Tôhoku Math. J.
46 (1994), 295–310

ULTIMATE BOUNDEDNESS OF SOLUTIONS FOR A GENERALIZED LIÉNARD EQUATION WITH FORCING TERM

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(Received January 19, 1993, revised May 10, 1993)

Abstract. A boundedness theorem is established for a second order differential equation describing parametric excitations in physics.

1. Introduction. The purpose of this paper is to establish a boundedness theorem for the equation

$$\ddot{x}+f(t, x, \dot{x})\dot{x}+g(t, x)=e(t)\,, \qquad (\cdot = d/dt) \tag{1}$$

where $f(t, x, y)$ is continuous for all t, x and y, $g(t, x)$ is continuous for all t and x and $e(t)$ is continuous and bounded for all t. It is a feature that $g(t, x)$, the restoring force in physics, depends not only on x but also on t, and then (1) describes parametric excitations (cf. [14], [23]). Although many boundedness theorems have been obtained for (1) in the case where $g(t, x)$ is independent of t, that is,

$$g(t, x)=g(x)\,, \tag{2}$$

(cf. [1]–[24] except [11], and [4, Theorem 4]), all of their proofs depend heavily on (2) and are not applicable to (1).

The first boundedness theorem for (1) was obtained by Cartwright and Swinnerton-Dyer [4, Theorem 4], but their theorem does not cover Rayleigh's equation

$$\ddot{x}+k(\frac{1}{3}\dot{x}^3-\dot{x})+g(x)=0\,, \tag{3}$$

except when $g(x)=x$.

In this paper we investigate the boundedness of solutions under the restriction

$$f(t, x, -y)\geq ky^{\lambda} \qquad \text{for all} \quad t, x\geq A \quad \text{and} \quad y\geq A\,, \tag{4}$$

where k, λ and A are positive constants, which was motivated by (3).

Theorem 1 in Section 2 shows that if $\lambda>1$ in (4), then the solutions are ultimately bounded under mild conditions on $g(t, x)$. However we can show that $\lambda=1$ is critical. Namely, when $\lambda=1$, Theorem 1 still guarantees the boundedness under some additional condition on $g(t,x)$, but the example in Section 4 shows that this condition cannot be

1991 *Mathematics Subject Classification*. Primary 34D40; Secondary 34C25.

dropped.

Our method is to construct a positively invariant Jordan curve in a phase plane corresponding to (1) in the same way as in many previous results.

ACKNOWLEDGEMENT. This study has begun while the author was visiting the Australian National University. The starting point was the relation between boundedness of solutions and the critical points at infinity of polynomial systems. The author would like to thank Professor Andrew W. Coppel for drawing his attention to this field and for his idea of using the integral norm in the argument of Section 4. He also would like to thank Professor Junji Kato of Tohoku University and Professor Grainger R. Morris of New England University for their comments on a draft of this paper.

2. Boundedness theorem. We assume (4), the uniqueness of solutions for (1) with arbitrary initial conditions and the following conditions (i) through (v):

(i) $e(t)$ is bounded for all t, and for any constant $N>0$

$$\sup\{|f(t, x, y)|\,;\,|x|\le N,\,|y|\le N,\ -\infty<t<\infty\}<+\infty\,,$$

(ii) there exists a constant $M>0$ such that $f(t, x, y)\ge -M$ for all t, x and y,

(iii) $f(t, x, y)\ge 0$ for all t and $|x|\ge 1$, $|y|\ge 1$,

(iv) there exist two functions $h_1(x)$ and $h_2(x)$, continuous for all x, such that $h_1(x)<(g(t, x)-e(t))\operatorname{sgn} x<h_2(x)$ for all x and all t and that $h_1(x)\ge M+b$ for $|x|\ge 1$ and for a constant $b>0$,

(v) if $0<\lambda\le 1$, then $h_2(x)$ satisfies

$$\text{(v.1)}\qquad \int_0^\infty e^{-2kx}h_2(x)dx<\infty \qquad \text{when}\quad \lambda=1\,,$$

$$\text{(v.2)}\qquad \limsup_{x\to\infty}\frac{h_2(x)}{x^\mu}\le I(\lambda) \qquad \text{when}\quad 0<\lambda<1\,,$$

where $\mu=(1+\lambda)/(1-\lambda)$, $I(\lambda)=(1-\lambda)(1-\lambda^2)^\mu(k/2)^{\mu+1}$ and k is the constant in (4).

