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ULAM STABILITY FOR THE ORTHOGONALLY GENERAL EULER - LAGRANGE TYPE FUNCTIONAL EQUATION

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ABSTRACT

In this paper, J. M. Rassias introduces the general Euler - Lagrange type functional equation of the form

$$f(mx+y)+f(mx-y)=2f(x+y)+2f(x-y)+2(m^2-2)f(x)-2f(y) \qquad (*)$$

for any arbitrary but fixed real constant m *with* $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ *. We investigate the Ulam stability for the orthogonally general Euler - Lagrange type functional equation (*) controlled by the mixed type product-sum function*

$$(x,y) \to \epsilon\left[\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| y \|_E^{2p}\right)\right]$$

introduced by the third author of this paper, and by a non-negative function with $x \perp y$*.*

Keywords: Hyers - Ulam - Rassias stability , Ulam - Gavurta - Rassias stability, Orthogonally Euler -Lagrange functional equation, Orthogonality space, Quadratic mapping.

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1 Introduction

In 1940, S. M. Ulam [27] raised the question concerning the stability of group homomorphisms:

Let G_1 *be a group and let* G_2 *be a metric group with the metric* $\rho(.,.)$*. Given* $\epsilon > 0$*, does there exists a* $\delta > 0$ *such that if a function* $h : G_1 \to G_2$ *satisfies the inequality* $\rho(h(xy), h(x)h(y)) < \delta$ *for all* $x, y \in G_1$*, then there exists a homomorphism* $H : G_1 \to G_2$ *with* $\rho(h(x), H(x)) < \epsilon$ *for all* $x \in G_1$*?*

D. H. Hyers [6] answered this problem under the assumption that the groups are Banach spaces. Th. M. Rassias [24] generalized the theorem of Hyers for approximately linear mappings. The stability phenomenon that was proved by Th. M. Rassias [24] is called the Hyers - Ulam - Rassias Stability.

J. M. Rassias [11-23] solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings. In 2005, J. M. Rassias [23] solved Euler- Lagrange type quadratic functional equation of the form

$$Q(m_1a_1x_1 + m_2a_2x_2) + m_1m_2Q(a_1x_1 - a_2x_2) = (m_1a_1^2 + m_2a_2^2)(m_1Q(x_1) + m_2Q(x_2))$$

and discussed its Ulam stability problem.

The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), x \perp y \tag{1.1}$$

in which $\perp$ is an abstract orthogonality symbol, was investigated by S. Gudder and D. Strawther [5]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.1) in [4].

We now introduce the concepts of orthogonality vector space, orthogonality space and orthogonality normed space and then proceed to prove our main results.

Definition 1.1. A vector space X is called an *orthogonality vector space* if there is a relation $x \perp y$ on X such that

(i) $x \perp 0, \ \ 0 \perp x$ for all $x \in X$;

(ii) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x \perp y$, $ax \perp by$ for all $a, b \in \mathbb{R}$;

(iv) if P is a two-dimensional subspace of X ; then

(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;

(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector $x, \ y$ define $x \perp y$ iff $x, \ y$ are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, \ y \in X$. The pair $(x, \perp)$ is called an *orthogonality space*. It becomes *orthogonality normed space* when the orthogonality space is equipped with a norm.

Definition 1.2. Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called *orthogonally quadratic* if it satisfies the so called orthogonally Euler-Lagrange (or Jordan - von Neumann) quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

for all $x, y \in X$ with $x \perp y$, (see [15]).

The orthogonality Hilbert space for the orthogonally quadratic functional equation (1.2) was first investigated by F. Vajzovic [28] . Recently Ulam - Gavruta - Rassias stability for the orthogonally Euler - Lagrange type functional equation of the form

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 4f(x) - 2f(y) \tag{1.3}$$

was investigated by Ravi and Arunkumar [26].

