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U–Duality and Central Charges in Various Dimensions Revisited

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Abstract

A geometric formulation which describes extended supergravities in any dimension in presence of electric and magnetic sources is presented. In this framework the underlying duality symmetries of the theories are manifest. Particular emphasis is given to the construction of central and matter charges and to the symplectic structure of all $D = 4$, N–extended theories. The latter may be traced back to the existence, for $N > 2$, of a flat symplectic bundle which is the $N > 2$ generalization of $N = 2$ Special Geometry.

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1 Introduction

Recent developments on duality symmetries [1] in supersymmetric quantum theories of fields and strings seem to indicate that the known different string theories are different manifestations, in different regions of the coupling constant space, of a unique more fundamental theory that, depending on the regime and on the particular compactification, may itself reveal extra (11 or 12) dimensions [2], [3]. A basic aspect that allows a comparison of different theories is their number of supersymmetries and their spectrum of massless and massive BPS states. Indeed, to explore a theory in the nonperturbative regime, the power of supersymmetry allows one to compute to a large extent all dynamical details encoded in the low energy effective action of a given formulation of the theory and to study the moduli (coupling constants) dependence of the BPS states. This latter property is important in order to study more dynamical questions such as phase transitions in the moduli space [4], [5], [6],[7], [8], [9] or properties of solitonic solutions of cosmological interest, such as extreme black holes [10] and their entropy. A major mathematical tool in these studies is the structure of supergravity theories in diverse dimensions [11] and with different numbers of supersymmetries. These theories have a central extension that gives an apparent violation of the Haag-Lopuszanski-Sohnius theorem [12], since they include "central charges" that are not Lorentz-invariant [13]. However, these charges are important because they are related to p-extended objects (for charges with p antisymmetrized indices) whose dynamics is now believed to be as fundamental as that of points and strings [1]. In fact point-like and string-like BPS states can be obtained by wrapping p (or $p-1$) of the dimensions of a p-extended object living in D dimensions when $d \geq p$ dimensions have been compactified. It is the aim of this paper to give a detailed analysis of central extensions of different supergravities existing in arbitrary dimensions $4 \leq D < 10$ in a unified framework and to study the moduli dependence of the BPS mass per unit of p-volume of generic BPS p-branes existing in a given theory. A basic tool in our investigation will be an exploitation of "duality symmetries"[14], [15] (rephrased nowadays as U–duality) of the underlying supergravity theory which, for a theory with more than 8 supercharges, takes the form of a discrete subgroup of the continuous isometries of the scalar field sigma model of the theory [16]. Duality symmetries, which rotate electric and magnetic charges, correspond, in a string context, to certain perturbative or nonperturbatives symmetries of the BPS spectrum, playing a crucial role in the study of string dynamics. In many respects the present investigation can be considered as a completion of the list of theories reported in the collected volume papers by A. Salam and E. Sezgin [11]. This paper will cover, for self-consistency, material covered but scattered in the literature, and then some new material, such as the details of some theories in $D = 6$ and $D = 8, 9$ dimensions. The basic focus of our approach is that the central extension of the supersymmetry algebra [12][17] is encoded in the supergravity transformation rules. The latter can be derived from supersymmetric Bianchi identities, even if the complete lagrangian has not yet been derived. Of course a careful study of these identities also allows a complete determination of the lagrangian, whenever it exists. Among the novelties of this analysis is a new formulation of $D = 4$, N–extended theories with $N > 2$, in which a manifest symplectic formulation is used. In particular all these theories have in common a flat symplectic bundle which encodes the differential relations among the symplectic sections and therefore among the central and matter charges. In this respect the $N = 2$

case, related to Special Geometry [18], [19], simply differs from the $N > 2$ cases by the fact that the base space is not necessarily a coset space. This is related to the physical fact that $N = 2$ Special Geometry suffers quantum corrections. For higher dimensional theories, relations between central and matter charges for different p–extended objects are derived in analogy with previous known results in $D = 4$ and $D = 5$. For $D > 4$ the embedding in a duality group that rotates electric into magnetic charges is only possible for $D/2 = p + 2$, which covers the cases of $D = 6, 8$.

A shorter version of this paper, with particular emphasis on the existence of duality invariant entropy formulae in higher dimensions, already appeared in the literature [20].

The paper is organized as follows:
In section 2 we recall how different supergravity theories are related to the low-energy limit of different string theories and their M-theory or F-theory extension.
In section 3, which is the main body of the paper, a general discussion of the geometric framework for all D and N is presented and, in particular, the symplectic embedding of all $D = 4$ theories is formulated. The subsequent sections, which may be skipped by a reader not interested in the details of a particular theory, consider the higher D and higher N theories in the formalism discussed in section 3. In section 4 the $D = 4$, $N > 2$ theories are presented. In section 5 matter coupled $D > 4$ supergravities are discussed. In sections 6 and 7 the maximally extended theories in odd and even dimensions respectively are reported.

2 Extended supergravities and their relations with superstrings, M-theory and F-theory

It is worth while to recall the various compactifications of superstrings in $4 \leq D < 10$ as well as of M-theory and their relation to extended supergravities and their duality symmetries. In the string context the latter symmetries are usually called S, T and U–dualities. S–duality means exchange of small with large coupling constant, i.e. strong–weak coupling duality. T–duality indicates the exchange of small with large volume of compactification while U–duality refers to the exchange of NS with RR scalars. The major virtue of space–time supersymmetry is that it links toghether these dualities; often some of them are interchanged in comparing dual theories in the nonperturbative regime. In this paper we will only consider compactifications on smooth manifolds since the analysis is otherwise more complicated (and richer) due to additional states concentrated at the singular points of the moduli space. The key ingredient to compare different theories in a given space-time dimension is Poincaré duality, which converts a theory with a $(p+2)$-form into one with a $(D-p-2)$-form (and inverse coupling constant). For example, at $D = 9$ Poincaré duality relates 4- and 5-forms, at $D = 7$ [2] 3- and 4-forms and at $D = 5$ 2- and 3-forms [21]. These relations are closely related to the fact that type IIA and type IIB are T-dual at $D = 9$ [22], heterotic on T_3 is dual to M-theory on $K3$ at $D = 7$ [2] and heterotic on $K3 \times S_1$ is dual to M-theory on CY_3 at $D = 5$ [23], [9]. Let us consider dualities by first comparing theories with maximal supersymmetry (32 supersymmetries). An example is the duality between M-theory on S_1 at large radius and type IIA in $D = 10$ at strong coupling. For the sequel we will omit the regime where these theories should be compared. We will just identify their low-energy effective action including BPS states.

Further compactifying type IIA on S_1, it becomes equivalent to IIB on S_1 with inverse radius. This is the T-duality alluded to before. It merely comes by Poincaré duality, exchanging the five form of one theory (IIB) with the 4-form of the other theory (IIA). The interrelation between M-theory and type IIA and type IIB theories at $D \leq 9$ explains most of the symmetries of all maximally extended supergravities. At $D = 8$ we have a maximal theory with U-duality [16] group $Sl(3, \mathbb{Z}) \times Sl(2, \mathbb{Z})$. The $Sl(3, \mathbb{Z})$ has a natural interpretation from an M-theory point of view, since the $D = 8$ theory is M-theory on T_3. On the other hand, the additional $Sl(2, \mathbb{Z})$ which acts on the 4-form and its dual has a natural interpretation from the type IIB theory on T_2, in which the $Sl(2, \mathbb{Z})$ is related to the complex structure of the 2-torus [24]. At $D = 7$ the $Sl(5, \mathbb{Z})$ U-duality has no obvious interpretation unless we move to an F–theory setting [24] This is also the case for $D = 6, 5, 4$, where the U-duality groups are $O(5, 5; \mathbb{Z})$, $E_{6,6}(\mathbb{Z})$ and $E_{7,7}(\mathbb{Z})$ respectively. However, they share the property that the related continuous group has, as maximal compact subgroup, the automorphism group of the supersymmetry algebra, i.e. $Sp(4)$, $Sp(4) \times Sp(4)$, $Usp(8)$ and $SU(8)$ respectively for $D = 7, 6, 5$ and 4. The U–duality group for any D corresponds to the series of E_{11-D} Lie algebras whose quotient with the above automorphism group of the supersymmetry algebra provides the local description of the scalar fields moduli space [25]. Recently, a novel way to unravel the structure of the U–duality groups in terms of solvable Lie algebras has been proposed in [26]. Moving to theories with lower (16) supersymmetries, we start to have dualities among heterotic, M-theory and type II theories on manifolds preserving 16 supersymmetries. For $D = 7$, heterotic theory on T_3 is "dual" to M-theory on $K3$ in the same sense that M-theory is "dual" to type IIA at $D = 10$. Here the coset space $O(1, 1) \times \frac{O(3,19)}{O(3)\times O(19)}$ identifies the dilaton and Narain lattice of the heterotic string with the classical moduli space of $K3$, the dilaton in one theory being related to the volume of $K3$ of the other theory [2]. The heterotic string on T_4 is dual to type IIA on $K3$. Here the coset space $O(1, 1) \times \frac{O(4,20)}{O(4)\times O(20)}$ identifies the Narain lattice with the "quantum" moduli space of $K3$ (including torsion). The $O(1, 1)$ factor again relates the dilaton to the $K3$ volume. A similar situation occurs for the theories at $D = 5$. At $D = 4$ a new phenomenon occurs since the classical moduli space $\frac{SU(1,1)}{U(1)} \times \frac{O(6,22)}{O(6)\times O(22)}$ interchanges S-duality of heterotic string with T-duality of type IIA theory and U-duality of type IIB theory [27]. If we compare theories with 8 supersymmetries, we may at most start with $D = 6$. On the heterotic side this would correspond to $K3$ compactification. However at $D = 6$ no M-theory or type II correspondence is possible because we have no smooth manifolds of dimension 5 or 4 which reduce the original supersymmetry (32) by one quarter. The least we can do is to compare theories at $D = 5$, where heterotic theory on $K3 \times S_1$ can be compared [23] and in fact is dual, to M-theory on a Calabi-Yau threefold which is a $K3$ fibration [28]. Finally, at $D = 4$ the heterotic string on $K3 \times T_2$ is dual to type IIA (or IIB) on a Calabi-Yau threefold (or its mirror) [29], [30], [31]. It is worth noticing that these "dualities" predict new BPS states as well as they identify perturbative BPS states of one theory with non-perturbative ones in the dual theory. A more striking correspondence is possible if we further assume the existence of 12 dimensional F-theory such that its compactification on T_2 gives type IIB at $D = 10$ [3]. In this case we can relate the heterotic string on T_2 at $D = 8$ to F-theory on $K3$ and the heterotic string at $D = 6$ on $K3$ with F-theory on a Calabi-Yau threefold [8], [32]. To make these comparisons one has to further assume that the smooth manifolds of F-theory are elliptically fibered [8]. An even larger correspondence arises if we also include type I strings [33] and D-branes [34] in the game. However we will not further comment on the other correspondences relating all string theories with M and F theory.

3 Duality symmetries and central charges in diverse dimensions

3.1 The general framework

All supergravity theories contain scalar fields whose kinetic Lagrangian is described by σ–models of the form G/H, with the exception of $D = 4$, $N = 1, 2$ and $D = 5$, $N = 2$. Here G is a non compact group acting as an isometry group on the scalar manifold while H, the isotropy subgroup, is of the form:

$$H = H_{Aut} \otimes H_{matter} \tag{3.1}$$

H_{Aut} being the automorphism group of the supersymmetry algebra while H_{matter} is related to the matter multiplets. (Of course $H_{matter} = \mathbb{1}$ in all cases where supersymmetric matter doesn't exist, namely $N > 4$ in $D = 4, 5$ and in general in all maximally extended supergravities). The coset manifolds G/H and the automorphism groups for various supergravity theories for any D and N can be found in the literature (see for instance [11], [35]). As it is well known, the group G acts linearly on the $(n = p+2)$–forms field strengths $H^\Lambda_{a_1 \cdots a_n}$ corresponding to the various $(p + 1)$–forms appearing in the gravitational and matter multiplets. Here and in the following the index Λ runs over the dimensions of some representation of the duality group G. The true duality symmetry (U–duality), acting on integral quantized electric and magnetics charges, is the restriction of the continuous group G to the integers [16]. The moduli space of these theories is $G(\mathbb{Z})\backslash G/H$.

All the properties of the given supergravity theories for fixed D and N are completely fixed in terms of the geometry of G/H, namely in terms of the coset representatives L satisfying the relation:

$$L(\phi') = gL(\phi)h(g, \phi) \tag{3.2}$$

where $g \in G$, $h \in H$ and $\phi' = \phi'(\phi)$, ϕ being the coordinates of G/H. Note that the scalar fields in G/H can be assigned, in the linearized theory, to linear representations R_H of the local isotropy group H so that dim R_H = dim G − dim H (in the full theory, R_H is the representation which the vielbein of G/H belongs to).

As explained in the following, the kinetic metric for the $(p + 2)$–forms H^Λ is fixed in terms of L and the physical field strengths of the interacting theories are "dressed" with scalar fields in terms of the coset representatives. This allows us to write down the central charges associated to the $(p + 1)$–forms in the gravitational multiplet in a neat way in terms of the geometrical structure of the moduli space. In an analogous way also the matter $(p + 1)$–forms of the matter multiplets give rise to charges which, as we will see, are closely related to the central charges. Note that when $p > 1$ the central charges do not appear in the usual supersymmetry algebra, but in the extended version of it containing central generators $Z_{a_1 \cdots a_p}$ associated to p–dimensional extended objects ($a_1 \cdots a_p$ are a set of space–time antisymmetric Lorentz indices) [36, 37, 13, 38, 39]

Our main goal is to write down the explicit form of the dressed charges and to find relations among them analogous to those worked out in $D = 4$, $N = 2$ by means of the Special Geometry relations [40][29].

To any $(p+2)$–form H^Λ we may associate a magnetic charge ($(D-p-4)$–brane) and an electric (p–brane) charge given respectively by:

$$g^\Lambda = \int_{S^{p+2}} H^\Lambda \qquad\qquad e_\Lambda = \int_{S^{D-p-2}} \mathcal{G}_\Lambda \tag{3.3}$$

where $\mathcal{G}_\Lambda = \frac{\partial \mathcal{L}}{\partial H^\Lambda}$.

These charges however are not the physical charges of the interacting theory; the latter ones can be computed by looking at the transformation laws of the fermion fields, where the physical field–strengths appear dressed with the scalar fields [20]. Let us first introduce the central charges: they are associated to the dressed $(p+2)$–forms T^i_{AB} appearing in the supersymmetry transformation law of the gravitino 1-form. Quite generally we have, for any D and N:

$$\delta\psi_A = D\epsilon_A + \sum_i c_i T^i_{AB|a_1\cdots a_{n_i}} \Delta^{aa_1\cdots a_{n_i}} \epsilon^B V_a + \cdots \tag{3.4}$$

where:

$$\Delta_{aa_1\cdots a_n} = \left(\Gamma_{aa_1\cdots a_n} - \frac{n}{n-1}(D-n-1)\delta^a_{[a_1}\Gamma_{a_2\cdots a_n]}\right). \tag{3.5}$$

Here D is the covariant derivative in terms of the space–time spin connection and the composite connection of the automorphism group H_{Aut}, c_i are coefficients fixed by supersymmetry, V^a is the space–time vielbein, $A = 1, \cdots, N$ is the index acted on by the automorphism group, $\Gamma_{a_1\cdots a_n}$ are γ–matrices in the appropriate dimensions, and the sum runs over all the $(p+2)$–forms appearing in the gravitational multiplet. Here and in the following the dots denote trilinear fermion terms. Each n-form field-strength T^i_{AB} is constructed by dressing the bare field-strengths H^Λ with the coset representative $L(\phi)$ of G/H, ϕ denoting a set of coordinates of G/H. In particular, for any p, except for $D/2 = p+2$, we have:

$$T^i_{AB} = L_{AB\Lambda_i}(\phi) H^{\Lambda_i} \tag{3.6}$$

where we have used the following decomposition of L:

$$L = (L^\Lambda_{AB}, L^\Lambda_I) \qquad L^{-1} = (L^{AB}_{\ \ \Lambda}, L^I_{\ \Lambda}) \tag{3.7}$$

Here L^Λ_Σ belongs to the representation of G under which the $(p+2)$–forms H^Λ transform irreducibly and the couple of indices AB and I refer to the transformation properties of L under the right action of $H_{Aut} \times H_{matter}$. More precisely, the couple of indices AB transform in the twofold tensor representation of H_{Aut}, which in general is a $Usp(N)$ group (except in $D = 8$ and $D = 9$ theories where H_{Aut} is $SU(N) \times U(1)$ or $O(N)$ respectively). and I is an index in the fundamental representation of H_{matter} which in general is an orthogonal group. Note that in absence of matter multiplets $L \equiv (L^\Lambda_{\ AB})$. In all these cases ($D/2 \neq p+2$) the kinetic matrix of the $(p+2)$–forms H^Λ is given in terms of the coset representatives as follows:

$$\frac{1}{2} L_{AB\Lambda} L^{AB}_{\ \ \Sigma} - L_{I\Lambda} L^I_{\ \Sigma} = \mathcal{N}_{\Lambda\Sigma} \tag{3.8}$$

with the indices of H_{Aut} raised and lowered with the appropriate metric of H_{Aut} in the given representation. For maximally extended supergravities $\mathcal{N}_{\Lambda\Sigma} = L_{AB\Lambda} L^{AB}_{\ \ \Sigma}$. Note that both for matter coupled and maximally extended supergravities we have:

$$L_{\Lambda AB} = \mathcal{N}_{\Lambda\Sigma} L^{\Sigma}_{\ AB} \tag{3.9}$$

When G contains an orthogonal factor $O(m,n)$, what happens for matter coupled supergravities in $D = 5,7,8,9$, where $G = O(10-D,n) \times O(1,1)$ and in all the matter coupled $D = 6$ theories, the coset representatives of the orthogonal group satisfy:

$$L^t \eta L = \eta \quad \rightarrow \quad L_{r\Lambda} L_{r\Sigma} - L_{I\Lambda} L_{I\Sigma} = \eta_{\Lambda\Sigma} \tag{3.10}$$

$$L^t L = \mathcal{N} \quad \rightarrow \quad L_{r\Lambda} L_{r\Sigma} + L_{I\Lambda} L_{I\Sigma} = \mathcal{N}_{\Lambda\Sigma} \tag{3.11}$$

where $\eta = \begin{pmatrix} 1\!\!1_{m\times m} & 0 \\ 0 & -1\!\!1_{n\times n} \end{pmatrix}$ is the $O(m,n)$ invariant metric and $A = 1,\cdots,m$; $I = 1,\cdots,n$ (In particular, setting the matter to zero, we have in these cases $\mathcal{N}_{\Lambda\Sigma} = \eta_{\Lambda\Sigma}$). In these cases we have:

$$L^{\Lambda}_{\ AB} = L^{\Lambda}_{\ r} (\gamma^r)_{AB}, \tag{3.12}$$

$(\gamma^r)_{AB}$ being the γ-matrices intertwining between orthogonal and $USp(N)$ indices.

When D is even and $D/2 = p+2$ the previous formulae in general require modifications, since in that case we have the complication that the action of G on the $p+2 = D/2$–forms (D even) is realized through the embedding of G in $Sp(2n, \mathbb{R})$ (p even) or $O(n,n)$ (p odd) groups [41], [42].
This happens for $D = 4$, $N > 1$, $D = 6$, $N = (2,2)$ and the maximally extended $D = 6$ and $D = 8$ supergravities. [1] (The necessary modifications for the embedding are worked out in section 3.2.)

Coming back to the case $D/2 \neq p+2$, it is now straightforward to compute the central charges.

Indeed, the magnetic central charges for BPS saturated $(D-p-4)$–branes can be now defined (modulo numerical factors to be fixed in each theory) by integration of the dressed field strengths as follows:

$$Z^{(i)}_{(m)AB} = \int_{S^{p+2}} T^i_{AB} = \int_{S^{p+2}} L_{\Lambda_i AB}(\phi) H^{\Lambda_i} = L_{\Lambda_i AB}(\phi_0) g^{\Lambda_i} \tag{3.13}$$

where ϕ_0 denote the $v.e.v.$ of the scalar fields, namely $\phi_0 = \phi(\infty)$ in a given background. The corresponding electric central charges are:

$$Z^{(i)}_{(e)AB} = \int_{S^{D-p-2}} L_{AB\Lambda_i}(\phi)\, {}^\star H^{\Lambda_i} = \int_{S^{D-p-2}} \mathcal{N}_{\Lambda_i\Sigma_i} L^{\Lambda_i}_{\ AB}(\phi)\, {}^\star H^{\Sigma_i} = L^{\Lambda_i}_{\ AB}(\phi_0) e_{\Lambda_i} \tag{3.14}$$

These formulae make it explicit that $L^{\Lambda}_{\ AB}$ and $L_{\Lambda AB}$ are related by electric–magnetic duality via the kinetic matrix.

Note that the same field strengths T^i_{AB} which appear in the gravitino transformation laws are also present in the dilatino transformation laws in the following way:

$$\delta\chi_{ABC} = \cdots + \sum_i b_i L_{\Lambda_i AB}(\phi) H^{\Lambda_i}_{a_1\cdots a_{n_i}} \Gamma^{a_1\cdots a_{n_i}} \epsilon_C + \cdots \tag{3.15}$$

[1] The 6 dimensional theories $N = (2,0)$ and $N = (4,0)$ do not require such embedding since the 3–forms H^{Λ} have definite self–duality and no lagrangian exists. For these theories the formulae of the previous odd dimensional cases are valid.

In an analogous way, when vector multiplets are present, the matter vector field strengths are dressed with the columns $L_{\Lambda I}$ of the coset element (3.7) and they appear in the transformation laws of the gaugino fields:

$$\delta\lambda^I_A = c_1 \Gamma^a P^I_{AB,i} \partial_a \phi^i \epsilon^B + c_2 L_\Lambda^{\ I}(\phi) F^\Lambda_{ab} \Gamma^{ab} \epsilon_A + \cdots \tag{3.16}$$

where $P^I_{AB} = P^I_{AB,i} d\phi^i$ (see eq. (3.23) in the following) is the vielbein of the coset manifold spanned by the scalar fields of the vector multiplets, F^Λ_{ab} is the field–strength of the matter photons and c_1, c_2 are constants fixed by supersymmetry (in $D = 6$, $N = (2,0)$ and $N = (4,0)$ the 2–form $F^\Lambda_{ab}\Gamma^{ab}$ is replaced by the 3–form $H^\Lambda_{abc}\Gamma^{abc}$). In the same way as for central charges, one finds the magnetic matter charges:

$$Z^{\ I}_{(m)} = \int_{S^{p+2}} L_\Lambda^{\ I} F^\Lambda = L_\Lambda^{\ I}(\phi_0) g^\Lambda \tag{3.17}$$

while the electric matter charges are:

$$Z_{(e)I} = \int_{S^{D-p-2}} L_{\Lambda I}(\phi)\ {}^\star F^\Lambda = \int_{S^{D-p-2}} \mathcal{N}_{\Lambda\Sigma} L^\Lambda_{\ I}(\phi)\ {}^\star F^\Sigma = L^\Lambda_{\ I}(\phi_0) e_\Lambda \tag{3.18}$$

The important fact to note is that the central charges and matter charges satisfy relations and sum rules analogous to those derived in $D = 4$, $N = 2$ using Special Geometry techniques [40]. They are inherited from the properties of the coset manifolds G/H, namely from the differential and algebraic properties satisfied by the coset representatives $L^\Lambda_{\ \Sigma}$. Indeed, for a general coset manifold we may introduce the left–invariant 1–form $\Omega = L^{-1}dL$ satisfying the relation (see for instance [35]):

$$d\Omega + \Omega \wedge \Omega = 0 \tag{3.19}$$

where

$$\Omega = \omega^i T_i + P^\alpha T_\alpha \tag{3.20}$$

T_i, T_α being the generators of G belonging respectively to the Lie subalgebra $\mathbb{H}$ and to the coset space algebra $\mathbb{K}$ in the Cartan decomposition

$$\mathbb{G} = \mathbb{H} + \mathbb{K} \tag{3.21}$$

$\mathbb{G}$ being the Lie algebra of G. Here ω^i is the $\mathbb{H}$ connection and P^α, in the representation R_H of H, is the vielbein of G/H. Since in all the cases we will consider G/H is a symmetric space ($[\mathbb{K},\mathbb{K}] \subset \mathbb{H}$), $\omega^i C_i^{\ \alpha\beta}$ ($C_i^{\ \alpha\beta}$ being the structure constants of G) can be identified with the Riemannian spin connection of G/H.