REMARK 1. In the case of (2), the above conditions (i) through (iv) are used as standard conditions (cf. [19]).

REMARK 2. We can see that $I(\lambda)\to 0$ as $\lambda\to 1$, because $\log I(\lambda)=(\mu+1)\log(1-\lambda^2)(k/2)-\log(1+\lambda)\to-\infty$ as $\lambda\to 1$.

Our main theorem is the following whose proof will be given in the next section.

THEOREM 1. *Under the above conditions, the solutions of* (1) *are ultimately bounded. Namely there exist positive constants B_1 and B_2 such that any solution $x(t)$ satisfies $|x(t)|<B_1$, $|\dot{x}(t)|<B_2$ for all $t\ge T$, where T is a constant depending on $x(t)$ while B_1 and B_2 are independent of the particular solution.*

We give examples of Theorem 1 when $\lambda=2$ and $\lambda=1/2$. The solutions of the following equations are ultimately bounded:

$$\ddot{x}+k\left(\frac{1}{3}\dot{x}^3-\dot{x}\right)+g(t, x)=e(t)$$

and

$$\ddot{x}+k(\sqrt{|\dot{x}|}-1)\dot{x}+(x+a(t)x^3)=e(t)\,,$$

where k is a positive constant, $g(t, x)$ satisfies (iv), $e(t)$ and $a(t)$ are continuous and bounded, $a(t)>0$ and

$$\sup\{a(t);\ -\infty<t<\infty\}<\frac{27}{2048}k^4\,.$$

Here we note that $I(1/2)=27k^4/2048$ in (v.2).

We now consider the periodic case of (1), that is, when $f(t, x, y)$, $g(t, x)$ and $e(t)$ have the same period, say 2π, in t. As a special case of Massera's theorem (cf. [20, p. 369]), it is known that if every solution of (1) is bounded in the future, then there exists at least one 2π-periodic solution. This is the case if (1) satisfies all conditions (ii)–(v) of Theorem 1. Hence both equations in the examples have at least one 2π-periodic solution if $g(t, x)$, $a(t)$ and $e(t)$ are 2π-periodic in t.

The existence of periodic solutions for equations of type (1) has already been studied by several authors. For example, Yoshizawa [24] considered it under the assumption of the existence of two constants a, b, $a<b$, such that $g(t, a)>e(t)>g(t, b)$ for $0\le t\le 2\pi$. Also, from the proof of Lazer's theorem [11] we can see that (1) has a periodic solution if $f(t, x, y)$ is constant and

$$\lim_{|x|\to\infty}\frac{g(t, x)}{x}=0\,.$$

However, both equations in the examples do not fall within the hypotheses of any of these results [11] and [24].

3. Proof of theorem. We consider the phase plane R^2 with coordinates x and $y=\dot{x}$. Then (1) is equivalent to the system

$$\dot{x}=y,\qquad \dot{y}=-f(t, x, y)y-h(t, x)\,, \tag{5}$$

where $h(t, x)=g(t, x)-e(t)$, and (iv) implies

$$h(t, x)\operatorname{sgn} x\ge M+b\qquad\text{for}\quad |x|\ge 1\ \text{and all}\ t\,. \tag{6}$$

From (5) we obtain by division

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=-f(t, x, y)-\frac{h(t, x)}{y} \qquad \text{for} \quad y \neq 0 .$$

We set $w(t)=(x(t), y(t)) \in R^2$. The following is a key lemma in our proof.