In this paper, we investigate the Ulam stability for the orthogonally general Euler - Lagrange type functional equation

$$f(mx+y)+f(mx-y)=2f(x+y)+2f(x-y)+2(m^2-2)f(x)-2f(y) \tag{1.4}$$

for all $x, y \in X$ with $x \perp y$, controlled by the mixed type product-sum function

$$(x,y) \to \epsilon\left\{\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| y \|_E^{2p}\right)\right\},$$

a concept introduced by the third author of this paper, and by a general non-negative function. Note that the general Euler - Lagrange type functional equation (1.4) is equivalent to the standard Euler - Lagrange equation (1.2).

A mapping $f : X \to Y$ is called orthogonally quadratic if it satisfies the quadratic functional equation (1.4) for all $x, y \in X$ with $x \perp y$ where X be an orthogonality space and Y be a real Banach space.

2 Stability of the Functional Equation (1.4)

In this section, let $(E, \perp)$ denote an orthogonality normed space with norm $\| \cdot \|_E$ and $(F, \| \cdot \|_F)$ is a Banach space.

Theorem 2.1. *Let $f : E \to F$ be a mapping which satisfying the inequality*

$$\begin{aligned} &\| f(mx+y)+f(mx-y)-2f(x+y)-2f(x-y)-2(m^2-2)f(x)+2f(y) \|_F \\ &\qquad \le \epsilon\left\{\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| x \|_E^{2p}\right)\right\} \end{aligned} \tag{2.1}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either

$$m > 1; p < 1 \ or \ m < 1; p > 1 \ with \ m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2} \ \ and - 1 \neq |m|^{p-1} < 1.$$

Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}} \tag{2.2}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler - Lagrange quadratic mapping such that

$$\| f(x) - Q(x) \|_F \le \frac{\epsilon}{2|m^2 - m^{2p}|} \|x\|_E^{2p} \tag{2.3}$$

for all $x \in E$.

Proof. Replacing (x, y) with $(0, 0)$ in (2.1), we obtain $2|2-m^2| \, \| f(0) \| = 0$ or $f(0) = 0$ if $m^2 \neq 2$. Again substituting (x, y) by $(x, 0)$ in (2.1), we get

$$\left\|f(mx) - m^2 f(x)\right\|_F \le \frac{1}{2}\epsilon \|x\|_E^{2p}$$

$$(i.e).,\qquad \left\|\frac{f(mx)}{m^2}-f(x)\right\|_F \leq \frac{1}{2}\frac{\epsilon}{m^2}\|x\|_E^{2p}\,(m\neq 0) \tag{2.4}$$

for all $x \in E$. Now replacing x by mx and dividing by m^2 in (2.4) and then adding the resulting inequality with (2.4), we obtain

$$\left\|\frac{f(m^2x)}{m^4}-f(x)\right\|_F \leq \frac{1}{2}\frac{\epsilon}{m^2}\left(1+\frac{m^{2p}}{m^2}\right)\|x\|_E^{2p} \tag{2.5}$$

for all $x \in E$. Using induction on n we obtain that

$$\begin{aligned}\left\|\frac{f(m^nx)}{m^{2n}}-f(x)\right\|_F &\leq \frac{1}{2}\frac{\epsilon}{m^2}\sum_{k=0}^{n-1}\frac{m^{2pk}}{m^{2k}}\|x\|_E^{2p} \\ &\leq \frac{1}{2}\frac{\epsilon}{m^2}\sum_{k=0}^{\infty}\frac{m^{2pk}}{m^{2k}}\|x\|_E^{2p}\end{aligned} \tag{2.6}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{f(m^nx)/m^{2n}\}$ replace x by m^lx and divide by m^{2l} in (2.6), for any $n, l > 0$, we obtain

$$\begin{aligned}\left\|\frac{f(m^{n+l}x)}{m^{2(l+n)}}-\frac{f(m^lx)}{m^{2l}}\right\|_F &= \frac{1}{m^{2l}}\left\|\frac{f(m^{n+l}x)}{m^{2n}}-f(m^lx)\right\|_F \\ &\leq \frac{1}{2}\frac{\epsilon}{m^2}\frac{1}{m^{2l(1-p)}}\sum_{k=0}^{\infty}\frac{m^{2pk}}{m^{2k}}\|x\|_E^{2p}.\end{aligned} \tag{2.7}$$

