no proper subset of B generates H, we refer to B as the minimal generating set of H. Any ideal of an affine semigroup has a unique minimal generating set. Note that for ideals  $H_1$  and  $H_2$  of S,  $H_1 + H_2$  is also an ideal of S. In particular, for an ideal H of S, n times summation of H which is denoted by nH, is again an ideal of S. Let  $M = S \setminus \{0\}$  be the maximal ideal of S.

A monomial in the semigroup ring  $\mathbb{K}[S]$  is an element of the form  $\mathbf{x}^{\mathbf{a}}$  for  $\mathbf{a} \in S$ . An ideal  $I \subseteq \mathbb{K}[S]$  is a monomial ideal if it is generated by monomials. For any subset H of S, let  $\mathbb{K}[H]$  denote the  $\mathbb{K}$ -linear span of the monomials  $\mathbf{x}^{\mathbf{a}}$  with  $\mathbf{a} \in H$ . Then I is a monomial ideal if and only if  $I = \mathbb{K}[H]$  for some ideal H of S, or equivalently, if I is homogeneous with respect to the tautological grading on  $\mathbb{K}[S]$ , which is defined by  $\deg(\mathbf{x}^{\mathbf{a}}) = \mathbf{a}$ . Note that  $\mathfrak{m} = \mathbb{K}[M]$ .

The affine semigroup  $S \subseteq \mathbb{N}^d$  is called *simplicial* if there exist d elements  $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_d} \in \operatorname{mgs}(S)$  such that they are linearly independent over the field of rational numbers  $\mathbb{Q}$  (equivalently, over the field of real numbers  $\mathbb{R}$ ), and for each element  $\mathbf{a} \in S$ , we have  $n\mathbf{a} \in \mathbb{N}\mathbf{a}_{i_1} + \cdots + \mathbb{N}\mathbf{a}_{i_d}$ , for some positive integer n. There is a geometrical interpretation for the simplicial property. Let

$$\operatorname{cone}(S) = \left\{ \sum_{i=1}^{e} \lambda_i \mathbf{a}_i \; ; \; \lambda_i \in \mathbb{R}_{\geq 0}, \; \text{for } i = 1, \dots, e \right\},\$$

denote the rational polyhedral cone generated by S. The dimension (or rank) of S is defined as the dimension of the affine subspace it generates, which is the same as the dimension of the subspace generated by  $\operatorname{cone}(S)$ . The  $\operatorname{cone}(S)$  is polyhedral i.e. it is the intersection of finitely many closed linear half-spaces in  $\mathbb{R}^d$ , each of whose bounding hyperplanes contains the origin, [27, Corollary 7.1(a)]. These half-spaces are called *support hyperplanes*. The integral vectors in each support hyperplane, is a face of S, and all maximal faces (called facets) are in this form. The intersection of any two adjacent support hyperplanes is a one-dimensional vector space, which is called an *extremal ray*. The  $\operatorname{cone}(S)$  has at least d facets and at least d extremal rays. It has d facets (equivalently, it has d extremal rays), precisely when S is simplicial.

Throughout this paper, we assume that S is a simplicial affine semigroup. On each extremal ray of cone(S), the componentwise smallest element from S, is called an *extremal ray* of S. Denote by  $\mathbf{a}_1, \ldots, \mathbf{a}_d$  the extremal rays of S. These form a basis for cone(S). For  $\mathbf{z} \in \mathbb{R}^d$  such that  $\mathbf{z} = \sum_{i=1}^d \lambda_i \mathbf{a}_i$  with  $\lambda_i \in \mathbb{Q}, i = 1, \ldots, d$ , we set  $[\mathbf{z}]_i = \lambda_i$  for  $i = 1, \ldots, d$ . For each element  $\mathbf{a} \in S$ , we have  $n\mathbf{a} \in \mathbb{N}\mathbf{a}_1 + \cdots + \mathbb{N}\mathbf{a}_d$ , for some positive integer n. In other words,  $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_d}\}$  provides a monomial system of parameters for  $\mathbb{K}[S]$ . The *fundamental (semi-open) parallelotope* of S is the set

$$P_S = \left\{ \mathbf{z} \in \mathbb{R}^d ; \ 0 \le [\mathbf{z}]_i < 1 \text{ for } i = 1, \dots, d \right\}$$
$$= \left\{ \sum_{i=1}^d \lambda_i \mathbf{a}_i ; \ \lambda_i \in \mathbb{Q} , \ 0 \le \lambda_i < 1 \text{ for } i = 1, \dots, d \right\}$$

Its closure in  $\mathbb{R}^d$  is the set  $\overline{P}_S = \{ \mathbf{z} \in \mathbb{R}^d ; 0 \leq [\mathbf{z}]_i \leq 1 \text{ for } i = 1, \dots, d \}$ . It is well known, and easy to see, that any  $\mathbf{a}$  in  $\operatorname{cone}(S) \cap \mathbb{N}^d$  decomposes uniquely as  $\mathbf{a} = \sum_{i=1}^d n_i \mathbf{a}_i + r(\mathbf{a})$ , with  $r(\mathbf{a}) \in P_S \cap \mathbb{N}^d$  and nonnegative integers  $n_1, \dots, n_d$ . We will call  $r(\mathbf{a})$  the remainder of  $\mathbf{a}$  in  $P_S$ .

**Remark 2.1.** Let  $\mathbf{a} \in \operatorname{cone}(S) \cap \mathbb{N}^d$  and let  $n_i = \lfloor [\mathbf{a}]_i \rfloor$  be the unique integer such that  $n_i \leq [\mathbf{a}]_i < n_i + 1$ , for  $i = 1, \ldots, d$ . Then  $r(\mathbf{a}) = \sum_{i=1}^d ([\mathbf{a}]_i - n_i)\mathbf{a}_i$ .

We consider the natural partial orderings  $\leq_S$  and  $\leq_c$  on  $\mathbb{N}^d$  where, for all elements **a** and **b** in  $\mathbb{N}^d$ , **b**  $\leq_S$  **a** (**b**  $\leq_c$  **a**), if there is an element **c**  $\in$  S (**c**  $\in$  cone(S)) such that **a** = **b** + **c**. The partial order  $\leq_c$  is indeed the coordinatewise order on cone(S)  $\cap \mathbb{N}^d$ . More precisely, for **a**, **b**  $\in$  cone(S)  $\cap \mathbb{N}^d$ , **a**  $\leq_c$  **b** if and only if  $[\mathbf{a}]_i \leq [\mathbf{b}]_i$  for  $i = 1, \ldots, d$ .

An element  $\mathbf{a} \in S$ , may be written as  $\mathbf{a} = \sum_{i=1}^{d+r} l_i \mathbf{a}_i$  for some nonnegative integers  $l_1, \ldots, l_{d+r}$ . The value  $\sum_{i=1}^{d+r} l_i$  is called the *length* of the expression  $\sum_{i=1}^{d+r} l_i \mathbf{a}_i$ . The maximum integer n such that nM contains  $\mathbf{a}$ , is called the *order* of  $\mathbf{a}$  and it is denoted by  $\operatorname{ord}(\mathbf{a})$ . In other words,  $\mathbf{a} \in nM \setminus (n+1)M$  if and only if  $n = \operatorname{ord}(\mathbf{a})$ . The expression of length  $\operatorname{ord}(\mathbf{a})$  of  $\mathbf{a}$ , is called a *maximal expression* of  $\mathbf{a}$ .

The Apéry set of an element  $\mathbf{b} \in S$  is defined as  $\operatorname{Ap}(S, \mathbf{b}) = {\mathbf{a} \in S ; \mathbf{a} - \mathbf{b} \notin S}$ . We will denote the zero vector of  $\mathbb{N}^d$  by 0. Since  $S \subseteq \mathbb{N}^d$ , for  $\mathbf{b} \neq 0$  we have  $0 \in \operatorname{Ap}(S, \mathbf{b})$ . Note that if  $\mathbf{a} \in \operatorname{Ap}(S, \mathbf{b})$  and  $\mathbf{z} \in S$  such that  $\mathbf{z} \preceq_S \mathbf{a}$ , then  $\mathbf{z} \in \operatorname{Ap}(S, \mathbf{b})$ . For a subset E, we set

$$\operatorname{Ap}(S, E) = \{ \mathbf{a} \in S ; \mathbf{a} - \mathbf{b} \notin S, \text{ for all } \mathbf{b} \in E \}.$$

Throughout the paper,  $E = \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$  will denote the set of extremal rays of S. Then  $\operatorname{Ap}(S, E) = \bigcap_{i=1}^d \operatorname{Ap}(S, \mathbf{a}_i)$ .

Let  $I_S$  denote the kernel of the K-algebra homomorphism  $\varphi : \mathbb{K}[z_1, \ldots, z_{d+r}] \longrightarrow \mathbb{K}[S]$ , defined by  $z_i \mapsto \mathbf{x}^{\mathbf{a}_i}$ , for  $i = 1, \ldots, d+r$ . Then  $I_S$  is a binomial prime ideal, [16, Proposition 1.4]. Note that  $\mathbb{K}[S] \cong \mathbb{K}[z_1, \ldots, z_{d+r}]/I_S$  has S-graded structure defined by  $\deg_S(z_1^{n_1} \ldots z_{d+r}^{n_{d+r}}) = \sum_{i=1}^{d+r} n_i \mathbf{a}_i$ . Let  $R' = \frac{\mathbb{K}[z_1, \ldots, z_{d+r}]}{I_S + (z_1, \ldots, z_d)}$ . Then, as a K-vector space, R' is generated by the set of monomials  $\mathbf{z}^{\mathbf{a}}$  such that  $\mathbf{a} \notin I_S + (z_1, \ldots, z_d)$ . Let B denote the monomial K-basis of R'. From [22, Theorem 3.3], we have

$$\operatorname{Ap}(S, E) = \{ \deg_S(u) \; ; \; u \in B \}.$$

Therefore, as an algorithm to find  $\operatorname{Ap}(S, E)$ , one may first compute  $I_S$ , using any of computer algebra systems GAP [8], Singular [7], CoCoA [1] or Macaulay2 [15], and then, find the monomial basis of  $\frac{\mathbb{K}[z_1,...,z_d+r]}{I_S+(z_1,...,z_d)}$ .

**Example 2.2.** Let  $\mathbf{a}_1 = (5,3,1), \mathbf{a}_2 = (1,5,2), \mathbf{a}_3 = (8,3,5), \mathbf{a}_4 = (2,1,1), \mathbf{a}_5 = (2,2,1)$ . A computation by Macaulay2 [15], shows that

$$I_S = (z_5^5 - z_1 z_2 z_4^2, z_4^{19} - z_1^2 z_3^3 z_5^2, z_4^{17} z_5^3 - z_1^3 z_2 z_3^3).$$

Consequently,  $I_S + (z_1, z_2, z_3) = (z_1, z_2, z_3, z_4^{19}, z_5^5, z_4^{17} z_5^3)$ . The image of

$$\{1, z_4^r z_5, z_4^r z_5^2, z_4^s z_5^3, z_4^s z_5^4, z_4^t ; 0 \le r \le 18, 0 \le s \le 16, 1 \le t \le 18\}$$

in  $\frac{\mathbb{K}[z_1,...,z_5]}{I_S + (z_1, z_2, z_3)}$ , provides a K-basis. Therefore, Ap(S, E) is equal to the set

 $\{0, r\mathbf{a}_4 + \mathbf{a}_5, r\mathbf{a}_4 + 2\mathbf{a}_5, s\mathbf{a}_4 + 3\mathbf{a}_5, s\mathbf{a}_4 + 4\mathbf{a}_5, t\mathbf{a}_4 \ ; \ 0 \le r \le 18, 0 \le s \le 16, 1 \le t \le 18\}.$ 

We write group(S) for the group of differences of S, i.e. group(S) is the smallest group (up to isomorphism) that contains S.

$$\operatorname{group}(S) = \{ \mathbf{a} - \mathbf{b} \mid \mathbf{a}, \mathbf{b} \in S \}.$$

By group  $(\mathbf{a}_1, \ldots, \mathbf{a}_d)$ , we mean the smallest group that contains  $\{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$ , equivalently group  $(\mathbf{a}_1, \ldots, \mathbf{a}_d) = \{\sum_{i=1}^d z_i \mathbf{a}_i ; z_i \in \mathbb{Z}\}.$ 

**Remark 2.3.** For  $\mathbf{c} \in \operatorname{group}(S)$ , there exists  $\mathbf{b} \in S$  such that  $\mathbf{c} + \mathbf{b} \in S$ . As S is simplicial,  $n\mathbf{b} \in \sum_{i=1}^{d} \mathbb{N}\mathbf{a}_i$  for some positive integer n. Therefore,  $\mathbf{c} + n\mathbf{b} = \mathbf{c} + \sum_{i=1}^{d} r_i \mathbf{a}_i \in S$ , for some  $r_i \in \mathbb{N}$ . Consider  $r_1, \ldots, r_d$ , as small as possible with this property, i.e.