Suppose now we have a matter coupled theory. Then, using the decomposition (3.21), from (3.19) and (3.20) we get:

$$\begin{aligned} dL^\Lambda_{\ AB} &= \frac{1}{2} L^\Lambda_{\ CD}\omega^{CD}_{\ \ AB} + L^\Lambda_{\ I} P^I_{AB} \\ dL^\Lambda_{\ I} &= L^\Lambda_{\ J}\omega^J_{\ I} + L^\Lambda_{\ AB} P^{AB}_I \end{aligned} \tag{3.22}$$

where P^I_{AB} is the vielbein on G/H and $\omega^{CD}_{\ \ AB}$ and $\omega^J_{\ I}$ are the $\mathbb{H}_{AUT}$ and $\mathbb{H}_{matter}$ connections respectively in the given representation. It follows:

$$\nabla^{(H)} L^\Lambda_{\ AB} = L^\Lambda_{\ I} P^I_{AB} \tag{3.23}$$

where the derivative is covariant with respect to the $\mathbb{H}$–connection $\omega^{CD}{}_{AB}$. Using the definition of the magnetic dressed charges given in (3.13) we obtain:

$$\nabla^{(H)} Z_{AB} = Z_I P^I_{AB} \tag{3.24}$$

This is a prototype of the formulae one can derive in the various cases for matter coupled supergravities [20]. To illustrate one possible application of this kind of formulae let us suppose that in a given background preserving some number of supersymmetries $Z_I = 0$ as a consequence of $\delta\lambda^I_A = 0$. Then we find:

$$\nabla^{(H)} Z_{AB} = 0 \to d(Z_{AB}\overline{Z}^{AB}) = 0 \tag{3.25}$$

that is the square of the central charge reaches an extremum with respect to the $v.e.v.$ of the moduli fields. Backgrounds with such fixed scalars describe the horizon geometry of extremal black holes and behave as attractor points for the scalar fields evolution in the black hole geometry [43].

For the maximally extended supergravities there are no matter field–strengths and the previous differential relations become differential relations among central charges only. As an example, let us consider $D = 5$, $N = 8$ theory. In this case the Maurer–Cartan equations become:

$$dL^\Lambda{}_{AB} = \frac{1}{2} L^\Lambda{}_{CD} \Omega^{CD}{}_{AB} + \frac{1}{2} \overline{L}^{\Lambda CD} P_{CDAB} \tag{3.26}$$

where the coset representative is taken in the 27×27 fundamental representation of E_6, $\Omega^{CD}{}_{AB} = 2Q^{[A}{}_{[C}\delta^{B]}_{D]}$, AB is a couple of antisymmetric symplectic–traceless $USp(8)$ indices, $Q^A{}_B$ is the $USp(8)$ connection and the vielbein P_{CDAB} is antisymmetric, $\mathbb{C}_{AB}$–traceless and pseudo–real. Note that $(L^\Lambda{}_{CD})^* = L^{\Lambda CD}$. Therefore we get:

$$\nabla^{(H)} L^\Lambda{}_{AB} = \frac{1}{2} \overline{L}^{\Lambda CD} P_{CDAB} \tag{3.27}$$

that is:

$$\nabla^{(H)} Z_{AB} = \frac{1}{2} \overline{Z}^{CD} P_{CDAB} \tag{3.28}$$

This relation implies that the vanishing of a subset of central charges forces the vanishing of the covariant derivatives of some other subset. Typically, this happens in some supersymmetry preserving backgrounds where the requirement $\delta\chi_{ABC} = 0$ corresponds to the vanishing of just a subset of central charges. Finally, from the coset representatives relations (3.8) (3.10) it is immediate to obtain sum rules for the central and matter charges which are the counterpart of those found in $N = 2$, $D = 4$ case using Special Geometry [40]. Indeed, let us suppose e.g. that the group G is $G = O(10 - D, n) \times O(1, 1)$, as it happens in general for all the minimally extended supergravities in $7 \leq D \leq 9$, $D = 6$ type IIA and $D = 5$, $N = 2$. The coset representative is now a tensor product $L \to e^\sigma L$, where e^σ parametrizes the $O(1, 1)$ factor.
We have, from (3.10)

$$L^t \eta L = \eta \tag{3.29}$$

where η is the invariant metric of $O(10 - D, n)$ and from (3.8)

$$e^{-2\sigma}(L^t L)_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma}. \tag{3.30}$$

Using eq.s (3.13) and (3.17) one finds:

$$\frac{1}{2}Z_{AB}Z_{AB} - Z_I Z_I = g^\Lambda \eta_{\Lambda\Sigma} g^\Sigma e^{-2\sigma} \tag{3.31}$$

$$\frac{1}{2}Z_{AB}Z_{AB} + Z_I Z_I = g^\Lambda \mathcal{N}_{\Lambda\Sigma} g^\Sigma \tag{3.32}$$

In more general cases analogous relations of the same kind can be derived.

3.2 The embedding procedure for $D/2 = p + 2$

In this subsection we work out the modifications to the formalism developed in the previous subsection in the case $D/2 = p + 2$, which derive from the embedding procedure [41] of the group G in $Sp(2n, \mathbb{R})$ (p even) or in $O(n, n)$ (p odd). We mainly concentrate on $D = 4$, while for $D = 6, 8$ we just outline the procedure, referring for more details to the next sections. Furthermore we show that the flat symplectic bundle formalism of the $D = 4$, $N = 2$ Special Geometry case [18], [19] can be extended to $N > 2$ theories. The $N = 2$ case differs from the other higher N extensions by the fact that the base space of the flat symplectic bundle is not in general a coset manifold.

Let us analyze the structure of the four dimensional theories.

In $D = 4$, $N > 2$ we may decompose the vector field–strengths in self–dual and anti self–dual parts:

$$F^{\mp} = \frac{1}{2}(F \mp \mathrm{i}\, {}^\star F) \tag{3.33}$$

According to the Gaillard–Zumino construction, G acts on the vector $(F^{-\Lambda}, \mathcal{G}^-_\Lambda)$ (or its complex conjugate) as a subgroup of $Sp(2n_v, \mathbb{R})$ (n_v is the number of vector fields) with duality transformations interchanging electric and magnetic field–strengths:

$$\mathcal{S}\begin{pmatrix} F^{-\Lambda} \\ \mathcal{G}^-_\Lambda \end{pmatrix} = \begin{pmatrix} F^{-\Lambda} \\ \mathcal{G}^-_\Lambda \end{pmatrix}' \tag{3.34}$$

where:

$$\begin{aligned} \mathcal{G}^-_\Lambda &= \overline{\mathcal{N}}_{\Lambda\Sigma} F^{-\Sigma} \\ \mathcal{G}^+_\Lambda &= \mathcal{N}_{\Lambda\Sigma} F^{+\Sigma} \end{aligned} \tag{3.35}$$

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \subset Sp(2n_v, \mathbb{R}) \rightarrow \begin{cases} A^t C - C^t A = 0 \\ B^t D - D^t B = 0 \\ A^t D - C^t B = 1 \end{cases} \tag{3.36}$$

and $\mathcal{N}_{\Lambda\Sigma}$, is the symmetric matrix appearing in the kinetic part of the vector Lagrangian:

$$\mathcal{L}_{kin} = \mathrm{i}\overline{\mathcal{N}}_{\Lambda\Sigma} F^{-\Lambda} F^{-\Sigma} + h.c. \tag{3.37}$$

If $L(\phi)$ is the coset representative of G in some representation, S represents the embedded coset representative belonging to $Sp(2n_v, \mathbb{R})$ and in each theory, A, B, C, D can be constructed in terms of $L(\phi)$. Using a complex basis in the vector space of $Sp(2n_v)$, we may rewrite the symplectic matrix as an $Usp(n_v, n_v)$ element:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{pmatrix} = \mathcal{A}^{-1} S \mathcal{A} \tag{3.38}$$

where:

$$
\begin{aligned}
f &= \frac{1}{\sqrt{2}}(A - \mathrm{i}B) \\
h &= \frac{1}{\sqrt{2}}(C - \mathrm{i}D) \\
\mathcal{A} &= \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}
\end{aligned}
\tag{3.39}
$$

The requirement $U \in Usp(n_v, n_v)$ implies:

$$
\begin{cases}
\mathrm{i}(f^\dagger h - h^\dagger f) = 1\!\!1 \\
(f^t h - h^t f) = 0
\end{cases}
\tag{3.40}
$$

The $n_v \times n_v$ subblocks of U are submatrices f, h which can be decomposed with respect to the isotropy group $H_{Aut} \times H_{matter}$ in the same way as L in equation (3.7), namely:

$$
\begin{aligned}
f &= (f^\Lambda_{AB}, f^\Lambda_I) \\
h &= (h_{\Lambda AB}, h_{\Lambda I})
\end{aligned}
\tag{3.41}
$$

where AB are indices in the antisymmetric representation of $H_{Aut} = SU(N) \times U(1)$ and I is an index of the fundamental representation of H_{matter}. Upper $SU(N)$ indices label objects in the complex conjugate representation of $SU(N)$: $(f^\Lambda_{AB})^* = f^{\Lambda AB}$ etc.

Note that we can consider $(f^\Lambda_{AB}, h_{\Lambda AB})$ and $(f^\Lambda_I, h_{\Lambda I})$ as symplectic sections of a $Sp(2n_v, \mathbb{R})$ bundle over G/H. We will see in the following that this bundle is actually flat. The real embedding given by S is appropriate for duality transformations of $F^\pm$ and their duals $\mathcal{G}^\pm$, according to equations (3.36), (3.35), while the complex embedding in the matrix U is appropriate in writing down the fermion transformation laws and supercovariant field–strengths. The kinetic matrix $\mathcal{N}$, according to Gaillard–Zumino [41], turns out to be:

$$
\mathcal{N} = hf^{-1}, \qquad \mathcal{N} = \mathcal{N}^t \tag{3.42}
$$

and transforms projectively under $Sp(2n_v, \mathbb{R})$ duality rotations:

$$
\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \tag{3.43}
$$

By using (3.40)and (3.42) we find that

$$
(f^t)^{-1} = \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})\overline{f} \tag{3.44}
$$

which is the analogous of equation (3.9), that is

$$
f_{AB\Lambda} \equiv (f^{-1})_{AB\Lambda} = \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\overline{f}^\Sigma_{AB} \tag{3.45}
$$

$$
f_{I\Lambda} \equiv (f^{-1})_{I\Lambda} = \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\overline{f}^\Sigma_I \tag{3.46}
$$

It follows that the dressing factor $(L^\Lambda)^{-1} = (L_{\Lambda AB}, L_{\Lambda I})$ in equation (3.4) which was given by the inverse coset representative in the defining representation of G has to be replaced by the analogous inverse representative $(f_{\Lambda AB}, f_{\Lambda I})$ when, as in the present $D = 4$ case, we have to embed G in $Sp(2n,\mathbb{R})$. As a consequence, in the transformation law of gravitino (3.4), dilatino (3.15) and gaugino (3.16) we perform the following replacement:

$$(L_{\Lambda AB}, L_{\Lambda I}) \to (\overline{f}_{\Lambda AB}, \overline{f}_{\Lambda I}) \tag{3.47}$$

In particular, the dressed graviphotons and matter self–dual field–strengths take the symplectic invariant form:

$$\begin{aligned} T^-_{AB} &= \mathrm{i}(\overline{f}^{-1})_{AB\Lambda} F^{-\Lambda} = f^\Lambda_{AB}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma} F^{-\Sigma} = h_{\Lambda AB} F^{-\Lambda} - f^\Lambda_{AB}\mathcal{G}^-_\Lambda \\ T^-_I &= \mathrm{i}(\overline{f}^{-1})_{I\Lambda} F^{-\Lambda} = f^\Lambda_I(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma} F^{-\Sigma} = h_{\Lambda I} F^{-\Lambda} - f^\Lambda_I\mathcal{G}^-_\Lambda \\ \overline{T}^{+AB} &= (T^-_{AB})^* \\ \overline{T}^{+I} &= (T^-_I)^* \end{aligned} \tag{3.48}$$

(Obviously, for $N > 4$, $L_{\Lambda I} = f_{\Lambda I} = T_I = 0$). To construct the dressed charges one integrates $T_{AB} = T^+_{AB} + T^-_{AB}$ and (for $N = 3,4$) $T_I = T^+_I + T^-_I$ on a large 2-sphere. For this purpose we note that

$$T^+_{AB} = h_{\Lambda AB} F^{+\Lambda} - f^\Lambda_{AB}\mathcal{G}^+_\Lambda = 0 \tag{3.49}$$

$$T^+_I = h_{\Lambda I} F^{+\Lambda} - f^\Lambda_I\mathcal{G}^+_\Lambda = 0 \tag{3.50}$$

as a consequence of eqs. (3.42), (3.35). Therefore we have:

$$Z_{AB} = \int_{S^2} T_{AB} = \int_{S^2}(T^+_{AB} + T^-_{AB}) = \int_{S^2} T^-_{AB} = h_{\Lambda AB} g^\Lambda - f^\Lambda_{AB} e_\Lambda \tag{3.51}$$

$$Z_I = \int_{S^2} T_I = \int_{S^2}(T^+_I + T^-_I) = \int_{S^2} T^-_I = h_{\Lambda I} g^\Lambda - f^\Lambda_I e_\Lambda \quad (N \le 4) \tag{3.52}$$

where:

$$e_\Lambda = \int_{S^2} \mathcal{G}_\Lambda, \quad g^\Lambda = \int_{S^2} F^\Lambda \tag{3.53}$$

and the sections (f^Λ, h_Λ) on the right hand side now depend on the *v.e.v.*'s of the scalar fields ϕ^i. We see that the central and matter charges are given in this case by symplectic invariants and that the presence of dyons in $D = 4$ is related to the symplectic embedding. In the case $D/2 \neq p+2$, D even, or D odd, we were able to derive the differential relations (3.24), (3.28) among the central and matter charges using the Maurer–Cartan equations (3.23), (3.27). The same can be done in the present case using the embedded coset representative U. Indeed, let $\Gamma = U^{-1}dU$ be the $Usp(n_v, n_v)$ Lie algebra left invariant one form satisfying:

$$d\Gamma + \Gamma \wedge \Gamma = 0 \tag{3.54}$$

In terms of (f,h) Γ has the following form:

$$\Gamma \equiv U^{-1}dU = \begin{pmatrix} \mathrm{i}(f^\dagger dh - h^\dagger df) & \mathrm{i}(f^\dagger d\overline{h} - h^\dagger d\overline{f}) \\ -\mathrm{i}(f^t dh - h^t df) & -\mathrm{i}(f^t d\overline{h} - h^t d\overline{f}) \end{pmatrix} \equiv \begin{pmatrix} \Omega^{(H)} & \overline{\mathcal{P}} \\ \mathcal{P} & \overline{\Omega}^{(H)} \end{pmatrix} \tag{3.55}$$

where the $n_v \times n_v$ subblocks $\Omega^{(H)}$ and $\mathcal{P}$ embed the H connection and the vielbein of G/H respectively . This identification folllows from the Cartan decomposition of the $Usp(n_v, n_v)$ Lie algebra. Explicitly, if we define the $H_{Aut} \times H_{matter}$–covariant derivative of a vector $V = (V_{AB}, V_I)$ as:

$$\nabla V = dV - V\omega, \quad \omega = \begin{pmatrix} \omega^{AB}{}_{CD} & 0 \\ 0 & \omega^I{}_J \end{pmatrix} \tag{3.56}$$

we have:

$$\Omega^{(H)} = \mathrm{i}[f^\dagger(\nabla h + h\omega) - h^\dagger(\nabla f + f\omega)] = \omega \mathbb{1} \tag{3.57}$$

where we have used:

$$\nabla h = \overline{\mathcal{N}} \nabla f; \quad h = \mathcal{N} f \tag{3.58}$$

and the fundamental identity (3.40). Furthermore, using the same relations, the embedded vielbein $\mathcal{P}$ can be written as follows:

$$\mathcal{P} = -\mathrm{i}(f^t \nabla h - h^t \nabla f) = \mathrm{i} f^t (\mathcal{N} - \overline{\mathcal{N}}) \nabla f \tag{3.59}$$

From (3.38) and (3.55), we obtain the $(n_v \times n_v)$ matrix equation:

$$\begin{aligned} \nabla(\omega)(f + \mathrm{i}h) &= (\overline{f} + \mathrm{i}\overline{h})\mathcal{P} \\ \nabla(\omega)(f - \mathrm{i}h) &= (\overline{f} - \mathrm{i}\overline{h})\mathcal{P} \end{aligned} \tag{3.60}$$

together with their complex conjugates. Using further the definition (3.41) we have:

$$\begin{aligned} \nabla(\omega) f^\Lambda_{AB} &= \overline{f}^\Lambda_I P^I_{AB} + \frac{1}{2} \overline{f}^{\Lambda CD} P_{ABCD} \\ \nabla(\omega) f^\Lambda_I &= \frac{1}{2} \overline{f}^{\Lambda AB} P_{ABI} + \overline{f}^{\Lambda J} P_{JI} \end{aligned} \tag{3.61}$$

where we have decomposed the embedded vielbein $\mathcal{P}$ as follows:

$$\mathcal{P} = \begin{pmatrix} P_{ABCD} & P_{ABJ} \\ P_{ICD} & P_{IJ} \end{pmatrix} \tag{3.62}$$

the subblocks being related to the vielbein of G/H, $P = L^{-1}\nabla^{(H)} L$, written in terms of the indices of $H_{Aut} \times H_{matter}$. Note that, since f belongs to the unitary matrix U, we have: $(f^\Lambda_{AB}, f^\Lambda_I)^\star = (\overline{f}^{\Lambda AB}, \overline{f}^{\Lambda I})$. Obviously, the same differential relations that we wrote for f hold true for the dual matrix h as well.

Using the definition of the charges (3.51), (3.52) we then get the following differential relations among charges:

$$\begin{aligned} \nabla(\omega) Z_{AB} &= \overline{Z}_I P^I_{AB} + \frac{1}{2} \overline{Z}^{CD} P_{ABCD} \\ \nabla(\omega) Z_I &= \frac{1}{2} \overline{Z}^{AB} P_{ABI} + \overline{Z}_J P^J_I \end{aligned} \tag{3.63}$$

Depending on the coset manifold, some of the subblocks of (3.62) can be actually zero. For example in $N = 3$ the vielbein of $G/H = \frac{SU(3,n)}{SU(3)\times SU(n)\times U(1)}$ [45] is P_{IAB} (AB antisymmetric), $I = 1, \cdots, n; A, B = 1, 2, 3$ and it turns out that $P_{ABCD} = P_{IJ} = 0$.

In $N = 4$, $G/H = \frac{SU(1,1)}{U(1)} \times \frac{O(6,n)}{O(6)\times O(n)}$ [46], and we have $P_{ABCD} = \epsilon_{ABCD} P$, $P_{IJ} = \overline{P} \delta_{IJ}$, where P is the Kählerian vielbein of $\frac{SU(1,1)}{U(1)}$, ($A, \cdots, D$ $SU(4)$ indices and I, J $O(n)$ indices) and P_{IAB} is the vielbein of $\frac{O(6,n)}{O(6)\times O(n)}$.

For $N > 4$ (no matter indices) we have that $\mathcal{P}$ coincides with the vielbein P_{ABCD} of the relevant G/H.

For the purpose of comparison of the previous formalism with the $N = 2$ Special Geometry case, it is interesting to note that, if the connection $\Omega^{(H)}$ and the vielbein $\mathcal{P}$

are regarded as data of G/H, then the Maurer–Cartan equations (3.61) can be interpreted as an integrable system of differential equations for a section $V = (V_{AB}, V_I, \overline{V}^{AB}, \overline{V}^I)$ of the symplectic fiber bundle constructed over G/H. Namely the integrable system:

$$\nabla \begin{pmatrix} V_{AB} \\ V_I \\ \overline{V}^{AB} \\ \overline{V}^I \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}P_{ABCD} & P_{ABJ} \\ 0 & 0 & \frac{1}{2}P_{ICD} & P_{IJ} \\ \frac{1}{2}P^{ABCD} & P^{ABJ} & 0 & 0 \\ \frac{1}{2}P^{ICD} & P^{IJ} & 0 & 0 \end{pmatrix} \begin{pmatrix} V_{CD} \\ V_J \\ \overline{V}^{CD} \\ \overline{V}^J \end{pmatrix} \tag{3.64}$$

has $2n$ solutions given by $V = (f^\Lambda{}_{AB}, f^\Lambda{}_I), (h_{\Lambda AB}, h_{\Lambda_I})$, $\Lambda = 1, \cdots, n$. The integrability condition (3.54) means that Γ is a flat connection of the symplectic bundle. In terms of the geometry of G/H this in turn implies that the $\mathbb{H}$–curvature, and hence the Riemannian curvature, is constant, being proportional to the wedge product of two vielbein.

Besides the differential relations (3.63), the charges also satisfy sum rules quite analogous to those found in [40] for the $N = 2$ Special Geometry case.

The sum rule has the following form:

$$\frac{1}{2} Z_{AB}\overline{Z}^{AB} + Z_I\overline{Z}^I = -\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P \tag{3.65}$$

where $\mathcal{M}(\mathcal{N})$ and P are:

$$\mathcal{M} = \begin{pmatrix} 1\!\!1 & -Re\mathcal{N} \\ 0 & 1\!\!1 \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & Im\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} 1\!\!1 & 0 \\ -Re\mathcal{N} & 1\!\!1 \end{pmatrix} \tag{3.66}$$

$$P = \begin{pmatrix} g^\Lambda \\ e_\Lambda \end{pmatrix} \tag{3.67}$$

In order to obtain this result we just need to observe that from the fundamental identities (3.40) and from the definition of the kinetic matrix given in (3.106) it follows:

$$ff^\dagger = -\mathrm{i}\left(\mathcal{N} - \overline{\mathcal{N}}\right)^{-1} \tag{3.68}$$

$$hh^\dagger = -\mathrm{i}\left(\overline{\mathcal{N}}^{-1} - \mathcal{N}^{-1}\right)^{-1} \equiv -\mathrm{i}\mathcal{N}\left(\mathcal{N} - \overline{\mathcal{N}}\right)^{-1}\overline{\mathcal{N}} \tag{3.69}$$

$$hf^\dagger = \mathcal{N} ff^\dagger \tag{3.70}$$

$$fh^\dagger = ff^\dagger \overline{\mathcal{N}} \tag{3.71}$$

We note that the matrix $\mathcal{M}$ is a symplectic tensor and in this sense it is quite analogous to the matrix $\mathcal{N}_{\Lambda\Sigma}$ of the odd dimensional cases defined in equation (3.8). Indeed, one sees that the matrix $\mathcal{M}$ can be written as the symplectic analogous of (3.8):

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} (f \quad h)^\dagger \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} \tag{3.72}$$

where $\begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix}$ is the embedded object corresponding to the L of equation (3.8)

The formalism we have developed so far for the $D = 4$, $N > 2$ theories is completely determined by the embedding of the coset representative of G/H in $Sp(2n, \mathbb{R})$ and by the $Usp(n, n)$ embedded Maurer–Cartan equations (3.61). We want now to show that this formalism, and in particular the identities (3.40), the differential relations among charges (3.63) and the sum rules (3.65), are completely analogous to the Special Geometry relations of $N=2$ matter coupled supergravity [47], [18]. This follows essentially from the fact that, though that the scalar manifold $\mathcal{M}_{N=2}$ of the $N=2$ theory is not in general a coset manifold, nevertheless it has a symplectic structure identical to the $N>2$ theories. Furthermore we will show that the analogous of the Maurer–Cartan equations of $N>2$ theories are given in the $N=2$ case by the Picard–Fuchs equations [19] for the symplectic sections which enter in the definition of the Special Geometry flat symplectic bundle.

Indeed, let us recall that Special Geometry can be defined in terms of the holomorphic flat vector bundle of rank $2n$ with structure group $Sp(2n,\mathbb{R})$ over a Kähler–Hodge manifold [18].