Since $m^{2(1-p)} < 1$, the R.H.S of (2.7) tends to 0 as $l \to \infty$ for all $x \in E$. Thus $\{f(m^nx)/m^{2n}\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n\to\infty}\frac{f(m^nx)}{m^{2n}} \quad \forall x \in E.$$

By letting $n \to \infty$ in (2.6), we arrive the formula (2.3) for all $x \in E$. To prove Q satisfies (1.4), replace $(x,\ y)$ by $(m^nx,\ m^ny)$ in (2.1) and divide by m^{2n} then it follows that

$$\begin{aligned}&\frac{1}{m^{2n}}\parallel f(m^n(mx+y))+f(m^n(mx-y))-2f(m^n(x+y))-2f(m^n(x-y)) \\ &-2(m^2-2)f(m^nx)+2f(m^ny)\parallel_F \leq \frac{\epsilon}{m^{2n}}\left\{\parallel m^nx\parallel_E^p\parallel m^ny\parallel_E^p+\left(\parallel m^nx\parallel_E^{2p}+\parallel m^ny\parallel_E^{2p}\right)\right\}.\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\parallel Q(mx+y)+Q(mx-y)-2Q(x+y)-2Q(x-y)-2(m^2-2)Q(x)+2Q(y)\parallel_F \leq 0.$$

which gives

$$Q(mx+y)+Q(mx-y) = 2Q(x+y)+2Q(x-y)+2(m^2-2)Q(x)-2Q(y)$$

for all $x, y \in E$ with $x \perp y$. Therefore $Q : E \to F$ is an orthogonally Euler - Lagrange quadratic mapping which satisfies (1.4). To prove the uniqueness of Q, let Q' be another orthogonally Euler - Lagrange quadratic mapping satisfying (1.4) and the inequality (2.3). We have

$$\begin{aligned}\left\|Q(x) - Q'(x)\right\|_F = &\quad \frac{1}{m^{2n}}\left\{\|Q(m^n x) - f(m^n x)\|_F + \left\|f(m^n x) - Q'(m^n x)\right\|_F\right\} \\ \leq &\quad \frac{1}{2}\frac{2\,\epsilon}{m^2}\sum_{j=0}^{\infty}\frac{1}{m^{2(k+n)(1-p)}}\,\|x\|_E^{2p} \\ &\to 0 \quad \text{as} \ \ n \to \infty\end{aligned}$$

for all $x \in E$. Therefore Q is unique. This completes the proof of the theorem. $\square$

Theorem 2.2. *Let $f : E \to F$ be a mapping which satisfying the inequality*

$$\begin{aligned}&\| \ f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y) \ \|_F \\ &\qquad \leq \epsilon\left\{\| \ x \ \|_E^p \| \ y \ \|_E^p + \left(\| \ x \ \|_E^{2p} + \| \ x \ \|_E^{2p}\right)\right\} \qquad (2.8)\end{aligned}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either

$$m > 1; p > 1 \ or \ m < 1; p < 1 \ with \ m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2} \quad and - 1 \neq |m|^{1-p} < 1.$$

Then the limit

$$Q(x) = \lim_{n\to\infty} m^{2n} f\left(\frac{x}{m^n}\right) \qquad (2.9)$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler - Lagrange quadratic mapping such that

$$\| \ f(x) - Q(x) \ \|_F \leq \ \frac{\epsilon}{2|m^{2p} - m^2|}\,\|x\|_E^{2p} \qquad (2.10)$$

for all $x \in E$.

