Mon. Not. R. astr. Soc. (1986) **223**, 69–77

Type I SNe from double degenerate CO dwarfs and their rate in the solar neighbourhood

A. Tornambè and F. Matteucci *European Southern Observatory, Karl-Schwarzschild-Str. 2, D-8046 Garching bei München, Federal Republic Germany*

Accepted 1986 May 22. Received 1986 May 1

Summary. An updated set of stellar models for various metallicities (from 10^{-6} to 10^{-2}) has been used to compute rates of Type I supernovae in the framework of the double degenerate CO dwarf model, which has been suggested to be a good candidate for explaining the origin of such supernovae. In particular, we have computed the Type I SN rate for several chemical compositions of the progenitor stars by means of a chemical-evolution model for the solar neighbourhood.

The main results can be summarized as follows:

(i) The relative number of binary systems giving rise to Type I SNe is a decreasing function of their metal content. However, this effect is not noticeable when the Type I SN rate is computed by means of a star-formation rate which is proportional to some power of the gas density, since the Type I SN rate mainly follows the same trend. The main difference between the rate computed by taking into account the various chemical compositions of the progenitors and the one computed for a unique chemical composition (solar), is the time at which Type I SNe begin to appear, which is shorter in the first case.

(ii) In both cases, the predicted Type I SN rate at the present time in the solar neighbourhood is lower by a factor of 10 with respect to the observed one. It is worth noting that the model of chemical evolution we used, reproduces all the main features of the solar neighbourhood as well as the observed Type II SN rate.

Therefore, although the many uncertainties present in the derivation of the Type I SN rate and in the observational estimate of SN rates do not allow firm conclusions, we suggest that Type I SNe are likely to come not only from the double degenerate CO dwarf model.

1 Introduction

In a previous paper (Tornambè & Chieffi 1986, hereafter referred to as TC), evolutionary results for metal deficient ($10^{-8} \leqslant Z \leqslant 10^{-6}$) intermediate–mass ($2.5 \leqslant M/M_\odot \leqslant 8$) stars were presented and used to compute the Type I SN rate during the halo phase of our Galaxy. The SNI rate was computed in the framework of the model proposed by Iben & Tutukov (1984a, b, 1985), which is a good theoretical candidate for a typical Type I SN progenitor. In such a model it is suggested that Type I SNe arise from the merging of two degenerate CO dwarfs in a binary system, due to the loss of angular momentum through gravitational wave radiation. This merging should give rise to a C-deflagration when the mass of the primary component reaches the Chandrasekhar limit ($\sim 1.4\,M_\odot$). This model can account for the uniformity, the energetics, the spectrum and the photospheric composition at maximum light, as well as the light curve (see Iben & Tutukov 1984b, hereafter referred to as IT, and references therein) of Type I SNe.

The evolutionary behaviour of the two stars before reaching the white dwarf stage is, of course, crucial in determining the conditions under which a Type I SN event is likely thus to occur (in the way previously described). In a previous paper (Tornambè & Matteucci 1985, hereafter referred to as TM), evolutionary results for intermediate-mass stars of different metallicites (Castellani *et al.* 1985; TC) were used to define simple considerations for the behaviour of the formation frequency of binary systems that will eventually produce Type I SN events in the solar neighbourhood. The main suggestion was that the formation frequency of these systems should be a decreasing function of metallicity (i.e. of time).

The aim of this paper is to use new evolutionary tracks computed for metallicities larger than in TC but with the same evolutionary code (Brocato 1986), as well as the results of TC, to derive the Type I SN rate, in the framework of Iben & Tutukov's model, as a function of the chemical composition of the progenitor stars. By means of a galactic-evolution model we want therefore to compute the Type I SN rate in the solar neighbourhood and to compare it with the Type I SN rate previously computed by Matteucci & Greggio (1986, hereafter referred to as MG) in the framework of the model proposed by Whelan & Iben (1973), as well as with the rate of Type II SNe. In fact, observational estimates (Tammann 1982) suggest that the present time rates of Type I and II SNe should be approximately the same.