If we introduce the Special Geometry symplectic and covariantly holomorphic section of $U(1)$-weight $p=-\overline{p}=1$:

$$V \equiv \left(f^{\Lambda}, h_{\Lambda}\right) = \left(f^{\Lambda}(z^i, z^{\bar{\imath}}), h_{\Lambda}(z^i, z^{\bar{\imath}})\right); \quad \Lambda = 1, ..., n \tag{3.73}$$

and its covariant derivatives with respect to the Kähler connection:

$$\begin{aligned} \nabla_i V &= \left(\partial_i + \frac{p}{2}\partial_i K\right) V \equiv \left(f_i^{\Lambda}, h_{\Lambda i}\right) \quad i = 1, \cdots, n-1 \\ \nabla_{\bar{\imath}} V &= \left(\partial_{\bar{\imath}} + \frac{\overline{p}}{2}\partial_{\bar{\imath}} K\right) V = 0 \quad \text{covariantly holomorphic} \end{aligned} \tag{3.74}$$

then, defining the $n \times n$ matrices:

$$f_{\Sigma}^{\Lambda} \equiv \left(f^{\Lambda}, \overline{f}^{\Lambda i}\right); \quad h_{\Lambda\Sigma} \equiv \left(h_{\Lambda}, \overline{h}_{\Lambda}^{\ i}\right) \tag{3.75}$$

where $\overline{f}^{\Lambda i} \equiv \overline{f}^{\Lambda}_{\ \bar{\jmath}} g^{\bar{\jmath} i}$, $\overline{h}_{\Lambda}^{\ i} \equiv \overline{h}_{\Lambda \bar{\jmath}} g^{\bar{\jmath} i}$, the set of algebraic relations of Special Geometry can be written in matrix form as:

$$\begin{cases} \mathrm{i}(f^{\dagger}h - h^{\dagger}f) &= \mathbb{1} \\ (f^{t}h - h^{t}f) &= 0 \end{cases} \tag{3.76}$$

Recalling equations (3.40) we see that the previous relations imply that the matrix U:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{pmatrix} \tag{3.77}$$

belongs to $Usp(n,n)$. In fact if we set $f^{\Lambda} \to f^{\Lambda}\epsilon_{AB} \equiv f^{\Lambda}_{AB}$ and flatten the world-indices of $(f_i^{\Lambda}, \overline{f}_{\bar{\imath}}^{\Lambda})$ (or $(\overline{h}_i, h_{\bar{\imath}})$) with the Kählerian vielbein $P_i^I, \overline{P}_{\bar{\imath}}^{\bar{I}}$:

$$(f_I^{\Lambda}, \overline{f}_{\bar{I}}^{\Lambda}) = (f_i^{\Lambda} P_I^i, \overline{f}_{\bar{\imath}}^{\Lambda} \overline{P}_{\bar{I}}^{\bar{\imath}}), \qquad P_i^I \overline{P}_{\bar{\jmath}}^{\bar{J}} \eta_{I\bar{J}} = g_{i\bar{\jmath}} \tag{3.78}$$

where $\eta_{I\bar{J}}$ is the flat Kählerian metric and $P^i_{\ I} = (P^{-1})^I_{\ i}$, the relations (3.76) are just a particular case of equations (3.40) since, for $N=2$, $H_{Aut} = SU(2) \times U(1)$, so that f^{Λ}_{AB} is actually an $SU(2)$ singlet.

Let us now consider the analogous of the embedded Maurer–Cartan equations of G/H. Defining as before the matrix one–form $\Gamma = U^{-1}dU$ valued in the $Usp(n,n)$ Lie algebra, we see that the relation $d\Gamma + \Gamma \wedge \Gamma = 0$ again implies a flat connection for the symplectic bundle over the Kähler–Hodge manifold. However, this does not imply anymore that the base manifold is a coset or a constant curvature manifold. Indeed, let us introduce the covariant derivative of the symplectic section $(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, \overline{f}^\Lambda, f^\Lambda_I)$ with respect to the $U(1)$–Kähler connection $\mathcal{Q}$ and the spin connection ω^{IJ} of $\mathcal{M}_{N=2}$:

$$\nabla(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, \overline{f}^\Lambda, f^\Lambda_I) = $$
$$d(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, f^\Lambda, \overline{f}^\Lambda_I) + (f^\Lambda, \overline{f}^\Lambda_{\overline{J}}, \overline{f}^\Lambda, f^\Lambda_J)\begin{pmatrix} -\mathrm{i}\mathcal{Q} & 0 & 0 & 0 \\ 0 & \mathrm{i}\mathcal{Q}\delta^{\overline{I}}_{\overline{J}} + \omega^{\overline{I}}{}_{\overline{J}} & 0 & 0 \\ 0 & 0 & \mathrm{i}\mathcal{Q} & 0 \\ 0 & 0 & 0 & -\mathrm{i}\mathcal{Q}\delta^I_J + \omega^I{}_J \end{pmatrix} \tag{3.79}$$

where:

$$\mathcal{Q} = -\frac{\mathrm{i}}{2}(\partial_i\mathcal{K}dz^i - \overline{\partial}_{\overline{\imath}}\mathcal{K}d\overline{z}^{\overline{\imath}}) \to d\mathcal{Q} = \mathrm{i}g_{i\overline{\jmath}}dz^i \wedge d\overline{z}^{\overline{\jmath}}, \tag{3.80}$$

($\mathcal{K}$ is the Kähler potential) the Kähler weight of $(f^\Lambda, \overline{f}^\Lambda_I)$ and $(\overline{f}^\Lambda, \overline{f}^\Lambda_{\overline{I}})$ being $p = 1$ and $p = -1$ respectively. Using the same decomposition as in equation (3.55) and eq.s (3.56), (3.20) we have in the $N = 2$ case:

$$\begin{aligned} \Gamma &= \begin{pmatrix} \Omega & \overline{\mathcal{P}} \\ \mathcal{P} & \overline{\Omega} \end{pmatrix}, \\ \Omega &= \omega = \begin{pmatrix} -\mathrm{i}\mathcal{Q} & 0 \\ 0 & \mathrm{i}\mathcal{Q}\delta^I_J + \overline{\omega}^I{}_J \end{pmatrix} \end{aligned} \tag{3.81}$$

For the subblocks $\mathcal{P}$ we obtain:

$$\mathcal{P} = -\mathrm{i}(f^t\nabla h - h^t\nabla f) = \mathrm{i}f^t(\mathcal{N} - \overline{\mathcal{N}})\nabla f = \begin{pmatrix} 0 & P_{\overline{I}} \\ P^J & P^J_{\overline{I}} \end{pmatrix} \tag{3.82}$$

where $\overline{P}^J \equiv \eta^{J\overline{I}}P_{\overline{I}}$ is the $(1,0)$–form Kählerian vielbein while $P^J_{\overline{I}} \equiv \mathrm{i}\left(f^t(\mathcal{N} - \overline{\mathcal{N}})\nabla f\right)^J_{\ \overline{I}}$ is a one–form which in general cannot be expressed in terms of the vielbein P^I and therefore represents a new geometrical quantity on $\mathcal{M}_{N=2}$. Note that we get zero in the first entry of equation (3.82) by virtue of the fact that the identity (3.76) implies $f^\Lambda(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}f^\Sigma_i = 0$ and that f^Λ is covariantly holomorphic. If Ω and $\mathcal{P}$ are considered as data on $\mathcal{M}_{N=2}$ then we may interpret $\Gamma = U^{-1}dU$ as an integrable system of differential equations, namely:

$$\nabla(V, \overline{V}_{\overline{I}}, \overline{V}, V_I) = (V, \overline{V}_{\overline{J}}, \overline{V}, V_J)\begin{pmatrix} 0 & 0 & 0 & \overline{P}_I \\ 0 & 0 & \overline{P}^{\overline{J}} & \overline{P}^{\overline{J}}{}_I \\ 0 & P_{\overline{I}} & 0 & 0 \\ P^J & P^J_{\overline{I}} & 0 & 0 \end{pmatrix} \tag{3.83}$$

where the flat Kähler indices $I, \overline{I}, \cdots$ are raised and lowered with the flat Kähler metric $\eta_{I\overline{J}}$. As it is well known, this integrable system describes the Picard–Fuchs equations for the periods $(V, \overline{V}_{\overline{I}}, \overline{V}, V_I)$ of the Calabi–Yau threefold with solution given by the $2n$ symplectic vectors $V \equiv (f^\Lambda, h_\Lambda)$. The integrability condition $d\Gamma + \Gamma \wedge \Gamma = 0$ gives three constraints on the Kähler base manifold:

$$d(\mathrm{i}\mathcal{Q}) + \overline{P}_I \wedge P^I = 0 \to \partial_{\overline{\jmath}}\partial_i\mathcal{K} = P^I\ , i\overline{P}_{I,\overline{\jmath}} = g_{i\overline{\jmath}} \tag{3.84}$$

$$(d\omega + \omega \wedge \omega)^{\overline{J}}_{\ \overline{I}} = P_{\overline{I}} \wedge \overline{P}^{\overline{J}} - \mathrm{i}d\mathcal{Q}\delta^{\overline{J}}_{\overline{I}} - \overline{P}^{\overline{J}}{}_L \wedge P^L_{\overline{I}} \tag{3.85}$$

$$\nabla P^J_{\overline{I}} = 0 \tag{3.86}$$

$$\overline{P}_J \wedge P^J_{\overline{I}} = 0 \tag{3.87}$$

Equation (3.84) implies that $\mathcal{M}_{N=2}$ is a Kähler–Hodge manifold. Equation (3.85), written with holomorphic and antiholomorphic curved indices, gives:

$$R_{\bar{i}j\bar{k}l} = g_{\bar{i}l}g_{j\bar{k}} + g_{\bar{k}l}g_{\bar{i}j} - P_{\bar{i}\bar{k}\bar{m}}\overline{P}_{jln}g^{\overline{m}n} \tag{3.88}$$

which is the usual constraint on the Riemann tensor of the special geometry. The further Special Geometry constraints on the three tensor $\overline{P}_{ijk}$ are then consequences of equations (3.86), (3.87), which imply:

$$\begin{aligned} \nabla_{[l}\overline{P}_{i]jk} &= 0 \\ \nabla_{\bar{l}}\overline{P}_{ijk} &= 0 \end{aligned} \tag{3.89}$$

In particular, the first of eq. (3.89) also implies that $\overline{P}_{ijk}$ is a completely symmetric tensor.

In summary, we have seen that the $N = 2$ theory and the higher N theories have essentially the same symplectic structure, the only difference being that since the scalar manifold of $N = 2$ is not in general a coset manifold the symplectic structure allows the presence of a new geometrical quantity which physically corresponds to the anomalous magnetic moments of the $N = 2$ theory. It goes without saying that, when $\mathcal{M}_{N=2}$ is itself a coset manifold [44], then the anomalous magnetic moments $\overline{P}_{ijk}$ must be expressible in terms of the vielbein of G/H. We give here two examples.

- Suppose $\mathcal{M}_{N=2} = \frac{SU(1,1)}{U(1)} \times \frac{O(2,n)}{O(2)\times O(n)}$. The symplectic sections entering the matrix U can be written as follows:

$$\begin{aligned} f &= \mathrm{i}e^{\frac{\mathcal{K}}{2}}(L^\Lambda, L^\Lambda_a) \\ h &= \mathrm{i}e^{\frac{\mathcal{K}}{2}}(SL^\Sigma\eta_{\Lambda\Sigma}, \overline{S}L^\Sigma_a\eta_{\Lambda\Sigma}) \end{aligned} \tag{3.90}$$

where $\Lambda = 1,\cdots,2n$, $a = 3,\cdots,n$, $\eta_{\Lambda\Sigma} = (1,1,-1,\cdots,-1)$ and we have set $L^\Lambda = \frac{1}{\sqrt{2}}(L^\Lambda_1 + \mathrm{i}L^\Lambda_2)$. In particular from the pseudoorthogonality of $O(2,n)$ we have:

$$L^\Lambda L^\Sigma\eta_{\Lambda\Sigma} = 0 \tag{3.91}$$

Furthermore we have parametrized $\frac{SU(1,1)}{U(1)}$ as follows:

$$M(S) = \frac{1}{\sqrt{\frac{4ImS}{1+|S|^2+2ImS}}}\begin{pmatrix} \mathbb{1} & \frac{\mathrm{i}-S}{\mathrm{i}+S} \\ \frac{\mathrm{i}+\overline{S}}{\mathrm{i}-\overline{S}} & \mathbb{1} \end{pmatrix} \tag{3.92}$$

We can then compute the embedded connection and vielbein using (3.55). In particular we find:

$$\mathcal{P} = \begin{pmatrix} 0 & P_I \\ P_I & -\overline{P}\delta_{ab} \end{pmatrix} \tag{3.93}$$

We see that the general $P_{I\overline{J}}$ matrix in this case can be expressed in terms of the vielbein of G/H and one finds that the only non vanishing anomalous magnetic moments are:

$$\overline{P}_{abS} = \delta_{ab}P_{,S} = e^{\mathcal{K}}\delta_{ab}. \tag{3.94}$$

- As a second example we consider the special coset manifold $\frac{SO^*(12)}{U(6)}$.

  Note that this manifold also appears as the scalar manifold of the $D=4$, $N=6$ theory, and we refer the reader to section 4 for notations and parametrization of G/H.

  The integrable system in this case can be written as follows:

$$\nabla \begin{pmatrix} V \\ V_{AB} \\ \overline{V} \\ \overline{V}^{AB} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}P_{CD} \\ 0 & 0 & P_{AB} & \frac{1}{2}\overline{P}_{ABCD} \\ 0 & \frac{1}{2}\overline{P}^{CD} & 0 & 0 \\ \overline{P}^{AB} & \frac{1}{2}P^{ABCD} & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ V_{CD} \\ \overline{V} \\ \overline{V}^{CD} \end{pmatrix} \tag{3.95}$$

  where P^{ABCD} is the Kählerian vielbein $(1,0)$–form ($\overline{P}_{ABCD} = (P^{ABCD})^*$ is a $(0,1)$–form) and:

$$P_{AB} = \frac{1}{4!}\epsilon_{ABCDEF}P^{CDEF}; \quad \overline{P}^{AB} = (P_{AB})^*. \tag{3.96}$$

  Moreover, f_{AB} transforms in the **15** of $SU(6)$, f is an $SU(6)$ singlet and $\overline{f}^{AB} = (f_{AB})^*$. It follows:

$$\nabla_i \begin{pmatrix} V \\ V_{AB} \\ \overline{V} \\ \overline{V}^{AB} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}P_{CD,i} \\ 0 & 0 & P_{AB,i} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}P^{ABCD}_{,i} & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ V_{CD} \\ \overline{V} \\ \overline{V}^{CD} \end{pmatrix} \tag{3.97}$$

  Hence:

$$\nabla_i \overline{V}^{AB} = \frac{1}{2}P^{ABCD}_{,i}V_{CD} \tag{3.98}$$

  If we set:

$$V_{CD} = P_{CD}^{\ \bar{l}}V_{\bar{l}}; \quad \overline{V}^{AB} = \overline{P}^{AB,k}V_k \tag{3.99}$$

  where the curved indices $k, \bar{l}$ are raised and lowered with the Kähler metric, one easily obtains:

$$\nabla_i V_j = \frac{1}{2}P_{AB,i}P_{CD,j}P^{ABCD}_{,k}g^{k\bar{l}}\overline{V}_{\bar{l}} = \frac{1}{4}\epsilon_{ABCDEF}P_{AB,i}P_{CD,j}P_{EF,k}g^{k\bar{l}}\overline{V}_{\bar{l}}. \tag{3.100}$$

  Therefore the anomalous magnetic moment is given, in terms of the vielbein, as:

$$\overline{P}_{ijk} = \frac{1}{4}\epsilon_{ABCDEF}P_{AB,i}P_{CD,j}P_{EF,k} \tag{3.101}$$

  The other two equations of the integral system give:

$$\begin{aligned} \nabla_i V = \frac{1}{2}P_{CD,i}V^{CD} \quad &\rightarrow \quad \nabla_i V = V_i \\ \nabla_i V_{AB} = P_{AB,i}\overline{V} \quad &\rightarrow \quad \nabla_i V_{\bar{j}} = g_{i\bar{j}}\overline{V} \end{aligned} \tag{3.102}$$

  which are the remaining equations defining $N=2$ Special Geometry.

To complete the analogy between the $N=2$ theory and the higher N theories in $D=4$, we also give for completeness in the $N=2$ case the central and matter charges, the differential relations among them and the sum rules.

Let us note that also in $N=2$ the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ which appears in the vector kinetic Lagrangian [2]:

$$\mathcal{L}^{(vector)}_{Kin} = \mathrm{i}\overline{\mathcal{N}}_{\Lambda\Sigma} F^{-\Lambda}_{\mu\nu} F^{-\Sigma\mu\nu} + h.c. \tag{3.103}$$

$$F^{\pm\Lambda} = \frac{1}{2}(\mathbb{1} \pm \mathrm{i}^{\star})\mathrm{F}^{\Lambda} \tag{3.104}$$

$$\mathcal{G}^{-}_{\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma} F^{-\Sigma} \tag{3.105}$$

is given in terms of f, h defined in eq. (3.75) by the formula:

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda\Gamma}(f^{-1})^{\Gamma}_{\Sigma} \tag{3.106}$$

The columns of the matrix f appear in the supercovariant electric field strength $\widehat{F}^{\Lambda}$:

$$\widehat{F}^{\Lambda} = F^{\Lambda} + f^{\Lambda}\overline{\psi}^{A}\psi^{B}\epsilon_{AB} - \mathrm{i}\overline{f}^{\Lambda}_{\bar{\imath}}\overline{\lambda}^{\bar{\imath}}_{A}\gamma_{a}\psi_{B}\epsilon^{AB}V^{a} + h.c. \tag{3.107}$$

(The columns of h^{Λ}_{I} would appear in the dual theory written in terms of the dual magnetic field strengths) .

The transformation laws for the chiral gravitino ψ_A and gaugino λ^{iA} fields are:

$$\delta\psi_{A\mu} = D_{\mu}\,\epsilon_{A} + \epsilon_{AB}T_{\mu\nu}\gamma^{\nu}\epsilon^{B} + \cdots \tag{3.108}$$

$$\delta\lambda^{iA} = \mathrm{i}\partial_{\mu}z^{i}\gamma^{\mu}\epsilon^{A} + \frac{\mathrm{i}}{2}T_{\bar{\jmath}\mu\nu}\gamma^{\mu\nu}g^{i\bar{\jmath}}\epsilon^{AB}\epsilon_{B} + \cdots \tag{3.109}$$

where:

$$T \equiv h_{\Lambda}F^{\Lambda} - f^{\Lambda}\mathcal{G}_{\Lambda} \tag{3.110}$$

$$T_{\bar{\imath}} \equiv \overline{h}_{\Lambda\bar{\imath}}F^{\Lambda} - \overline{f}^{\Lambda}_{\bar{\imath}}\mathcal{G}_{\Lambda} \tag{3.111}$$

are respectively the graviphoton and the matter-vectors z^i $(i = 1, \cdots, n)$ are the complex scalar fields and the position of the $SU(2)$ automorphism index A (A,B=1,2) is related to chirality (namely (ψ_A, λ^{iA}) are chiral, $(\psi^A, \lambda^{\bar{\imath}}_A)$ antichiral). In principle only the (anti) self dual part of F and $\mathcal{G}$ should appear in the transformation laws of the (anti)chiral fermi fields; however, exactly as in eqs. (3.49),(3.50) for $N > 2$ theories, from equations (3.105), (3.106) it follows that :

$$T^{+} = h_{\Lambda}F^{+\Lambda} - f^{\Lambda}\mathcal{G}^{+}_{\Lambda} = 0 \tag{3.112}$$

so that $T = T^{-}$ (and $\overline{T} = \overline{T}^{+}$). Note that both the graviphoton and the matter vectors are $Usp(n,n)$ invariant according to the fact that the fermions do not transform under the duality group (except for a possible R-symmetry phase). To define the physical charges let us note that in presence of electric and magnetic sources we can write:

$$\int_{S^2} F^{\Lambda} = g^{\Lambda}, \quad \int_{S^2} \mathcal{G}_{\Lambda} = e_{\Lambda} \tag{3.113}$$

[2]The same normalization for the vector kinetic lagrangian will be used in section 4 when discussing $D=4$, $N>2$ theories

The central charges and the matter charges are now defined as the integrals over a S^2 of the physical graviphoton and matter vectors:

$$Z = \int_{S^2} T = \int_{S^2} (h_\Lambda F^\Lambda - f^\Lambda \mathcal{G}_\Lambda) = (h_\Lambda(z,\overline{z}) g^\Lambda - f^\Lambda(z,\overline{z}) e_\Lambda) \tag{3.114}$$

where $z^i, \overline{z}^{\overline{\imath}}$ denote the v.e.v. of the moduli fields in a given background. Owing to eq (3.74) we get immediately:

$$Z_i = \nabla_i Z \tag{3.115}$$

We observe that if in a given background $Z_i = 0$ the BPS states in this configuration have a minimum mass. Indeed

$$\nabla_i Z = 0 \to \partial_i |Z|^2 = 0. \tag{3.116}$$

As a consequence of the symplectic structure, one can derive two sum rules for Z and Z_i:

$$|Z|^2 \pm |Z_i|^2 \equiv |Z|^2 \pm Z_i g^{i\overline{\jmath}} \overline{Z}_{\overline{\jmath}} = -\frac{1}{2} P^t \mathcal{M}_\pm P \tag{3.117}$$

where:

$$\mathcal{M}_+ = \begin{pmatrix} 1\!\!1 & -Re\mathcal{N} \\ 0 & 1\!\!1 \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & Im\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} 1\!\!1 & 0 \\ -Re\mathcal{N} & 1\!\!1 \end{pmatrix} \tag{3.118}$$

$$\mathcal{M}_- = \begin{pmatrix} 1\!\!1 & -ReF \\ 0 & 1\!\!1 \end{pmatrix} \begin{pmatrix} ImF & 0 \\ 0 & ImF^{-1} \end{pmatrix} \begin{pmatrix} 1\!\!1 & 0 \\ -ReF & 1\!\!1 \end{pmatrix} \tag{3.119}$$

and:

$$P = \left(g^\Lambda, e_\Lambda\right) \tag{3.120}$$

Equation (3.118) is obtained by using exactly the same procedure as in (3.66). The sum rule (3.119) involves a matrix $\mathcal{M}_-$, which has exactly the same form as $\mathcal{M}_+$ provided we perform the substitution $\mathcal{N}_{\Lambda\Sigma} \to F_{\Lambda\Sigma} = \frac{\partial^2 F}{\partial X^\Lambda \partial X^\Sigma}(X^\Lambda = e^{-\frac{K}{2}} f^\Lambda)$. It can be derived in an analogous way by observing that, when a prepotential $F = F(X)$ exists, Special Geometry gives the following extra identity:

$$f_I^\Lambda (F - \overline{F})_{\Lambda\Sigma} f_J^\Sigma = -\mathrm{i}\eta_{IJ} \qquad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1\!\!1_{n\times n} \end{pmatrix} \tag{3.121}$$

from which it follows:

$$f\eta f^\dagger = \mathrm{i}\left(F - \overline{F}\right)^{-1} \tag{3.122}$$

$$h\eta h^\dagger = \mathrm{i}\left(\overline{F}^{-1} - F^{-1}\right)^{-1} \equiv \mathrm{i}\overline{F}\left(F - \overline{F}\right)^{-1} F \tag{3.123}$$

Note that while $Im\mathcal{N}$ has a definite (negative) signature, ImF is not positive definite.

To conclude this section, we outline the embedding procedure in $D = 6$ and in $D = 8$ maximal supergravities. More details are given in sections 5.2 and 7.

In $D = 8$, $N = 2$ the situation is exactly similar to the $D = 4$ case, where instead of 2–forms field–strengths we now have 4–forms. In the case at hand the 4–form in the gravitational multiplet and its dual are a doublet under the duality group $Sl(2,\mathbb{R})$. The embedding procedure and the relevant relations are discussed in section 7.

Finally in $D = 6$ the 3–form field strengths H^Λ which appear in the gravitational and/or tensor multiplet have a definite self–duality

$$H^{\pm\Lambda} = \frac{1}{2}(H^\Lambda \pm^\star H^\Lambda) \tag{3.124}$$

In this case we have a T-duality group of the form $G = O(m, n)$. In the chiral $N = (2, 0)$ and $N = (4, 0)$ theories, the number of self–dual tensors $H^{+\Lambda_1}$ in the gravitational multiplet, $\Lambda_1 = 1, \cdots m$ and antiself–dual tensors $H^{-\Lambda_2}$ in the matter multiplet, $\Lambda_2 = 1, \cdots n$ are different in general and G acts in its fundamental representation on $(H^{+\Lambda_1}, H^{-\Lambda_2})$ so that no embedding is required. The procedure to find the charges and their relations is thus completely analogous to the odd dimensional case. One finds e.g. for the magnetic charges: $(Z, Z_I) = (L_\Lambda g^\Lambda, L_{\Lambda I} g^\Lambda)$ for(2,0) theory $(I = 1, \cdots, n)$, n being the number of vector multiplets, and $(Z_r, Z_I) = (L_{\Lambda r} g^\Lambda, L_{\Lambda I} g^\Lambda)$ $(r = 1, \cdots, 5)$ for(4,0) theory, r being the number of self–dual 2–forms in the gravitational multiplet (see sect.5.2). However, due to the relation:

$$\mathcal{N}_{\Lambda\Sigma}{}^\star H^\Sigma = \eta_{\Lambda\Sigma} H^\Sigma, \tag{3.125}$$

where η and $\mathcal{N}$ are defined in terms of the coset representatives of $\frac{O(n,m)}{O(m)\times O(n)}$ as in (3.10), (3.8), we have no distinction among electric and magnetic charges. Indeed

$$e_\Lambda = \int \mathcal{N}_{\Lambda\Sigma}{}^\star H^\Sigma = \int \eta_{\Lambda\Sigma} H^\Sigma = \eta_{\Lambda\Sigma} g^\Sigma \tag{3.126}$$

The six dimensional $N = (2, 0)$ and $N = (4, 0)$ matter coupled theories are discussed in section 7.