Proof. Replacing x by $\frac{x}{m}(m \neq 0)$ in (2.4), we get

$$\left\|f(x) - m^2 f\left(\frac{x}{m}\right)\right\|_F \leq \ \frac{1}{2}\frac{\epsilon}{m^{2p}}\,\|x\|_E^{2p}\,(m \neq 0) \qquad (2.11)$$

for all $x \in E$. Now replacing x by $\frac{x}{m}$ and multiply by m^2 in (2.11) and summing the resultant inequality with (2.11), we arrive

$$\left\|f(x) - m^4 f\left(\frac{x}{m^2}\right)\right\|_F \leq \frac{1}{2}\frac{\epsilon}{m^{2p}}\left(1 + \frac{m^2}{m^{2p}}\right)\|x\|_E^{2p} \qquad (2.12)$$

for all $x \in E$. Using induction on n we obtain that

$$\begin{aligned}\left\|f(x) - m^{2n} f\left(\frac{x}{m^n}\right)\right\|_F &\leq \frac{1}{2}\frac{\epsilon}{m^{2p}}\sum_{k=0}^{n-1}\frac{m^{2k}}{m^{2pk}}\,\|x\|_E^{2p} \qquad (2.13) \\ &\leq \frac{1}{2}\frac{\epsilon}{m^{2p}}\sum_{k=0}^{\infty}\frac{m^{2k}}{m^{2pk}}\,\|x\|_E^{2p}\end{aligned}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{m^{2n} f\left(\frac{x}{m^n}\right)\}$, replace x by $\frac{x}{m^l}$ and multiply by m^{2l} in (2.13), for any $n, l > 0$,we obtain

$$\left\| m^{2(n+l)} f\left(\frac{x}{m^{l+n}}\right) - m^{2l} f\left(\frac{x}{m^l}\right) \right\|_F = m^{2l} \left\| m^{2n} f\left(\frac{x}{m^{l+n}}\right) - f\left(\frac{x}{m^l}\right) \right\|_F$$
$$\leq \frac{1}{2} \frac{\epsilon}{m^{2p}} \frac{1}{m^{2l(p-1)}} \sum_{k=0}^{\infty} \frac{m^{2k}}{m^{2pk}} \|x\|_E^{2p}. \qquad (2.14)$$

Since $m^{2(p-1)} < 1$, the R.H.S of (2.14) tends to 0 as $l \to \infty$ for all $x \in E$. Thus $\{m^{2n} f\left(\frac{x}{m^n}\right)\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n \to \infty} m^{2n} f\left(\frac{x}{m^n}\right) \qquad \forall\, x \in E.$$

By letting $n \to \infty$ in (2.13), we arrive the formula (2.10) for all $x \in E$. To show that Q is unique and it satisfies (1.4), the proof is similar to that of Theorem 2.1 □

Theorem 2.3. *Let E be a real orthogonality normed linear space and F be a real complete normed linear space. Assume in addition that $f : E \to F$ is an approximately quadratic mappings for which there exists a constant $\theta > 0$ such that f satisfies*

$$\| f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2(m^2 - 2)f(x) + 2f(y) \|_F$$
$$\leq \theta H(x, y), \quad x \perp y \qquad (2.15)$$

for all $(x, y) \in E^2, x \perp y$ and $H : E^2 \to \mathbb{R}^+ \cup \{0\}$ is a non negative real valued function, such that

$$R(x) = \sum_{j=0}^{\infty} \frac{H(m^j x, 0)}{m^{2j}} (< \infty)(m \neq 0) \qquad (2.16)$$

is a non negative function on x, with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ and the condition

$$\lim_{k \to \infty} \frac{H(m^k x, m^k y)}{m^{2k}} = 0 \qquad (2.17)$$

holds. Then there exists a unique orthogonally Euler - Lagrange quadratic mappings $Q : E \to F$ such that

$$\| f(x) - Q(x) \|_F \leq \frac{\theta}{2m^2} R(x) + \frac{||f(0)||_F}{|m^2 - 1|} \qquad (2.18)$$

for all $x \in E$. In addition $f : E \to F$ is a mapping such that the transformation $t \to f(tx)$ is continuous in real t for each fixed $x \in E$, then Q is $\mathbb{R}-$ linear mapping.