The results, which will be presented and discussed in Sections 2 and 3, suffer from several uncertainties due to the poorly known physics concerning the processes leading to the formation of the two degenerate CO dwarfs (like the common-envelope phases), as well as the merging process (see also Nomoto & Iben 1985). For this reason, the assumptions we have made are the most simple ones, without *ad hoc* hypotheses in order to reproduce the observed Type I SN rate in the solar neighbourhood. Our results indicate that the number of Type I SNe that can arise per unit time from Iben & Tutukov's scenario, once a reasonable initial mass function and rate of star formation have been chosen, seems to be lower by a factor of ~ 10 with respect to the predicted Type II SN rate which instead agrees with the observational estimate. Therefore, we will suggest that the double CO dwarf model is not the only one responsible for the production of Type I SNe.

2 Type I SN progenitors

The new evolutionary results we have used here will be discussed elsewhere (Brocato 1986). We only recall that they have been obtained for He content $Y=0.25$ and metallicity content $Z=0.01$; semi-convection treatment during central He-burning phases was also included, as discussed by Castellani *et al.* (1985). Since the same code and input physics as in TC have been used, this set of computations is consistent with those of TC for $Z=10^{-8}$, 10^{-6}, 10^{-4} and $Y=0.20$.

Some more comments on the assumed He abundance are necessary. The He abundance is important because, for a fixed Z and mass, it determines the efficacy of the envelope runaway towards the Hayashi track at the end of the central H-burning phase. This occurrence and its efficacy are well known to be crucial features for the evolution of stars in close binary systems. Although the value of the primordial He and He variation during the galactic lifetime are still uncertain, we assumed as reasonable choices a primordial helium abundance of ~ 0.20 and a solar one of ~ 0.25 (see Peimbert 1986).

As a consequence of this, we have interpolated our input data between $Z=10^{-6}$, $Y=0.20$ and $Z=0.01$, $Y=0.25$. As pointed out by TC, when the metallicity of the star rises above $Z=10^{-3}$ the first envelope runaway toward the Hayashi track occurs after the H exhaustion at the centre of the star. This is, of course, the case for stars with $Z=0.01$. This occurrence results in substantial differences in the evolutionary history of binary systems with $Z=0.01$ compared to those of lower metallicity discussed in TC. Those systems in the separation range of interest here ($10\leqslant A/R_\odot\leqslant 2000$), should, in principle, undergo four common-envelope episodes, two at the end of central H-burning and two at the end of central He-burning in both the primary and secondary component.

In this paper, as in TC, we have used the formalism of Iben & Tutukov (1984a, b) for the treatment of the common-envelope stages. If A is the initial separation of the system, the new separation after a common envelope is $A'=\alpha A M_{1r} M_2/M_1^2$, where M_1 and M_2 are the masses of the primary and secondary component, respectively, M_{1r} is the remnant left by the primary after the common envelope action and α is a parameter which accounts for the efficacy of the common-envelope action and is here assumed to be equal to 1 (see Iben & Tutukov 1984a, b). The mass ratio of the two components has been chosen to be peaked around 1 (see TC).

Our analysis concerning the frequency of occurrence of double degenerate systems, with a total mass larger than $1.4\,M_\odot$ which can merge in one Hubble time, leads to values that, for solar metallicity, are three times smaller than the ones found by IT. This result is mainly due to the fact that we estimate that systems, which experience the first common envelope when the primary fills its Roche lobe for the first time during its red giant phase, are not good candidates to develop a double degenerate CO system. If we require the common-envelope action to be able to shorten the separation of the initial system before the gravitational wave radiation can act, we are forced to conclude that such systems will produce two He dwarfs which merge (due to gravitational wave radiation) before the CO core could be formed. In fact, the gravitational wave emission time-scale is always much shorter than the evolutionary time-scale of the two He components, or, more generally, of the secondary He star. These systems of two He dwarfs can still produce a SN event but presumably of the e-capture type, and, in any case not a Type I event with a remarkable production of Fe nuclei.

Therefore we identify, as more promising candidates to form double degenerate CO systems, those systems which undergo the first common envelope when the primary fills its Roche lobe when reaching the Hayashi track for the second time, after the end of central He-burning. In this case, however, there exists a narrow range of separations for which an as yet not evolved He star and a degenerate CO star will merge. Again a classical Type I event (e.g. the explosion is triggered by central carbon ignition in a highly electron-degenerate environment) is not likely to occur.