(2.1) 
$$\mathbf{c} + (r_j - 1)\mathbf{a}_j + \sum_{i=1, i \neq j}^d r_i \mathbf{a}_i \notin S,$$

for  $j = 1, \ldots, d$ . Let  $\mathbf{c} + \sum_{i=1}^{d} r_i \mathbf{a}_i = \mathbf{w} + \sum_{i=1}^{d} s_i \mathbf{a}_i$  for some  $\mathbf{w} \in \operatorname{Ap}(S, E)$  and  $s_1, \ldots, s_d \in \mathbb{N}$ . If  $r_j > 0$ , for some  $1 \le j \le d$ , then

$$\mathbf{c} + (r_j - 1)\mathbf{a}_j + \sum_{i=1, i \neq j}^d r_i \mathbf{a}_i = \mathbf{w} + (s_j - 1)\mathbf{a}_j + \sum_{i=1, i \neq j}^d r_i \mathbf{a}_i.$$

By (2.1), we get  $s_i = 0$ .

As an advantage of considering  $\operatorname{Ap}(S, E)$ , we recall the following criteria for the Cohen-Macaulay property of  $\mathbb{K}[S]$ .

Proposition 2.4. [23, Corollary 1.6] The following statements are equivalent.

- (1)  $\mathbb{K}[S]$  is Cohen-Macaulay.
- (2) For all  $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{Ap}(S, E)$ , if  $\mathbf{w}_1 \mathbf{w}_2 \in \operatorname{group}(\mathbf{a}_1, \dots, \mathbf{a}_d)$ , then  $\mathbf{w}_1 = \mathbf{w}_2$ .

The Cohen-Macaulay property is indeed equivalent to have a one to one correspondence between elements in Ap(S, E) and their remainders. More precisely, let

$$r(\operatorname{Ap}(S, E)) = \{r(\mathbf{w}) ; \mathbf{w} \in \operatorname{Ap}(S, E)\} = \{0 = \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_k\}.$$

Then the family of subsets  $C_j = {\mathbf{w} \in \operatorname{Ap}(S, E) ; r(\mathbf{w}) = \mathbf{b}_j}$ , for  $j = 0, \dots, k$ , defines a partition of  $\operatorname{Ap}(S, E)$ .

Lemma 2.5. The following statements hold.

- (1)  $\mathbb{K}[S]$  is Cohen-Macaulay if and only if  $C_i$  is a singleton for i = 1, ..., k.
- (2)  $\mathbb{K}[S]$  is Buchsbaum if and only if, either  $C_i$  is a singleton or  $C_i = \{\mathbf{c} + \mathbf{a}_1, \dots, \mathbf{c} + \mathbf{a}_d\}$  for some  $\mathbf{c} \in \mathbb{N}^d$  such that  $\mathbf{c} + (S \setminus \{0\}) \subset S$ , for  $i = 1, \dots, d$ .

Proof. Let  $\mathbf{v}, \mathbf{w} \in \operatorname{Ap}(S, E)$ . Then  $\mathbf{v} - \mathbf{w} = r(\mathbf{v}) + \sum_{i=1}^{d} r_i \mathbf{a}_i - r(\mathbf{w}) - \sum_{i=1}^{d} s_i \mathbf{a}_i$ , for some  $r_i, s_i \in \mathbb{N}$ . In particular,  $\mathbf{v} - \mathbf{w} = r(\mathbf{v}) - r(\mathbf{w}) + \mathbf{b}$ , where  $\mathbf{b} = \sum_{i=1}^{d} (r_i - s_i)\mathbf{a}_i \in \operatorname{group}(\mathbf{a}_1, \ldots, \mathbf{a}_d)$ . Therefore,  $\mathbf{v} - \mathbf{w} \in \operatorname{group}(\mathbf{a}_1, \ldots, \mathbf{a}_d)$  if and only if  $r(\mathbf{v}) = r(\mathbf{w})$ , equivalently  $\mathbf{v}, \mathbf{w} \in C_i$ , for some  $0 \le i \le k$ . Thus,  $C_i$ s are precisely the equivalence classes under the equivalence relation ~ on  $\operatorname{Ap}(S, E)$ , where  $\mathbf{w}_i \sim \mathbf{w}_j$ if  $\mathbf{w}_i - \mathbf{w}_j \in \operatorname{group}(\mathbf{a}_1, \ldots, \mathbf{a}_d)$ . Now, the statement (1) follows by Proposition 2.4, and the statement (2) is a consequence of [10, Theorem 11].

When d = 1, equivalently S is isomorphic to a numerical semigroup, then  $\mathbb{K}[S]$  is a one-dimesional domain and consequently it is Cohen-Macaulay. From another point of view, since all elements of S belong to the real line, one may easily use Proposition 2.4 or Lemma 2.5 to check that  $\mathbb{K}[S]$  is Cohen-Macaulay. The following lemma, extends this property to simplicial affine semigroups in  $\mathbb{N}^d$ .

**Lemma 2.6.** Let S be of embedding dimension d+r. If the vectors  $\mathbf{a}_{d+1}, \ldots, \mathbf{a}_{d+r}$  belong to the same line passing through the origin of coordinates, then  $\mathbb{K}[S]$  is Cohen-Macaulay.

*Proof.* As  $\mathbf{a}_{d+1}, \ldots, \mathbf{a}_{d+r}$  belong to the same line passing through the origin of coordinates, there exist nonnegative rational numbers  $l_i \in \mathbb{Q}$  such that  $\mathbf{a}_{d+i} = l_i \mathbf{a}_{d+1}$ , for  $i = 1, \ldots, r$ . Let  $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{Ap}(S, E)$ , then  $\mathbf{w}_j = \lambda_j \mathbf{a}_{d+1} = \lambda_j (\sum_{i=1}^d \mu_i \mathbf{a}_i)$ , for some nonnegative rational numbers  $\lambda_1, \lambda_2, \mu_1, \ldots, \mu_d$ . If  $\mathbf{w}_1 - \mathbf{w}_2 \in \operatorname{group}(\mathbf{a}_1, \ldots, \mathbf{a}_d)$ , then

$$(\lambda_1 - \lambda_2)\left(\sum_{i=1}^d \mu_i \mathbf{a}_i\right) = \mathbf{w}_1 - \mathbf{w}_2 = \sum_{i=1}^d z_i \mathbf{a}_i,$$

for some integers  $z_1, \ldots, z_d$ . Hence, for all  $i, z_i = (\lambda_1 - \lambda_2)\mu_i$ , which forces the signs of  $z_i$  to be the same for all i. If they are all nonnegative (nonpositive), the equation  $\mathbf{w}_1 = \mathbf{w}_2 + \sum_{i=1}^d z_i \mathbf{a}_i$  ( $\mathbf{w}_2 = \mathbf{w}_1 + \sum_{i=1}^d (-z_i)\mathbf{a}_i$ ), implies  $z_i = 0$ , since  $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{Ap}(S, E)$ . Therefore,  $\mathbb{K}[S]$  is Cohen-Macaulay by Proposition 2.4.

The following lemma states an easy but useful property about maximal expressions of Apéry elements in Ap(S, E), when S has only two nonextremal generators that belong to the same line passing through the origin of coordinates.

**Lemma 2.7.** Let S be of embedding dimension d + 2. Then

- (1) There are no elements in Ap(S, E) having two different expressions with the same length. In particular, each element in Ap(S, E) has a unique maximal expression.
- (2) Assume that  $r\mathbf{a}_{d+1} = s\mathbf{a}_{d+2}$ , where r and s are relatively prime positive integers with r > s. For  $\mathbf{b} \in \operatorname{Ap}(S, E)$ , an expression  $\mathbf{b} = n_1\mathbf{a}_{d+1} + n_2\mathbf{a}_{d+2}$  is maximal if and only if  $n_2 < s$ .

*Proof.* (1). Let  $mgs(S) = \{\mathbf{a}_1, \ldots, \mathbf{a}_d, \mathbf{a}_{d+1}, \mathbf{a}_{d+2}\}$ . Assume on the contrary that, an element  $\mathbf{b} \in Ap(S, E)$  has two expressions of the same length

$$\mathbf{b} = n_1 \mathbf{a}_{d+1} + n_2 \mathbf{a}_{d+2} = n'_1 \mathbf{a}_{d+1} + n'_2 \mathbf{a}_{d+2}.$$

Then  $(n_1 - n'_1)\mathbf{a}_{d+1} + (n_2 - n'_2)\mathbf{a}_{d+2} = 0$ . Since  $n_1 + n_2 = n'_1 + n'_2$ ,  $n_1 - n'_1 = n'_2 - n_2 \neq 0$ . Therefore  $\mathbf{a}_{d+1} = -\mathbf{a}_{d+2}$ , a contradiction.

(2). If  $n_2 \ge s$ , then  $\mathbf{b} = (n_1 + r)\mathbf{a}_{d+1} + (n_2 - s)a_{d+2}$ . Since  $(n_1 + r + n_2 - s) > n_1 + n_2$ , the expression  $n_1\mathbf{a}_{d+1} + n_2\mathbf{a}_{d+2}$  is not maximal.

Now assume that  $n_2 < s$ . Let  $\mathbf{b} = n'_1 \mathbf{a}_{d+1} + n'_2 \mathbf{a}_{d+2}$  be a maximal expression. Then  $n'_2$  is also smaller than s, by our first argument. Comparing the two expressions of  $\mathbf{b}$ , we derive

$$(n_1 - n'_1)\mathbf{a}_{d+1} = (n'_2 - n_2)\mathbf{a}_{d+2} = \frac{r(n'_2 - n_2)}{s}\mathbf{a}_{d+1}.$$

Since r and s are relatively prime, this follows  $(n'_2 - n_2)$  is a multiple of s, which contradicts  $|n'_2 - n_2| < s$ , unless  $n'_2 = n_2$ .

#### 3. The type of simplicial affine semigroups

In this section, we are looking for a characterization of the Cohen-Macaulay type of the affine semigroup ring  $\mathbb{K}[S]$ , in terms of some numerical invariants of S. All over the section, as thorough the paper, S is a *d*-dimensional simplicial affine semigroup with  $\operatorname{mgs}(S) = \{\mathbf{a}_1, \ldots, \mathbf{a}_{d+r}\}$ , where  $\mathbf{a}_1, \ldots, \mathbf{a}_d$  are the extremal rays of S.