In $D = 6$ maximally extended theory we have an equal number (five) of self–dual and anti self–dual field strengths and therefore a Lagrangian exists. The group $G = O(5, 5)$ rotates among themselves H^+ and H^- in the representation $\underline{10}$. The analogous of the Gaillard–Zumino construction in this case would define an $O(5, 5)$ embedding of $O(5)$ rotating among themselves $(H^+, \mathcal{G}^+)$ or $(H^-, \mathcal{G}^-)$ where

$$\mathcal{G}^\pm = \mathcal{N}_\mp H^\pm \tag{3.127}$$

$\mathcal{N}_+, \mathcal{N}_- = -(\mathcal{N}_+)^t$ is the kinetic metric of the 3-forms in the Lagrangian: $\mathcal{L}_{kin} = \mathcal{N}^+_{\Lambda\Sigma} H^{+\Lambda} \wedge H^{-\Sigma} + \mathcal{N}^-_{\Lambda\Sigma} H^{-\Lambda} \wedge H^{+\Sigma}$. In this case we obtain a formula analogous to (3.51), (3.52) which is however invariant under $O(5, 5)$ instead of $Usp(n, n)$. Namely the central charges of $D = 6$, $N = (4, 4)$ have the following dyonic form:

$$Z_{\pm AB} = f^\Lambda_\pm e_\Lambda + h_{\Lambda\pm} g^\Lambda \tag{3.128}$$

which is $O(5, 5)$ invariant with respect to the off diagonal metric $\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. Even in this case the relation (3.125) holds so that only ten independent charges exist. The maximally extended $D = 6$ supergravity is discussed in detail in section 8.

Finally, in the $N = (2, 2)$ (Type IIA) six dimensional theory we have a single 2–form in the gravitational multiplet. Here the embedding group is $O(1, 1)$ and acts in the two dimensional representation on the self–dual and antiself–dual parts of the 3–form $H = dB$. More details on this case are given in section 7.

4 $N > 2$ four dimensional supergravities revisited

In this section and in the following ones we apply the general considerations of section 3 to the various ungauged supergravity theories with scalar manifold G/H for any D and N. This excludes $D = 4$ $N = 2$, already discussed in section 3.2, and $D = 5$ $N = 2$ for which we refer to the literature [50]. Our aim is to write down the group theoretical structure of each theory, their symplectic or orthogonal embedding, the vector kinetic matrix, the supersymmetric transformation laws, the structure of the central and matter charges, the differential relations originating from the Maurer–Cartan equations and the sum rules the charges satisfy. For each theory we give the group–theoretical assignments for the fields, their supersymmetry transformation laws, the $(p+2)$–forms kinetic metrics and the relations satisfied by central and matter charges. As far as the boson transformation rules are concerned we prefer to write down the supercovariant definition of the field strengths (denoted by a superscript hat), from which the susy–laws are immediately retrieved. As it has been mentioned in section 3 it is here that the symplectic sections $(f^\Lambda_{AB}, f^\Lambda_I, \overline{f}^\Lambda_{AB}, \overline{f}^\Lambda_I)$ appear as coefficients of the bilinear fermions in the supercovariant field–strengths while the analogous symplectic section $(h_{\Lambda AB}, h_{\Lambda I}, \overline{h}_{\Lambda AB}, \overline{h}_{\Lambda I})$ would appear in the dual magnetic theory. We include in the supercovariant field–strengths also the supercovariant vielbein of the G/H manifolds. Again this is equivalent to giving the susy transformation laws of the scalar fields. The dressed field strengths from which the central and matter charges are constructed appear instead in the susy transformation laws of the fermions for which we give the expression up to trilinear fermion terms. It should be stressed that the numerical coefficients in the aforementioned susy transformations and supercovariant field strengths are fixed by supersymmetry (or ,equivalently, by Bianchi identities in superspace), but we have not worked out the relevant computations being interested in the general structure rather that in the precise numerical expressions. However the numerical factors could also be retrieved by comparing our formulae with those written in the standard literature on supergravity and performing the necessary redefinitions. The same kind of considerations apply to the central and matter charges whose precise normalization has not been fixed. In the Tables of the present and of the following sections, we give the group assignements for the supergravity fields; in particular, we quote the representation R_H under which the scalar fields of the linearized theory (or the vielbein of G/H of the full theory) transform. Furthermore, for $D = 4, 8$ only the left–handed fermions are quoted. Right handed fermions transform in the complex conjugate representation of H. In the present section we apply the cosiderations given in section 3.2 to the 4D–Supergravities for $N > 2$. Throughout the section we denote by $A, B, \cdots$ indices of $SU(N)$, $SU(N) \otimes U(1)$ being the automorphism group of the N–extended supersymmetry algebra. Lower and upper $SU(N)$ indices on the fermion fields are related to their left or right chirality respectively. If some fermion is a $SU(N)$ singlet chirality is denoted by the usual (L) or (R) suffixes. Right–handed fermions of $D = 4$ transform in the complex f representation of $SU(N) \times U(1) \times H_{matter}$.

Furthermore for any boson field v carrying $SU(N)$ indices we have that lower and upper indices are related by complex conjugation, namely:

$$(v_{AB\cdots})^\star \sim \overline{v}^{AB\cdots} \tag{4.1}$$

- Let us first consider the $N=3$ case [45]. The coset space is:

$$G/H = \frac{SU(3,n)}{SU(3)\otimes SU(n)\otimes U(1)} \tag{4.2}$$

and the field content is given by:

$$(V^a_\mu, \psi_{A\mu}, A^{AB}_\mu, \chi_{(L)}) \qquad A=1,2,3 \qquad \text{(gravitational multiplet)} \tag{4.3}$$
$$(A_\mu, \lambda_A, \lambda_{(R)}, 3z)^I \qquad I=1,\cdots,n \qquad \text{(vector multiplets)} \tag{4.4}$$

The transformation properties of the fields are given in the following Table 1 [3] The

Table 1: Transformation properties of fields in $D=4$, $N=3$

	V^a_μ	$\psi_{A\mu}$	A^Λ_μ	$\chi_{(L)}$	λ^I_A	$\lambda^I_{(L)}$	L^Λ_{AB}	L^Λ_I	R_H
$SU(3,n)$	1	1	$3+n$	1	1	1	$3+n$	$3+n$	-
$SU(3)$	1	3	1	1	3	1	3	1	3
$SU(n)$	1	1	1	1	n	n	1	n	n
$U(1)$	0	$\frac{n}{2}$	0	$3\frac{n}{2}$	$3+\frac{n}{2}$	$-3(1+\frac{n}{2})$	n	-3	$3+n$

embedding of $SU(3,n)$ in $Usp(3+n,3+n)$ allows to express the section (f,h) in terms of L as follows::

$$f^\Lambda_\Sigma \equiv \frac{1}{\sqrt{2}}(L^\Lambda{}_{AB}, \overline{L}^\Lambda{}_I) \tag{4.5}$$
$$h_{\Lambda\Sigma} = -\mathrm{i}(JfJ)_{\Lambda\Sigma} \qquad J = \begin{pmatrix} 1\!\!1_{3\times 3} & 0 \\ 0 & -1\!\!1_{n\times n}\end{pmatrix} \tag{4.6}$$

where AB are antisymmetric $SU(3)$ indices, I is an index of $SU(n)\otimes U(1)$ and $\overline{L}^\Lambda{}_I$ denotes the complex conjugate of the coset representative. We have:

$$\mathcal{N}_{\Lambda\Sigma} = (hf^{-1})_{\Lambda\Sigma} = -\mathrm{i}(JfJf^{-1})_{\Lambda\Sigma} \tag{4.7}$$

The supercovariant field-strengths and the supercovariant scalar vielbein $\widehat{P}^A_I = (L^{-1}\nabla^{(H)}L)_I{}^A$ are:

$$\widehat{F}^\Lambda = dA^\Lambda - \frac{1}{2}f^\Lambda_{AB}\overline{\psi}^A\psi^B + \frac{\mathrm{i}}{2}f^\Lambda_I\overline{\lambda}^I_A\gamma_a\psi^A V^a + \mathrm{i}f^\Lambda_{AB}\overline{\chi}_{(R)}\gamma_a\psi_C\epsilon^{ABC}V^a$$
$$+\ h.c. \tag{4.8}$$
$$\widehat{P}_I{}^A = P_I{}^A - \overline{\lambda}^I_B\psi_C\epsilon^{ABC} - \overline{\lambda}_{I(R)}\psi^A \tag{4.9}$$

where:

$$P^A_I = \frac{1}{2}\epsilon^{ABC}P_{IBC} = \frac{1}{2}\epsilon^{ABC}(L^{-1}\nabla^{(SU(3)\times U(1))}L)_{IBC}$$
$$= P^A_{I,i}dz^i \tag{4.10}$$
$$\overline{P}^{IA} = P_{IA} \tag{4.11}$$

[3] We recall that R_H denotes the representation which the vielbein of the scalar manifold belongs to.

z^i being the (complex) coordinates of G/H and $H = H_{Aut} = SU(3) \times U(1)$. The chiral fermions transformation laws are given by:

$$\begin{aligned}
\delta\psi_A &= D\epsilon_A + 2\mathrm{i}T^-_{AB|ab}\Delta^{abc}V_c\epsilon^B + \cdots && (4.12)\\
\delta\chi_{(L)} &= 1/2T^-_{AB|ab}\gamma^{ab}\epsilon_C\epsilon^{ABC} + \cdots && (4.13)\\
\delta\lambda_{IA} &= -\mathrm{i}P_I{}^B{}_{,i}\partial_a z^i\gamma^a\epsilon^C\epsilon_{ABC} + T_{I|ab}\gamma^{ab}\epsilon_A + \cdots && (4.14)\\
\delta\lambda^I_{(L)} &= \mathrm{i}P_I{}^A{}_{,i}\partial_a z^i\gamma^a\epsilon_A + \cdots && (4.15)
\end{aligned}$$

where T_{AB} and T_I have the general form given in equation (3.48). Therefore, the general form of the dyonic charges (Z_{AB}, Z_I) are given by eqns. (3.51)– (3.53). From the general form of the Maurer-Cartan equations for the embedded coset representatives $U \in Usp(n,n)$, we find:

$$\nabla^{(H)}\begin{pmatrix} f^\Lambda_{AB} \\ h_{\Lambda AB}\end{pmatrix} = \begin{pmatrix} \overline{f}^\Lambda_I \\ \overline{h}_{\Lambda I}\end{pmatrix} P_I{}^C\epsilon_{ABC} \tag{4.16}$$

According to the discussion given in section 3, using (3.51), (3.52) one finds:

$$\begin{aligned}
\nabla^{(H)}Z_{AB} &= \overline{Z}^I P_I{}^C\epsilon_{ABC} && (4.17)\\
\nabla^{(H)}Z_I &= \frac{1}{2}\overline{Z}^{AB}P_I{}^C\epsilon_{ABC} && (4.18)
\end{aligned}$$

and the sum rule:

$$\frac{1}{2}\overline{Z}^{AB}Z_{AB} + Z_I\overline{Z}_I = -\frac{1}{2}P^t\mathcal{M}(\mathcal{N})P \tag{4.19}$$

where the matrix $\mathcal{M}(\mathcal{N})$ has the same form as in equation (3.66) in terms of the kinetic matrix $\mathcal{N}$ of eq.(4.7) and P is the charge vector $P^t = (g, e)$.

- For $N = 4$ [46], the coset space is a product:

$$G/H = \frac{SU(1,1)}{U(1)} \otimes \frac{O(6,n)}{O(6)\otimes O(n)} \tag{4.20}$$

The field content is given by: Gravitational multiplet:

$$(V^a_\mu, \psi_{A\mu}, A^{AB}_\mu, \chi_{ABC}, S) \qquad (A, B = 1, \cdots, 4) \tag{4.21}$$

Vector multiplets:

$$(A_\mu, \lambda^A, 6\phi)^I \qquad (I = 1, \cdots, n) \tag{4.22}$$

The coset representative can be written as:

$$L^\Lambda{}_\Sigma \to M(S)L^\Lambda{}_\Sigma \tag{4.23}$$

where $L^\Lambda{}_\Sigma$ parametrizes the coset manifold $\frac{O(6,n)}{O(6)\otimes O(n)}$ and

$$M(S) = \frac{1}{\sqrt{\frac{4ImS}{1+|S|^2+2ImS}}}\begin{pmatrix} \mathbb{1} & \frac{\mathrm{i}-S}{\mathrm{i}+S} \\ \frac{\mathrm{i}+\overline{S}}{\mathrm{i}-\overline{S}} & \mathbb{1}\end{pmatrix} \tag{4.24}$$

Table 2: $D=4$, $N=4$ transformation properties

	V^a_μ	$\psi_{A\vert\mu}$	A^Λ_μ	χ_{ABC}	λ_{IA}	$M(S)L^\Lambda_{AB}$	$M(S)L^\Lambda_I$	R_H
$SU(1,1)$	1	1	-	1	1	2×1	2×1	-
$O(6,n)$	1	1	$6+n$	1	1	$1\times(6+n)$	$1\times(6+n)$	-
$O(6)$	1	4	1	$\overline{4}$	$\overline{4}$	1×6	1	6
$O(n)$	1	1	1	1	n	1	n	n
$U(1)$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	1	0

The group assignments of the fields are given in Table 2. With the given coset parametrizations the symplectic embedded section $(f^\Lambda_\Sigma, h_{\Lambda\Sigma})$ is (apart from a unessential phase $\frac{\mathrm{i}+S}{\mathrm{i}-\overline{S}}$):

$$
\begin{aligned}
f^\Lambda_\Sigma &= \mathrm{i}e^{\frac{K}{2}}(L^\Lambda{}_{AB}, L^\Lambda{}_I) && (4.25)\\
h_{\Lambda\Sigma} &= \mathrm{i}e^{\frac{K}{2}}(SL^\Gamma{}_{AB}\eta_{\Lambda\Gamma}, \overline{S}L^\Gamma{}_I\eta_{\Lambda\Gamma}) && (4.26)
\end{aligned}
$$

where $K=-\log[\mathrm{i}(S-\overline{S})]$ is the Kähler potential of $\frac{SU(1,1)}{U(1)}$, and the kinetic matrix $\mathcal{N}=hf^{-1}$ takes the form:

$$
\mathcal{N}_{\Lambda\Sigma}=\frac{1}{2}(S-\overline{S})\overline{L}_\Lambda^{AB}L_{\Sigma AB}+\overline{S}\eta_{\Lambda\Sigma} \tag{4.27}
$$

The supercovariant field strengths and the vielbein of the coset manifold are:

$$
\begin{aligned}
\widehat{F}^\Lambda &= dA^\Lambda+[f^\Lambda_{AB}(c_1\overline{\psi}^A\psi^B+c_2\overline{\psi}_C\gamma_a\chi^{ABC}V^a)\\
&+ f^\Lambda_I(c_3\overline{\psi}^A\gamma_a\lambda^I_AV^a+c_4\overline{\chi}^{ABC}\gamma_{ab}\lambda^{ID}\epsilon_{ABCD}V^aV^b)+h.c.] && (4.28)\\
\widehat{P} &= P-\overline{\psi}_A\chi_{BCD}\epsilon^{ABCD} && (4.29)\\
\widehat{P}^I_{AB} &= P^I_{AB}-(\overline{\psi}_A\lambda^I_B+\epsilon_{ABCD}\overline{\psi}^C\lambda^{ID}) && (4.30)\\
& && (4.31)
\end{aligned}
$$

where $P=P_{,S}dS$ and $P^I_{AB}=P^I_{AB,i}d\phi^i$ are the vielbein of $\frac{SU(1,1)}{U(1)}$ and $\frac{O(6,n)}{O(6)\times O(n)}$ respectively. The fermion transformation laws are:

$$
\begin{aligned}
\delta\psi_A &= D\epsilon_A+a_1T^-_{AB|ab}\Delta^{abc}\epsilon^BV_c+\cdots && (4.32)\\
\delta\chi_{ABC} &= a_2P_{,S}\partial_aS\gamma^a\epsilon^D\epsilon_{ABCD}+a_3T^-_{[AB|ab}\gamma^{ab}\epsilon_{C]}+\cdots && (4.33)\\
\delta\lambda^I_A &= a_4P^I_{AB,i}\partial_a\phi^i\gamma^a\epsilon^B+a_5T^{-I}_{ab}\gamma^{ab}\epsilon_A+\cdots && (4.34)
\end{aligned}
$$

where the 2–forms T_{AB} and T_I are defined in eq.(3.48) By integration of these two-forms, using eq.(3.50)–(3.53) we find the central and matter dyonic charges given in eq.s (3.51), (3.52). From the Maurer-Cartan equations for f,h and the definitions of the charges one easily finds:

$$
\begin{aligned}
\nabla^{SU(4)\otimes U(1)}Z_{AB} &= \overline{Z}^IP_{IAB}+\frac{1}{2}\epsilon_{ABCD}\overline{Z}^{CD}P && (4.35)\\
\nabla^{SO(n)}Z_I &= \frac{1}{2}\overline{Z}^{AB}P_{IAB}+Z_I\overline{P} && (4.36)
\end{aligned}
$$

In terms of the kinetic matrix (4.27) the sum rule for the charges is given by eqs.(3.65)–(3.67):

$$\frac{1}{2}Z_{AB}\overline{Z}^{AB} + Z_I\overline{Z}_I = -\frac{1}{2}P^t\mathcal{M}(\mathcal{N})P \qquad (4.37)$$

For $N > 4$ the only available supermultiplet is the gravitational one, so that $H_{matter} = \mathbb{1}$. The embedding procedure is much simpler than in the matter coupled supergravities since for each $N > 4$ there exists a representation of the scalar manifold isometry group G given in terms of $Usp(n_v, n_v)$ matrices.

- For the $N = 5$ theory [48] the coset manifold is:

$$G/H = \frac{SU(1,5)}{U(5)} \qquad (4.38)$$

The field content and the group assignments are displayed in table 3. Here $x, y, \cdots =$

Table 3: Transformation properties of fields in $D = 4$, $N = 5$

	V^a	ψ_A;	χ_{ABC}, χ_L	$A^{\Lambda\Sigma}$	L^x_A	R_H
$SU(1,5)$	1	1	1	-	6	-
$SU(5)$	1	5	(10, 1)	1	5	$\overline{5}$
$U(1)$	0	$\frac{1}{2}$	$(\frac{3}{2}, -\frac{5}{2})$	0	1	2

$1, \cdots, 6$ and $A, B, C \cdots = 1, \cdots, 5$ are indices of the fundamental representations of $SU(1,5)$ and $SU(5)$,respectively. L^x_A denote as usual the coset representative in the fundamental representation of $SU(1,5)$. The antisymmetric couple $\Lambda\Sigma$, $\Lambda, \Sigma = 1, \cdots, 5$, enumerates the ten vectors. The embedding of $SU(1,5)$ into the Gaillard-Zumino group $Usp(10,10)$ is given in terms of the three-times antisymmetric representation of $SU(1,5)$, a generic element t^{xyz} satisfying:

$$t^{xyz} = \frac{1}{3!}\epsilon^{xyzuvw}t_{uvw} \qquad (4.39)$$

We may decompose t^{xyz} as follows:

$$t^{xyz} = \begin{pmatrix} t^{\Lambda\Sigma 6} \\ t^{\Lambda\Sigma\Gamma} = \epsilon^{\Lambda\Sigma\Gamma\Delta\Pi 6}t_{\Delta\Pi 6} \end{pmatrix} \qquad (\Lambda, \Sigma, \cdots = 1, \cdots, 5) \qquad (4.40)$$

In the following we write $t^{\Lambda\Sigma 6} \equiv t^{\Lambda\Sigma}$. The 20 dimensional vector $(F^{\mp\Lambda\Sigma}, \mathcal{G}^{\mp}_{\Lambda\Sigma})$ transforms under $Sp(20, \mathbb{R})$, as well as, for fixed AB, each of the 20– dimensional vectors $(f^{\Lambda\Sigma}_{AB}, h_{\Lambda\Sigma AB})$ of the embedding matrix:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{pmatrix} \qquad (4.41)$$

The supercovariant field-strengths and vielbein are:

$$\widehat{F}^{\Lambda\Sigma} = dA^{\Lambda\Sigma} + (f^{\Lambda\Sigma}{}_{AB}(a_1\overline{\psi}^A\psi^B + a_2\overline{\psi}_C\gamma_a\chi^{ABC}V^a) + h.c.) \qquad (4.42)$$

$$\widehat{P}_{ABCD} = P_{ABCD} - \overline{\chi}_{[ABC}\psi_{D]} - \epsilon_{ABCDE}\overline{\chi}^{(R)}\psi^E \qquad (4.43)$$

where $P_{ABCD} = \epsilon_{ABCDF} P^F$ is the complex vielbein, completely antisymmetric in $SU(5)$ indices and $(P_{ABCD})^\star = \overline{P}^{ABCD}$.
The fermion transformation laws are:

$$\begin{aligned}
\delta\psi_A &= D\epsilon_A + a_3 T^-_{AB|ab}\Delta^{abc}\epsilon^B V_c + \cdots && (4.44)\\
\delta\chi_{ABC} &= a_4 P_{ABCD,i}\partial_a\phi^i\gamma^a\epsilon^D + a_5 T^-_{[AB|ab}\gamma^{ab}\epsilon_{C]} + \cdots && (4.45)\\
\delta\chi_{(L)} &= a_6 \overline{P}^{ABCD}_{,\bar{\imath}}\partial_a\overline{\phi}^{\bar{\imath}}\gamma^a\epsilon^E\epsilon_{ABCDE} + \cdots && (4.46)
\end{aligned}$$

where:

$$\begin{aligned}
T_{AB} &= -\frac{\mathrm{i}}{2}(\overline{f}^{-1})_{\Lambda\Sigma AB}F^{\Lambda\Sigma} = \frac{1}{4}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma,\Gamma\Delta} f^{\Gamma\Delta}{}_{AB}F^{\Lambda\Sigma}\\
&= \frac{1}{2}(h_{\Lambda\Sigma AB}F^{\Lambda\Sigma} - f^{\Lambda\Sigma}{}_{AB}\mathcal{G}_{\Lambda\Sigma}) && (4.47)\\
\mathcal{N}_{\Lambda\Sigma,\Delta\Pi} &= \frac{1}{2}h_{\Lambda\Sigma|AB}(f^{-1})^{AB}{}_{\Delta\Pi} && (4.48)\\
\mathcal{G}^{\pm}_{\Lambda\Sigma} &= -\mathrm{i}/2\frac{\partial\mathcal{L}}{\partial F^{\pm\Lambda\Sigma}}; \quad \mathcal{G}_{\Lambda\Sigma} = \mathcal{G}^+_{\Lambda\Sigma} + \mathcal{G}^-_{\Lambda\Sigma} && (4.49)
\end{aligned}$$

With a by now familiar procedure one finds the following (complex) central charges:

$$Z_{AB} = \frac{1}{2}(h_{\Lambda\Sigma|AB}g^{\Lambda\Sigma} - f^{\Lambda\Sigma}{}_{AB}e_{\Lambda\Sigma}) \tag{4.50}$$

where:

$$\begin{aligned}
g^{\Lambda\Sigma} &= \int_{S^2} F^{\Lambda\Sigma} && (4.51)\\
e_{\Lambda\Sigma} &= \int_{S^2} \mathcal{G}_{\Lambda\Sigma} && (4.52)
\end{aligned}$$

From the Maurer–Cartan equation

$$\nabla^{(U(5))} f^{\Lambda\Sigma} \quad AB = \frac{1}{2}\overline{f}^{\Lambda\Sigma|CD} P_{ABCD} \tag{4.53}$$

and the analogous one for h we find:

$$\nabla^{(U(5))} Z_{AB} = \frac{1}{2}\overline{Z}^{CD} P_{ABCD} \tag{4.54}$$

Finally, the sum rule for the central charges is:

$$\frac{1}{2}Z_{AB}\overline{Z}^{AB} = -\frac{1}{2}(g^{\Lambda\Sigma}, e_{\Lambda\Sigma})\mathcal{M}(\mathcal{N})_{\Lambda\Sigma,\Gamma\Delta}\begin{pmatrix} g^{\Gamma\Delta}\\ e_{\Gamma\Delta}\end{pmatrix} \tag{4.55}$$

where the matrix $\mathcal{M}(\mathcal{N})$ has exactly the same form as in eq (3.66).