The second main reason for which we find a lower frequency of occurrence of double degenerate dwarf systems with respect to Iben & Tutukov is that we adopted a lower value of M_{up}. The value of M_{up}, namely the upper mass for non-degenerate carbon ignition, has been recently reduced to $\sim 6\,M_\odot$ (the correct value depending on the He and Z abundances) due to improved treatment of central convection (see e.g. Castellani *et al.* 1985, Bertelli, Bressan & Chiosi 1985; Renzini *et al.* 1985; Brocato 1986). Following Brocato (1986) we chose, for $Y=0.25$ and $Z=0.01$, $M_{up}=6\,M_\odot$. As a consequence, primary components with $M>M_{up}$ which will not fill their Roche lobe prior to carbon ignition (i.e. systems with separations $A\sim 560\,R_\odot$), will avoid the common-envelope phases and presumably will end their life as neutron stars after an e-capture SN event (see, e.g. Nomoto 1984). Our estimate for CO degenerate systems with total mass larger than $1.4\,M_\odot$ which will merge in one Hubble time is summarized for $Z=0.01$ in the hatched region of Fig. 1 which shows the mass of the components versus the semi-major axis of the system. Our results of Fig. 1 have to be compared with fig. 29 of IT.

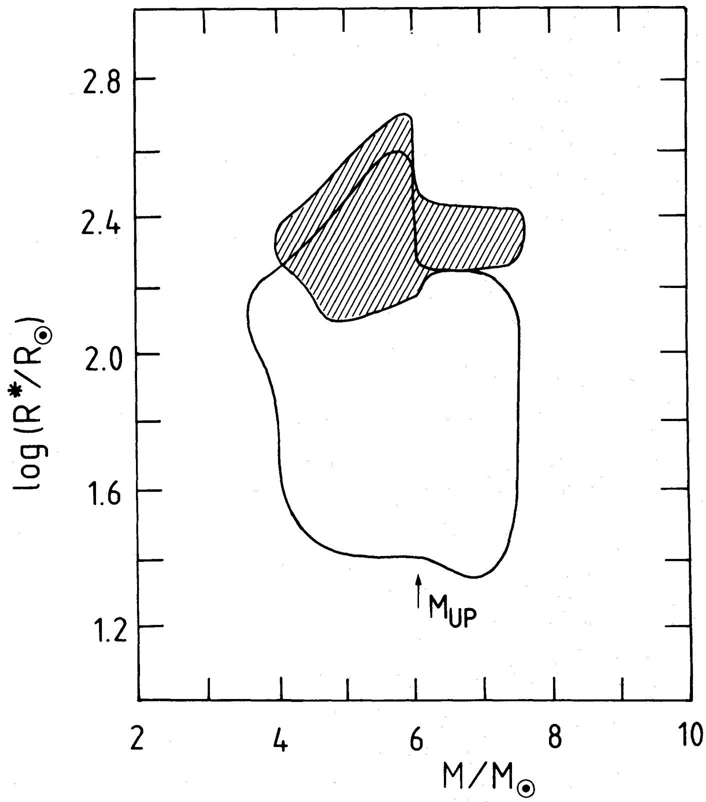

Figure 1. The areas in the $M-\log R^*$ (R^*=semi-major axis) diagram for which a double degenerate merging event is possible for the same chemical compositions as in Fig. 2. The smaller shaded area refers to $Z=0.01$, $Y=0.25$, whereas the larger one refers to $Z=10^{-6}$, $Y=0.20$.

From now on we will assume as bona fide Type I SN progenitors only those systems for which we are reasonably convinced that a double degenerate CO dwarf system, sufficiently bound to give rise to a merging of the two dwarfs at least in a Hubble time, will be formed. It should be noted that different choices of the parameter α do not substantially affect the double degenerate merging frequency of occurrence but simply shift the values of the separation for which such a merging will occur.

Fig. 1 shows the areas (in the mass-separation diagram of the original binary system) for which double degenerate merging events will be produced for two different chemical compositions ($Z=10^{-6}$, $Y=0.20$ and $Z=10^{-2}$, $Y=0.25$) according to the model and the assumptions we have used. The less extended shaded one refers to the case $Z=10^{-2}$, $Y=0.25$.