If d = 1, then dividing elements of S by the greatest common divisor of  $\mathbf{a}_1, \ldots, \mathbf{a}_{d+r}$ , we obtain an isomorphic semigroup. So, we may assume that S is a numerical semigroup, equivalently S is a submonid of  $\mathbb{N}$  such that  $\mathbb{N}\setminus S$  is a finite set. If  $S \neq \mathbb{N}$ , the elements of  $\operatorname{PF}(S) := \max_{\leq S} \mathbb{N} \setminus S = \{f \in \mathbb{N} \setminus S ; f+s \in S, \text{ for all } s \in S \setminus \{0\}\}$  are called pseudo-Frobenius numbers of S. The cardinality of  $\operatorname{PF}(S)$  is called the type of S, by Fröberg, Gottlieb and Häggkvist, in [9]. This notion of type coincides with the Cohen-Macaulay type of the numerical semigroup ring  $\mathbb{K}[S]$ , see [28] for a detailed proof.

In general case that  $d \ge 1$ , the pseudo-Frobenius elements of S are defined by analogy, in [12], to be  $PF(S) = \{ \mathbf{a} \in \mathbb{N}^d \setminus S ; \mathbf{a} + S \setminus \{0\} \subseteq S \}$ . As  $\mathbb{N}^d \setminus S$  is not

necessarily a finite set, PF(S) might be an empty set. Indeed  $PF(S) \neq \emptyset$  if and only if depth  $\mathbb{K}[S] = 1$ , [12, Theorem 6]. The pseudo-Frobenius numbers of a numerical semigroup, can be described in terms of Apéry sets. Let **a** be a nonzero element of a numerical semigroup S. Then  $PF(S) = \{\mathbf{w} - \mathbf{a} ; \mathbf{w} \in \operatorname{Max}_{\leq S} \operatorname{Ap}(S, \mathbf{a})\}$ , see [25, 2.20]. First, we present a generalization of PF(S), for simplicial affine semigroups  $S \subseteq \mathbb{N}^d$ , in terms of some Apéry sets. Let  $E = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$  and let  $l_i$  be the smallest positive integer such that  $l_i \mathbf{a}_{d+i} \in \sum_{j=1}^d \mathbb{N}\mathbf{a}_j$ , for  $i = 1, \ldots, r$ . Then

$$\operatorname{Ap}(S, E) = \bigcap_{i=1}^{d} \operatorname{Ap}(S, \mathbf{a}_i) \subseteq \{\sum_{i=1}^{r} n_i \mathbf{a}_{d+i} ; 0 \le n_i < l_i\},\$$

is a finite set. The last set in the above equation is called  $\Gamma$  in [23].

**Definition 3.1.** The element  $\mathbf{b} - \sum_{i=1}^{d} \mathbf{a}_i$ , where  $\mathbf{b} \in \operatorname{Max}_{\leq S} \operatorname{Ap}(S, E)$ , is called a *quasi-Frobenius* element. The set of quasi-Frobenius elements of S is denoted by QF(S). The number of quasi-Frobenius elements, is called the *type* of S and is denoted by type(S).

**Remark 3.2.** Let d > 1. If  $\mathbf{f} \in QF(S) \cap PF(S)$ , then  $\mathbf{f} + \mathbf{a}_1 = \mathbf{m} - \sum_{i=2}^{d} \mathbf{a}_i$ , where  $\mathbf{m} \in \operatorname{Max}_{\leq S} \operatorname{Ap}(S, E)$ . Since  $\mathbf{f} \in PF(S)$ , this follows  $\mathbf{f} + \mathbf{a}_1 \in S$ , which contradicts  $\mathbf{m} - \sum_{i=2}^{d} \mathbf{a}_i \notin S$ .

The type of a d-dimensional Cohen-Macaulay local ring  $(R, \mathfrak{m})$  is type $(R) = \dim_{R/\mathfrak{m}} \operatorname{Ext}_{R}^{d}(R/\mathfrak{m}, R)$ . For a Cohen-Macaulay ring R, the type is defined as the maximum of type $(R_{\mathfrak{p}})$ , where  $\mathfrak{p}$  ranges in the set of maximal ideals of R.

The ring  $\mathbb{K}[S]$  is  $\mathbb{N}$ -graded by setting deg $(\mathbf{x}^{\mathbf{a}}) = |\mathbf{a}|$ , for all  $\mathbf{a} \in S$ , where  $|(a_1, \ldots, a_d)| = \sum_{i=1}^d a_i$ , denotes the total degree. Therefore,

$$\operatorname{type}(\mathbb{K}[S]) = \operatorname{type}(\mathbb{K}[S]_{\mathfrak{m}}),$$

by [2, Theorem]. The following result can be also deduced from [5, Theorem 4.2(ii)], where the authors provide a combinatorial method to study some simplicial complexes associated to the elements of S. We bring a different algebraic proof here.

**Proposition 3.3.** If  $\mathbb{K}[S]$  is a Cohen-Macaulay ring, then type $(S) = type(\mathbb{K}[S]_m) = type(\mathbb{K}[S])$ .

*Proof.* The ring map  $\mathbb{K}[S]_{\mathfrak{m}} \longrightarrow \mathbb{K}[S]$  is flat and has only one trivial fiber which is the field  $\mathbb{K}$ . Since  $\mathbb{K}[S]/\mathfrak{m}\mathbb{K}[S]$ , as an  $\mathbb{K}[S]$ -module, has type 1 and depth 0, we have

$$type(\mathbb{K}[\![S]\!] \underset{\mathbb{K}[S]_{\mathfrak{m}}}{\otimes} \mathbb{K}[S]_{\mathfrak{m}}) = type(\mathbb{K}[S]_{\mathfrak{m}}),$$
$$depth(\mathbb{K}[\![S]\!] \underset{\mathbb{K}[S]_{\mathfrak{m}}}{\otimes} \mathbb{K}[S]_{\mathfrak{m}}) = depth(\mathbb{K}[S]_{\mathfrak{m}}),$$

by [4, Proposition 1.2.16]. Thus,  $\mathbb{K}[\![S]\!]$  is Cohen-Macaulay and type( $\mathbb{K}[\![S]\!]_{\mathfrak{m}}$ ) = type( $\mathbb{K}[\![S]\!]$ ). Let  $R = \mathbb{K}[\![S]\!]$ . Then R is a local ring with maximal ideal  $\mathfrak{m} = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{d+r}})$ . Note that  $\mathfrak{q} = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_d})$  is a parameter ideal of R, since S is simplicial. As R is Cohen-Macaulay,  $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_d}$  provide a maximal R-regular sequence. By [4, Lemma 1.2.19],

type
$$(R) = \dim_{R/\mathfrak{m}}(\operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{q})).$$

Since  $\operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{q}) \cong (0:_{R/\mathfrak{q}} \mathfrak{m}) = \{r \in R/\mathfrak{q} ; r\mathfrak{m} = 0\}$ , it is enough to show that  $(0:_{R/\mathfrak{q}} \mathfrak{m})$  is the  $R/\mathfrak{m}$ -vector space generated by residue classes of  $\mathbf{x}^{\mathbf{s}}$ , where

 $\mathbf{s} \in \max_{\leq S} \operatorname{Ap}(S, E)$ . For an element  $\mathbf{f} \in R$ , the residue of  $\mathbf{f}$  in  $R/\mathfrak{q}$  is equal to the residue of  $\sum_{i\geq 1} r_i \mathbf{x}^{\mathbf{s}_i}$ , for some  $r_i \in \mathbb{K}$  and  $\mathbf{s}_i \in \operatorname{Ap}(S, E)$ . If the residue of  $\mathbf{f}$  in  $R/\mathfrak{q}$ , belongs to  $(0:_{R/\mathfrak{q}}\mathfrak{m})$ , then we derive  $\mathbf{x}^{\mathbf{s}_i+\mathbf{a}_j} \in \mathfrak{q}$ , for  $i\geq 1$  and  $1\leq j\leq d+r$  which implies  $\mathbf{s}_i \in \max_{\leq S} \operatorname{Ap}(S, E)$ . Conversely, let  $\mathbf{s} \in \max_{\leq S} \operatorname{Ap}(S, E)$ . Since  $\mathbf{s} + \mathbf{a}_i \notin \operatorname{Ap}(S, E)$ , for  $i = d + 1, \ldots, d + r$ , we get  $\mathbf{x}^{\mathbf{s}_i+\mathbf{a}_i} \in \mathfrak{q}R$ .

Recall that a Cohen-Macaulay ring is Gorenstein precisely when its Cohen-Macaulay type is one. As an immediate consequence of Proposition 3.3, we derive the following

**Corollary 3.4.** [23, 4.6 and 4.8] [5, 4.2]  $\mathbb{K}[S]$  is a Gorenstein ring if and only if it is Cohen-Macaulay and Ap(S, E) has a single maximal element with respect to  $\leq_S$ .

If S is a numerical semigroup of embedding dimension three, then type(S)  $\leq 2$ , [9, Theorem 11]. The following result is a generalization of this fact, to simplicial affine semigroups of embedding dimension d + 2.

**Theorem 3.5.** Let S be of embedding dimension d+2. If  $\mathbb{K}[S]$  is Cohen-Macaulay, then type $(S) \leq 2$ .

*Proof.* Let  $\operatorname{Max}_{\leq S} \operatorname{Ap}(S, E) = \{\mathbf{m}_1, \dots, \mathbf{m}_t\}$ . By Lemma 2.7,  $\mathbf{m}_i$  has a unique maximal expression

$$\mathbf{m}_i = r_{i_1} \mathbf{a}_{d+1} + r_{i_2} \mathbf{a}_{d+2},$$

for  $i = 1, \ldots, t$ . Let s, k be such that  $r_{s_1} = \max\{r_{i_1} ; i = 1, \ldots, t\}$  and  $r_{k_2} = \max\{r_{i_2} ; i = 1, \ldots, t\}$ . Assume on the contrary, that  $t \ge 3$  and let  $l \in \{1, \ldots, t\} \setminus \{s, k\}$ . Since  $\mathbf{m}_l \not\leq_S \mathbf{m}_s$  and  $\mathbf{m}_l \not\leq_S \mathbf{m}_k$ , we have

$$r_{k_1} < r_{l_1} < r_{s_1}, r_{s_2} < r_{l_2} < r_{k_2}.$$

Moreover,

(3.1) 
$$\mathbf{m}_l + \mathbf{a}_{d+1} = (1 + r_{l_1})\mathbf{a}_{d+1} + r_{l_2}\mathbf{a}_{d+2} = \mathbf{w}_1 + \sum_{i=1}^d n_i \mathbf{a}_i,$$

(3.2) 
$$\mathbf{m}_l + \mathbf{a}_{d+2} = r_{l_1}\mathbf{a}_{d+1} + (1+r_{l_2})\mathbf{a}_{d+2} = \mathbf{w}_2 + \sum_{i=1}^d m_i \mathbf{a}_i$$

where  $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{Ap}(S, E)$  and  $n_i, m_i \in \mathbb{N}$  for  $i = 1, \ldots, d$ . Note that

$$\mathbf{w}_1 = c_1 \mathbf{a}_{d+1} + c_2 \mathbf{a}_{d+2} , \ \mathbf{w}_2 = e_1 \mathbf{a}_{d+1} + e_2 \mathbf{a}_{d+2},$$

for some  $c_1, c_2, e_1, e_2 \in \mathbb{N}$ . Since  $\mathbf{m}_l + \mathbf{a}_{d+i} \notin \operatorname{Ap}(S, E)$ , for i = 1, 2, both  $\sum_{i=1}^d n_i \mathbf{a}_i$ and  $\sum_{i=1}^d m_i \mathbf{a}_i$  are nonzero. As  $r_{l_1} \mathbf{a}_{d+1} + r_{l_2} \mathbf{a}_{d+2} \in \operatorname{Ap}(S, E)$ , we get  $c_1 = e_2 = 0$ . Note that,  $1 + r_{l_1} \leq r_{s_1}$  and  $1 + r_{l_2} \leq r_{k_2}$ . Therefore,  $(1 + r_{l_1})\mathbf{a}_{d+1}$  and  $(1 + r_{l_2})\mathbf{a}_{d+2}$ belong to  $\operatorname{Ap}(S, E)$ . Consequently,

$$(3.3) r_{l_1} > e_1 \text{ and } r_{l_2} > c_2,$$

which implies  $(1 + e_1)\mathbf{a}_{d+1}$  and  $(1 + c_2)\mathbf{a}_{d+2}$  are also in Ap(S, E).