LEMMA 1. *Under* (iv) *and* (v), *there exists a continuously differentiable function* $y=-\varphi(x)$ *for* $A \leq x < \omega$, where $\omega=\infty$ *if* $0<\lambda \leq 1$ *and* $\omega < A+1$ *if* $\lambda > 1$, *such that*

$$\varphi(x) \geq A , \qquad \varphi(x) \to \infty \qquad \textit{as} \quad x \to \omega \tag{7}$$

and $y=-\varphi(x)$ *satisfies*

$$\frac{dy}{dx} > -f(t, x, y)-\frac{h(t, x)}{y} \qquad \textit{for} \quad \omega > x > A \ \textit{and all} \ t . \tag{8}$$

Consequently the curve $w(t)$ *crosses* $y=-\varphi(x)$ *from below to above as* t *increases.*

PROOF. In order to prove (7) and (8), it is sufficient to show that

$$-\varphi(x)\frac{d\varphi(x)}{dx} \geq -k\varphi^{\lambda+1}(x)+h_2(x) \tag{9}$$

for $A \leq x < \omega$, as well as (7), because of (iv) and (4), and (9) is reduced to

$$\frac{dz}{dx}(x)-2kz^{\nu}+2h_2(x) \leq 0 \tag{10}$$

by $z(x)=\varphi^2(x)$ and $\nu=(1+\lambda)/2$.

We first consider the case where $\lambda > 1$. Letting N to be the constant such that

$$N=\sup\{h_2(x) ; A \leq x \leq A+1\} ,$$

we consider the equation

$$\frac{dz}{dx}(x)-2kz^{\nu}(x)+2N=0 . \tag{11}$$

Because of $\nu > 1$, we can see that if $z(A)$ is sufficiently large, then the solution $z(x)$ of (11) is monotone increasing for $x > A$, but noncontinuable until $x=A+1$. Namely there is a constant ω, $A<\omega<A+1$, such that $z(x)$ is defined on $[A, \omega)$, $z(x)>A^2$ and $z(x) \to \infty$ as $x \to \omega$. Therefore, $\varphi(x)=\sqrt{z(x)}$ satisfies (7) and (9).

Secondly we consider the case where $\lambda=1$. Then (10) is reduced to

$$\frac{dz}{dx}(x)-2kz(x)+2h_2(x) \leq 0 ,$$

which is satisfied by

$$z(x)=e^{2k(x-A)}\{z(A)-2\int_A^x e^{-2k(u-A)}h_2(u)du\}\ .$$

Because of (v.1), if $z(A)$ is sufficiently large, then $z(x)$ is defined for all $x \geq A$, $z(x) \geq A^2$ and $z(x) \to \infty$ as $x \to \infty$. Therefore $\varphi(x)=\sqrt{z(x)}$ satisfies (7) and (9).

Finally, for the case where $0<\lambda<1$, we can see that

$$\varphi(x)=(1-\lambda)^{1/(1-\lambda)}\left\{\frac{(1+\lambda)k}{2}(x-A)+c\right\}^{1/(1-\lambda)} \quad \text{for some constant } c>0$$

satisfies (7) and (9) by (v.2). The proof of Lemma 1 is complete.

We denote the straight line segment joining any two points P and Q of R^2 by $\overline{PQ}$ and the curve $y=-\varphi(x)$ $(x \geq A)$ in R^2 by Γ. We will prove Theorem 1 by constructing a Jordan domain $D_\alpha \subset R^2$, $\alpha \geq A$, such that

$$\bigcup_{\alpha>A} D_\alpha = R^2 \tag{12}$$

and the closure $\bar{D}_\alpha$ of D_α is positively invariant, that is, if $w(t_0) \in \bar{D}_\alpha$ for some $t_0 \geq 0$, then $w(t) \in \bar{D}_\alpha$ for $t \geq t_0$ and that if $w(t_0) \in D_\alpha - D_A$ for some $\alpha > A$, then $w(T) \in J_A$ for some $T > t_0$, where J_A is the boundary of D_A.