The upper panel of Fig. 2 gives the cumulative fraction of binary systems for $Y=0.25$ and $Z=0.01$ with total mass between 6 and 18 $M_\odot$ and separations in the range $1\leq\log A\leq4$, which can give rise to a merging of two CO degenerate dwarfs, as a function of time. The computations of the upper panel of Fig. 2 refer to a single stellar generation and have been performed by assuming that, in the mass-separation (M–A) space, binary systems are distributed according to $dN\approx d\log A\, dM/M^{2.3}$ (Iben & Tutukov 1984a). For any fixed couple M_i, A_i it is possible to define the number of systems which, in the range (M_i, $M_i+\delta M$) and (A_i, $A_i+\delta A$) will be able to develop the conditions to produce Type I SN events

$$N_i=\int_{M_i}^{M_i+\delta M_i}\int_{A_i}^{A_i+\delta A_i} d\log A\, dM/M^{2.3}. \quad (1)$$

To every value of N_i is associated an explosion time t_i which is the sum of the evolutionary

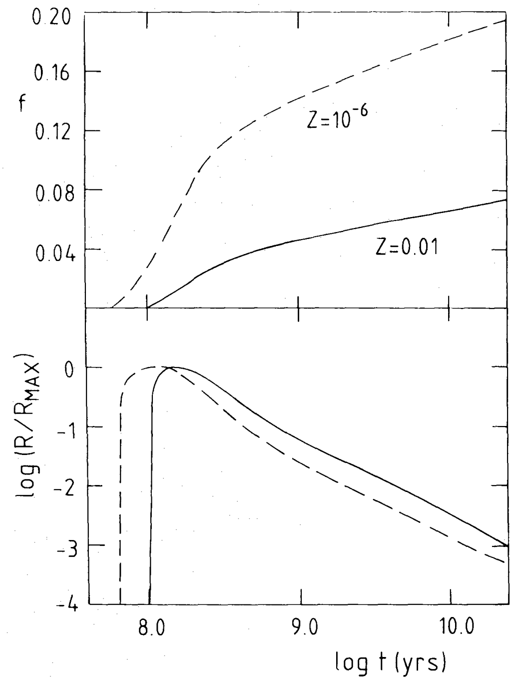

Figure 2. Upper panel: The cumulative fractionary number of merging events of double degenerate dwarfs as a function of time. The fraction is computed with respect to all binary systems with total mass between 6 and 18 $M_\odot$. The continuous line refers to a stellar generation with $Z=0.01$ and $Y=0.25$, whereas the broken line refers to a stellar generation with $Z=10^{-6}$ and $Y=0.20$. Lower panel: The logarithm of the rate of merging events, normalized to the maximum computed rate, as a function of time for the two stellar generations discussed above.

time-scale for M_i plus the gravitational wave emission time (namely the time necessary to reach the merging; see Landau & Lifshitz 1962). For uniformity with TC, in Fig. 2, instead of N_i we present the cumulative fractionary number $f(t)$. Let us define the fractional number q_i as

$$q_i = \frac{N_i}{N_{\rm TOT}} \tag{2}$$

where $N_{\rm TOT}$ is the total number of binary systems of mass between 6 and 18 $M_\odot$ and separation between 10 and $10^4\,R_\odot$. The cumulative number (as a fraction of the total $N_{\rm TOT}$) of Type I SNe exploding until the time, t, will be

$$f(t) = \sum_t \sum_i q_i \delta_{t_i,\,t}. \tag{3}$$

The Kronecker delta, $\delta_{t_i,\,t}$, accounts for the fact that only those systems exploding at time t have to be summed.

The bottom panel of Fig. 2 gives the logarithm of the rate (namely the fraction of systems per unit time), normalized to the maximum rate, $R/R_{\rm max}$, of Type I SN events as a function of time.

At any time t, a certain fraction of systems, with respect to the total number, will explode according to their evolutionary lifetimes plus the gravitational delay times.

As already predicted in TM and TC, the number of systems involved in SNI production appears to be a decreasing function of metallicity.

Finally, in this section we want to briefly discuss how the quoted results are sensitive to the He-abundance variations. Evolutionary results suggest that the higher the He content, the larger is the number of binary systems able to develop the conditions which can give rise to a SN I event. There are essentially two reasons for this. First, at a fixed mass and metallicity, the higher the He content, the smaller is the radius attained by the primary at the end of the first envelope runaway toward the Hayashi track at the end of central H-burning. As a consequence, the first common envelope phase can be avoided by a greater number of systems, and their primaries will be able to develop rather massive CO cores (and therefore the possibility of a C-deflagration is higher). The second reason is that binary systems of the same mass, metallicity and separation but different He content undergo the first common-envelope phase with He cores of different mass; the higher the He content, the more massive the He cores. As a consequence, more massive CO cores will be developed in the subsequent evolutionary phases.