Now, subtracting (3.2) from (3.1), we derive

$$(1+e_1)\mathbf{a}_{d+1} - (1+c_2)\mathbf{a}_{d+2} = \sum_{i=1}^{a} (n_i - m_i)\mathbf{a}_i.$$

Since  $\mathbb{K}[S]$  is Cohen-Macaulay, Proposition 2.4 implies that

$$(1+e_1)\mathbf{a}_{d+1} = (1+c_2)\mathbf{a}_{d+2}.$$

Without loss of generality we may assume that  $e_1 > c_2$ . Then

$$r_{l_2}\mathbf{a}_{d+2} = (r_{l_2} - c_2 - 1)\mathbf{a}_{d+2} + (1 + e_1)\mathbf{a}_{d+1}.$$

Consequently,  $\operatorname{ord}(r_{l_2}\mathbf{a}_{d+2}) \ge r_{l_2} - c_2 + e_1 > r_{l_2}$ , in contradiction to our choice of maximal expression of  $\mathbf{m}_l$ .

The converse of Theorem 3.5 is not true.

**Example 3.6.** Let  $\mathbf{a}_1 = (2,0), \mathbf{a}_2 = (0,2), \mathbf{a}_3 = (4,1), \mathbf{a}_4 = (2,3)$ . Since  $\mathbf{a}_1 - \mathbf{a}_2 = \mathbf{a}_3 - \mathbf{a}_4$ ,  $\mathbb{K}[S]$  is not Cohen-Macaulay, by Proposition 2.4. However,

$$\operatorname{Ap}(S, E) \setminus \{0\} = \max_{\leq s} \operatorname{Ap}(S, E) = \{(4, 1), (2, 3)\}.$$

The following two examples show that either if  $\mathbb{K}[S]$  is not Cohen-Macaulay or if  $r \geq 3$ , then type(S) does not have any upper bound in terms of its embedding dimension.

**Example 3.7.** Let  $\mathbf{a}_1 = (3,0), \mathbf{a}_2 = (0,3^n), \mathbf{a}_3 = (5,2), \mathbf{a}_4 = (2,2+3^n)$ , where n is a positive integer. Since  $\mathbf{a}_4 - \mathbf{a}_3 = \mathbf{a}_2 - \mathbf{a}_1$ ,  $\mathbb{K}[S]$  is not Cohen-Macaulay, by Proposition 2.4. First, we show that

(3.4) 
$$\operatorname{Ap}(S, E) = \{ r\mathbf{a}_3 + s\mathbf{a}_4; \text{ for all } r, s \in \mathbb{N} \text{ such that } r + s < 3^n \}.$$

Assume on the contrary that,  $r\mathbf{a}_3 + s\mathbf{a}_4 \notin \operatorname{Ap}(S, E)$  for some  $r, s \in \mathbb{N}$  such that  $r + s < 3^n$ . Then  $r\mathbf{a}_3 + s\mathbf{a}_4 = \sum_{i=1}^4 n_i \mathbf{a}_i$  for some integers  $n_i \ge 0$  such that either  $n_1 > 0$  or  $n_2 > 0$ . Consequently

$$(r - n_3)(5, 2) + (s - n_4)(2, 2 + 3^n) = n_1(3, 0) + n_2(0, 3^n),$$

which implies

(3.5) 
$$2(r+s-n_3-n_4)+3^n(s-n_4)=3^nn_2$$

$$(3.6) 5(r-n_3) + 2(s-n_4) = 3n_1$$

Therefore,  $r + s = 3^n k + n_3 + n_4$  for some integer k. Since  $r + s < 3^n$ , we get  $k \le 0$ . Then  $r + s - n_3 - n_4 \le 0$ . Since  $n_2 \ge 0$ , we get by (3.5) that  $s - n_4 \ge 0$  and  $r - n_3 \le 0$ . Note that

$$3(r - n_3) + 2(r + s - n_3 - n_4) = 3n_1,$$

from (3.6). Since  $r - n_3 \leq 0$ ,  $r + s - n_3 - n_4 \leq 0$  and  $n_1 \geq 0$ , we get

$$r - n_3 = r + s - n_3 - n_4 = n_1 = 0.$$

Then  $n_2 = 0$ , from (3.5), a contradiction. We have shown that, the set on the right hand side of (3.4) is a subset of Ap(S, E). Now, let  $r, s \in \mathbb{N}$  such that  $r + s = 3^n$ . So

$$r\mathbf{a}_3 + s\mathbf{a}_4 - \mathbf{a}_1 = r(5,2) + s(2,2+3^n) - (3,0)$$
  
= (2(r+s) + 3(r-1), 2(r+s) + 3^n s)  
= (2 × 3^{n-1} + (r-1))(3,0) + (2+s)(0,3^n).

Therefore  $r\mathbf{a}_3 + s\mathbf{a}_4 \notin \operatorname{Ap}(S, E)$ . If  $r + s > 3^n$ , then

$$r\mathbf{a}_3 + s\mathbf{a}_4 = r'\mathbf{a}_3 + s'\mathbf{a}_4 + (r - r')\mathbf{a}_3 + (s - s')\mathbf{a}_4$$

for some nonnegative integers r', s' such that r > r', s > s' and  $r' + s' = 3^n$ . Consequently,  $r\mathbf{a}_3 + s\mathbf{a}_4 \notin \operatorname{Ap}(S, E)$ . Therefore, (3.4) is proved. Now, one can easily see that

$$\max_{\prec_S} \operatorname{Ap}(S, E) = \{ r\mathbf{a}_3 + s\mathbf{a}_4; \text{ for all } r, s \in \mathbb{N} \text{ such that } r + s = 3^n - 1 \}$$

and so type $(S) = |\{(r,s) \in \mathbb{N}^2 ; r+s = 3^n - 1\}| = 3^n$ .

**Example 3.8.** For an integer  $a \ge 3$ , let S be the affine semigroup generated by  $\mathbf{a}_1 = (a^2, 0), \mathbf{a}_2 = (0, a^2), \mathbf{a}_3 = (a^2 - a, a^2 - a), \mathbf{a}_4 = (a^2 - a + 1, a^2 - a + 1), \mathbf{a}_5 = (a^2 - 1, a^2 - 1)$ . Then S is simplicial with extremal rays  $\mathbf{a}_1, \mathbf{a}_2$ . Let T be the numerical semigroup generated by  $\{a^2 - a, a^2 - a + 1, a^2 - 1, a^2\}$ . Then

(3.7) 
$$\operatorname{Ap}(S, E) = \operatorname{Ap}(S, \mathbf{a}_1 + \mathbf{a}_2) = \{(s, s) : s \in \operatorname{Ap}(T, a^2)\}.$$

Therefore, type(S) = type(T) = 2a - 4, by [6, (3.4)Proposition]. Note that  $\mathbb{K}[S]$  is Cohen-Macaulay, by Lemma 2.6.

## 4. The conductor ideal of simplicial affine semigroups

Thorough this section,  $S \subseteq \mathbb{N}^d$  is a simplicial affine semigroup with  $\operatorname{mgs}(S) = \{\mathbf{a}_1, \ldots, \mathbf{a}_{d+r}\}$ , where  $\mathbf{a}_1, \ldots, \mathbf{a}_d$  are the extremal rays of S. Let  $R = \mathbb{K}[S]$  be the affine semigroup ring. Recall that the normalization of an integral domain R is the set of elements in its field of fractions satisfying a monic polynomial in R[y]. Then  $R = \mathbb{K}[S]$  is an integral domain with normalization  $\overline{R} = \mathbb{K}[\operatorname{group}(S) \cap \operatorname{cone}(S)]$  [21, Proposition 7.25]. Recall that the conductor of R,  $C_R = (R :_T \overline{R})$ , where T denotes the total ring of fractions of R, is the largest common ideal of R and  $\overline{R}$ , [19, Exercise 2.11]. The purpose of this section, is to investigate the normality of R and the generating set of  $C_R$  as an ideal of  $\overline{R}$ .

The integral closure of S in group(S),  $\overline{S} = \{\mathbf{a} \in \text{group}(S) ; n\mathbf{a} \in S \text{ for some } n \in \mathbb{N}\}$ , is called the *normalization* of S. As a geometrical interpretation, one can see that  $\overline{S} = \text{cone}(S) \cap \text{group}(S)$ . The semigroup S is *normal* when  $S = \overline{S}$ , equivalently  $\mathbb{K}[S]$  is a normal ring, [3, 4]. Since S is finitely generated, cone(S) is generated by finitely many rational vectors, i.e. it is the intersection of finitely many rational vector halfspaces, [27, Corollary 7.1(a)]. By Gordan's lemma,  $\overline{S}$  is also finitely generated.

The conductor of S is defined as  $\mathfrak{c}(S) = \{\mathbf{b} \in S ; \mathbf{b} + \overline{S} \subseteq S\}$ . The conductor,  $\mathfrak{c}(S)$ , is the largest ideal of S that is also an ideal of  $\overline{S}$ , [3, Exercise 2.9].

**Remark 4.1.** As  $\mathfrak{c}(S)$  is an ideal of S, we have  $S = \overline{S}$  precisely when  $0 \in \mathfrak{c}(S)$ . In other words, S is normal if and only if  $\mathfrak{c}(S) = S$ .

When S is fully embedded in  $\mathbb{N}^d$ , that is the affine subspace it generates coincides with  $\mathbb{R}^d$ , we have group(S)  $\cong \mathbb{Z}^d$ . In the case that group(S)  $= \mathbb{Z}^d$ , we have  $\bar{S} = \mathbb{N}^d \cap \operatorname{cone}(S)$ . The later property may happen also for affine semigroups that group(S)  $\neq \mathbb{Z}^d$ . For instance, it holds when  $(\operatorname{cone}(S) \setminus S) \cap \mathbb{N}^d$  is finite, such semigroups are considered in [11].

**Lemma 4.2.** If  $(\operatorname{cone}(S) \setminus S) \cap \mathbb{N}^d$  is finite, then  $\operatorname{group}(S) \cap \operatorname{cone}(S) = \mathbb{N}^d \cap \operatorname{cone}(S)$ . In particular, S is normal if and only if  $S = \mathbb{N}^d \cap \operatorname{cone}(S)$ .

*Proof.* All vectors in cone(S) have nonnegative components. Thus group(S)  $\cap$  cone(S)  $\subseteq \mathbb{N}^d \cap \operatorname{cone}(S)$ . Let  $\mathbf{a} \in \mathbb{N}^d \cap \operatorname{cone}(S)$ . Then  $\mathbf{a} + \sum_{i=1}^d l_i \mathbf{a}_i \in \operatorname{cone}(S) \cap \mathbb{N}^d$ , for all  $l_1, \ldots, l_d \in \mathbb{N}$ . Since  $(\operatorname{cone}(S) \cap \mathbb{N}^d) \setminus S$  is a finite set, we have  $\mathbf{a} + \sum_{i=1}^d l_i \mathbf{a}_i \in S$ ,

for some  $l_1, \ldots, l_d \in \mathbb{N}$ , which implies  $\mathbf{a} \in \operatorname{group}(S)$ . Therefore,  $\mathbb{N}^d \cap \operatorname{cone}(S) = \operatorname{group}(S) \cap \operatorname{cone}(S)$ .