Without loss of generality, assume that

$$M \geq 1 \quad \text{and} \quad \alpha \geq A \geq 2M\ .$$

We construct a Jordan curve J_α, which will be the boundary of D_α, starting at $Q_1=(\alpha, -\varphi(\alpha))$ (see Figure 1).

Step 1. From (iv) we can choose a constant $K(\alpha)>M$ such that

$$K(\alpha) > \sup\{|h(t, x)|;\ -\infty<t<\infty\ ,\ -1 \leq x \leq \alpha\}\ ,$$

$$K(\alpha) > \sup\left\{\left|\frac{d\varphi(x)}{dx}\right|;\ A \leq x \leq \alpha\right\} \qquad \text{when} \quad 0<\lambda \leq 1\ ,$$

and consider the equation

$$\frac{dy}{dx}=K(\alpha)\left(1-\frac{1}{y}\right), \qquad y(\alpha)=-\varphi(\alpha)\ . \tag{13}$$

Since the right hand side is positive for $y<0$ and $y(\alpha)<0$, it follows that $y(x)$ is defined for $x \leq \alpha$ and $y(x) \leq -\varphi(\alpha)<0$ for $x \leq \alpha$. Indeed, $y(x)$ is given by the equation

$$y+\log|y-1|-K(\alpha)x=\text{const}\ .$$

Let $C_1(\alpha)$ be the arc of $y(x)$ from Q_1 to the point of intersection with the line $x=-1$, say $Q_2=(-1, -\beta)$ for some $\beta>0$. Since $dy/dx>K(\alpha)>1$, the gradient of the vector $\overrightarrow{Q_2Q_1}$ is larger than 1, that is, $(\beta-\varphi(\alpha))/(\alpha+1)>1$, which implies

$$\beta>\varphi(\alpha)+\alpha>2 \tag{14}$$

and, therefore, by (7)

$$\beta \to +\infty \qquad \text{as} \quad \alpha \to \omega\,. \tag{15}$$

Since

$$\frac{d^2y}{dx^2}=\frac{K(\alpha)^2}{y^2}\left(1-\frac{1}{y}\right)>0 \qquad \text{for} \quad y<0\,,$$

the arc $C_1(\alpha)$ is concave with respect to the x-axis, and hence $\overline{Q_1Q_2}$ is above $C_1(\alpha)$. Moreover $C_1(\alpha)$ is below Γ for $A\leq x\leq\alpha$, because

$$\frac{dy(x)}{dx}>K(\alpha)>\left|\frac{d\varphi(x)}{dx}\right| \quad (A\leq x\leq\alpha) \qquad \text{when} \quad 0<\lambda\leq 1$$

and

$$\frac{dy(x)}{dx}>0>-\frac{d\varphi(x)}{dx} \quad (A\leq x<\omega) \qquad \text{when} \quad \lambda>1\,.$$

Step 2. The equation

$$\frac{dy}{dx}=\frac{M}{y}\,, \qquad y(-1)=-\beta \tag{16}$$

has a solution $y(x)$ such that

$$y^2-2Mx=\beta^2+2M\,.$$

Since $\beta>1$, $y(x)$ meets the line $y=-1$, say at $Q_3=(-\gamma,\,-1)$, where

$$\gamma>\frac{\beta^2}{2M}\,. \tag{17}$$

Then the arc $C_2(\alpha)=\widehat{Q_2Q_3}$ is contained in the region $x\leq-1$, $y\leq-1$ and is concave with respect to the x-axis.

Step 3. We set $Q_4=(-\delta, 0)$, where $\delta=\gamma+2/b$. Then $\overline{Q_3Q_4}$ is contained in the region $x\leq-1$, $-1\leq y\leq 0$ and the gradient of the vector $\overrightarrow{Q_3Q_4}$ is $-b/2$.

Step 4. We choose a constant $L(\alpha)\geq M$ such that

$$L(\alpha)\geq \sup\{|h(t, x)|\,;\ -\infty<t<\infty,\ -\delta\leq x\leq 1\}\,,$$

and consider the equation

$$\frac{dy}{dx}=L(\alpha)\left(1+\frac{1}{y}\right). \tag{18}$$

A positive valued solution $y(x)$ such that $y(x)\to 0$ as $x\to -\delta+0$ uniquely defined for $x>-\delta$.