3 Type I SN rates

The results predicted in the previous section suffer from the poorly understood physics employed to derive them. First, the parameter α used in computing the separation reduction during the common envelope phases, could in reality be different from one and varying from one case to another. Other uncertainties arise from the evolutionary stellar models themselves. In fact, the efficacy of the envelope runaway towards the Hayashi track depends on the opacity of matter which is only known approximately, and finally the value of $M_{\rm up}$ depends on the treatment of semi-convection during the central He-burning phase. Other sources of uncertainties result from our poor knowledge of the binary system properties (e.g. mass function and its cut-offs, $q(q=M_1/M_2)$ value distribution, separation function and so on . . .).

A reasonable way of calibrating the results of the previous section is to insert them in a galactic-evolution model, to assess if they can reproduce the presently observed rate of such SNe. In particular, we assumed the model for the evolution of the solar neighbour of MG, where a detailed description can be found.

We will only recall here the main assumptions of MG's model: (i) one zone model with complete instantaneous mixing of gas; (ii) no instantaneous recycling approximation, namely the lifetimes of the stars are taken into account; (iii) infall of material of primordial chemical composition is considered; (iv) the rate of star formation is assumed to be proportional to the surface gas density, according to the formulation of Talbot & Arnett (1975) but adapted to an open model; and (v) the initial mass function (IMF) is assumed to be constant in space and time. By means of this model, detailed rates of Type I and II SNe as well as the temporal evolution of several chemical elements can be computed.

The Type I SN rate (in number) can be expressed as

$$R_{\rm SNI}(t_N)=B\times C\int_0^{t_N}\psi(t)\left(\frac{dN}{dt}\right)_{Z(t),\,t_N}dt, \tag{4}$$

where the integration is performed on all those systems exploded until the time t_N. The quantity $\psi(t)$ is the star-formation rate, expressed in $M_\odot\,{\rm pc}^{-2}\,{\rm Gyr}^{-1}$, at the time when the primordial binary system was born, $t=t_N-t^*$, where t^* represents the evolutionary lifetime plus the gravitational time delay for the merging of the system. $(dN/dt)_{Z(t),\,t_N}$ represents the fraction of the systems born per unit time with metallicity $Z(t)$ and dying at t_N. B is a normalization constant which takes into account the assumption that binary systems, in the IMF $[\phi(m)\propto m^{-(1+x)}]$, are $\sim$50 per cent of the total number of stars which are defined in the mass interval 0.1–100 $M_\odot$. The slope of the IMF has been assumed to be $x\sim1.3$ (Salpeter 1955) as in the previous section.

The value of B is therefore derived in the following way

$$B=0.5\bigg/\int_{0.1}^{100}m^{-1.3}\,dm\approx0.086. \tag{5}$$

The constant C is connected with the assumed function of distribution of separations and is given

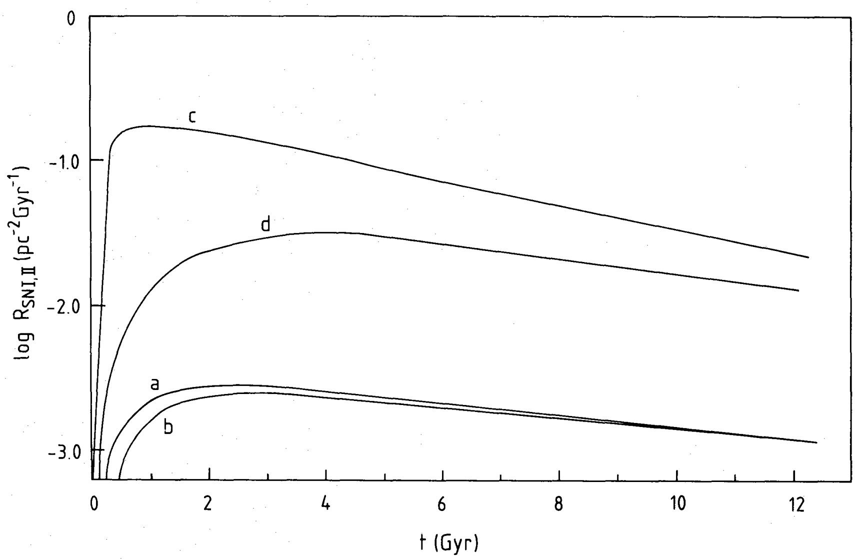

Figure 3. The predicted Type I SN rates in the framework of the double degenerate models (curves a and b). Curve a refers to different chemical compositions for the progenitor stars. Curve b refers to a solar chemical composition. The curves labelled c and d refer respectively to the predicted Type II SN rate (both here and in MG) and Type I SN rate as obtained in MG in the framework of the Whelan & Iben (1973) model.