**Lemma 4.3.** As an affine semigroup,  $\overline{S}$  is generated by  $(P_S \cap \operatorname{group}(S)) \cup \{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$ , and  $P_S \cap \operatorname{group}(S) = \{r(\mathbf{w}) ; \mathbf{w} \in \operatorname{Ap}(S, E)\}.$ 

Proof. An element  $\mathbf{a} \in \overline{S}$  can be written as  $\mathbf{a} = r(\mathbf{a}) + \sum_{i=1}^{d} r_i \mathbf{a}_i$  for some  $r_1, \ldots, r_d \in \mathbb{N}$ . Therefore,  $(P_S \cap \operatorname{group}(S)) \cup \{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$  provides a generating set for  $\overline{S}$ . For the last statement, let  $\mathbf{b} \in P_S \cap \operatorname{group}(S)$ . By Remark 2.3, there exist  $r_1, \ldots, r_d, s_1, \ldots, s_d \in \mathbb{N}$  and  $\mathbf{w} \in \operatorname{Ap}(S, E)$  such that  $\mathbf{b} + \sum_{i=1}^{d} r_i \mathbf{a}_i = \mathbf{w} + \sum_{i=1}^{d} s_i \mathbf{a}_i$ . Since  $\mathbf{b} \in P_S$ , it follows that  $\mathbf{b} = r(\mathbf{w})$ .

As an immediate consequence of Lemma 4.3,

(4.1) 
$$\mathbf{c}(S) = \{\mathbf{a} \in S ; \ \mathbf{a} + r(\mathbf{w}) \in S \text{ for all } \mathbf{w} \in \operatorname{Ap}(S, E)\}$$

**Example 4.4.** Let S be a numerical semigroup, that is d = 1. Then  $\overline{S} = \mathbb{N}$  and  $\mathfrak{c}(S) = \{n ; n \ge F+1\}$ , where F is the maximal integer in  $\mathbb{Z} \setminus S$ . Therefore,  $\mathfrak{c}(S)$  is generated by  $\{F+1\}$  as an ideal of N. In this regard, F+1 is called the conductor of S.

Lemma 4.5. The following statements are equivalent.

(1)  $-\operatorname{QF}(S) \subseteq \operatorname{cone}(S);$ (2)  $\operatorname{Ap}(S, E) \subseteq \overline{P}_S.$ 

*Proof.* (1)  $\Longrightarrow$  (2): Let  $\mathbf{w} \in \operatorname{Ap}(S, E)$ . Then  $\mathbf{m} - \mathbf{w} \in S$ , for some  $\mathbf{m} \in \operatorname{Max}_{\preceq S} \operatorname{Ap}(S, E)$ . As  $\mathbf{m} - \sum_{i=1}^{d} \mathbf{a}_{i} \in \operatorname{QF}(S)$ , we get  $\sum_{i=1}^{d} (1 - [\mathbf{m}]_{i}) \mathbf{a}_{i} \in \operatorname{cone}(S)$  precisely when  $[\mathbf{w}]_{i} \leq [\mathbf{m}]_{i} \leq 1$ .

 $(2) \Longrightarrow (1): \text{ Let } \mathbf{f} = \mathbf{m} - \sum_{i=1}^{d} \mathbf{a}_i \in QF(S) \text{ for some } \mathbf{m} \in \text{Max}_{\leq S} \operatorname{Ap}(S, E). \text{ As}$  $[\mathbf{m}]_i \leq 1 \text{ for } i = 1, \dots, d, \text{ we get } -\mathbf{f} = \sum_{i=1}^{d} (1 - [\mathbf{m}]_i) \mathbf{a}_i \in \operatorname{cone}(S). \square$ 

If we replace  $\bar{P}_S$  in the above lemma, with  $P_S$ , then we derive an equivalent condition for S to be normal. Roughly speaking, having more negative coefficients  $[\mathbf{f}]_i$  for  $\mathbf{f} \in \mathrm{QF}(S)$ , makes the semigroup more close to being normal. In this order, we need to consider the relative interior of  $\mathrm{cone}(S)$ . Let  $\mathrm{relint}(S)$  denote the elements of  $\mathbb{R}^d$  that belong to the relative interior of  $\mathrm{cone}(S)$ ,

relint(S) = 
$$\left\{ \mathbf{b} \in \operatorname{cone}(S) ; \mathbf{b} = \sum_{i=1}^{d} \lambda_i \mathbf{a}_i \text{ with } \lambda_i \in \mathbb{R}_{>0} \text{ for all } i = 1, \dots, d \right\}.$$

**Theorem 4.6.** The following statements are equivalent.

- (1) S is normal; (2)  $-\operatorname{QF}(S) \subseteq S \cap \operatorname{relint}(S);$ (3)  $-\operatorname{QF}(S) \subseteq \operatorname{relint}(S);$
- (4)  $\operatorname{Ap}(S, E) \subseteq P_S$ .

*Proof.* (1)  $\implies$  (2): Let  $\mathbf{f} \in \operatorname{QF}(S)$ . Then  $\mathbf{f} = \mathbf{m} - \sum_{i=1}^{d} \mathbf{a}_{i}$ , for some  $\mathbf{m} \in \operatorname{Max}_{\leq S} \operatorname{Ap}(S, E)$ . If  $[\mathbf{m}]_{j} \geq 1$ , for some  $1 \leq j \leq d$ , then  $\mathbf{m} = \mathbf{a}_{j} + ([\mathbf{m}]_{j} - 1)\mathbf{a}_{j} + \sum_{i=1, i \neq j}^{d} [\mathbf{m}]_{i}\mathbf{a}_{i}$ , which implies  $\mathbf{m} - \mathbf{a}_{j} \in \overline{S} = S$ , which contradicts  $\mathbf{m} \in \operatorname{Ap}(S, E)$ . Therefore  $[\mathbf{m}]_{i} < 1$ , for  $i = 1, \ldots, d$ . Now,  $-\mathbf{f} = \sum_{i=1}^{d} (1 - [\mathbf{m}]_{i})\mathbf{a}_{i} \in \operatorname{relint}(S) \cap \operatorname{group}(S) \subseteq S$ .

 $(2) \Longrightarrow (3)$  is clear.

(3)  $\Longrightarrow$  (4): Let  $\mathbf{w} \in \operatorname{Ap}(S, E)$ . Then  $\mathbf{m} - \mathbf{w} \in S$ , for some  $\mathbf{m} \in \operatorname{Max}_{\preceq S} \operatorname{Ap}(S, E)$ . As  $\mathbf{m} - \sum_{i=1}^{d} \mathbf{a}_i \in \operatorname{QF}(S)$ , we get  $\sum_{i=1}^{d} (1 - [\mathbf{m}]_i) \mathbf{a}_i \in \operatorname{relint}(S)$ . Thus,  $1 - [\mathbf{m}]_i > 0$  which implies  $[\mathbf{w}]_i \leq [\mathbf{m}]_i < 1$ .

(4)  $\implies$  (1): Ap $(S, E) \subseteq P_S$  is equivalent to Ap(S, E) = r(Ap(S, E)). As the later equality holds precisely when  $r(Ap(S, E)) \subseteq S$ , the result follows by Lemma 4.3.

Our next aim in this section is to find a generating set for  $\mathbf{c}(S)$  as an ideal of  $\overline{S}$ . Recall from Section 2, that  $C_j = \{\mathbf{w} \in \operatorname{Ap}(S, E) ; r(\mathbf{w}) = \mathbf{b}_j\}$ , for  $j = 0, \ldots, k$ , where  $r(\operatorname{Ap}(S, E)) = \{r(\mathbf{w}) ; \mathbf{w} \in \operatorname{Ap}(S, E)\} = \{0 = \mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_k\}$ . For any  $(\mathbf{w}_1, \ldots, \mathbf{w}_k) \in C_1 \times \cdots \times C_k$ , we consider the vector

$$\mathbf{f}_{(\mathbf{w}_1,\ldots,\mathbf{w}_k)} = \sum_{i=1}^d f_i \mathbf{a}_i,$$

where  $f_i = \max\{[\mathbf{w}_j - r(\mathbf{w}_j)]_i ; j = 1, ..., k\}$ , for i = 1, ..., d. Note that

 $f_i = \max\{\lfloor [\mathbf{w}_j]_i \rfloor \; ; \; j = 1, \dots, k\},\$ 

for i = 1, ..., d, where  $\lfloor [\mathbf{w}_j]_i \rfloor$  denotes the greatest integer less than or equal to  $[\mathbf{w}_j]_i$ .

**Lemma 4.7.** Given  $(\mathbf{w}_1, \ldots, \mathbf{w}_k) \in C_1 \times \cdots \times C_k$ , the vector  $\mathbf{f}_{(\mathbf{w}_1, \ldots, \mathbf{w}_k)}$  belongs to  $\mathfrak{c}(S)$ .

*Proof.* Let  $\mathbf{f} = \mathbf{f}_{(\mathbf{w}_1,\ldots,\mathbf{w}_k)}$ . By (4.1), it is enough to show that  $\mathbf{f} + \mathbf{b}_j \in S$  for  $j = 1,\ldots,k$ . Let  $\mathbf{b}_j = r(\mathbf{w}_j)$ . Note that  $\mathbf{f} = \sum_{i=1}^d ([\mathbf{w}_j - \mathbf{b}_j]_i + r_i)\mathbf{a}_i$ , for some  $r_i \in \mathbb{N}$ . Therefore,  $\mathbf{f} + \mathbf{b}_j = \mathbf{w}_j + \sum_{i=1}^d r_i \mathbf{a}_i \in S$ , for  $j = 1,\ldots,k$ .

Corollary 4.8. The following statements are equivalent.

- (1) S is normal;
- (2)  $\mathbf{f}_{(\mathbf{w}_1,...,\mathbf{w}_k)} = 0$ , for all  $\mathbf{w}_i \in C_i$  and i = 1,...,k;
- (3)  $\mathbf{f}_{(\mathbf{w}_1,\ldots,\mathbf{w}_k)} = 0$ , for some  $\mathbf{w}_i \in C_i$  and  $i = 1,\ldots,k$ .

*Proof.*  $(1) \Longrightarrow (2)$  follows from Theorem 4.6.

 $(2) \Longrightarrow (3)$  is clear.

(3)  $\Longrightarrow$  (1): By Lemma 4.7,  $0 = \mathbf{f}_{(\mathbf{w}_1,...,\mathbf{w}_k)}$  belongs to  $\mathfrak{c}(S)$  which is an ideal of S. Therefore, S is normal, see Remark 4.1.

**Theorem 4.9.** Let **c** be a minimal generator of  $\mathbf{c}(S)$ . Then there exist  $(\mathbf{w}_1, \ldots, \mathbf{w}_k) \in C_1 \times \cdots \times C_k$  such that  $\mathbf{c} = \mathbf{f}_{(\mathbf{w}_1, \ldots, \mathbf{w}_k)} - \mathbf{b}_j + \sum_{i=1}^d l_i \mathbf{a}_i$  for some  $l_i \in \{0, 1\}$  and  $j \in \{0, \ldots, k\}$ . Moreover, at least for one i, we have  $l_i = 0$ .