Proof. Letting $y = 0$ in (2.15), we get

$$\left\| \frac{f(mx)}{m^2} - f(x) + \frac{f(0)}{m^2} \right\|_F \leq \frac{\theta}{2\, m^2} H(x, 0) \quad (m \neq 0)$$
$$\left\| f(x) - \frac{f(mx)}{m^2} \right\|_F \leq \frac{\theta}{2\, m^2} H(x, 0) + \frac{||f(0)||_F}{m^2} \quad (m \neq 0) \qquad (2.19)$$

for all $x \in E$. Now replacing x by mx divide by m^2 in (2.19), we obtain

$$\left\|\frac{f(mx)}{m^2} - \frac{f(m^2x)}{m^4}\right\|_F \leq \frac{\theta}{2\,m^4} H(mx, 0) + \frac{||f(0)||_F}{m^4}.$$

Using (2.19) and the above inequality, we arrive

$$\left\|f(x) - \frac{f(m^2x)}{m^4}\right\|_F \leq \frac{\theta}{2m^2}\left[H(x,0) + \frac{H(mx,0)}{m^2}\right] + \frac{||f(0)||_F}{m^2}\left[1 + \frac{1}{m^2}\right] \tag{2.20}$$

for all $x \in E$. Using the induction on n we obtain that

$$\left\|f(x) - \frac{f(m^nx)}{m^{2n}}\right\|_F \leq \frac{\theta}{2m^2}\sum_{j=0}^{n-1}\frac{H(m^jx,0)}{m^{2j}} + \frac{||f(0)||_F}{m^2}\sum_{j=0}^{n-1}\frac{1}{m^{2j}} \tag{2.21}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{\frac{f(m^nx)}{m^{2n}}\}$ replace x by m^lx and divided by m^{2l} in (2.21), for any $n, l > 0$, we obtain

$$\begin{aligned}\left\|\frac{f(m^lx)}{m^{2l}} - \frac{f(m^{n+l}x)}{m^{2(n+l)}}\right\|_F &= \frac{1}{m^{2l}}\left\|f(m^lx) - \frac{f(m^{n+l}x)}{m^{2n}} - \right\|_F \\ &\leq \frac{\theta}{2m^2}\sum_{j=0}^{n-1}\frac{H(m^{j+l}x,0)}{m^{2(j+l)}} + \frac{||f(0)||_F}{m^2}\sum_{j=0}^{n-1}\frac{1}{m^{2(j+l)}} \\ &\to 0 \quad \text{as} \quad l \to \infty\end{aligned}$$

for all $x \in E$. Thus $\{\frac{f(m^nx)}{m^{2n}}\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ such that

$$Q(x) = \lim_{n\to\infty}\frac{f(m^nx)}{m^{2n}}, \quad \forall x \in E.$$

Letting $n \to \infty$ in (2.21) and using the definition of $Q(x)$ and (2.16), we arrive at the formula (2.18). Indeed

$$\begin{aligned}\|f(x) - Q(x)\|_F \leq& \quad \frac{\theta}{2m^2}\sum_{j=0}^{\infty}\frac{H(m^jx,0)}{m^{2j}} + \frac{||f(0)||_F}{m^2}\sum_{j=0}^{\infty}\frac{1}{m^{2j}} \\ \leq& \quad \frac{\theta}{2m^2}R(x) + \frac{||f(0)||_F}{m^2}\left[\frac{m^2}{m^2-1}\right] \\ \leq& \quad \frac{\theta}{2m^2}R(x) + \frac{||f(0)||_F}{|m^2-1|}\end{aligned}$$