by

$$C = 1 \Big/ \int_{10}^{10^4} d \log A \approx 0.25. \tag{6}$$

Finally the rate of Type II SNe (in number) is simply defined as

$$R_{\rm SNII} = \int_8^{100} \psi(t - \tau m) \phi(m)\, dm. \tag{7}$$

In Fig. 3 are shown the Type I and II SN rates as computed here and in MG. In MG the Type I SN rate was computed in the framework of the model proposed by Whelan & Iben (1973) for precursors of Type I SNe (e.g. a CO white dwarf plus a red giant) and the same IMF and rate of star formation (curve d).

The new Type I SN rate has been computed in two cases: (i) for different chemical compositions (curve labelled a), and (ii) for the solar chemical composition (curve labelled b). It is worth noting that we have not imposed any constraint in order to reproduce the present-time Type I SN observed rate as well as the Type II one, which should be the same as Type I (see Tammann 1982), whereas in MG the number of binary systems, in the appropriate mass range, giving rise to SNI events was constrained to reproduce those conditions. To do that MG had to assume that a fraction of ~0.09 of binary systems with total mass between 3 and 16 $M_\odot$ were suitable for producing Type I SNe in the way proposed by Whelan & Iben's model. Actually, it should be noted that the systems proposed by Whelan & Iben, as progenitors of Type I SNe, have a lower probability of occurrence with respect to the double CO dwarf model (see IT). In the present computations, the number of Type I SNe per unit time is computed in detail and therefore it depends only on the assumptions made in the previous section to derive the frequency of occurrence of such events, namely the stellar evolution results, the parameters of the common-envelope process, the distribution of separations etc. Fig. 3 shows that the rate of Type I for the double degenerate dwarf model (both curves a and b) is lower by about a factor of 10 with respect to that obtained by MG and to the observed one. In particular, the predicted present-time solar neighbourhood value of the Type I SN rate is ~0.001 SNe pc^{-2} Gyr^{-1}.

The general behaviour of the Type I SN rate obtained by MG and here is not very different, apart from the fact that the new ones start later than MG's one, and this is mostly due to the additional gravitational time delay present in the double degenerate dwarf model, and the maximum of the new rates is reached at ~2 billion years whereas in MG it was reached at ~4 billion years. This is probably due to the predominance, in the latter case, of stars of lower masses, with respect to the former case, as Type I SNe progenitors.

Finally, the difference between curve (a) and (b) consists only in the epoch at which the SNI rate starts to be noticeable, and this is due to the effect of different chemical compositions on the evolution and lifetimes of Type I SN progenitors.

4 Discussions and conclusions

The value of the present-time Type I SN rate we obtain for the solar neighbourhood, if translated into units of SNe yr^{-1} in the whole Galaxy, can be compared with the frequency of occurrence of Type I SN events estimated by IT. In order to do that we multiply our Type I SN rate by the area of the galactic disc, because our results are all projected on the galactic plane, which is assumed to be $\sim 10^9\,pc^2$ and we obtain $R_{SNI}=0.001$ SNe yr^{-1}, whereas IT estimate ~0.01 SNe yr^{-1}.

In our opinion, one of the main reasons for this discrepancy is the different normalization of the IMF and function of distribution of separations. In fact, our normalization constants in expression (4) are $BC \simeq 0.02$, whereas IT give a normalization constant of 0.2 which refers to the distribution of binary stars over the major axis A. In the normalization they assume for the IMF the equivalent of our constant B is equal to ~2/3, the star-formation rate is constant in time and equal to $\approx 2/3\,M_\odot\,yr^{-1}$ and they do not assume that only 50 per cent of the total number of stars is in the form of binaries. Part of the discrepancy between our rate and that of IT is also due to the fact that we find a range of masses of binary components, which can give rise to the bulk of SNI events, of 5–7 $M_\odot$, whereas their range is 5–9 $M_\odot$. Moreover, we do not account, in our computations, for those systems whose gravitational wave emission time-scales are shorter than the evolutionary time-scales of the He secondary, whereas IT do.