*Proof.* Since  $\mathbf{c} + \mathbf{b}_j \in S$ , for  $j = 0, \ldots, k$ ,

(4.2) 
$$\mathbf{c} + \mathbf{b}_j = \mathbf{w}_{tj} + \sum_{i=1}^d r_{ji} \mathbf{a}_i,$$

for some  $\mathbf{w}_{t_j} \in \operatorname{Ap}(S, E)$  and  $r_{j_i} \in \mathbb{N}$ . Note that  $r(\mathbf{w}_{t_j}) \neq r(\mathbf{w}_{t_i})$ , for  $0 \leq i \neq j \leq k$ , since otherwise  $\mathbf{b}_j - \mathbf{b}_i \in \operatorname{group}(\mathbf{a}_1, \ldots, \mathbf{a}_d)$  which is not possible. Let  $\{t_1, \ldots, t_k\} = \{1, \ldots, k\}$  such that  $\mathbf{w}_i \in C_i$ , and let  $\mathbf{f} = \mathbf{f}_{(\mathbf{w}_1, \ldots, \mathbf{w}_k)}$ . Then

$$[\mathbf{f}]_j = \max\{\lfloor [\mathbf{w}_i]_j \rfloor ; i = 1, \dots, k\},\$$

for  $j = 1, \ldots, d$ . Let  $1 \leq s \leq d$ . If  $r_{j_s} \geq 1$  for  $j = 0, \ldots, k$ , then

$$\mathbf{c} - \mathbf{a}_s + \mathbf{b}_i = \mathbf{w}_{t_i} + \sum_{j=1, j \neq s}^d r_{i_j} \mathbf{a}_j + (r_{i_s} - 1) \mathbf{a}_s \in S,$$

for i = 0, ..., k, which implies  $\mathbf{c} - \mathbf{a}_s \in \mathfrak{c}(S)$ , a contradiction. Thus,  $r_{j_s} = 0$  for some  $0 \le j \le k$ . Consider  $0 \le h \le k$  such that

$$[\mathbf{b}_h]_s = \max\{[\mathbf{b}_j]_s ; r_{j_s} = 0, 0 \le j \le k\}.$$

Then  $[\mathbf{w}_{t_h}]_s = [\mathbf{c} + \mathbf{b}_h]_s = \max\{[\mathbf{w}_{t_j}]_s ; r_{j_s} = 0, 0 \le j \le k\}$ . If  $r_{i_s} > 0$  for some  $0 \le i \le k$ , then

$$[\mathbf{w}_{t_h}]_s - [\mathbf{b}_h]_s = [\mathbf{c}]_s = [\mathbf{w}_{t_i}]_s + r_{i_s} - [\mathbf{b}_i]_s > [\mathbf{w}_{t_i}]_s.$$

Consequently,

$$[\mathbf{w}_{t_h}]_s = \max\{[\mathbf{w}_i]_s ; i = 1, \dots, k\},\$$

and thus

$$\lfloor [\mathbf{w}_{t_h}]_s \rfloor = [\mathbf{f}]_s.$$

Let  $c_s = \lceil [\mathbf{c}]_s \rceil$ , where  $\lceil [\mathbf{c}]_s \rceil = \operatorname{ceil}([\mathbf{c}]_s)$  denotes the least integer greater than or equal to  $[\mathbf{c}]_s$ , for  $s = 1, \ldots, d$ . Now, we distinguish the following two possibilities:

(1) If  $[\mathbf{b}_h]_s \ge c_s - [\mathbf{c}]_s$ , then  $[\mathbf{w}_{t_h}]_s = [\mathbf{b}_h + \mathbf{c}]_s \ge c_s$ . As  $c_s$  is an integer, it follows that  $\lfloor [\mathbf{w}_{t_h}]_s \rfloor \ge c_s$ . If  $\lfloor [\mathbf{w}_{t_h}]_s \rfloor > c_s$ , then  $\lfloor [\mathbf{w}_{t_h}]_s \rfloor \ge 1 + c_s = 1 + \lceil [\mathbf{c}]_s \rceil$ , which implies  $[\mathbf{b}_h]_s = [\mathbf{w}_{t_h}]_s - [\mathbf{c}]_s \ge 1$ , a contradiction. Therefore,

$$c_s = \lfloor [\mathbf{c} + \mathbf{b}_h]_s \rfloor = \lfloor [\mathbf{w}_{t_h}]_s \rfloor = [\mathbf{f}]_s$$

(2) If  $[\mathbf{b}_h]_s < c_s - [\mathbf{c}]_s$ , then  $\lfloor [\mathbf{b}_h + \mathbf{c}]_s \rfloor = \lfloor [\mathbf{c}]_s \rfloor$ . Note that,  $c_s - [\mathbf{c}]_s > 0$  which means  $[\mathbf{c}]_s$  is not an integer. Thus,  $\lfloor [\mathbf{c}]_s \rfloor = c_s - 1$ . Therefore,

$$c_s = \lfloor [\mathbf{c}]_s \rfloor + 1 = \lfloor [\mathbf{b}_h + \mathbf{c}]_s \rfloor + 1 = \lfloor [\mathbf{w}_{t_h}]_s \rfloor + 1 = [\mathbf{f}]_s + 1.$$

If  $c_s$  satisfies (1), then let  $l_s = 0$ , and otherwise let  $l_s = 1$ . Then

$$\mathbf{c} = \mathbf{f} - \sum_{s=1}^{d} (c_s - [\mathbf{c}]_s) \mathbf{a}_s + \sum_{i=1}^{d} l_i \mathbf{a}_i.$$

Note that  $\sum_{s=1}^{d} (c_s - [\mathbf{c}]_s) \mathbf{a}_s = \sum_{s=1}^{d} c_s \mathbf{a}_s - \mathbf{c}$  belongs to group $(S) \cap P_S = r(\operatorname{Ap}(S, E))$ , see Lemma 4.3.

For the last statement, let  $1 \leq j \leq k$  and let

$$\{i_1,\ldots,i_t\} = \{i \ ; \ 1 \le i \le d, [\mathbf{b}_j]_i > 0\}.$$

Then  $\sum_{s=1}^{t} \mathbf{a}_{i_s} - \mathbf{b}_j \in \operatorname{group}(S) \cap P_S = r(\operatorname{Ap}(S, E))$ . In particular,  $\mathbf{f} + \sum_{i=1}^{d} \mathbf{a}_i - \mathbf{b}_j \in {\mathbf{f}} + \overline{S}$ . As  $\mathbf{f} \in \mathfrak{c}(S)$  by Lemma 4.7, it means that  $\mathbf{f} + \sum_{i=1}^{d} \mathbf{a}_i - \mathbf{b}_j$  is not in the minimal generating set of  $\mathfrak{c}(S)$ .

**Corollary 4.10.** If  $\mathbf{c} \in \mathfrak{c}(S)$  such that  $[\mathbf{c}]_i \in \mathbb{N}$  for  $i = 1, \ldots, d$ , then  $[\mathbf{c}]_i \geq [\mathbf{f}_{(\mathbf{w}_1,\ldots,\mathbf{w}_k)}]_i$  for some  $\mathbf{w}_1,\ldots,\mathbf{w}_k \in C_1 \times \cdots \times C_k$ .

*Proof.* Note that  $\mathbf{c} = \mathbf{c}' + \mathbf{b}$  for a minimal generator  $\mathbf{c}'$  of  $\mathbf{c}(S)$  and some  $\mathbf{b} \in S$ . By Theorem 4.9, there exist  $(\mathbf{w}_1, \ldots, \mathbf{w}_k) \in C_1 \times \cdots \times C_k$  and  $l_1, \ldots, l_d \in \{0, 1\}$  such that  $\mathbf{c}' = \mathbf{f}_{(\mathbf{w}_1, \ldots, \mathbf{w}_k)} - \mathbf{b}_j + \sum_{i=1}^d l_i \mathbf{a}_i$  for some  $j \in \{0, \ldots, k\}$ . Thus

$$[\mathbf{c}]_i \ge [\mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_k)}]_i - [\mathbf{b}_j]_i.$$

As  $[\mathbf{c}]_i$  is an integer, and  $0 \leq [\mathbf{b}_j]_i < 1$ , we have  $[\mathbf{c}]_i \geq [\mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_k)}]_i$ .

**Example 4.11.** Let  $\mathbf{a}_1 = (3,0), \mathbf{a}_2 = (0,3), \mathbf{a}_3 = (5,2), \mathbf{a}_4 = (2,5)$ . As we have seen in Example 3.7,

$$\begin{aligned} \operatorname{Ap}(S, E) &= \{0, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_3 + \mathbf{a}_4, 2\mathbf{a}_3, 2\mathbf{a}_4\} \\ &= \{0, \mathbf{w}_1 = (5, 2), \mathbf{w}_2 = (2, 5), \mathbf{w}_3 = (7, 7), \mathbf{w}_4 = (10, 4), \mathbf{w}_5 = (4, 10)\} \\ \text{and } r(\operatorname{Ap}(S, E)) &= \{0, \mathbf{b}_1 = (1, 1), \mathbf{b}_2 = (2, 2)\}. \text{ Note that } C_1 = \{\mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\} \\ C_2 &= \{\mathbf{w}_1, \mathbf{w}_2\} \text{ and } \mathbf{f}_{(\mathbf{w}_3, \mathbf{w}_i)} = 2\mathbf{a}_1 + 2\mathbf{a}_2 = (6, 6), \ \mathbf{f}_{(\mathbf{w}_4, \mathbf{w}_i)} = 3\mathbf{a}_1 + \mathbf{a}_2 = (9, 3) \\ \mathbf{f}_{(\mathbf{w}_5, \mathbf{w}_i)} = \mathbf{a}_1 + 3\mathbf{a}_2 = (3, 9), \text{ for } i = 1, 2. \end{aligned}$$

As 
$$\{(6,6) - (1,1), (9,3) - (1,1), (3,9) - (1,1)\} + r(\operatorname{Ap}(S,E)) \subset S$$
, we have  $\{(5,5), (8,2), (2,8)\} \subset \mathfrak{c}(S)$ .

If  $\mathfrak{c}(S) \neq \{(5,5), (8,2), (2,8)\} + \overline{S}$ , the other generators of  $\mathfrak{c}(S)$  are among

$$\{(9,3), (3,9), (6,6)\} + \{l_i \mathbf{a}_i - (2,2) ; l_i \in \{0,1\}, i = 1,2\},\$$

by Theorem 4.9. Since the above set which equals

 $\{(7,1), (1,7), (4,4), (10,1), (1,10), (4,7), (7,4)\},\$ 

has no element in S,  $\mathfrak{c}(S)$  is generated by  $\{(5,5), (8,2), (2,8)\} = \{\mathbf{f}_{(\mathbf{w}_3,\mathbf{w}_1)} - \mathbf{b}_1, \mathbf{f}_{(\mathbf{w}_4,\mathbf{w}_1)} - \mathbf{b}_1, \mathbf{f}_{(\mathbf{w}_5,\mathbf{w}_1)} - \mathbf{b}_1\}$ , as an ideal of  $\bar{S} = \langle (3,0), (0,3), (1,1) \rangle$ .

The following example shows that the summand  $\sum_{i=1}^{d} l_i \mathbf{a}_i$  in the statement of Theorem 4.9 can not be removed.

**Example 4.12.** Let  $\mathbf{a}_1 = (5,2), \mathbf{a}_2 = (2,2), \mathbf{a}_3 = (2,1), \mathbf{a}_4 = (5,3)$ . Then Ap $(S, E) = \{0, \mathbf{w}_1 = (2,1), \mathbf{w}_2 = (4,2), \mathbf{w}_3 = (6,3), \mathbf{w}_4 = (8,4), \mathbf{w}_5 = (5,3)\}$  and  $r(\operatorname{Ap}(S, E)) = \{0, \mathbf{b}_1 = (2,1), \mathbf{b}_2 = (4,2), \mathbf{b}_3 = (1,1), \mathbf{b}_4 = (3,2), \mathbf{b}_5 = (5,3)\}$ . Note that  $C_i = \{\mathbf{w}_i\}$  for  $i = 1, \ldots, 5$  and  $\mathbf{f}_{(\mathbf{w}_1, \ldots, \mathbf{w}_5)} = \mathbf{a}_1$ . By Theorem 4.9, the generators of  $\mathbf{c}(S)$  are among

$$\mathbf{a}_1 - \mathbf{b}_i, 2\mathbf{a}_1 - \mathbf{b}_i, \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{b}_i ; i = 0, \dots, 5\}.$$

The only elements of the above set, that belong also to S are

 $\{(5,2), (10,4), (5,3), (2,1), (7,4), (4,2), (6,3)\}.$ 

Note that  $(2,1)+(1,1) \notin S$ ,  $\{(5,2), (4,2)\}+r(\operatorname{Ap}(S,E)) \subseteq S$ ,  $\{(10,4), (7,4), (6,3)\} \subset (5,2) + \overline{S}$  and  $(5,3) = (4,2) + \mathbf{b}_3$ . Therefore,  $\mathfrak{c}(S)$  is generated by  $\{(5,2), (4,2)\} = \{\mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_5)}, \mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_5)} + \mathbf{a}_2 - \mathbf{b}_4\}$ , as an ideal of  $\overline{S} = \langle (1,1), (2,1), (5,2) \rangle$ .