Let $C_3(\alpha)$ be the arc of $y(x)$ from Q_4 to be point of intersection with $x=1$, say $Q_5=(1, \xi)$. Since

$$\frac{dy}{dx}>L(\alpha)>M ,$$

the gradient of the vector $\overrightarrow{Q_4Q_5}$ is larger than M, and hence $\xi/(1+\delta)>M$, which implies

$$\xi>\frac{\beta^2}{2}>1 . \tag{19}$$

Step 5. The equation

$$\frac{dy}{dx}=-\frac{M}{y}, \qquad y(1)=\xi , \tag{20}$$

has a solution $y(x)$ such that

$$y^2+2Mx=\xi^2+2M .$$

Since $\xi>1$, $y(x)$ meets the line $y=1$, say at $Q_6=(\eta, 1)$, where

$$\eta>\frac{\xi^2}{2M}>\frac{\alpha^4}{8M}>\alpha M^2>\alpha \tag{21}$$

by (19) and (14). We set $C_4(\alpha)=\widehat{Q_5Q_6}$.

Step 6. We set $Q_7=(\zeta, 0)$, where $\zeta=\eta+2/b>\alpha$. The segment $\overline{Q_6Q_7}$ is contained in the region $x\geq A$, $0\leq y\leq 1$ and the gradient of the vector $\overrightarrow{Q_6Q_7}$ is $-b/2$.

Step 7. Let $L^*(\alpha)>M$ be a constant such that

$$L^*(\alpha)>h(t, x)+\frac{\varphi(\alpha)}{\zeta-\alpha} \qquad \text{for} \quad \alpha\leq x\leq\zeta \text{ and all } t$$

and let $y=y(x)$ be the negative valued solution of the equation

$$\frac{dy}{dx}=L^*(\alpha)\left(1-\frac{1}{y}\right) \tag{22}$$

with $y(x)\to 0$ as $x\to\zeta-0$. Since $y(x)$ exists for all $x<\zeta$, we have

$$\frac{dy}{dx}(x)>L^*(\alpha)>\frac{\varphi(\alpha)}{\zeta-\alpha} \qquad (x<\zeta)$$

and hence $y(\alpha)<-\varphi(\alpha)$, which implies that $y=y(x)$ intersects Γ at a point Q_8 to the right of Q_1 on Γ. We set $C_5(\alpha)=\widehat{Q_7Q_8}$.

Now we take J_α to be the Jordan curve consisting of the arc $\widehat{Q_1Q_8}$ of Γ, $C_1(\alpha)$,

$C_2(\alpha)$, $\overline{Q_3Q_4}$, $C_3(\alpha)$, $C_4(\alpha)$, $\overline{Q_6Q_7}$ and $C_5(\alpha)$, and we take D_α to be the interior of J_α.

We will prove (12). Indeed, taking E_α to be the interior of the polygon of Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_1, we can see that

$$E_\alpha \subset D_\alpha \tag{23}$$

by the convexity of $C_k(\alpha)$ $(1 \le k \le 5)$ to the x-axis. Since each of the y-coordinates of Q_1, Q_2 and Q_5 and each of the x-coordinates of Q_3, Q_4, Q_6 and Q_7 tend to ∞ or $-\infty$ as $\alpha \to \omega$ by (7), (15), (17), (19) and (21), we have

$$\bigcup_{\alpha > A} E_\alpha = R^2 ,$$

which implies (12), together with (23).