Finally, IT have used an He abundance of $Y \sim 0.28$ whereas we have at maximum $Y \sim 0.25$, and a lower He abundance contributes to restrict the area of possible Type I SN precursors, as we have already discussed in Section 2. However this assumption only marginally affects the final results.

Although the number of uncertainties involved in the computation of the Type I SN rate is such that no firm conclusions can be drawn, we would like to suggest that:

(i) The number of Type I SNe predicted for the solar neighbourhood in the framework of the double degenerate CO dwarf model is likely to be not enough to reproduce the observational constraints, unless an IMF quite different from the Salpeter one is adopted. Actually, an IMF which favours intermediate-mass stars with respect to massive ones more than the Salpeter one is that of Miller & Scalo (1979). The adoption of such an IMF would improve the situation but it would never account for the factor of 10 difference between predictions and observations. It seems, therefore, more likely that SNI are a mixture of products of different scenarios (e.g. Iben & Tutukov 1984a, b).

(ii) There is not much difference between the predicted chemical-composition-dependent Type I SN rate and that for the solar chemical composition. This is due to the fact that we adopted a continuous star formation until the present time. In such a framework in fact, the increase of metallicity is very fast at the beginning and the solar value is reached ~7 billion year ago. On the other hand, we would expect that the effect of the chemical composition of the progenitors of Type I SNe on their rate would be more noticeable in systems less evolved than the solar neighbourhood like irregular galaxies and the external regions of the galactic disc.

Acknowledgments

We thank I. Iben Jr for critically reading the manuscript and giving us very useful suggestions. We are grateful to L. Lucy for his critical discussion of the first version of the manuscript. Useful discussions with K. Nomoto are acknowledged. We also thank E. Brocato for giving us his results before publication. Finally, we thank Miss P. Bristow for her patience in the preparation of the typed version of the manuscript.

References

Bertelli, G., Bressan, A. & Chiosi, C., 1985. *Astr. Astrophys.,* **150,** 32.
Brocato, E., 1986. *PhD thesis,* University of Rome.
Castellani, V., Chieffi, A., Pulone, L. & Tornambè, A., 1985. *Astrophys. J.,* **294,** L31.
Iben, I. Jr & Tutokov, A., 1984a. In: *Stellar Nucleosynthesis,* p. 181, eds Chiosi, C. & Renzini, A., Reidel, Dordrecht, Holland.
Iben, I. Jr & Tutukov, A., 1984b. *Astrophys. J. Suppl.,* **54,** 335.
Iben, I. Jr & Tutukov, A., 1985. *Astrophys. J. Suppl.,* **58,** 661.
Landau, L. & Lifschitz, E., 1962. In: *The Classical Theory of Fields,* Addison-Wesley, Reading, Massachusetts.
Matteucci, F. & Greggio, L., 1986. *Astr. Astrophys.,* **154,** 279.
Miller, G. E. & Scalo, J. H., 1979. *Astrophys. J. Suppl.,* **41,** 513.
Nomoto, K., 1984. *Astrophys. J.,* **277,** 791.
Nomoto, K. & Iben, I. Jr, 1985. *Astrophys. J.,* **297,** 531.
Peimbert, M., 1986. *IAU Symp. No. 115,* eds Peimbert, M. & Jugaku, J., Reidel, Dordrecht, Holland, in press.
Renzini, A., Bernazzani, M., Buonanno, R. & Corsi, C. E., 1985. *Astrophys. J.,* **294,** L7.
Salpeter, E. E., 1955. *Astrophys. J.,* **121,** 161.
Talbot, R. J. & Arnett, D. W., 1975. *Astrophys. J.,* **197,** 551.
Tammann, G. A., 1982. In: *Supernovae: A Survey of Current Research,* p. 371, eds Rees, M. J. & Stoneham, R. J., Reidel, Dordrecht, Holland.
Tornambè, A. & Chieffi, A., 1986. *Mon. Not. R. astr. Soc.,* **220,** 529.
Tornambè, A. & Matteucci, F., 1985. *Astr. Astrophys.,* **151,** 447.
Whelan, J. C. & Iben, I. Jr, 1973. *Astrophys. J.,* **186,** 1007.