**Proposition 4.13.** Assume that there is a fixed class  $C_j$  such that for any  $\mathbf{w} \in C_j$ and  $\mathbf{w}' \in \operatorname{Ap}(S, E) \setminus C_j$ , one has  $\max_{\leq c}(\mathbf{w}, \mathbf{w}') = \mathbf{w}$ . If either  $C_j$  is a singleton or  $\mathbf{b}_j = \min_{\leq c} (r(\operatorname{Ap}(S, E)) \setminus \{0\})$ , then  $\mathbf{c}(S)$  is generated by

$$\{\mathbf{w} - \mathbf{b} ; \mathbf{w} \in C_j , \mathbf{b} \in \max_{\prec_c} \{\mathbf{b}_1, \dots, \mathbf{b}_k\}\},\$$

as an ideal of  $\bar{S}$ .

*Proof.* First we show that

 $\mathbf{w} - \mathbf{b}_l + \mathbf{b}_s \in S,$ 

for any  $\mathbf{w} \in C_j$  and  $0 \le l, s \le k$ .

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As  $\mathbf{w} - \mathbf{b}_l + \mathbf{b}_s \in \operatorname{group}(S)$ , we have by Remark 2.3, that  $\mathbf{w} - \mathbf{b}_l + \mathbf{b}_s + \sum_{i=1}^{d} r_i \mathbf{a}_i = \mathbf{w}' + \sum_{i=1}^{d} s_i \mathbf{a}_i$  for some  $\mathbf{w}' \in \operatorname{Ap}(S, E)$  and nonnegative integers  $r_1, \ldots, r_d, s_1, \ldots, s_d$ . If  $r_1 = \cdots = r_d = 0$ , then (4.3) is clear. Assume that  $r_h > 0$ , for some  $1 \leq h \leq d$ . Then  $s_h = 0$  by our choice in Remark 2.3, and  $[\mathbf{w}]_h < [\mathbf{w}']_h$ .

If  $\mathbf{w}' \notin C_j$ , then  $\max_{\leq_c}(\mathbf{w}, \mathbf{w}') = \mathbf{w}$  which implies  $[\mathbf{w}]_h \geq [\mathbf{w}']_h$ , a contradiction. Thus  $\mathbf{w}' \in C_j$ . In other words,  $\mathbf{w}$  and  $\mathbf{w}'$  have the same remainder  $\mathbf{b}_j$ . Therefore,  $\mathbf{b}_s = \mathbf{b}_l$ , and (4.3) holds. Consequently,  $\mathbf{w} - \mathbf{b}_r \in \mathfrak{c}(S)$  for  $r = 0, \ldots, k$ .

Let **c** be a minimal generator of c(S). By Theorem 4.9, **c** can be written as

(4.4) 
$$\mathbf{c} = \mathbf{f}_{(\mathbf{w}_1,\dots,\mathbf{w}_k)} - \mathbf{b}_t + \sum_{i=1}^d l_i \mathbf{a}_i,$$

for some  $(\mathbf{w}_1, \ldots, \mathbf{w}_k) \in C_1 \times \cdots \times C_k$ ,  $0 \le t \le k$  and some  $l_1, \ldots, l_d \in \{0, 1\}$ . Note that  $\mathbf{f}_{(\mathbf{w}_1, \ldots, \mathbf{w}_k)} = \mathbf{w}_j - \mathbf{b}_j$  and  $\mathbf{b}_j + \mathbf{b}_t = \mathbf{b}_s + \sum_{i=1}^d l'_i \mathbf{a}_i$  for some  $l'_1, \ldots, l'_d \in \{0, 1\}$  and  $0 \le s \le k$ . Therefore, we get

(4.5) 
$$\mathbf{c} = \mathbf{w}_j - (\mathbf{b}_j + \mathbf{b}_t) + \sum_{i=1}^d l_i \mathbf{a}_i = \mathbf{w}_j - \mathbf{b}_s + \sum_{i=1}^d (l_i - l'_i) \mathbf{a}_i,$$

from (4.4). If  $\mathbf{b}_s = 0$ , then  $\mathbf{c}$  and  $\mathbf{w}_j$  are in the same congruence class modulo the group spanned by the extremal rays. In other words,  $r(\mathbf{c}) = r(\mathbf{w}_j) = \mathbf{b}_j$ . Thus,  $\mathbf{c} = \mathbf{w} + \sum_{i=1}^d h_i \mathbf{a}_i$ , for some  $\mathbf{w} \in C_j$  and  $h_1, \ldots, h_d \in \mathbb{N}$ . This provides a contradiction with minimality of  $\mathbf{c}$ , as  $\mathbf{w} - \mathbf{b}_i \in \mathfrak{c}(S)$ , for  $1 \leq i \leq k$ . Therefore,  $\mathbf{b}_s \neq 0$ . Now, we distinguish the following two cases:

Case 1,  $\mathbf{b}_j = \min_{\leq c} (r(\operatorname{Ap}(S, E)) \setminus \{0\})$ . Then  $[\mathbf{b}_j]_i \leq [\mathbf{b}_s]_i = [\mathbf{b}_j]_i + [\mathbf{b}_t]_i - l'_i$ , which implies  $l'_i = 0$ , for  $i = 1, \ldots, d$ .

Case 2:  $C_j$  is a singleton. From equation (4.5), we have  $r(\mathbf{c} + \mathbf{b}_s) = r(\mathbf{w}_j) = \mathbf{b}_j$ which implies  $\mathbf{c} + \mathbf{b}_s = \mathbf{w}_j + \sum_{i=1}^d r_i \mathbf{a}_i$  for some  $r_1, \ldots, r_d \in \mathbb{N}$ . Now, looking again at (4.5), we derive

$$\mathbf{w}_j + \sum_{i=1}^d r_i \mathbf{a}_i = \mathbf{w}_j + \sum_{i=1}^d (l_i - l'_i) \mathbf{a}_i.$$

As  $\mathbf{a}_1, \ldots, \mathbf{a}_d$  are linearly independent, we get  $l_i - l'_i = r_i \ge 0$ , for  $i = 1, \ldots, d$ .

Thus in both cases,  $l_i - l'_i \ge 0$ , for i = 1, ..., d. Since  $\mathbf{w}_j - \mathbf{b}_s \in \mathfrak{c}(S)$  and  $\mathbf{c}$  is a minimal generator of  $\mathfrak{c}(S)$ , we derive from (4.5), that  $l_i - l'_i = 0$  for i = 1, ..., d and  $\mathbf{c} = \mathbf{w}_j - \mathbf{b}_s$ .

If  $\mathbf{b}_s \preceq_c \mathbf{b}_r$  for some  $1 \leq r \leq k$ , then  $\mathbf{b}_r - \mathbf{b}_s \in \operatorname{cone}(S) \cap \operatorname{group}(S) = \overline{S}$  and  $\mathbf{w}_j - \mathbf{b}_s = \mathbf{w}_j - \mathbf{b}_r + \mathbf{b}_r - \mathbf{b}_s$ . As **c** is a minimal generator, we get  $\mathbf{b}_s = \mathbf{b}_r$ . Thus,  $\mathbf{b}_s \in \max_{\leq c} {\mathbf{b}_1, \ldots, \mathbf{b}_k}$ .

Applying the above proposition to the semigroup in Example 4.11, provides an easier argument to find the minimal generating set of  $\mathfrak{c}(S)$ .

**Example 4.14.** Let  $\mathbf{a}_1 = (3,0), \mathbf{a}_2 = (0,3), \mathbf{a}_3 = (5,2), \mathbf{a}_4 = (2,5)$ . As we have seen in Example 4.11,  $\operatorname{Ap}(S, E) = \{0, \mathbf{w}_1 = (5,2), \mathbf{w}_2 = (2,5), \mathbf{w}_3 = (7,7), \mathbf{w}_4 = (10,4), \mathbf{w}_5 = (4,10)\}, r(\operatorname{Ap}(S,E)) = \{0, \mathbf{b}_1 = (1,1), \mathbf{b}_2 = (2,2)\}, C_1 = \{\mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$  and  $C_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ . Note that  $\max_{\leq_c} \{\mathbf{w}_i, \mathbf{w}_j\} = \mathbf{w}_j$  for i = 1, 2 and j = 3, 4, 5. Therefore,  $\mathfrak{c}(S)$  is generated by  $\{\mathbf{w}_3 - (2,2), \mathbf{w}_4 - (2,2), \mathbf{w}_5 - (2,2)\} = \{(5,5), (8,2), (2,8)\}$ , as an ideal of  $\overline{S} = \langle (3,0), (0,3), (1,1) \rangle$ .

**Example 4.15.** Let  $\mathbf{a}_1 = (2,0), \mathbf{a}_2 = (0,2), \mathbf{a}_3 = (4,1), \mathbf{a}_4 = (2,3)$ . We have  $\operatorname{Ap}(S, E) = \{0, \mathbf{w}_1 = (4,1), \mathbf{w}_2 = (2,3)\}$  and  $r(\operatorname{Ap}(S, E)) = \{0, (0,1)\}$ . More precisely,  $\mathbf{w}_1 - 2\mathbf{a}_1 = \mathbf{w}_2 - \mathbf{a}_1 - \mathbf{a}_2 = 1/2\mathbf{a}_2$ . Thus k = 1 and  $C_1 = \{(4,1), (2,3)\}$ . By Proposition 4.13,  $\mathfrak{c}(S)$  is generated by  $\{(4,1) - (0,1), (2,3) - (0,1)\} = \{(4,0), (2,2)\}$  as an ideal of  $\overline{S} = \langle (2,0), (0,1) \rangle$ .

**Remark 4.16.** If  $C_i$  is a singleton for i = 1, ..., k, then the hypothesis on the existence of j in Proposition 4.13 is equivalent to existence of a single maximal element of  $\operatorname{Ap}(S, E)$  with respect to  $\preceq_c$ . This condition can not be removed. For instance, let S be the semigroup defined in Example 4.12. Then  $\max_{\preceq_c} \operatorname{Ap}(S, E) = \{\mathbf{w}_4, \mathbf{w}_5\} = \{(8, 4), (5, 3)\}$  and  $\max_{\preceq_c} (r(\operatorname{Ap}(S, E))) = \{\mathbf{b}_5 = (5, 3)\}$ . But, (8, 4) - (5, 3) is not in S and in particular, it is not in  $\mathfrak{c}(S)$ .

**Corollary 4.17.** If  $\mathbb{K}[S]$  is a Cohen-Macaulay ring, and  $\max_{\leq_c} \operatorname{Ap}(S, E) = \{\mathbf{w}\}$ , then  $\mathfrak{c}(S)$  is generated by  $\{\mathbf{w} - \mathbf{b} ; \mathbf{b} \in \max_{\leq_c} r(\operatorname{Ap}(S, E))\}$ , as an ideal of  $\overline{S}$ . In particular, if  $\max_{\leq_c} (r(\operatorname{Ap}(S, E))) = \{\mathbf{b}\}$ , then  $\mathfrak{c}(S)$  is a principal ideal of  $\overline{S}$ generated by  $\mathbf{w} - \mathbf{b}$ .