- The scalar manifold of the $N = 6$ theory has the coset structure:

$$G/H = \frac{SO^\star(12)}{U(6)} \tag{4.56}$$

Table 4: Transformation properties of fields in $D=4$, $N=6$

	V^a	ψ_A	χ_{ABC}, χ_A	A^Λ	S_r^α	R_H
$SO^\star(12)$	1	1	1	-	32	-
$SU(6)$	1	6	$(20+6)$	1	$(15,1)+(\overline{15},\overline{1})$	$\overline{15}$
$U(1)$	0	$\frac{1}{2}$	$(\frac{3}{2},-\frac{5}{2})$	0	$(1,-3)+(-1,3)$	2

We recall that $SO^\star(2n)$ is defined as the subgroup of $O(2n,\mathbb{C})$ that preserves the sesquilinear antisymmetric metric:

$$L^\dagger C L = C, \qquad C = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} \tag{4.57}$$

The field content and transformation properties are given in Table 4, where $A,B,C = 1,\cdots,6$ are $SU(6)$ indices in the fundamental representation and $\Lambda = 1,\cdots,16$. As it happens in the $N=5$ theory, the $\underline{32}$ spinor representation of $SO^\star(12)$ can be given in terms of a $Usp(16,16)$ matrix, which we denote by $S_r^\alpha(\alpha, r = 1,\cdots,32)$, so that the embedding is automatically realized in terms of the spinor representation. Employing the usual notation we may set:

$$S_r^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} f_I^\Lambda + \mathrm{i}h_{\Lambda I} & \overline{f}_I^\Lambda + \mathrm{i}\overline{h}_{\Lambda I} \\ f_I^\Lambda - \mathrm{i}h_{\Lambda I} & \overline{f}_I^\Lambda - \mathrm{i}\overline{h}_{\Lambda I} \end{pmatrix} \tag{4.58}$$

where $\Lambda, I = 1,\cdots,16$. With respect to $SU(6)$,the sixteen symplectic vectors $(f_I^\Lambda, h_{\Lambda I})$, $(I = 1,\cdots,16)$ are reducible into the antisymmetric 15– dimensional representation plus a singlet of $SU(6)$:

$$(f_I^\Lambda, h_{\Lambda I}) \to (f_{AB}^\Lambda, h_{\Lambda AB}) + (f^\Lambda, h_\Lambda) \tag{4.59}$$

It is precisely the existence of a $SU(6)$ singlet which allows for the Special Geometry structure of $\frac{SO^*(12)}{U(6)}$ as discussed in section 3.2 Note that the coset element S_r^α has no definite $U(1)$ weight since the submatrices $f_{AB}^\Lambda, f^\Lambda$ have the weights 1 and -3 respectively. The supercovariant field-strenghts and the coset manifold vielbein have the following expression:

$$\begin{aligned} \widehat{F}^\Lambda &= dA^\Lambda + [f^\Lambda_{AB}(a_1\overline{\psi}^A\psi^B + a_2\overline{\psi}_C\gamma_a\chi^{ABC}V^a) \\ &+ a_3 f^\Lambda\overline{\psi}_C\gamma_a\chi^C V^a + h.c.] \end{aligned} \tag{4.60}$$

$$\widehat{P}_{ABCD} = P_{ABCD} - \overline{\chi}_{[ABC}\psi_{D]} - \epsilon_{ABCDEF}\overline{\chi}^E\psi^F \tag{4.61}$$

where $P_{ABCD} = P_{ABCD,i}dz^i$ is the Kähler vielbein of the coset. The fermion transformation laws are:

$$\delta\psi_A = D\epsilon_A + b_1 T^-_{AB|ab}\Delta^{abc}\epsilon^B V_c + \cdots \tag{4.62}$$

$$\delta\chi_{ABC} = b_2 P_{ABCD|i}\partial_a z^i\gamma^a\epsilon^D + b_3 T^-_{[AB|ab}\gamma^{ab}\epsilon_{C]} + \cdots \tag{4.63}$$

$$\delta\chi_A = b_4 P^{BCDE}_{,i}\partial_a z^i\gamma^a\epsilon^F\epsilon_{ABCDEF} + b_5 T^-_{ab}\gamma^{ab}\epsilon_A + \cdots \tag{4.64}$$

where:

$$T_{AB} = -\mathrm{i}(\overline{f}^{-1})_{\Lambda AB} F^{-\Lambda} \quad (4.65)$$
$$T = -\mathrm{i}(\overline{f}^{-1})_{\Lambda} F^{-\Lambda} \quad (4.66)$$

With the usual procedure we have the following complex dyonic central charges:

$$Z_{AB} = h_{\Lambda AB} g^{\Lambda} - f^{\Lambda}_{AB} e_{\Lambda} \quad (4.67)$$
$$Z = h_{\Lambda} g^{\Lambda} - f^{\Lambda} e_{\Lambda} \quad (4.68)$$

in the 15 and singlet representation of $SU(6)$ respectively. Notice that although we have 16 graviphotons, only 15 central charges are present in the supersymmetry algebra. The singlet charge plays a role analogous to a "matter" charge. From the Maurer–Cartan equations:

$$\nabla f^{\Lambda}{}_{AB} = \frac{1}{2}\overline{f}^{\Lambda|CD} P_{ABCD} + \frac{1}{4!}\overline{f}^{\Lambda} \epsilon_{ABCDEF} P^{CDEF} \quad (4.69)$$
$$\nabla f^{\Lambda} = \frac{1}{2!4!} f^{\Lambda|AB} \epsilon_{ABCDEF} P^{CDEF} \quad (4.70)$$

and the relation (3.42) one finds:

$$\nabla^{(U(6))} Z_{AB} = \frac{1}{2}\overline{Z}^{CD} P_{ABCD} + \frac{1}{4!}\overline{Z} \epsilon_{ABCDEF} P^{CDEF} \quad (4.71)$$
$$\nabla^{(U(1))} Z = \frac{1}{2!4!}\overline{Z}^{AB} \epsilon_{ABCDEF} P^{CDEF} \quad (4.72)$$

and the sum-rule (3.65):

$$\frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z\overline{Z} = -\frac{1}{2}\left(g^{\Lambda}, \quad e_{\Lambda} \right) \mathcal{M}(\mathcal{N})_{\Lambda\Sigma} \begin{pmatrix} g^{\Sigma} \\ e_{\Sigma} \end{pmatrix} \quad (4.73)$$

with the usual meaning for $\mathcal{M}(\mathcal{N})$ (see eq.(3.66)).

- In the $N = 8$ case [49] the coset manifold is:

$$G/H = \frac{E_{7(-7)}}{SU(8)}. \quad (4.74)$$

The field content and group assignments are given in the following Table 5:

Table 5: Field content and group assignments in $D = 4$, $N = 8$ supergravity

	V^a_μ	ψ_A	$A^{\Lambda\Sigma}_\mu$	χ_{ABC}	S^α_r	R_H
$E_{7(-7)}$	1	1	-	1	56	-
$SU(8)$	1	8	1	56	$28 + \overline{28}$	70

As in $N=5,6$, the embedding is automatically realized in terms of the 56 defining representation for E_7 which belongs to $Usp(28,28)$ and it is given by the usual coset element (3.38) where

$$f + \mathrm{i}h \equiv f^{\Lambda\Sigma}{}_{AB} + \mathrm{i}h_{\Lambda\Sigma AB} \tag{4.75}$$
$$\overline{f} - \mathrm{i}\overline{h} \equiv \overline{f}^{\Lambda\Sigma AB} - \mathrm{i}\overline{h}_{\Lambda\Sigma}{}^{AB} \tag{4.76}$$

$\Lambda\Sigma, AB$ are couples of antisymmetric indices, with Λ, Σ, A, B running from 1 to 8 . The supercovariant field-strengths and coset manifold vielbein are:

$$\widehat{F}^{\Lambda\Sigma} = dA^{\Lambda\Sigma} + [f^{\Lambda\Sigma}_{AB}(a_1\overline{\psi}^A\psi^B + a_2\overline{\chi}^{ABC}\gamma_a\psi_C V^a) + h.c.] \tag{4.77}$$
$$\widehat{P}_{ABCD} = P_{ABCD} - \overline{\chi}_{[ABC}\psi_{D]} + h.c. \tag{4.78}$$

where $P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}\overline{P}^{EFGH} \equiv (L^{-1}\nabla^{SU(8)}L)_{AB|CD} = P_{ABCD,i}d\phi^i$ (ϕ^i coordinates of G/H). The fermion transformation laws are given by:

$$\delta\psi_A = D\epsilon_A + a_3 T^-_{AB|ab}\Delta^{abc}\epsilon^B V_c + \cdots \tag{4.79}$$
$$\delta\chi_{ABC} = a_4 P_{ABCD,i}\partial_a\phi^i\gamma^a\epsilon^D + a_5 T^-_{[AB|ab}\gamma^{ab}\epsilon_{C]} + \cdots \tag{4.80}$$

where:

$$\begin{aligned} T_{AB} &= -\frac{\mathrm{i}}{2}(\overline{f}^{-1})_{\Lambda\Sigma AB}F^{\Lambda\Sigma} = \frac{1}{4}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma,\Gamma\Delta}f^{\Lambda\Sigma}_{AB}F^{\Gamma\Delta} \\ &= \frac{1}{2}(h_{\Lambda\Sigma AB}F^{\Lambda\Sigma} - f^{\Lambda\Sigma}_{AB}\mathcal{G}_{\Lambda\Sigma}) \end{aligned} \tag{4.81}$$

with:

$$\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta} = \frac{1}{2}h_{\Lambda\Sigma AB}(f^{-1})^{AB}{}_{\Gamma\Delta} \tag{4.82}$$
$$\mathcal{G}_{\Lambda\Sigma} = -\mathrm{i}/2\frac{\partial\mathcal{L}}{\partial F^{\Lambda\Sigma}} \tag{4.83}$$

With the usual manipulations we obtain the central charges:

$$Z_{AB} = \frac{1}{2}(h_{\Lambda\Sigma AB}g^{\Lambda\Sigma} - f^{\Lambda\Sigma}{}_{AB}e_{\Lambda\Sigma}), \tag{4.84}$$

the differential relations:

$$\nabla^{SU(8)}Z{}_{AB} = \frac{1}{2}\overline{Z}{}^{CD}P_{ABCD} \tag{4.85}$$

and the sum rule:

$$\frac{1}{2}Z_{AB}\overline{Z}^{AB} = -\frac{1}{8}(g^{\Lambda\Sigma}, e_{\Lambda\Sigma})\mathcal{M}(\mathcal{N})_{\Lambda\Sigma,\Gamma\Delta}\begin{pmatrix} g^{\Gamma\Delta} \\ e_{\Gamma\Delta} \end{pmatrix} \tag{4.86}$$

5 Matter coupled higher dimensional supergravities

With the exception of $D = 5$, $N = 2$ supergravity in which the vector multiplets moduli space is described by Very Special Geometry [50][51], all the higher dimensional supergravities exhibit a coset structure G/H as in $D = 4$, $N \geq 3$. As we are going to see, their structure is completely fixed in terms of the coset representative L and in particular the "dressed" central and matter charges satisfy relations quite analogous to those discussed in the four dimensional case for $N \geq 2$.

In this section we discuss the matter coupled supergravities deferring the maximally extended cases to next section.

We begin by considering all the matter coupled supergravities which have the coset structure:

$$G/H = \frac{O(10-D, n)}{O(10-D) \otimes O(n)} \times O(1,1) \tag{5.1}$$

Actually this class covers all the cases except $D = 6$, $N = (2, 0)$ and $D = 6$, type IIB.

It is convenient to treat separately the odd and the even dimensional cases.

5.1 Odd dimensional theories

The three cases $D = 5, 7, 9$ can be discussed at the same time [52], [53], [54], [55], [56], [57] if one takes in $D = 5$ one of the vectors dualized with a two form and disregard the presence of hypermultiplets. With these assumptions the field content for all the three theories is given by:
Gravitational multiplet:

$$(V^a_\mu, \psi_{A\mu}, B_{\mu\nu}, A^\alpha_\mu, \chi_A, \sigma) \qquad (\mu = 1, \cdots, D) \tag{5.2}$$

Vector multiplet:

$$(A_\mu, \lambda_A, (10-D)\phi)^I \qquad (I = 1, \cdots, n) \tag{5.3}$$

where α runs from 1 to $10 - D$, the coset manifold of the scalar fields being given in equation (5.1).

The transformation properties of the fields are given in Table 6, where:

Table 6: Transformation properties of fields in matter coupled $D = 5$ ($N = 4$), $D = 7$ ($N = 2$), $D = 9$ ($N = 1$)

	V^a_μ	$H_{\mu\nu\rho}$	$F^\Lambda_{\mu\nu}$	$L^\Lambda_{\ \Sigma}$	e^σ	ψ^A_μ	χ^A	λ^{IA}	R_H
$O(10-D,n)$	1	1	$10-D+n$	$10-D+n$	1	1	1	1	-
$O(1,1)$	0	2	1	0	1	0	0	0	-
H_{Aut}	1	1	1	$10-D$	0	N	N	N	$10-D$
H_{matter}	1	1	1	n	0	1	1	n	n

$$H_{Aut} \times H_{matter} = O(10-D) \times O(n). \tag{5.4}$$

H_{Aut} acts on the index $A = 1, \cdots, N$ of the spinors as a unitary symplectic group in $D = 5, 7$ according to:

$$\begin{aligned} O(10-D) &\sim Usp(4) \qquad & D=5 \\ &\sim Usp(2) \qquad & D=7. \end{aligned} \tag{5.5}$$

In $D = 9$ $H_{Aut} \equiv 1\!\!1$. H_{matter} always acts in the vector representation labelled by the index I. The coset representative of (5.1) is:

$$e^\sigma L^\Lambda{}_\Sigma \equiv e^\sigma (L^\Lambda{}_{AB}, L^\Lambda{}_I) \tag{5.6}$$

where e^σ parametrizes $O(1,1)$, σ being the real scalar field of the gravitational multiplet and $L^\Lambda{}_\Sigma$ is the representative of $\frac{O(10-D,n)}{O(10-D)\otimes O(n)}$. The indices AB of $L^\Lambda{}_{AB}$ are $Usp(N) \equiv H_{(Aut)}$ antisymmetric indices for $D = 5$ ($N = 4$) and symmetric indices for $D = 7$ ($N = 2$) intertwining between the vector representation of $O(10-D)$ and the representation of $Usp(N)$:

$$L^\Lambda{}_{AB} = L^\Lambda_r \gamma^r_{AB} \tag{5.7}$$

where γ^r_{AB} are gamma–matrices of $O(10-D)$. For $D = 9$ $L^\Lambda{}_{AB} \to L^\Lambda \delta_{AB} = L^\Lambda$ (A,B=1). The index I of L^Λ_I is an index of $O(n) = H_{(matter)}$ in the vector representation. As usual, the set of n_g gravitational and n_v matter field–strengths F^Λ of the vectors (A^α, A^I) $(\Lambda = 1, \cdots, n_g + n_v)$ transform among themselves under $O(10-D,n) \times O(1,1)$ while $H \equiv dB$ is charged under $O(1,1)$ only (we have labelled the $O(1,1)$ representations by the "charge" under the shift $e^\sigma \to e^{\sigma+c}$).

The supercovariant field–strengths contain, as in 4 dimensions, the coset representatives, but in the defining representation of G. We have [4]:

$$\begin{aligned} \widehat{H} &= dB + a_1 \eta_{\Lambda\Sigma} dA^\Lambda \wedge A^\Sigma + a_2 e^{2\sigma} \overline{\psi}^A \Gamma_a \psi^B \mathbb{C}_{AB} V^a \\ &+ a_3 e^{2\sigma} \overline{\psi}^A \Gamma_{ab} \chi^B \mathbb{C}_{AB} V^a V^b \end{aligned} \tag{5.8}$$

$$\begin{aligned} \widehat{F}^\Lambda &= dA^\Lambda + b_1 e^\sigma \overline{\psi}^A \psi^B L^\Lambda{}_{AB} + b_2 e^\sigma \overline{\psi}^A \Gamma_a \chi^B V^a L^\Lambda{}_{AB} \\ &+ b_3 e^\sigma \overline{\psi}^A \Gamma_a \lambda^I_A V^a L^\Lambda{}_I \end{aligned} \tag{5.9}$$

$$\widehat{P}^I_r = (\gamma_r)^{AB} \widehat{P}^I_{AB} = (\gamma_r)^{AB} (P^I_{AB} - \overline{\psi}_A \lambda^I_B) \tag{5.10}$$

$$\widehat{d\sigma} = d\sigma - \overline{\psi}^A \chi_A \tag{5.11}$$

$$\tag{5.12}$$

where $\mathbb{C}_{AB}$ is the invariant metric of $Sp(N)$ in $D = 5, 7$ while in $D = 9$ we set $\mathbb{C}_{AB} \equiv \delta_{AB}$, $A, B \equiv 1$. $P^I_{AB} = P^I_{AB,i} d\phi^i$ and $d\sigma$ are the vielbein 1–forms of $\frac{O(10-D,n)}{O(10-D)\otimes O(n)}$ and of $O(1,1)$ respectively.

The supercovariant field strengths appear in the transformation laws of the fermions dressed with coset representatives:

$$\delta\psi_A = D\epsilon_A + c_1 T_{AB|ab} \Delta^{abc} \epsilon^B V_c + c_2 T_{|abc} \Delta^{abcd} \epsilon_A V_d + \cdots \tag{5.13}$$

$$\delta\chi_A = d_1 \Gamma^a \partial_a \sigma \epsilon_A + d_2 T_{AB|ab} \Gamma^{ab} \epsilon^B + d_3 T_{|abc} \Gamma^{abc} \epsilon_A + \cdots \tag{5.14}$$

$$\delta\lambda^I_A = f_1 \Gamma^a P^I_{AB,i} \partial_a \phi^i \epsilon^B + f_2 T^I_{|ab} \Gamma^{ab} \epsilon_A + \cdots \tag{5.15}$$

[4]In all higher dimensional theories the space–time γ–matrices are denoted as Γ^a ($a = 0, \cdots, D-1$).

where $\Delta_{aa_1\cdots a_n}$ is given in equation (3.5), ϕ^i are the coordinates of $\frac{O(10-D,n)}{O(10-D)\otimes O(n)}$ and

$$\begin{array}{rclcl} T_{AB} & = & e^{\sigma}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}{}_{AB}F^{\Lambda} & \equiv & e^{-\sigma}L_{\Lambda AB}F^{\Lambda} \\ T_I & = & e^{\sigma}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}{}_{I}F^{\Lambda} & \equiv & e^{-\sigma}L_{\Lambda I}F^{\Lambda} \\ T & = & \mathcal{N}^{(H)}e^{2\sigma}H & \equiv & e^{-2\sigma}H \end{array} \tag{5.16}$$

are the dressed graviphotons, matter vectors and 3–form. Here the vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ and the 3–form metric $\mathcal{N}^{(H)}$ are:

$$\mathcal{N}_{\Lambda\Sigma} = e^{-2\sigma}(L_{AB\Lambda}L^{AB}_{\Sigma} + L_{I\Lambda}L_{I\Sigma}) \tag{5.17}$$

$$\mathcal{N}^{(H)} = e^{-4\sigma} \tag{5.18}$$

and we have used the pseudo–orthogonality relations of $O(10-D,n)$:

$$\eta_{\Lambda\Sigma} = (L_{\Lambda AB}L^{AB}_{\Sigma} - L_{\Lambda I}L_{\Sigma I}) \tag{5.19}$$

where η is the $O(10-D,n)$ invariant metric. At this point it is very easy to construct the dressed charges for the gravitational and matter multiplets. Defining the magnetic charge as:

$$g^{\Lambda} = \int_{S^2} F^{\Lambda} \tag{5.20}$$

$$g = \int_{S^3} H \tag{5.21}$$

we have the vector central charge:

$$Z_{AB} = \int_{S^2} T_{AB} = e^{-\sigma}L_{\Lambda AB}(\phi)g^{\Lambda} \tag{5.22}$$

the 3–form central charge:

$$Z = \int_{S^3} T = e^{-2\sigma}g \tag{5.23}$$

and the gaugino matter charge:

$$Z_I = \int_{S^2} T_I = e^{-\sigma}L_{\Lambda I}(\phi)g^{\Lambda} \tag{5.24}$$

where on the r.h.s. the scalar fields σ, ϕ^i are understood as $v.e.v.$'s on a given background. From the Maurer Cartan equation for the coset representative $e^{\sigma}L^{\Lambda}{}_{\Sigma}$ we now have:

$$\nabla(e^{\sigma}L^{\Lambda}{}_{AB}) = e^{\sigma}L^{\Lambda}{}_{I}P^{I}{}_{AB} \tag{5.25}$$

where ∇ denotes the derivative covariant under the composite connection of $H_{Aut} \times H_{matter} \equiv O(10-D)\otimes O(n)$. It follows:

$$\nabla_i Z_{AB} = Z_I P^{I}{}_{AB,i} \tag{5.26}$$

$$\nabla Z_I = \frac{1}{4}(P^{AB}_{I}Z_{AB} + \overline{P}_{IAB}\overline{Z}^{AB}) - Z_I d\sigma \tag{5.27}$$

In particular, if $Z_I = 0$, the invariant

$$\mathcal{M}(\phi) = \frac{1}{2}Z_{AB}\overline{Z}^{AB} \tag{5.28}$$

reaches a minimum.
Furthermore, using eq.s (5.17) and (5.19) one also finds:

$$\frac{1}{2}Z_{AB}\overline{Z}^{AB} + Z_I Z_I = g^\Lambda \mathcal{N}_{\Lambda\Sigma} g^\Sigma \tag{5.29}$$

$$\frac{1}{2}Z_{AB}\overline{Z}^{AB} - Z_I Z_I = g^\Lambda \eta_{\Lambda\Sigma} g^\Sigma e^{-2\sigma} \tag{5.30}$$

$$\tag{5.31}$$

These "obvious" sum rules are the counterpart of the sum rules found in $N = 2$, $D = 4$ for the central and matter charges.
Finally we observe that the electric charges associated to F^Λ and H are defined as:

$$e_\Lambda = \int \mathcal{N}_{\Lambda\Sigma}{}^\star F^\Sigma \qquad\qquad e = \int e^{-4\sigma}\ {}^\star H \tag{5.32}$$

Let us note that $N = 4$, $D = 5$ is usually formulated dualizing the 3–form $H^{(3)}$ in terms of an extra 2–form $F = dB$

$$dB =^\star He^{-4\sigma} \tag{5.33}$$

In this case, instead of (5.8), we have:

$$\widehat{F} = dB + f_1 e^{-2\sigma}\overline{\psi}^A \psi^B \mathbb{C}_{AB} + f_2 e^{-2\sigma}\overline{\psi}^A \gamma_a \chi^B V^a \mathbb{C}_{AB} \tag{5.34}$$

and the transformation rules of $\delta\psi_A, \delta\chi_A$ are changed accordingly:

$$\delta\psi_A = D\epsilon_A + a(T_{AB|ab}\Delta^{abc}\epsilon^B V_c + bT_{|ab}\Delta^{abc}\mathbb{C}_{AB}\epsilon^B V_c) + \cdots \tag{5.35}$$

$$\delta\chi_A = c\Gamma^a\partial_a\sigma\epsilon_A + d(T_{AB|ab}\Gamma^{ab}\epsilon^B - 2bT_{|ab}\Gamma^{ab}\epsilon_A) + \cdots \tag{5.36}$$

$$\tag{5.37}$$

where $T_{ab} = e^{2\sigma}F_{ab}$. Note that in this case the kinetic matrix for the singlet is:

$$\mathcal{N} = e^{4\sigma} \tag{5.38}$$

We now have that the central charges are:

$$Z_{AB} = e^{-\sigma} g^\Lambda L_{\Lambda AB} - be^{2\sigma} m\mathbb{C}_{AB} \tag{5.39}$$

$$Z = e^{2\sigma} m \tag{5.40}$$

where $m = \int_{S^2} F$, while the matter charges are the same as in (5.24).