for all $x \in E$. To prove Q satisfies (1.4), replace $(x,\ y)$ by $(m^nx,\ m^ny)$ in (2.15) and divide by m^{2n} then it follows that

$$\begin{aligned}&\frac{1}{m^{2n}} \parallel f(m^n(mx+y)) + f(m^n(mx-y)) - 2f(m^n(x+y)) - 2f(m^n(x-y)) \\ &- 2(m^2-2)f(m^nx) + 2f(m^ny) \parallel_F \leq \frac{\theta}{m^{2n}}H(m^nx, m^ny)\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\parallel Q(mx+y) + Q(mx-y) - 2Q(x+y) - 2Q(x-y) - 2(m^2-2)Q(x) + 2Q(y) \parallel_F \leq 0.$$

which gives

$$Q(mx+y)+Q(mx-y)=2Q(x+y)+2Q(x-y)+2(m^2-2)Q(x)-2Q(y)$$

for all $x, y \in E$ with $x \perp y$. Therefore $Q : E \to F$ is an orthogonally Euler - Lagrange quadratic mapping which satisfies (1.4). To prove the uniqueness of Q, let Q' be another orthogonally Euler - Lagrange quadratic mapping satisfying (1.4) and the inequality (2.18). We have

$$\begin{aligned}\left\|Q(x)-Q'(x)\right\|_F = & \frac{1}{m^{2n}}\left\{\|Q(m^n x)-f(m^n x)\|_F+\left\|f(m^n x)-Q'(m^n x)\right\|_F\right\}\\ \leq & \frac{1}{m^{2n}}\left\{\frac{\theta}{m^2}R(x)+\frac{2||f(0)||_F}{|m^2-1|}\right\}\\ & \to 0 \quad \text{as} \quad n\to\infty\end{aligned}$$

for all $x \in E$. Therefore Q is unique. This completes the proof of the theorem. □

Theorem 2.4. *Let E be a real orthogonality normed linear space and F be a real complete normed linear space. Assume in addition that $f : E \to F$ is an approximately quadratic mappings for which there exists a constant $\theta > 0$ such that f satisfies*

$$\begin{aligned}&\| f(mx+y)+f(mx-y)-2f(x+y)-2f(x-y)-2(m^2-2)f(x)+2f(y) \|_F\\ &\leq \theta H(x,y), \quad x\perp y \qquad (2.22)\end{aligned}$$

for all $(x, y) \in E^2, x \perp y$ and $H : E^2 \to \mathbb{R}^+ \cup \{0\}$ is a non negative real valued function, such that

$$R(x)=\sum_{j=0}^{\infty} m^{2j} H\left(\frac{x}{m^{j+1}},0\right)(<\infty)(m\neq 0) \qquad (2.23)$$

is a non negative function on x, with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ and the condition

$$\lim_{k\to\infty} m^{2k} H\left(\frac{x}{m^k},\frac{y}{m^k}\right)=0 \qquad (2.24)$$

holds. Then there exists a unique orthogonally Euler - Lagrange quadratic mappings $Q : E \to F$ such that

$$\| f(x)-Q(x) \|_F \leq \frac{\theta}{2}R(x)+\frac{||f(0)||_F}{|1-m^2|} \qquad (2.25)$$

for all $x \in E$. In addition $f : E \to F$ is a mapping such that the transformation $t \to f(tx)$ is continuous in real t for each fixed $x \in E$, then Q is $\mathbb{R}-$ linear mapping.

Proof. Replacing x by $\frac{x}{m}$ in (2.19) and using the proof of Theorem 2.3, we arrive at the desired result. □

The following two analogous Theorems 2.5 and 2.6 can be obtained as two special cases: either $m = 1$ or $m = -1$. In these two cases the pertinent functional equations are obviously equivalent to the classical quadratic equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.26}$$

for all $x, y \in E$ with $x \perp y$.