As Example 4.12 shows, for an affine semigroup with Cohen-Macaulay semigroup ring,  $\max_{\leq_c} \operatorname{Ap}(S, E)$  is not necessarily a singleton. Here, we have an example of an affine semigroup satisfying the conditions of Corollary 4.17.

**Example 4.18.** Let  $\mathbf{a}_1 = (1, 5), \mathbf{a}_2 = (5, 1), \mathbf{a}_3 = (2, 2), \mathbf{a}_4 = (3, 3)$ . Then

$$Ap(S, E) = \{0, \mathbf{a}_3, \mathbf{a}_4, 2\mathbf{a}_3, \mathbf{a}_3 + \mathbf{a}_4, 2\mathbf{a}_3 + \mathbf{a}_4\} \\ = \{0, \mathbf{w}_1 = (2, 2), \mathbf{w}_2 = (3, 3), \mathbf{w}_3 = (4, 4), \mathbf{w}_4 = (5, 5), \mathbf{w}_5 = (7, 7)\},\$$

and  $r(\operatorname{Ap}(S, E) = \{0, \mathbf{b}_1 = (2, 2), \mathbf{b}_2 = (3, 3), \mathbf{b}_3 = (4, 4), \mathbf{b}_4 = (5, 5), \mathbf{b}_5 = (1, 1)\}$ . As  $C_i = \{\mathbf{w}_i\}$ , for  $i = 1, \ldots, 5$ ,  $\mathbb{K}[S]$  is Cohen-Macaulay, by Lemma 2.5. Moreover,  $\max_{\leq_c} \operatorname{Ap}(S, E) = \{\mathbf{w}_5\}$  and  $\max_{\leq_c} r(\operatorname{Ap}(S, E)) = \{\mathbf{b}_4\}$ . Therefore, by Corollary 4.17,  $\mathfrak{c}(S)$  is generated by  $\mathbf{w}_5 - \mathbf{b}_4 = (2, 2)$ , as an ideal of  $\overline{S} = \langle (1, 1), (1, 5), (5, 1) \rangle$ .

For an affine semigroup S with Cohen-Macaulay semigroup ring, if  $\max_{\leq c} \operatorname{Ap}(S, E)$  is a singleton, it does not imply that  $\max_{\leq c} r(\operatorname{Ap}(S, E))$  is also a singleton.

**Example 4.19.** Let  $\mathbf{a}_1 = (2, 1), \mathbf{a}_2 = (1, 5), \mathbf{a}_3 = (1, 1), \mathbf{a}_4 = (4, 5)$ . As the computation in [20, Example 2] shows,  $\max_{\leq_c} \operatorname{Ap}(S, E) = (6, 7), \max_{\leq_c} r(\operatorname{Ap}(S, E)) = \{(2, 2), (2, 3), (2, 4), (2, 5)\}$  and  $\mathbb{K}[S]$  is Cohen-Macaulay. Therefore,  $\mathfrak{c}(S)$  is generated by  $\{(4, 5), (4, 4), (4, 3), (4, 2)\}$  as an ideal of  $\overline{S} = \langle (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1)\rangle$ , by Corollary 4.17.

Note that  $\max_{\leq_c} \operatorname{Ap}(S, E) \subseteq \max_{\leq_S} \operatorname{Ap}(S, E)$ . In particular, if  $\mathbb{K}[S]$  is Gorenstein, then  $\max_{\leq_c} \operatorname{Ap}(S, E)$  has a single element. The converse is not true, for instance  $\max_{\leq_S} \operatorname{Ap}(S, E) = \{(6, 7), (5, 5)\}$  in Example 4.19, while  $\max_{\leq_c} \operatorname{Ap}(S, E)$  has a single element.

**Corollary 4.20.** If  $\mathbb{K}[S]$  is a Gorenstein ring and  $\max_{\leq c}(r(\operatorname{Ap}(S, E)))$  has a single element, then  $\mathfrak{c}(S)$  is a principal ideal of  $\overline{S}$ .

**Remark 4.21.** Let  $S \subseteq \mathbb{N}$  be a numerical semigroup with multiplicity e, that is  $e = \min(S \setminus \{0\})$ . As we mentioned in Example 4.4,  $\mathfrak{c}(S)$  is generated by F + 1 as an ideal of  $\overline{S} = \mathbb{N}$ , where  $F = \max(\mathbb{Z} \setminus S)$ . Note that  $r(\operatorname{Ap}(S, e)) = \{0, 1, \ldots, e-1\}$ . As an immediate consequence of Corollary 4.17, we derive that F = w - e, where w is the maximal number in  $\operatorname{Ap}(S, e)$ . This is a fact already proved differently in [26], see also [25, Theorem 2.12]. As Example 4.12 shows, the conductor of a Cohen-Macaulay affine semigroup S is not necessarily a principal ideal of  $\overline{S}$ .

The following is an example of a Cohen-Macaulay simplicial affine semigroup, for which  $\max_{\leq c} \operatorname{Ap}(S, E)$  is a singleton but  $\mathfrak{c}(S)$  is not principal.

**Example 4.22.** Let  $\mathbf{a}_1 = (3,0), \mathbf{a}_2 = (0,3), \mathbf{a}_3 = (2,1)$ . Then  $\operatorname{Ap}(S, E) = \{0, \mathbf{w}_1 = (2,1), \mathbf{w}_2 = (4,2)\}$  and  $r(\operatorname{Ap}(S,E)) = \{0, \mathbf{b}_1 = (2,1), \mathbf{b}_2 = (1,2)\}$ . Since  $C_i = \{\mathbf{w}_i\}$  for  $i = 1, 2, \mathbb{K}[S]$  is Cohen-Macaulay by Lemma 2.5. Moreover,  $\max_{\leq c} \{\mathbf{w}_1, \mathbf{w}_2\} = \{\mathbf{w}_2\}$  and  $\max_{\leq c} r(\operatorname{Ap}(S,E)) = \{\mathbf{b}_1, \mathbf{b}_2\}$ . By Proposition 4.13,  $\mathfrak{c}(S)$  is generated by  $\{\mathbf{w}_2 - \mathbf{b}_1, \mathbf{w}_2 - \mathbf{b}_2\} = \{(2,1), (3,0)\}$  as an ideal of  $\overline{S} = \langle (3,0), (0,3), (1,2), (2,1) \rangle$ .

**Example 4.23.** Let S be the affine semigroup presented in Example 2.2. Based on a computation by Macaulay 2 [15],  $\mathbb{K}[S]$  is Cohen-Macaulay. Note that Ap(S, E) has two maximal elements  $18\mathbf{a}_4 + 2\mathbf{a}_5 = (40, 22, 20)$  and  $16\mathbf{a}_4 + 4\mathbf{a}_5 = (40, 24, 20)$ . Computing the coordinates of these two elements, we find that  $\mathbf{f}_{(\mathbf{w}_1,...,\mathbf{w}_k)} = 3\mathbf{a}_1 + \mathbf{a}_2 + 3\mathbf{a}_3 = (40, 23, 20)$ . One can use GAP [8] with an implementation based on Theorem 4.9 (provided to us by the referee), to find that  $\mathfrak{c}(S)$  is minimally generated by {(29, 15, 14), (29, 16, 14), (29, 18, 15), (30, 15, 14), (30, 16, 14), (31, 15, 15), (31, 16, 14), (31, 18, 16), (33, 18, 17), (33, 19, 15), (35, 18, 18)}.

We close this section by an example in dimension three.

**Example 4.24.** Let  $\mathbf{a}_1 = (1, 2, 1), \mathbf{a}_2 = (2, 3, 1), \mathbf{a}_3 = (2, 1, 3), \mathbf{a}_4 = (2, 3, 2), \mathbf{a}_5 = (2, 2, 2), \mathbf{a}_6 = (3, 3, 3)$ . Then Ap $(S, E) = \{0, \mathbf{w}_1 = \mathbf{a}_4, \mathbf{w}_2 = \mathbf{a}_5, \mathbf{w}_3 = \mathbf{a}_6, \mathbf{w}_4 = \mathbf{a}_5 + \mathbf{a}_6\}$  and  $r(\operatorname{Ap}(S, E)) = \{0, \mathbf{b}_1 = (1, 1, 1), \mathbf{b}_2 = (2, 2, 2), \mathbf{b}_3 = (3, 3, 3)\}$ . Moreover,  $C_1 = \{\mathbf{w}_1 = \mathbf{a}_1 + \mathbf{b}_1, \mathbf{w}_4 = \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{b}_1 = 2\mathbf{a}_5 + \mathbf{b}_1\}, C_2 = \{\mathbf{w}_2 = \mathbf{b}_2\}$  and  $C_3 = \{\mathbf{w}_3 = \mathbf{b}_3\}$ . As  $\mathbf{b}_2, \mathbf{b}_3 \in S$ , we have  $\mathbf{c}(S) = \{\mathbf{a} \in S \ ; \ \mathbf{a} + \mathbf{b}_1 \in S\}$ . Since  $\mathbf{a}_i + \mathbf{b}_1 \in S$  for  $i \in \{1, 4, 5, 6\}$ , we have  $\mathbf{a}_1, \mathbf{a}_4 = \mathbf{a}_1 + \mathbf{b}_1, \mathbf{a}_5, \mathbf{a}_6 = \mathbf{a}_5 + \mathbf{b}_1$  are in  $\mathbf{c}(S)$ . If  $\mathbf{c}$  is a minimal generator of  $\mathbf{c}(S)$  that is not in  $\{\mathbf{a}_1, \mathbf{a}_5\}$ , then  $\mathbf{c} = t\mathbf{a}_2 + s\mathbf{a}_3$  for some  $t, s \in \mathbb{N}$ . Thus,  $r(\mathbf{c} + \mathbf{b}_1) = \mathbf{b}_1$ , and we get  $\mathbf{c} + \mathbf{b}_1 = \mathbf{w} + \sum_{i=1}^3 l_i \mathbf{a}_i$  for some  $\mathbf{w} \in C_1$  and  $l_1, l_2, l_3 \in \mathbb{N}$ .

If  $\mathbf{w} = \mathbf{w}_1$ , then  $\mathbf{c} + \mathbf{b}_1 = \mathbf{a}_1 + \mathbf{b}_1 + \sum_{i=1}^3 l_i \mathbf{a}_i$ , consequently  $\mathbf{c} \in \mathbf{a}_1 + \overline{S}$ , which is a contradiction.

If  $\mathbf{w} = \mathbf{w}_4$ , then  $\mathbf{c} + \mathbf{b}_1 = 2\mathbf{a}_5 + \mathbf{b}_1 + \sum_{i=1}^3 l_i \mathbf{a}_i$ , which implies  $\mathbf{c} \in \mathbf{a}_5 + \bar{S}$ , a contradiction.

Thus, as an ideal of  $\overline{S}$ ,  $\mathfrak{c}(S)$  is minimally generated by  $\{\mathbf{a}_1, \mathbf{a}_5\}$ . Note that  $\mathbf{f}_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)} = \mathbf{a}_1, \mathbf{f}_{(\mathbf{w}_4, \mathbf{w}_2, \mathbf{w}_3)} = \mathbf{a}_2 + \mathbf{a}_3$  and  $\mathbf{a}_5 = \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{b}_2$ . In particular,  $\mathfrak{c}(S)$  is generated by  $\{\mathbf{f}_{(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)}, \mathbf{f}_{(\mathbf{w}_4, \mathbf{w}_2, \mathbf{w}_3)} - \mathbf{b}_2\}$  as an ideal of  $\overline{S} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1 \rangle$ .

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