Note that the 2–forms T_{AB} and T appear in $\delta\chi_A$ with a relative weight, $-2b$, which is fixed by supersymmetry. By $Usp(4)$–covariant differentiation one obtains:

$$\nabla Z_{AB} = Z^I P_{IAB} - 2Z^{(\chi)}_{AB} d\sigma \tag{5.41}$$

$$\nabla Z = 2Zd\sigma \tag{5.42}$$

$$\nabla Z_I = \frac{1}{4}(Z_{AB}P_I^{AB} + \overline{Z}^{AB}P_{ABI}) - Z_I d\sigma \tag{5.43}$$

where:

$$Z^{(\chi)}_{AB} = \frac{1}{2}(Z_{AB} + 3bZ\mathbb{C}_{AB}) \tag{5.44}$$

The sum rules are:

$$\frac{1}{2}Z_{AB}\overline{Z}^{AB} - 2Z^2 - Z_I Z^I = g^\Lambda \mathcal{N}_{\Lambda\Sigma} g^\Sigma \tag{5.45}$$

$$Z^2 = m^2\mathcal{N} \tag{5.46}$$

5.2 Even dimensional matter coupled theories

- In $D=8$, $N=1$ [58], [59] we have again a coset manifold of the class (5.1), namely:

$$G/H=\frac{O(2,n)}{O(2)\times O(n)}\times O(1,1) \tag{5.47}$$

The field content is:
Gravitational multiplet:

$$(V^a_\mu,\psi_\mu,B_{\mu\nu},A^\alpha_\mu,\chi,\sigma) \qquad (\mu=1,\cdots,8) \tag{5.48}$$

Vector multiplets:

$$(A_\mu,\lambda,2\phi)^M \qquad (M=1,\cdots,n_v) \tag{5.49}$$

where ψ,χ,λ are complex Weyl spinors. The group assignments can be read from Table 7, where the first entries in the last row are the weight under $H_{Aut}=U(1)$.

Table 7: Transformation properties of fields in matter coupled $D=8$ ($N=1$)

	V^a_μ	$H_{\mu\nu\rho}$	$F^\Lambda_{\mu\nu}$	$L^\Lambda_{\ \Sigma}$	e^σ	$\psi_{\mu L}$	χ_L	λ^I_L	R_H
$O(2,n)$	1	1	$2+n$	$2+n$	1	1	1	1	-
$O(1,1)$	0	2	1	0	1	0	0	0	-
$U(1)\times O(n)$	$(0,1)$	$(0,1)$	$(0,1)$	$(2,n)$	$(0,0)$	$(1,1)$	$(-1,1)$	$(1,n)$	$(2,n)$

From the coset representative decomposition:

$$e^\sigma L^\Lambda_\Sigma=e^\sigma(L^\Lambda_A,L^\Lambda_I) \qquad (A=1,2) \tag{5.50}$$

setting $L^\Lambda=L^\Lambda_1+\mathrm{i}L^\Lambda_2$ we have the following supercovariant field–strenghts:

$$\begin{aligned}
\widehat{H} &= dB+a_1\eta_{\Lambda\Sigma}dA^\Lambda\wedge A^\Sigma+(a_2e^{2\sigma}\overline{\psi}_R\Gamma_a\psi_L+h.c.)V^a\\
&+ (a_3e^{2\sigma}\overline{\psi}_L\Gamma_{ab}\chi_L+h.c.)V^aV^b && (5.51)\\
\widehat{F}^\Lambda &= dA^\Lambda+(b_1e^\sigma\overline{\psi}_L\psi_L\overline{L}^\Lambda+h.c.)\\
&+ (b_2e^\sigma\overline{\psi}_L\Gamma_a\chi_RL^\Lambda+h.c.)V^a+(b_3e^\sigma\overline{\psi}_L\Gamma_a\lambda^I_RL^\Lambda{}_I+h.c.)V^a && (5.52)\\
\widehat{P}^I &= P^I-\overline{\psi}_L\lambda^I_L-\overline{\psi}_R\lambda^I_R && (5.53)\\
\widehat{d\sigma} &= d\sigma-\overline{\psi}_L\chi_L-\overline{\psi}_R\chi_R && (5.54)\\
& && (5.55)
\end{aligned}$$

where $P^I\equiv P^I_i dz^i$ is the Kählerian vielbein of $\frac{O(2,n)}{O(2)\times O(n)}$ and $d\sigma$ the einbein of $O(1,1)$. We have now a complex graviphoton:

$$T^{(2)}=e^{-\sigma}L_\Lambda F^\Lambda \tag{5.56}$$

while the dressed vector matter and tensor charges are:

$$\begin{aligned}
T^I &= L_\Lambda{}^I F^\Lambda && (5.57)\\
T^{(3)} &= e^{-2\sigma}H && (5.58)
\end{aligned}$$

$T^{(2)}$,$T^{(3)}$ and T^I appear in the susy transformation laws of the fermions:

$$\delta\psi_L = D\epsilon_L + c_1 T^{(2)}_{ab}\Delta^{abc}\epsilon_R V_c + c_2 T^{(3)}_{abc}\Delta^{abcd}\epsilon_L V_d + \cdots \quad (5.59)$$
$$\delta\chi_L = d_1\Gamma^a\partial_a\sigma\epsilon_R + d_2 T^{(2)}_{ab}\Gamma^{ab}\epsilon_R + d_3 T^{(3)}_{abc}\Gamma^{abc}\epsilon_L + \cdots \quad (5.60)$$
$$\delta\lambda^I_L = f_1\Gamma^a P^I_i\partial_a z^i\epsilon_R + f_2 T^I_{ab}\Gamma^{ab}\epsilon_L + \cdots \quad (5.61)$$

Taking into account:

$$\eta_{\Lambda\Sigma} = L_{(\Lambda}\overline{L}_{\Sigma)} - L_{\Lambda I}L_{\Sigma I} \quad (5.62)$$
$$\mathcal{N}_{\Lambda\Sigma} = \left(L_{(\Lambda}\overline{L}_{\Sigma)} + L_{\Lambda I}L_{\Sigma I}\right) e^{-2\sigma} \quad (5.63)$$

we find:

$$Z^{(2)} = e^{-\sigma}L_\Lambda(z)g^\Lambda \quad (5.64)$$
$$Z^{(3)} = e^{-2\sigma}g \quad (5.65)$$
$$Z_I = L_{\Lambda I}(z)g^\Lambda \quad (5.66)$$

where

$$g = \int_{S^3} H \qquad g^\Lambda = \int_{S^2} F^\Lambda \quad (5.67)$$

The differential relations and sum rules among the charges are:

$$\nabla Z^{(2)} = Z_I P^I - Z^{(2)}d\sigma \quad (5.68)$$
$$\nabla Z^{(3)} = -2Z^{(3)}d\sigma \quad (5.69)$$
$$\nabla Z_I = \frac{1}{2}(\overline{Z}P_I + Z\overline{P}_I) \quad (5.70)$$
$$Z^{(2)}\overline{Z}^{(2)} - Z_I Z_I = g^\Lambda\eta_{\Lambda\Sigma}g^\Sigma \quad (5.71)$$
$$Z^{(2)}\overline{Z}^{(2)} + Z_I Z_I = g^\Lambda\mathcal{N}_{\Lambda\Sigma}g^\Sigma \quad (5.72)$$

Let us now consider the $D = 6$ matter coupled theories.

- For type IIA [60], [61] ($(N_+, N_-) = (2,2)$) the coset manifold is

$$G/H = \frac{O(4,n)}{O(4)\times O(n)} \times O(1,1) \quad (5.73)$$

and the field content is:
Gravitational multiplet:

$$(V^a_\mu, \psi_{A\mu}, \psi_{\dot{A}\mu}, B_{\mu\nu}, A^\alpha_\mu, \chi_A, \chi_{\dot{A}}, \sigma) \qquad (\alpha = 1, \cdots, 4) \quad (5.74)$$

Vector multiplets:

$$(A_\mu, \lambda_A, \lambda_{\dot{A}}, 4\phi)^i \qquad (i = 1, \cdots, n) \quad (5.75)$$

The group theoretical assignments under the duality group $O(4,n)\times O(1,1)$ can be read from Table 8. In the present case $H_{Aut} = O(4)$ acts as $SU_L(2)\times SU_R(2)$ on the

Table 8: Transformation properties of fields in matter coupled $D=6$ $N=(2,2)$

	V^a_μ	$H_{\mu\nu\rho}$	$F^\Lambda_{\mu\nu}$	$L^\Lambda_{\ \Sigma}$	e^σ	$\psi_{A\mu}$	χ_A	λ^I_A	R_H
$O(4,n)$	1	1	$4+n$	$4+n$	1	1	1	1	-
$O(1,1)$	0	2	1	0	1	0	0	0	-
$SU_L(2)\times SU_R(2)$	$(1,1)$	$(1,1)$	$(1,1)$	4	$(1,1)$	$(2,1)$	$(2,1)$	$(2,1)$	$(2,2)$
$O(n)$	1	1	1	n	1	1	1	n	n

two left–handed and two right–handed fermions. In particular, left–handed spinors transform as doublets under $SU_L(2)$ and as singlets $SU_R(2)$ (the opposite happens for right–handed ones, which are not quoted in the Table). The two chiralities are distinguished by indices $A, \dot{A}$ respectively ($A, \dot{A}=1,2$). $SU_L(2), SU_R(2)$ indices are raised and lowered with the metric ϵ_{AB}, $\epsilon_{\dot{A}\dot{B}}$ respectively. The coset representative is decomposed as follows:

$$e^\sigma L \equiv e^\sigma (L^\Lambda_r, L^\Lambda_{\ I}) \tag{5.76}$$

where $r=1,\cdots,4$ is a vector index of $O(4)$, and we also define

$$L^\Lambda_{\ A\dot{A}} \equiv L^\Lambda_r \mathrm{e}^r_{A\dot{A}} \tag{5.77}$$

where $\mathrm{e}^r_{A\dot{A}}$ is a quaternion converting $O(4)$ into $SU(2)\times SU(2)$ indices. The supercovariant field–strengths have the following form:

$$\begin{aligned} \widehat{H} &= dB + a_1 \eta_{\Lambda\Sigma} dA^\Lambda \wedge A^\Sigma \\ &+ e^{2\sigma}\left(a_2 \overline{\psi}^A \Gamma_a \psi^B \epsilon_{AB} + \overline{a}_2 \overline{\psi}^{\dot{A}} \Gamma_a \psi^{\dot{B}} \epsilon_{\dot{A}\dot{B}}\right) V^a \\ &+ e^{2\sigma}\left(a_3 \overline{\psi}^A \Gamma_{ab} \chi^B \epsilon_{AB} + \overline{a}_3 \overline{\psi}^{\dot{A}} \Gamma_{ab} \chi^{\dot{B}} \epsilon_{\dot{A}\dot{B}}\right) V^a V^b \end{aligned} \tag{5.78}$$

$$\begin{aligned} \widehat{F}^\Lambda &= dA^\Lambda + e^\sigma\left(b_1 \overline{\psi}^A \psi^{\dot{B}} L^\Lambda_{\ A\dot{B}} + \overline{b}_1 \overline{\psi}^{\dot{A}} \psi^B L^\Lambda_{\ \dot{A}B}\right) \\ &+ e^\sigma\left(b_2 \overline{\psi}^A \Gamma_a \chi^{\dot{B}} L^\Lambda_{\ A\dot{B}} + \overline{b}_2 \overline{\psi}^{\dot{A}} \Gamma_a \chi^B L^\Lambda_{\ \dot{A}B}\right) \\ &+ e^\sigma\left(b_3 \overline{\psi}^A \Gamma_a \lambda^I_A + \overline{b}_3 \overline{\psi}^{\dot{A}} \Gamma_a \lambda^I_{\dot{A}}\right) V^a L^\Lambda_{\ I} \end{aligned} \tag{5.79}$$

$$\widehat{P}^{Ir} = P^{Ir} - \left(\overline{\psi}^A \lambda^{I\dot{B}} (e_r)_{A\dot{B}} + h.c.\right) \tag{5.80}$$

$$\widehat{d\sigma} = d\sigma - \left(\overline{\psi}^A \chi_A + h.c.\right) \tag{5.81}$$

where $P^{Ir} \equiv (L^{-1}\nabla^{(H)}L)^{Ir} = P^{Ir}_{,i} d\phi^i$. The supersymmetry transformation laws for the chiral fermions are as follows:

$$\delta\psi_A = D\epsilon_A + c_1 T_{A\dot{B}|ab} \Delta^{abc} \epsilon^{\dot{B}} V_c + c_2 T_{|abc} \Delta^{abcd} \epsilon_A V_d + \cdots \tag{5.82}$$

$$\delta\chi_A = d_1 \Gamma^a \partial_a \sigma \epsilon_A + d_2 T_{A\dot{B}|ab} \Gamma_{ab} \epsilon^{\dot{B}} + d_3 T_{|abc} \Gamma^{abc} \epsilon_A + \cdots \tag{5.83}$$

$$\delta\lambda^I_A = f_1 \Gamma^a P^I_{A\dot{B},i} \partial_a \phi^i \epsilon^{\dot{B}} + f_2 T^I_{|ab} \Gamma^{ab} \epsilon_A + \cdots \tag{5.84}$$

with analogous expressions for the antichiral ones. Here

$$\begin{array}{rcl} T_{A\dot{B}} & = & e^{-\sigma} L_{\Lambda A\dot{B}} F^{\Lambda} \\ T_I & = & e^{-\sigma} L_{\Lambda I} F^{\Lambda} \\ T & = & e^{-2\sigma} H \end{array} \tag{5.85}$$

By integration of the above 2– and 3–forms on S^2 and S^3 respectively we find the central charges. Turning back to the $O(4)$ vector indices we have:

$$Z_a = e^{-\sigma} L_{\Lambda a} g^{\Lambda} \tag{5.86}$$

$$Z = e^{-2\sigma} g \tag{5.87}$$

$$Z_I = e^{-\sigma} L_{\Lambda I} g^{\Lambda} \tag{5.88}$$

Furthermore, from Maurer–Cartan equations and:

$$\mathcal{N}_{\Lambda\Sigma} = e^{-2\sigma}(L_{r\Lambda} L^r{}_{\Sigma} + L_{I\Lambda} L_{I\Sigma}) \tag{5.89}$$

$$\eta_{\Lambda\Sigma} = L_{r\Lambda} L^r{}_{\Sigma} - L_{I\Lambda} L^I{}_{\Sigma} \tag{5.90}$$

we obtain:

$$\nabla Z_r = Z_I P^I_r - Z_r d\sigma \tag{5.91}$$

$$\nabla Z = -2 Z d\sigma \tag{5.92}$$

$$\nabla Z_I = Z^r P_{Ir} - Z_I d\sigma \tag{5.93}$$

$$Z_r Z^r + Z_I Z^I = g^{\Lambda} \mathcal{N}_{\Lambda\Sigma} g^{\Sigma} \tag{5.94}$$

$$Z_r Z^r - Z_I Z^I = g^{\Lambda} \eta_{\Lambda\Sigma} g^{\Sigma} \tag{5.95}$$

The last two cases in $D = 6$ are $N = (2,0)$ and Type IIB ($N = (4,0)$) theories [62].

- For Type IIB the field content is:
  Gravitational multiplet:

$$(V^a_{\mu}, \psi_{A\mu}, 5B^{+}_{\mu\nu}) \qquad (\mu = 1, \cdots, 6\,;\ A = 1, \cdots, 4) \tag{5.96}$$

  Tensor multiplets:

$$(B^{-}_{\mu\nu}, \lambda_A, 5\phi)^I \qquad (I = 1, \cdots, n) \tag{5.97}$$

  and the scalar coset manifold is:

$$G/H = \frac{O(5,n)}{O(5) \times O(n)} \tag{5.98}$$

  The transformation properties of the fields are encoded in the following Table 9 where $H_{Aut} = O(5)$ acts as $Usp(4)$ on the indices A, B of the left–handed gravitino and right–handed spin 1/2 fermions λ^{IA} of the tensor multiplet. $H_{matter} \equiv O(n)$ acts on the indices I. As usual we decompose the coset representative of G/H as follows:

$$L^{\Lambda}{}_{\Sigma} = (L^{\Lambda}_r, L^{\Lambda}_I) \qquad r = 1, \cdots, 5 \tag{5.99}$$

Table 9: Transformation properties of the fields in $D = 6$, $N = (4,0)$

$D = 6$, $N = (4,0)$	V^a_μ	$H^{+\Lambda}_{\mu\nu\rho}, H^{-\Lambda}_{\mu\nu\rho}$	L^Λ_Σ	$\psi_{A\mu}$	λ^{IA}	R_H
$O(5,n)$	1	$5+n$	$5+n$	1	1	-
$O(5)\times O(n)$	$(1,1)$	$(1,1)$	$(5,n)$	$(4,1)$	$(4,n)$	$(5,n)$

and, setting $L^\Lambda_{AB} \equiv L^\Lambda_r \gamma^r_{AB}$ ($\gamma^r_{AB}$are gamma–matrices of $O(5)$), the supercovariant three–form and the vielbein can be written as follows:

$$\widehat{H}^\Lambda = dB^\Lambda + c_1 L^\Lambda_{AB} \overline{\psi}^A \Gamma_a \psi^B V^a + c_2 L^\Lambda_I \overline{\lambda}^I_A \Gamma_{ab} \psi_B \mathbb{C}^{AB} V^a V^b \qquad (5.100)$$

$$\widehat{P}^{Ir} = P^{Ir} - \overline{\psi}^A \lambda^{IB} (\gamma^r)_{AB} \qquad (5.101)$$

where $P^{Ir} = P^{Ir}_{,i} d\phi^i$ is the vielbein of G/H. The transformation laws of the fermions are:

$$\delta\psi_A = D\epsilon_a + a_1 T_{AB|abc} \Delta^{abcd} \mathbb{C}^{BC} \epsilon_C V_d + \cdots \qquad (5.102)$$

$$\delta\lambda^{IA} = P^I_{r,i} \partial_a \phi^i \Gamma^a (\gamma^r)^{AB} \epsilon_B + b_2 T^I_{abc} \Gamma^{abc} \mathbb{C}^{AB} \epsilon_B + \cdots \qquad (5.103)$$

where the 3– form dressed field–strenghts appearing in the gravitino and dilatino transformation laws are:

$$T_{AB} = L_{\Lambda AB}\, H^\Lambda \qquad (5.104)$$

$$T_I = L_{\Lambda I}\, H^\Lambda. \qquad (5.105)$$

We get the following central and matter charges ($Z_r \equiv \frac{1}{8} Z_{AB} \gamma_r^{AB}$):

$$Z_r = L_{\Lambda r} g^\Lambda \qquad (5.106)$$

$$Z_I = L_{\Lambda I} g^\Lambda \qquad (5.107)$$

We note that in this case there is no distinction between magnetic and electric charges. Indeed from the previous definitions it follows:

$$\mathcal{N}_{\Lambda\Sigma} {}^\star H^\Sigma = \eta_{\Lambda\Sigma} H^\Sigma \qquad (5.108)$$

Integrating both sides on a three–sphere we get:

$$e_\Lambda = \eta_{\Lambda\Sigma} g^\Sigma \qquad (5.109)$$

The differential relation derived from the Maurer–Cartan equations and the sum rules are:

$$\nabla Z_r = Z_I P^I_r \qquad (5.110)$$

$$\nabla Z_I = Z^r P_{rI} \qquad (5.111)$$

$$Z_r Z_r + Z_I Z_I = g^\Lambda \mathcal{N}_{\Lambda\Sigma} g^\Sigma \qquad (5.112)$$

$$Z_r Z_r - Z_I Z_I = g^\Lambda \eta_{\Lambda\Sigma} g^\Sigma \qquad (5.113)$$

- Finally, we consider shortly the $N=(2,0)$, $D=6$ theory [63], [64], [32].
The field content is:
Gravitational multiplet:

$$(V^a_\mu, \psi_{A\mu}, B^+_{\mu\nu}) \qquad (\mu = 1,\cdots,6;\ A=1,2) \tag{5.114}$$

Tensor multiplets:

$$(B^-_{\mu\nu}, \chi^A, 5\phi)^I \qquad (I=1,\cdots,n) \tag{5.115}$$

Vector multiplets:

$$(A_\mu, \lambda_A)^\alpha \qquad (\alpha=1,\cdots,m) \tag{5.116}$$

Hypermultiplets:

$$(\xi^A, 4q)^l \qquad (l=1,\cdots,p) \tag{5.117}$$

The coset manifold is in this case:

$$G/H = \frac{O(1,n)}{O(n)} \times \mathcal{Q} \tag{5.118}$$

where $\mathcal{Q}$ is a quaternionic manifold parametrized by the hypermultiplet scalars q. Notice that in this theory the automorphism group coincides with the holonomy factor $SU(2)$ of $\mathcal{Q}$. However, since the hypermultiplets do not enter in the definition of the supersymmetry charges, we forget about them in the following. Table 10 shows the transformation properties of the fields where ψ_A and λ^α_A are chiral while χ^{IA} are antichiral. We set

Table 10: Transformation properties of fields in D=6, N=(2,0)

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D=6, N=(2,0)	V^a_μ	$H^{+\Lambda}_{\mu\nu\rho}, H^{-\Lambda}_{\mu\nu\rho}$	$F^\alpha_{\mu\nu}$	L^Λ_Σ	$\psi_{A\mu}$	χ^{IA}	λ^α_A	R_H
$O(1,n)$	1	$n+1$	1	$n+1$	1	1	1	-
$O(n)$	1	1	1	$(n,1)$	1	n	1	n
$Usp(2)$	1	1	1	1	2	2	2	1

$$L = (L^\Lambda, L^\Lambda_I) \tag{5.119}$$

and write down the supercovariant field–strenghts as:

$$\begin{aligned}\widehat{H}^\Lambda &= dB^\Lambda + a_1 C^\Lambda_{\alpha\beta} dA^\alpha \wedge A^\beta + a_2 L^\Lambda \overline{\psi}^A \Gamma_a \psi_A V^a \\ &+ a_3 L^\Lambda_I \overline{\psi}^A \Gamma_{ab} \chi_A V^a V^b \end{aligned}\tag{5.120}$$

$$\widehat{F}^\alpha = dA^\alpha + b_1 \overline{\psi}^A \Gamma_a \lambda^\alpha_A V^a \tag{5.121}$$

$$\widehat{P}^I = P^I - \overline{\psi}_A \chi^{IA} \tag{5.122}$$

where $P^I = P^I_{,i} d\phi^i$ is the vielbein of G/H, $C^\Lambda_{\alpha\beta}$ are constants and the $SU(2)$ indices $A, B, \cdots = 1,2$ are contracted with ϵ_{AB}. The fermions transformation laws are:

$$\delta\psi_A = D\epsilon_A + b_1 T_{abc}\Gamma^{ab}\epsilon_A V^c + \cdots \tag{5.123}$$

$$\delta\chi^{IA} = b_2 P^I_{,i}\partial_a\phi^i\Gamma^a\epsilon^{AB}\epsilon_B + b_3 T^I_{abc}\Gamma^{abc}\epsilon^{AB}\epsilon_B + \cdots \tag{5.124}$$

$$\delta\lambda^\alpha_A = b_4 F^\alpha_{ab}\Gamma^{ab}\epsilon_A + \cdots \tag{5.125}$$

with:

$$T = L_\Lambda H^{+\Lambda}, \quad T^I = L^I{}_\Lambda H^{-\Lambda} \tag{5.126}$$

Using a by now familiar procedure we may construct the charges associated to three–form $H^\Lambda \equiv (L^\Lambda H^+ + L^\Lambda_I H^{-I})$:

$$\begin{aligned} Z &= L_\Lambda g^\Lambda & (5.127) \\ Z_I &= L_{\Lambda I} g^\Lambda & (5.128) \end{aligned}$$

satisfying:

$$\begin{aligned} \nabla Z &= Z_I P^I & (5.129) \\ \nabla Z_I &= Z P_I & (5.130) \\ ZZ + Z_I Z_I &= g^\Lambda \mathcal{N}_{\Lambda\Sigma} g^\Sigma & (5.131) \\ ZZ - Z_I Z_I &= g^\Lambda \eta_{\Lambda\Sigma} g^\Sigma & (5.132) \end{aligned}$$

As in Type IIB theory there is no distinction between electric and magnetic charges. We have of course also the charges associated to the vector two–form F^α, namely the magnetic charge:

$$m^\alpha = \int_{S^2} F^\alpha \tag{5.133}$$

and the electric charge:

$$e_\alpha = \int_{S^4} L_\Lambda C^\Lambda_{\alpha\beta} \, {}^\star F^\alpha \tag{5.134}$$

since in this case the kinetic matrix for the vector is:

$$\mathcal{N}_{\alpha\beta} = L_\Lambda C^\Lambda_{\alpha\beta}. \tag{5.135}$$

6 Maximally extended supergravities in odd dimensions

The common feature of all maximally extended supergravities is that the relations between central and matter charges are now substituted by relations among central charges only, in the same way as it happens in $D = 4$, $N = 5, 6, 8$. By abuse of language we also include in this section the $D = 5$, $N = 6$ theory which, though not maximally extended, does not admit matter coupling.