Theorem 2.5. *Let $f : E \to F$ be a mapping satisfying the inequality*

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \|_F \leq \epsilon \left[\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| x \|_E^{2p} \right) \right] \tag{2.27}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} \tag{2.28}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique Euler - Lagrange quadratic mapping such that

$$\| f(x) - Q(x) \|_F \leq \frac{3\,\epsilon}{4 - 2^{2p}} \|x\|_E^{2p} \tag{2.29}$$

for all $x \in E$.

Proof. Letting $y = x$ in (2.27), we get

$$\left\| \frac{f(2x)}{4} - f(x) \right\|_F \leq \frac{3\,\epsilon}{4} \|x\|_E^{2p} \tag{2.30}$$

for all $x \in E$. Now Replacing x by $2x$ and dividing by 4 in (2.30) and summing the resultant inequality with (2.30), we arrive

$$\left\| \frac{f(2^2 x)}{4^2} - f(x) \right\|_F \leq \frac{3\,\epsilon}{4} \left(1 + \frac{2^{2p}}{4} \right) \|x\|_E^{2p} \tag{2.31}$$

for all $x \in E$. Using induction on n, we obtain that

$$\begin{aligned} \left\| \frac{f(2^n x)}{4^n} - f(x) \right\|_F &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{n-1} \frac{2^{2pk}}{4^k} \|x\|_E^{2p} \\ &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{\infty} \frac{2^{2pk}}{4^k} \|x\|_E^{2p} \end{aligned} \tag{2.32}$$

for all $x \in E$. In order to prove the convergence of the sequence $\{f(2^n x)/4^n\}$, replace x by $2^l x$ and divide by 4^l in (2.32), for $n, l > 0$,we obtain

$$\begin{aligned} \left\| \frac{f(2^{n+l} x)}{4^{l+n}} - \frac{f(2^l x)}{4^l} \right\|_F &= \frac{1}{4^l} \left\| \frac{f(2^{n+l} x)}{4^n} - f(2^l x) \right\|_F \\ &\leq \frac{1}{4^l} \frac{3\,\epsilon}{4} \sum_{k=0}^{n-1} \frac{2^{2pk}}{4^k} \left\| 2^l x \right\|_E^{2p} \\ &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{\infty} \frac{2^{2p(k+l)}}{4^{(k+l)}} \|x\|_E^{2p} \\ &\leq \frac{3\,\epsilon}{4} \sum_{k=0}^{\infty} \frac{1}{2^{2(1-p)(k+l)}} \|x\|_E^{2p} . \end{aligned} \tag{2.33}$$

As $p < 1$, the R.H.S of (2.33) tends to 0 as $l \to \infty$. Thus $\{f(2^n x)/4^n\}$ is a Cauchy sequence. Since F is complete, there exists a mapping $Q : E \to F$ and define

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \quad \forall x \in E.$$

Letting $n \to \infty$ in (2.32), we arrive the formula (2.29) for all $x \in E$. To prove Q satisfies (1.4) and it is unique the proof is similar to that of Theorem 2.1. Hence the proof is complete. □

Theorem 2.6. *Let $f : E \to F$ be a mapping satisfying the inequality*

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \|_F \quad \le \epsilon \left[\| x \|_E^p \| y \|_E^p + \left(\| x \|_E^{2p} + \| x \|_E^{2p} \right) \right] \quad (2.34)$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon > 0$ and $p > 1$. Then the limit

$$Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \quad (2.35)$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler - Lagrange quadratic mapping such that

$$\| f(x) - Q(x) \|_F \le \frac{3\,\epsilon}{2^{2p} - 4} \|x\|_E^{2p} \quad (2.36)$$

for all $x \in E$.

Proof. Replacing x by $\frac{x}{m}$ in (2.30) and using the proof of Theorem 2.5, we arrive at the desired result. □

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