- $D = 5$, $N = 6$ [53] Coset manifold:

$$G/H = \frac{SU^\star(6)}{Usp(6)} \tag{6.1}$$

Field group assignments:

where $A, B; \Lambda, \Sigma = 1, \cdots, 6$, the spinors are pseudo-Majorana and the dilatino field χ_{ABC} can be decomposed as follows:

$$\chi_{ABC} = \overset{\circ}{\chi}_{ABC} + \mathbb{C}_{[AB} \chi_{C]} \tag{6.2}$$

Table 11: Transformation properties of fields in $D=5$, $N=6$

	V^a	ψ_A	$F^{\Lambda\Sigma}$	χ_{ABC}	$L^{\Lambda\Sigma}{}_{AB}$	R_H
$SU^\star(6)$	1	1	15	1	15	-
$Usp(6)$	1	6	1	$14' + 6$	$14 + 1$	14

$\overset{\circ}{\chi}_{ABC}$ being the antisymmetric $\mathbb{C}_{AB}$-traceless representation of $Usp(6)$. In $D=5$, $N=6$, we have 15 vectors in the antisymmetric irrep of $SU^*(6)$. We take the coset representatives in the same representation, namely $L^{\Lambda\Sigma}{}_{AB} = L^{[\Lambda}_{[A}L^{\Sigma]}_{B]}$, where L^Λ_A is in the fundamental representation of $SU^*(6)$ and Λ, Σ and A, B are $SU^*(6)$ and $Usp(6)$ indices respectively. Note that with respect to $Usp(6)$ we have:

$$L^{\Lambda\Sigma}{}_{AB} = \overset{\circ}{L}{}^{\Lambda\Sigma}{}_{AB} + \mathbb{C}_{AB} L^{\Lambda\Sigma} \tag{6.3}$$

where $\overset{\circ}{L}{}^{\Lambda\Sigma}{}_{AB}$ is $\mathbb{C}_{AB}$-traceless. The coset representative L^Λ_Σ satisfies:

$$\begin{cases} L^\dagger & = & L \\ \mathbb{C}L^t\mathbb{C}^{-1} & = & L \end{cases} \tag{6.4}$$

The supercovariant vector field-strengths and vielbein are:

$$\widehat{F}^{\Lambda\Sigma} = dA^{\Lambda\Sigma} + \left[L^{\Lambda\Sigma}{}_{AB}(a_1\overline{\psi}^A\psi^B + a_2\overline{\psi}_C\gamma_a\chi^{ABC}V^a) + h.c.\right] \tag{6.5}$$

$$\widehat{P}_{AB} = P_{AB} - \overline{\psi}^C \overset{\circ}{\chi}_{ABC} \tag{6.6}$$

where $P_{AB} = P_{AB,i}d\phi^i$ belongs to the **14** irrep. of $USp(6)$ (that is it is antisymmetric and $\mathbb{C}_{AB}$ traceless). The fermion transformation laws define the physical graviphotons T_{AB}:

$$\delta\psi_A = D\epsilon_A + a_3 T_{AB|ab}\Delta^{abc}\epsilon^B V_c + \cdots \tag{6.7}$$

$$\delta\chi_{ABC} = a_4 P_{[AB}\gamma_a\epsilon_{C]} + a_5\gamma^{ab}T_{ab|[AB}\epsilon_{C]} + \cdots \tag{6.8}$$

where the two-form T_{AB} is given by:

$$T_{AB} = \frac{1}{2}L_{\Lambda\Sigma|AB}F^{\Lambda\Sigma} \tag{6.9}$$

The magnetic central charges are:

$$Z_{AB} = \int_{S^2} T_{AB} = \frac{1}{2}L_{\Lambda\Sigma AB}g^{\Lambda\Sigma} \tag{6.10}$$

and with the usual procedure, setting

$$Z_{AB} = \overset{\circ}{Z}_{AB} + \mathbb{C}_{AB}Z \tag{6.11}$$

we get [5]:

$$\nabla Z = \frac{1}{4}\overset{\circ}{Z}_{AB}P^{AB} \tag{6.13}$$

$$\nabla \overset{\circ}{Z}_{AB} = \overset{\circ}{Z}_{C[A}\mathbb{C}^{CD}P_{B]D} + \frac{1}{6}\mathbb{C}_{AB}\overset{\circ}{Z}_{LM}P^{LM} + \frac{2}{3}ZP_{AB} \tag{6.14}$$

and the sum rule:

$$\frac{1}{4}g^{\Lambda\Sigma}\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}g^{\Gamma\Delta} = \frac{1}{2}Z_{AB}\overline{Z}^{AB} \tag{6.15}$$

where:

$$\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta} = \frac{1}{2}L_{\Lambda\Sigma AB}\overline{L}^{AB}{}_{\Gamma\Delta} \tag{6.16}$$

- $\underline{D = 5, N = 8}$ [65]
  The coset manifold is $E_{6(6)}/Usp(8)$ and the coset representative $L^{\Lambda\Sigma}_{AB}$, antisymmetric and $\mathbb{C}_{AB}$ traceless in the $USp(8)$ indices AB, is taken in the 27×27 defining representation of $E_{6(6)}$. The $Usp(8)$ indices A, B are raised and lowered with the symplectic metric $\mathbb{C}_{AB}$ (*i.e.* $Z_{AB} = \mathbb{C}_{AL}\mathbb{C}_{BM}\overline{Z}^{LM}$). Spinors are symplectic–Majorana.
  The field content and the transformation properties are given in Table 12. As usual

Table 12: Transformation properties of fields in $D = 5$, $N = 8$

	V^a	ψ_A	$F^{\Lambda\Sigma}$	χ_{ABC}	$L^{\Lambda\Sigma}{}_{AB}$	R_H
$E_{6(6)}$	1	1	27	1	27	-
$Usp(8)$	1	8	1	48	$\overline{27}$	42

the coset representatives appear in the supercovariant field strengths as follows:

$$\widehat{F}^{\Lambda\Sigma} = F^{\Lambda\Sigma} + a_1\overline{\psi}^A\psi^B L^{\Lambda\Sigma}{}_{AB} + a_2\overline{\psi}^A\Gamma_a\chi_{ABC}L^{\Lambda\Sigma BC}V^a \tag{6.17}$$

while the vielbein $P_{ABCD} = P_{ABCD,i}d\phi^i$, completely antisymmetric and pseudoreal, is related to its supercovariant part $\widehat{P}_{ABCD}$ as:

$$\widehat{P}_{ABCD} = P_{ABCD} - \overline{\psi}_{[A}\chi_{BCD]} \tag{6.18}$$

The physical graviphotons appear in the transformation laws for the fermions:

$$\delta\psi_A = D\epsilon_A + b_1 T_{AB|ab}\Delta^{abc}\epsilon^B V_c + \cdots \tag{6.19}$$

$$\delta\chi_{ABC} = b_2 P_{ABCD,i}\partial_a\phi^i\Gamma^a\epsilon^D + b_3(T_{[AB|ab}\Gamma^{ab}\epsilon_{C]} - \frac{1}{3}\mathbb{C}_{[AB}T_{C]D|ab}\Gamma^{ab}\epsilon^D) \tag{6.20}$$

where

$$T_{AB} = \frac{1}{2}L_{AB\Lambda\Sigma}F^{\Lambda\Sigma} \tag{6.21}$$

[5]We have adopted the convention, for lowering and raising indices, that:

$$\mathbb{C}_{AB}V^B = V_A; \qquad \mathbb{C}^{AB}V_B = -V^A. \tag{6.12}$$

and the vectors kinetic matrix is:

$$\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} = \frac{1}{2} L_{AB\Lambda\Sigma} L^{AB}{}_{\Gamma\Delta} \equiv \frac{1}{2} L_{AB\Lambda\Sigma} L_{CD\Gamma\Delta} \mathbb{C}^{AC} \mathbb{C}^{BD} \tag{6.22}$$

where $L_{AB\Lambda\Sigma} = (L^{\Lambda\Sigma}{}_{AB})^{-1}$.
Defining the magnetic and electric charges

$$g^{\Lambda\Sigma} = \int_{S^2} F^{\Lambda\Sigma} \quad ; \quad e_{\Lambda\Sigma} = \int_{S^3} \frac{1}{2} \mathcal{N}_{\Lambda\Sigma\Gamma\Delta} F^{\Gamma\Delta} \tag{6.23}$$

the magnetic central charges are:

$$Z_{AB} = \frac{1}{2} L_{\Lambda\Sigma AB}(\phi) g^{\Lambda\Sigma} \tag{6.24}$$

The Maurer Cartan equations for the L's are:

$$dL^{\Lambda\Sigma}{}_{AB} = \frac{1}{2} L^{\Lambda\Sigma}{}_{CD} \Omega^{CD}{}_{AB} + \frac{1}{2} L^{\Lambda\Sigma CD} P_{CDAB} \tag{6.25}$$

where $\Omega^{CD}{}_{AB} = 2Q^{[C}{}_{[A}\delta^{D]}_{B]}$, $Q^C{}_A$ belongs to the Lie algebra of $Usp(8)$. It follows:

$$\nabla^{(Usp(8))} Z_{AB} = \frac{1}{2} \overline{Z}^{CD} P_{CDAB} \tag{6.26}$$

Furthermore, from (6.22) we find

$$\frac{1}{2} Z_{AB} \overline{Z}^{AB} = \frac{1}{4} g^{\Lambda\Sigma} \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} g^{\Gamma\Delta} \tag{6.27}$$

The electric central charges are given by:

$$Z^{(e)}_{AB} = \frac{1}{2} L^{\Lambda\Sigma}{}_{AB}(\phi) e_{\Lambda\Sigma} \tag{6.28}$$

- $D = 7, N = 4$ [66]
  The coset space is:

$$G/H = \frac{Sl(5, \mathbb{R})}{O(5)} \tag{6.29}$$

and the field content with the transformation properties under G and H is given in Table 13, where ψ^A and χ^{IA} are symplectic–Majorana spinors and χ^{IA} satisfies the

Table 13: Transformation properties of fields in $D = 7$, $N = 4$

	V^a_μ	$B_{\Lambda\mu\nu}$	$A^{\Lambda\Sigma}_\mu$	$L^\Lambda{}_I$	ψ^A_μ	χ^{IA}	R_H
$Sl(5)$	1	5	10	5	1	1	-
$O(5)$	1	1	1	5	4	16	14

irreducibility constraint

$$(\gamma_I)_{AB} \chi^{IB} = 0 \tag{6.30}$$

γ_I being the $O(5)$ gamma matrices. $USp(4)$ symplectic indices $A, B, \cdots$ are raised and lowered with $\mathbb{C}_{AB}$. Besides $L^\Lambda{}_I$ in the fundamental representation of $Sl(5)$ we also introduce the coset representatives $L^{\Lambda\Sigma}{}_{IJ} = L^{[\Lambda}{}_{[I} L^{\Sigma]}{}_{J]}$ in the representation $\underline{10}$ of $O(5)$.

Then the supercovariant field strengths are:

$$\begin{aligned} \widehat{H}^\Lambda &\equiv dB_\Lambda + a_1 L_{\Lambda I} (\gamma^I)_{AB} \overline{\psi}^A \Gamma_a \psi^B V^a + a_2 \epsilon_{\Lambda\Sigma_1 \cdots \Sigma_4} F^{\Sigma_1\Sigma_2} \wedge A^{\Sigma_3\Sigma_4} \\ &+ a_3 L_{\Lambda I} \psi_A \Gamma_{ab} \chi^{IA} V^a V^b \end{aligned} \tag{6.31}$$

$$\begin{aligned} \widehat{F}^{\Lambda\Sigma} &\equiv dA^{\Lambda\Sigma} + b_1 L^{\Lambda\Sigma}{}_{IJ} (\gamma^{IJ})_{AB} \overline{\psi}^A \psi^B \\ &+ b_2 L^{\Lambda\Sigma}{}_{IJ} (\gamma^J)_{AB} \overline{\psi}^A \Gamma_a \chi^{IB} V^a \end{aligned} \tag{6.32}$$

$$\widehat{P}_{IJ} \equiv P_{IJ} - \overline{\psi}^A \chi_{(I}{}^B (\gamma_{J)})_{AB} \tag{6.33}$$

where $P_{IJ} = P_{IJ,i} d\phi^i$ (I, J symmetric and traceless) is the coset vielbein.

The fermion transformation laws are:

$$\delta\psi_A = D\epsilon_A + c_1 T^{(3)}_{AB|abc} \Delta^{abcd} \epsilon^B V_d + c_2 T^{(2)}_{AB|ab} \Delta^{abc} \epsilon^B V_c + \cdots \tag{6.34}$$

$$\delta\chi^I_A = P^{IJ}_{,i} \partial_a \phi^i \Gamma^a (\gamma_J)_{AB} \epsilon^B + c_3 T^{(2)I}_{AB|ab} \Gamma^{ab} \epsilon^B + c_4 T^{(3)I}_{AB|abc} \epsilon^B + \cdots \tag{6.35}$$

where ($F^{\Lambda\Sigma} \equiv dA^{\Lambda\Sigma}$, $H_\Lambda \equiv dB_\Lambda$)

$$T^{(3)}_{[AB]} = L^\Lambda_I (\gamma^I)_{AB} H_\Lambda \tag{6.36}$$

$$T^{(3)I}_{(AB)} = L^\Lambda_J (\gamma^{IJ} - 4\delta^{IJ} \mathbb{1})_{AB} H_\Lambda \tag{6.37}$$

$$T^{(2)}_{(AB)} = L_{\Lambda\Sigma}{}^{IJ} (\gamma_{IJ})_{AB} F^{\Lambda\Sigma} \tag{6.38}$$

$$T^{(2)}_{IAB} = L_{\Lambda\Sigma}{}^{KJ} (\gamma_{IKJ} - 3\delta_{IK} \gamma_J)_{AB} F^{\Lambda\Sigma} \tag{6.39}$$

By integrating the 3–form $T^{(3)}_{AB}, T^{I(3)}_{AB}$ on S^3 and the 2–form $T^{(2)}_{AB}, T^{I(2)}_{AB}$ on S^2 we get the following set of central and matter charges:

$$Z^{(2)}_{(AB)} = \frac{1}{2} L_{\Lambda\Sigma}{}^{IJ} (\gamma_{IJ})_{AB} g^{\Lambda\Sigma} \tag{6.40}$$

$$Z^{(3)}_{[AB]} = L^\Lambda_I (\gamma^I)_{AB} g^\Lambda \tag{6.41}$$

$$Z^{(3)\ I}_{(AB)} = L^\Lambda_J (\gamma^{IJ} - 4\delta^{IJ} \mathbb{1})_{AB} g^\Lambda \tag{6.42}$$

$$Z^{(2)}_{AB|K} = \frac{1}{2} L_{\Lambda\Sigma}{}^{IJ} (\gamma_{KIJ} - 3\delta_{KI} \gamma_J)_{AB} g^{\Lambda\Sigma} \tag{6.43}$$

where

$$g^{\Lambda\Sigma} = \int_{S^2} F^{\Lambda\Sigma} \qquad ; \qquad g_\Lambda = \int_{S^3} H_\Lambda \tag{6.44}$$

and $Z^{I(3)}_{(AB)}$ and $Z^{(2)}_{K|AB}$ satisfy the constraint:

$$(\gamma_I Z^{I(3)})_{AB} = (\gamma_I Z^{I(2)})_{AB} = 0 \tag{6.45}$$

From the Maurer Cartan equations

$$\nabla^{(O(5))} L_{\Lambda I} = L_\Lambda{}^J P_{JI} \tag{6.46}$$

and from the definition of the kinetic matrix

$$\mathcal{N}_{\Lambda\Sigma} = L_\Lambda^{\ I} L_{\Sigma I} \tag{6.47}$$

we find:

$$\begin{aligned}
\nabla^{(O(5))} Z^{I(3)} &= Z^{(3)}_J P^{IJ} & (6.48)\\
\nabla^{(O(5))} Z^{IJ(2)} &= Z^{(2)K[J} P^{I]}_K & (6.49)\\
Z^{(3)I} Z^{(3)}_I &= g^\Lambda \mathcal{N}_{\Lambda\Sigma} g^\Sigma & (6.50)
\end{aligned}$$

where we have traded the $Usp(4)$ indices of the dressed charges with $O(5)$ indices according to:

$$\begin{aligned}
Z^{I(3)} &= Z^{(3)}_{AB} (\gamma^I)^{AB} & (6.51)\\
Z^{IJ(2)} &= Z^{(2)}_{AB} (\gamma^{IJ})^{AB} & (6.52)
\end{aligned}$$

- $\underline{D = 9, N = 2}$ The coset manifold is:

$$G/H = \frac{Sl(2,\mathbb{R})}{O(2)} \times O(1,1) \tag{6.53}$$

and the field content and group assignements are given in Table 14. Here A, B, C are

Table 14: Transformation properties of fields in $D = 9$, $N = 2$

	V^a_μ	$C_{\mu\nu\rho}$	$B^\Lambda_{\mu\nu}$	A^Λ_μ	A_μ	$L^\Lambda{}_{AB}$	e^σ	ψ^A_μ	χ^{ABC}	R_H
$Sl(2,\mathbb{R})$	1	1	2	2	1	2	1	1	1	-
$O(1,1)$	0	1	1	0	1	0	1	0	0	-
$O(2)$	1	1	1	1	1	2	1	2	2 + 2	2

$O(2)$ vector indices, $L^\Lambda{}_{AB}$ is the coset representative of $\frac{Sl(2,\mathbb{R})}{O(2)}$ symmetric and traceless in A, B, e^σ parametrizes $O(1,1)$, $\Lambda = 1, 2$ are indices of $Sl(2,\mathbb{R})$ in the defining representation. ψ_A, χ_{ABC} are Majorana spinors. χ_{ABC} is completely symmetric and can be decomposed as

$$\chi_{ABC} = \overset{\circ}{\chi}_{ABC} + \delta_{(AB}\chi_{C)} \tag{6.54}$$

The supercovariant field strengths are defined as follows:

$$\begin{aligned}
\widehat{H}^{\Lambda(3)} &\equiv dB^\Lambda + a_1 \epsilon_{\Lambda\Sigma}(A^\Lambda \wedge dA^\Sigma + A^\Sigma \wedge dA^\Lambda) + a_2 e^\sigma \overline{\psi}_A \Gamma_a \psi_B V^a L^\Lambda{}_{AB} \\
&+ a_3 e^\sigma \overline{\psi}_C \Gamma_{ab} \chi^{ABC} V^a V^b \mathbb{C}_{AB} & (6.55)\\
\widehat{H}^{(4)} &\equiv dC + b_1(B^\Lambda \wedge dA_\Lambda + A \wedge A^\Lambda \wedge dA^\Sigma \epsilon_{\Lambda\Sigma}) + b_2 e^\sigma \overline{\psi}^A \Gamma_{ab} \psi^B V^a \wedge V^b \epsilon_{AB} \\
&+ b_3 e^\sigma \overline{\chi}_C \Gamma_{abc} \psi^C V^a V^b V^c & (6.56)\\
\widehat{F}^\Lambda &\equiv dA^\Lambda + c_1 L^\Lambda{}_{AB} \overline{\psi}^A \psi^B + c_2 L^\Lambda{}_{AB} \overline{\chi}^{ABC} \Gamma_a \psi_C V^a & (6.57)\\
\widehat{F} &\equiv dA + d_1 e^\sigma \overline{\psi}^A \psi_A + d_2 e^\sigma \overline{\chi}_A \Gamma_a \psi^A V^a & (6.58)\\
\widehat{P}_{AB} &\equiv P_{AB} - \overline{\psi}^C \overset{\circ}{\chi}_{ABC} & (6.59)\\
\widehat{d\sigma} &= d\sigma - \overline{\psi}^A \chi_A & (6.60)
\end{aligned}$$

where $P_{AB} = P_{AB,i} d\phi^i$ is the real vielbein of $\frac{SL(2,\mathbf{R})}{O(2)}$. The fermion transformation laws are:

$$\begin{aligned}
\delta\psi_A &= D\epsilon_A + l_1 T^{(4)}_{abcd}\Delta^{abcdf}\epsilon_A V_f + l_2 T^{(3)}_{AB|abc}\Delta^{abcd}\epsilon^B V_d \\
&+ l_3 T^{(2)}_{AB|ab}\Delta^{abc}\epsilon^B V_c + l_4 T^{(2)}_{ab}\Delta^{abc}\epsilon_A V_c + \cdots \qquad (6.61)\\
\delta\chi_{ABC} &= h_1 P_{(AB,i}\partial_a\phi^i\Gamma^a\epsilon_{C)} + h_2 T_{abcd}\Gamma^{abcd}\delta_{(AB}\epsilon_{C)} \\
&+ h_3 T_{(AB|abc}\Gamma^{abc}\epsilon_{C)} + h_4 T_{(AB|ab}\Gamma^{ab}\epsilon_{C)} + h_5 T_{ab}\Gamma^{ab}\delta_{(AB}\epsilon_{C)} + \cdots \qquad (6.62)
\end{aligned}$$

where

$$T^{(4)} = e^{-\sigma}H^{(4)} \qquad (6.63)$$
$$T^{(3)}_{AB} = e^{-\sigma}L_{\Lambda AB}H^{(3)\Lambda} \qquad (6.64)$$
$$T^{(2)}_{(AB)} = L_{\Lambda AB}F^{(2)\Lambda} \qquad (6.65)$$
$$T^{(2)} = e^{-\sigma}F^{(2)} \qquad (6.66)$$

By integration of the previous $(p+2)$–forms on S^{p+2} we get the magnetic central charges:

$$Z^{(4)} = e^{-\sigma}g; \qquad g = \int H^{(4)} \qquad (6.67)$$
$$Z^{(3)}_{AB} = e^{-\sigma}L_{\Lambda AB}g^\Lambda; \qquad g^\Lambda = \int H^{(3)\Lambda} \qquad (6.68)$$
$$Z^{(2)}_{(AB)} = L_{\Lambda AB}m^\Lambda \qquad m^\Lambda = \int F^{(2)\Lambda} \qquad (6.69)$$
$$Z^{(2)} = e^{-\sigma}m; \qquad m = \int F^{(2)} \qquad (6.70)$$

Using now the Maurer Cartan equations for the coset representative $e^{-\sigma}L_{\Lambda AB}$ we find:

$$\nabla^{O(2)}(e^{-\sigma}L_{\Lambda AB}) = e^{-\sigma}(L_{\Lambda C(A}P_{B)C} - d\sigma L_{\Lambda AB}) \qquad (6.71)$$

where the indices between brackets are symmetric and traceless. Therefore:

$$\partial_\sigma \begin{pmatrix} Z^{(4)} \\ Z^{(3)}_{AB} \\ Z^{(2)} \end{pmatrix} = -\begin{pmatrix} Z^{(4)} \\ Z^{(3)}_{AB} \\ Z^{(2)} \end{pmatrix} \qquad (6.72)$$
$$\nabla_i \begin{pmatrix} Z^{(3)}_{AB} \\ Z^{(2)}_{AB} \end{pmatrix} = \begin{pmatrix} Z^{(3)}_{C(A} \\ Z^{(2)}_{C(A} \end{pmatrix} P_{B)C,i} \qquad (6.73)$$

Finally the kinetic matrices for the p–forms are:

$$\mathcal{N}^{(4)} = e^{-2\sigma} \qquad (6.74)$$
$$\mathcal{N}^{(3)}_{\Lambda\Sigma} = \frac{1}{2}e^{-2\sigma}L_{\Lambda AB}L_\Sigma{}^{AB} \qquad (6.75)$$
$$\mathcal{N}^{(2)}_{\Lambda\Sigma} = \frac{1}{2}L_{\Lambda AB}L_\Sigma{}^{AB} \qquad (6.76)$$
$$\mathcal{N}^{(2)} = e^{-2\sigma} \qquad (6.77)$$
$$(6.78)$$

It follows:

$$\begin{aligned} (Z^{(4)})^2 &= e^{-2\sigma} g^2 && (6.79)\\ \frac{1}{2} Z^{(3)}_{AB} Z^{(3)AB} &= g^\Lambda \mathcal{N}^{(3)}_{\Lambda\Sigma} g^\Sigma && (6.80)\\ \frac{1}{2} Z^{(2)}_{AB} Z^{(2)AB} &= m^\Lambda \mathcal{N}^{(2)}_{\Lambda\Sigma} m^\Sigma && (6.81)\\ (Z^{(2)})^2 &= e^{-2\sigma} m^2 && (6.82)\\ & && (6.83) \end{aligned}$$

7 $D = 6$ and $D = 8$ maximally extended supergravities

- The field content of $D = 6$ $N = (4,4)$ supergravity [67] is given by the following gravitational multiplet:

$$(V^a_\mu, \psi_{A\mu}, \psi_{\dot{A}\mu}, B^{+I}_{\mu\nu}, B^{-\dot{I}}_{\mu\nu}, A^{\alpha\dot{\alpha}}_\mu, \chi_{AI}, \chi_{\dot{A}\dot{I}}, L^x{}_y) \tag{7.1}$$

$(I, \dot{I} = 1, \cdots, 5;\ \alpha, \dot{\alpha} = 1, \cdots, 4;\ A, \dot{A} = 1, \cdots, 4;\ x, y = 1, \cdots, 10)$ where $L^x{}_y$ is the coset representative of:

$$G/H = \frac{O(5,5)}{O(5) \times O(5)}, \tag{7.2}$$

and

$$H_{Aut} \equiv Usp(4) \times Usp(4) \sim O(5) \times O(5) \tag{7.3}$$

The group theoretical assignments for the fields are defined in Table 15. Here, $\psi_A, \psi_{\dot{A}}$

Table 15: Transformation properties of fields in maximally extended $D = 6$, $N = (4,4)$ supergravity

$D = 6,\ N = (4,4)$	V^a_μ	$H^{+\Lambda}_{\mu\nu\rho}, H^{-\Lambda}_{\mu\nu\rho}$	$F^{\alpha\dot{\alpha}}_{\mu\nu}$	$L^\Lambda{}_I, L^\Lambda{}_{\dot{I}},$	$\psi^A_\mu, \psi^{\dot{A}}_\mu$	$\chi^{I\dot{A}}, \chi^{\dot{I}A}$	R_H
$O(5,5)$	1	$\underline{10}$	$\underline{16}$	$\underline{10}$	$1; 1$	$1; 1$	-
$O(5) \times O(5)$	1	$(1,1)$	$(1,1)$	$(5,1); (1,5)$	$(4,1); (1,4)$	$(5,4); (4,5)$	$(5,5)$

are chiral and antichiral gravitinos and transform under H_{Aut} in the fundamental representation of the two $Usp(4)$ factors respectively. The dilatinos $\chi_{I\dot{A}}, \chi_{\dot{I}A}$ are instead spinor–vectors under each $O(5)$ with the couple of indices $I\dot{A}$ $(\dot{I}A)$ transforming in the $\underline{\mathbf{5}}$ and $\underline{\mathbf{4}}$ of the two $O(5)$ factors respectively. The dilatinos chiralities are related to dotted and undotted indices in the opposite way as for the gravitinos. The conversion between $O(5)$ and $USp(4)$ indices is performed via the $O(5)$ γ–matrices $(\gamma^I)_{AB}$, $(\gamma^{\dot{I}})_{\dot{A}\dot{B}}$. Raising and lowering of the $Usp(4)$ indices are performed with the symplectic metric $\mathbb{C}_{AB}$ according to the rule: $\psi_A = \mathbb{C}_{AB}\psi^B$ (and the same for dotted indices).
Note that we have labelled the index of the spinorial representation $\underline{16}$ of $O(5,5)$

with a couple of indices $\alpha, \dot{\alpha}$. The coset representative in the $\underline{1}6$ irrep will be denoted in the following $U^{\alpha\dot{\alpha}}_{\ \ A\dot{A}}$ Let us recall that in $D = 6$ we can decompose H^Λ into real self–dual and antiself–dual parts which transform irreducibly under the Lorentz group. Defining

$$H^x = (H^{\Lambda+}, H^{\Lambda-}) \quad (x = 1, \cdots, 10; \Lambda = 1, \cdots, 5) \tag{7.4}$$

$O(5,5)$ acts as a T–duality group on H^x in the fundamental representation. However, in $D/2 = p+2$, p odd, the $p+2$–forms are also acted on by the Gaillard–Zumino duality group $O(n,n)$ (n being the number of $p+2$ tensors). In $D = 6$, $p = 1$ and $n = 5$, so that the Gaillard–Zumino duality group is again $O(5,5)$. Quite generally, when $D/2 = p+2$, p odd, the kinetic lagrangian for the $p+2$–forms $H^{\pm\Lambda}$ has the following form:

$$\mathcal{L}_{kin} = H^{+\Lambda}\mathcal{N}^+_{\Lambda\Sigma}H^{-\Sigma} + H^{-\Lambda}\mathcal{N}^-_{\Lambda\Sigma}H^{+\Sigma} \tag{7.5}$$

where the kinetic matrices $\mathcal{N}^\pm$ satisfy:

$$\mathcal{N}^\pm = -(\mathcal{N}^\mp)^t \tag{7.6}$$

and transform projectively under $O(n,n)$:

$$\mathcal{N}^{\pm\prime} = (C + D\mathcal{N}^\pm) \times (A + B\mathcal{N}^\pm)^{-1}. \tag{7.7}$$

Here we have used a notation where a generic element of $O(n,n)$ is decomposed in $n \times n$ blocks as follows:

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n,n) \tag{7.8}$$

satisfying:

$$\begin{aligned} C^tA + A^tC &= 0 && (7.9)\\ C^tB + A^tD &= 1 && (7.10)\\ D^tA + B^tC &= 1 && (7.11)\\ D^tB + B^tD &= 0 && (7.12)\end{aligned}$$

where the $O(n,n)$ invariant metric has the off diagonal form:

$$\eta = \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix}. \tag{7.13}$$

Identifying S with the coset representative of $O(n,n)$ it is convenient, in analogy with the 4 dimensional case, to express the $n \times n$ subblocks of L in the following way:

$$S = \sqrt{2}\begin{pmatrix} f_+ + f_- & f_+ - f_- \\ h_+ + h_- & h_+ - h_- \end{pmatrix} \tag{7.14}$$

and the orthogonality relations become:

$$\begin{aligned} h^t_+ f_- + f^t_+ h_- &= 0 && (7.15)\\ h^t_\pm f_\pm + f^t_\pm h_\pm &= \pm 1 && (7.16)\end{aligned}$$

The kinetic matrices $\mathcal{N}_+, \mathcal{N}_-$ can be expressed in terms of $f_\pm$ and $h_\pm$ as follows:

$$h_- = \mathcal{N}_+ f_- \quad (7.17)$$
$$h_+ = \mathcal{N}_- f_+ \quad (7.18)$$

so that (7.15) can be also rewritten as:

$$f^t_\pm (\mathcal{N}_- - \mathcal{N}_+) f_\pm = 1 \quad (7.19)$$

As in the four dimensional case, the matrices $\mathcal{N}_\pm$ transform projectively under the S-duality group $O(5,5)$:

$$\mathcal{N}'_\pm = (C + D\mathcal{N}_\pm)(A + B\mathcal{N}_\pm)^{-1} \quad (7.20)$$

The supercovariant field-strengths are:

$$\begin{aligned}
\widehat{H}^{\Lambda +} &= (dB^\Lambda)^+ + f^\Lambda_{I+}(\gamma^I \otimes 1\!\!1)_{\alpha\dot\alpha\beta\dot\beta} F^{\alpha\dot\alpha} A^{\beta\dot\beta} & (7.21)\\
&+ f^\Lambda_{I+}\left((\gamma^I)_{AB}\overline{\psi}^A \Gamma_a \psi^B V^a + \overline{\chi}^{\dot A I}\Gamma_{ab}\psi_{\dot A} V^a V^b\right) & (7.22)\\
\widehat{H}^{\Lambda -} &= (dB^\Lambda)^- + f^\Lambda_{\dot I -}(\gamma^{\dot I} \otimes 1\!\!1)_{\alpha\dot\alpha\beta\dot\beta} F^{\alpha\dot\alpha} A^{\beta\dot\beta} & (7.23)\\
&+ f^\Lambda_{\dot I -}(\gamma^{\dot I})_{\dot A\dot B}(\overline{\psi}^{\dot A}\Gamma_a \psi^{\dot B} V^a + \overline{\chi}^{A\dot I}\Gamma_{ab}\psi_A V^a V^b) & (7.24)\\
\widehat{F}^{\alpha\dot\alpha} &= dA^{\alpha\dot\alpha} + U^{\alpha\dot\alpha}_{A\dot A}(\overline{\psi}^A \psi^{\dot A} + \overline{\chi}^{\dot A I}\Gamma_a \psi^B (\gamma^I)^A{}_B V^a & \\
&+ \overline{\chi}^{A\dot I}\gamma_a \psi^{\dot B}(\gamma^{\dot I})^{\dot A}{}_{\dot B} V^a) & (7.25)\\
\widehat{P}_{IJ} &= P_{IJ} - \overline{\psi}^A \chi_{\dot I}{}^B (\gamma_J)_{AB} - \overline{\psi}^{\dot A}\chi_I{}^{\dot B}(\gamma_{\dot J})_{\dot A\dot B} & (7.26)
\end{aligned}$$

The transformation laws of the chiral fermions are:

$$\begin{aligned}
\delta\psi_A &= D\epsilon_A + T_{I+|abc}\Delta^{abcd}(\gamma_I)^{AB}\epsilon_B V_d + T_{A\dot A|ab}\Delta^{abc}\epsilon^{\dot A} V_c + \cdots & (7.27)\\
\delta\chi^{I\dot A} &= P^{I\dot I}_{,a}\Gamma^a \epsilon_{\dot B}(\gamma_{\dot I})^{\dot A\dot B} + T_{I+|abc}\Gamma^{abc}\epsilon_{\dot A} + T^{A\dot A}_{|ab}\Gamma^{ab}\epsilon^B(\gamma^I)_{AB} + \cdots & (7.28)
\end{aligned}$$

where:

$$T_{I+} = f_{\Lambda I+} H^{+\Lambda} \quad (7.29)$$
$$T_{A\dot A} = U_{\alpha\dot\alpha A\dot A} F^{\alpha\dot\alpha} \quad (7.30)$$

The central charges for the 3– and 2– forms are found by integration of the corresponding 3– and 2–forms $T_{+I}, T_{A\dot A}$ on S^3 and S^2 respectively. Defining:

$$\int_{S^3} H^\Lambda = g^\Lambda \quad (7.31)$$
$$\int_{S^3} \mathcal{G}_\Lambda = e_\Lambda \qquad \left(\mathcal{G}^\mp_\Lambda \equiv \frac{\partial \mathcal{L}}{\partial H^{\Lambda\pm}} = \mathcal{N}^\pm_{\Lambda\Sigma} H^{\Sigma\mp}\right) \quad (7.32)$$
$$\int_{S^2} F^{\alpha\dot\alpha} = g^{\alpha\dot\alpha} \quad (7.33)$$

where g and e are magnetic and electric charges respectively, we find:

$$\begin{aligned}
Z_{+I} &= \int f^\Lambda_{I+}(\mathcal{N}_- - \mathcal{N}_+)_{\Lambda\Sigma} H^{\Lambda +}\\
&= \int \left(h_{\Lambda I+} H^{\Lambda +} + f^\Lambda_{I+}\mathcal{G}^+_\Lambda + h_{\Lambda I+} H^{\Lambda -} + f^\Lambda_{I+}\mathcal{G}^-_\Lambda\right)\\
&= h_{\Lambda I+} g^\Lambda + f^\Lambda_{I+} e_\Lambda & (7.34)
\end{aligned}$$

Note that in the second line of eq. (7.32) we have inserted the last two terms which sum up to zero using the relations (7.18) and (7.30).

In an analogous way one gets:

$$Z_{-\dot{I}} = h_{\Lambda\dot{I}-}g^{\Lambda} + f^{\Lambda}_{\dot{I}-}e_{\Lambda} \tag{7.35}$$

For the magnetic charge of the dressed vector fields we get:

$$Z_{A\dot{A}} = U_{\alpha\dot{\alpha}A\dot{A}}g^{\alpha\dot{\alpha}} \tag{7.36}$$

The Maurer–Cartan equations give the following relations:

$$\begin{aligned} \nabla Z_{J+} &= P_{J\dot{J}}Z_{\dot{J}-} && (7.37)\\ \nabla Z_{A\dot{A}} &= Z_{B\dot{B}}(\gamma^{I})_{A}^{\ \dot{B}}(\gamma^{\dot{J}})_{\dot{A}}^{\ B}P_{I\dot{J}} && (7.38) \end{aligned}$$

The sum rules for the dyonic charges associated to the 3–forms can be obtained by using the same procedure as in the 4-dimensional case, provided we use the (7.19) instead of (3.40). One gets:

$$Z_{+I}Z_{+I} = Z_{-\dot{I}}Z_{-\dot{I}} = (g,e)\mathcal{M}(\mathcal{N}_{+},\mathcal{N}_{-})\begin{pmatrix} g \\ e \end{pmatrix} \tag{7.39}$$

where:

$$\mathcal{M}(\mathcal{N}_{+},\mathcal{N}_{-}) = \begin{pmatrix} \mathcal{N}_{-}(\mathcal{N}_{-}-\mathcal{N}_{+})^{-1}\mathcal{N}_{+} & \mathcal{N}_{-}(\mathcal{N}_{-}-\mathcal{N}_{+})^{-1} \\ (\mathcal{N}_{-}-\mathcal{N}_{+})^{-1}\mathcal{N}_{+} & (\mathcal{N}_{-}-\mathcal{N}_{+})^{-1} \end{pmatrix} \tag{7.40}$$

For the vector central charges one gets:

$$Z_{A\dot{A}}Z^{A\dot{A}} = g^{\alpha\dot{\alpha}}\mathcal{N}_{\alpha\dot{\alpha}\beta\dot{\beta}}g^{\beta\dot{\beta}}. \tag{7.41}$$

We observe that we have obtained five electric charges e_{Λ} and five magnetic charges g^{Λ}. However, as in the $D = 6$, $N = 1$ case, electric and magnetic charges are not independent. Indeed, if we use the 3–form H^{x} previously defined, we have the T–duality relations:

$$N_{xy}H^{y} = \eta_{xy}{}^{\star}H^{y} \tag{7.42}$$

where:

$$\begin{aligned} N_{xy} &= L_{xI}L_{yI} + L_{x\dot{I}}L_{y\dot{I}} && (7.43)\\ \eta_{xy} &= L_{xI}L_{yI} - L_{x\dot{I}}L_{y\dot{I}} && (7.44) \end{aligned}$$

and $L^{x}_{I}, L^{x}_{\dot{I}}$ are the representatives of $O(5,5)$ in the fundamental representation. Therefore, as in the $D = 6$, $N = (2,0)$ case:

$$g^{x} = -e_{x} \tag{7.45}$$

so that the distinction between electric and magnetic charge is immaterial. From (7.42) it follows that $L_{xI}H^{x}$ has definite self-duality. Indeed we have:

$$L_{xI}H^{x} = L_{xI}(N^{-1})^{xy}\eta_{yz}{}^{\star}H^{z} = L^{y}_{I}\eta_{yz}{}^{\star}H^{z} = L_{xI}{}^{\star}H^{x} \tag{7.46}$$

The analogous expression $L_{x\dot{I}}H^{x}$ is antiself-dual. Finally, we observe that the matrix N_{xy}, contrary to what happens for $\mathcal{N}_{\pm}$, transforms tensorially under $O(5,5)$.

- Let us now discuss the $N=2$, $D=8$ theory [68].
Coset manifold:

$$G/H = \frac{Sl(3,\mathrm{I\!R})}{O(3)} \times \frac{Sl(2,\mathrm{I\!R})}{U(1)} \tag{7.47}$$

Field content:

$$(V_\mu^A, C_{\mu\nu\rho}, B_{\mu\nu}^\Lambda, A_\mu^{\Lambda\alpha}, L_I^\Lambda, L_i^\alpha, \psi_A, \chi^{AI}) \tag{7.48}$$

where L_I^Λ, $(\Lambda, I = 1,2,3)$ is the coset representative of $\frac{Sl(3,\mathrm{I\!R})}{O(3)}$ and Λ_i^α $(\alpha, i = 1,2)$ is the representative of $\frac{Sl(2,\mathrm{I\!R})}{O(2)}$. $A = 1,2$ is a $SU(2)$ index of $H_{Aut} = SU(2)\times U(1)$ and ψ_A, χ^{AI} are left handed spinors, the right handed parts being denoted by ψ^A, χ_A^I. It is convenient to decompose χ^{AI} into the $\frac{3}{2}$ and $\frac{1}{2}$ $SU(2)$ representations according to:

$$\chi^{IA} = \overset{\circ}{\chi}{}^{IA} + (\sigma^I)^A{}_B \chi^B \tag{7.49}$$

The group theoretical assignments are displayed in Table 16

Table 16: Transformation properties of fields in D=8, N=2

$D=8,\ N=2$	V_μ^a	$C_{\mu\nu\rho}$	$B_{\mu\nu}^\Lambda$	$A_\mu^{\Lambda\alpha}$	$L^\Lambda{}_I$	$L^\alpha{}_i$	ψ_μ^A	$\overset{\circ}{\chi}{}^{IA}$	χ^A	R_H
$Sl(3,\mathrm{I\!R})$	1	1	3	3	3	1	1	1	1	-
$Sl(2,\mathrm{I\!R})$	1	1	1	2	1	2	1	1	1	-
$SU(2)$	1	1	1	1	3	1	2	4	2	5
$U(1)$	0	0	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	2

We parametrize the coset representative of $\frac{Sl(2,\mathrm{I\!R})}{U(1)}$ in the following way:

$$L_i^\alpha = \begin{pmatrix} L_+^1 & L_-^2 \\ L_+^2 & L_-^1 \end{pmatrix} \tag{7.50}$$

where:

$$\eta_{\alpha\beta} L_+^\alpha L_-^\beta = 1 \tag{7.51}$$

$$\overline{L}_+^\alpha = L_-^\alpha \tag{7.52}$$

and we define:

$$f = L_+^1 + L_+^2 \quad ; \quad h = L_+^1 - L_+^2 \tag{7.53}$$

The $Sl(2,\mathrm{I\!R})$ group acts as a Gaillard–Zumino S–duality group on the $Sl(2,\mathrm{I\!R})$ vector $(H^{(4)\pm}, \mathcal{G}^{(4)\pm})$ where

$$\mathcal{G}_{abcd} = -\mathrm{i}/2 \frac{\delta \mathcal{L}}{\delta H^{abcd}} \tag{7.54}$$

the kinetic 1×1 "matrix" $\mathcal{N} \equiv S$ being given by:

$$S = \frac{L_+^1 - L_+^2}{L_+^1 + L_+^2}. \tag{7.55}$$

Furthermore, the same group $Sl(2,\mathbb{R})$ acts as a T–duality group on the doublet of 2–forms F^α_Λ, ($\Lambda = 1,2,3$) so that in the supercovariant expression of F^α_Λ there appear naturally $L^\alpha_\pm$, while in the supercovariant 4–form there appear naturally the expressions $f = L^1_+ + L^2_+$. The supercovariant field strengths and vielbein are:

$$\begin{aligned}
\widehat{H}^{(4)} &= dC + f\left(a_1\overline{\psi}^A\Gamma_{ab}\psi^B\epsilon_{AB} + a_2\overline{\psi}_A\Gamma_{abc}\chi^I_B(\sigma_I)^{AB} + h.c.\right) \\
&+ (a_3 B^\Lambda \wedge F_\Lambda + h.c.) && (7.56) \\
\widehat{H}^{(3)\Lambda} &= dB^\Lambda + b_1\epsilon^{\Lambda\Delta\Sigma}F^\alpha_\Delta A_{\alpha\Sigma} \\
&+ b_2(\overline{\psi}_A\Gamma_a\psi^B L^\Lambda_I(\sigma_I)_B^{\ A}V^a + b_3\overline{\psi}_A\Gamma_{ab}\chi^{IA}L^\Lambda_I V^a V^b + h.c.) && (7.57) \\
\widehat{F}^\alpha_\Lambda &= dA^\alpha_\Lambda + (c_1 L^\alpha_+\overline{\psi}^A\psi^B L_{\Lambda AB} \\
&+ c_2\overline{\psi}^A\Gamma_a\chi^B_I L_{\Lambda J}(\sigma_K)_{AB}\epsilon^{IJK}V^a + h.c.) && (7.58) \\
\widehat{P}^{IJ} &= P^{IJ} - (\overline{\overset{\circ}{\chi}}{}^{IA}\psi_B(\sigma^J)_A^{\ B} + h.c.) && (7.59) \\
\widehat{P} &= P - (\overline{\chi}^A\psi_A + h.c.) && (7.60)
\end{aligned}$$

where $P^{IJ} = P^{IJ}_{,i}d\phi^i$ is the $Sl(3,\mathbb{R})/O(3)$ vielbein, symmetric and traceless in the $O(3)$ indices I,J, while $P = P_{,S}dS$ is the complex 1–bein of $SU(1,1)/U(1) \sim Sl(2,\mathbb{R})/O(2)$.

The transformation rules for the fermions are:

$$\begin{aligned}
\delta\psi_A &= D\epsilon_A + d_1 T^{(4)-}_{abcd}\Delta^{abcde}\epsilon^B V_e\epsilon_{AB} \\
&+ d_2 T^{(3)}_{Iabc}\Delta^{abcd}(\sigma^I)_A^{\ B}\epsilon_B V_d + d_3 T^{(2)}_{I-ab}(\sigma^I)_{AB}\Delta^{abc}\epsilon^B V_c + \cdots && (7.61) \\
\delta\overset{\circ}{\chi}{}^{IA} &= f_1 P^{IJ}_{,i}\partial_a\phi^i\Gamma^a(\sigma_J)^A_{\ B}\epsilon^B + f_2 T^{(3)}_{Jabc}\Gamma^{abc}(\sigma^{IJ} - 2\delta^{IJ})^A_{\ B}\epsilon^B \\
&+ f_3 T^{(2)}_{J+ab}(\sigma^{IJ} - 2\delta^{IJ})^{AB}\epsilon_B + \cdots && (7.62) \\
\delta\chi^A &= g_1 P_{,S}\partial_a S\Gamma^a\epsilon^A + g_2 T^{-(4)}_{abcd}\Gamma^{abcd}\epsilon^{AB}\epsilon_B \\
&+ g_3 T^{(2)}_{ab-}\Gamma^{ab}(\sigma^I)^{AB}\epsilon_B + \cdots && (7.63)
\end{aligned}$$

where:

$$\begin{aligned}
T^{(4)-} &= \overline{f}^{-1}H^- && (7.64) \\
T^{(3)}_I &= L_{\Lambda I}H^\Lambda && (7.65) \\
T^{(2)}_{I\pm} &= L_{\alpha\pm}L^\Lambda{}_I F^\alpha_\Lambda && (7.66)
\end{aligned}$$

Note that as in $D = 4$ we have:

$$T^{(4)-} = \overline{f}^{-1}H^- = (\mathcal{N} - \overline{\mathcal{N}})fH^- = hH^- - f\mathcal{G}^- = hH - f\mathcal{G} = T^{(4)} \qquad (7.67)$$

since:

$$hH^+ - f\mathcal{G}^+ = 0. \qquad (7.68)$$

Integrating on S^{p+2} the $(p+2)$–forms $T^{(4)}, T^{(3)}_I, T^{(2)}_{I\pm}$ appearing in the fermions transformation laws we obtain:

$$\begin{aligned}
Z^{(4)} &= hg - fe && (7.69) \\
Z^{(3)}_I &= L^\Lambda_I g^\Lambda && (7.70) \\
Z^{(2)}_{I\pm} &= L_{\alpha\pm}L^\Lambda_I g^\alpha_\Lambda && (7.71)
\end{aligned}$$

Note that the central charge associated to the 4–form is dyonic while those associated to the three and two forms are magnetic. The corresponding electric charges are retrieved as usual by integrating $\mathcal{N}_{\Lambda\Sigma} H^{\Sigma}$ and $\mathcal{N}^{\Lambda\Sigma}_{\alpha\beta} F^{\beta}_{\Sigma}$.

The Maurer–Cartan equations give rise to the following differential relations among the charges:

$$\begin{aligned}
\nabla Z_{+I} &= Z_{+I} P_{++} + Z_{-I} P_{-+} && (7.72)\\
\nabla Z_I &= Z_J P^J_{\ I} && (7.73)\\
\nabla Z &= -\overline{Z} P && (7.74)
\end{aligned}$$

where P_{++} and P_{-+} are the components of the zweibein of $\frac{Sl(2,\mathbf{R})}{O(2)}$ in a real notation. Finally, from the explicit expression of the kinetic matrices:

$$\begin{aligned}
\mathcal{N}^{(4)} &= h f^{-1} && (7.75)\\
\mathcal{N}_{\Lambda\Sigma} &= L_{\Lambda I} L_{\Sigma I} && (7.76)\\
\mathcal{N}^{\Lambda\Sigma}_{\alpha\beta} &= (L_{\alpha+} L_{\beta+} + L_{\alpha-} L_{\beta-}) L^{\Lambda}_{I} L^{\Sigma}_{I} && (7.77)
\end{aligned}$$

we get the following sum rules:

$$\begin{aligned}
Z_I Z_I &= g^{\Lambda} \mathcal{N}_{\Lambda\Sigma} g^{\Sigma} && (7.78)\\
Z_{+I} Z_{-I} &= g^{\alpha}_{\Lambda} \mathcal{N}^{\Lambda\Sigma}_{\alpha\beta} g^{\beta}_{\Sigma} && (7.79)\\
|Z|^2 &= (g,e) \mathcal{M} \begin{pmatrix} g \\ e \end{pmatrix} && (7.80)
\end{aligned}$$

$$\mathcal{M} = \begin{pmatrix} 1 & -Re\mathcal{N} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Im\mathcal{N} & 0 \\ 0 & Im\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Re\mathcal{N} & 1 \end{pmatrix}. \qquad (7.81)$$

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