In order to prove the positive invariance of D_α, assuming that $w(t_0) \in J_\alpha$ for some $t_0 \ge 0$, we show that $w(t)$ crosses $J_\alpha - \{Q_4, Q_7\}$ transversely from outside to inside as t increases through t_0 and that if $w(t_0) \in \{Q_4, Q_7\}$, there is a constant $\varepsilon > 0$ such that

$$w(t) \in D_\alpha \qquad \text{for} \quad t_0 < t < t_0 + \varepsilon . \tag{24}$$

Step 8. Suppose $w(t_0) \in C_5(\alpha) - \{Q_7, Q_8\}$. From (5) we obtain by division

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -f(t, x, y) - \frac{h(t,x)}{y} \qquad \text{for} \quad y \ne 0 . \tag{25}$$

By the definition of $L^*(\alpha)$, we have

$$\frac{dy}{dx} < L^*(\alpha)\left(1 - \frac{1}{y}\right) \qquad \text{for} \quad y < 0 . \tag{26}$$

Since $\dot{x}(t_0) = y(t_0) < 0$, we can see from the comparison of the above with (22) that $w(t)$ crosses $C_5(\alpha) - \{Q_7, Q_8\}$ transversely from below to above as t increases through t_0. For $w(t_0) = Q_7$, we have

$$\dot{x}(t_0) = y(t_0) = 0$$

and

$$\ddot{x}(t_0) = \dot{y}(t_0) = -h(t_0, x(t_0)) < 0 ,$$

because $x(t_0) \ge 1$. Therefore, for a small constant $\varepsilon > 0$,

$$x(t) < x(t_0) \quad \text{and} \quad y(t) < 0 \qquad \text{for} \quad t_0 < t < t_0 + \varepsilon .$$

Considering (26) again for $y = y(t)$ $(t_0 < t < t_0 + \varepsilon)$, $w(t)$ must be above $C_5(\alpha)$. Thus we obtain (24). The case where $w(t_0) = Q_8$ is treated in Step 9.

In the following discussion, t is always assumed to increase through t_0.

Step 9. Suppose $w(t_0) \in \widehat{Q_1Q_8} - \{Q_1\}$. By Lemma 1, $w(t)$ crosses Γ from below to above at $t = t_0$, and since

$$\dot{x}(t_0)=y(t_0)<0\,,$$

we can see that $w(t)$ crosses $\widehat{Q_1Q_8}-\{Q_1\}$ from outside to inside at t_0. The case where $w(t_0)=Q_1$ is treated in Step 10.

Step 10. Suppose $w(t_0)\in C_1(\alpha)-\{Q_2\}$. From the definition of $K(\alpha)$, (25) implies

$$\frac{dy}{dx}<K(\alpha)\left(1-\frac{1}{y}\right)\qquad\text{for}\quad y<0\,.$$

Since $\dot{x}(t_0)=y(t_0)<0$, we can see from the comparison of the above with (13) that $w(t)$ crosses $C_1(\alpha)-\{Q_2\}$ from below to above at $t=t_0$. The case where $w(t_0)=Q_2$ is treated in Step 11.

Step 11. Suppose $w(t_0)\in C_2(\alpha)-\{Q_3\}$. Since

$$f(t,x,y)\geq 0\qquad\text{for}\quad x\leq -1\ \text{ and }\ y\leq -1$$

and

$$h(t,x)\leq -M-b\qquad\text{for}\quad x\leq -1\,,$$

(25) implies that

$$\frac{dy}{dx}\leq\frac{M+b}{y}<\frac{M}{y}\qquad\text{for}\quad y<0\,.$$

Since $\dot{x}(t_0)=y(t_0)<0$, we can see from the comparison of the above with (16) that $w(t)$ crosses $C_2(\alpha)-\{Q_3\}$ from below to above at $t=t_0$. The case where $w(t_0)=Q_3$ is treated in Step 12.

Step 12. Suppose $w(t_0)\in\overline{Q_3Q_4}-\{Q_4\}$. Since

$$h(t,x)\leq -M-b\qquad\text{for}\quad x\leq -1\,,$$

(25) implies

$$\frac{dy}{dx}\leq M+\frac{M+b}{y}\leq -b\qquad\text{for}\quad -1\leq y<0\,.$$

On the other hand, the gradient of the vector $\overrightarrow{Q_3Q_4}$ is $-b/2$. Therefore, since $\dot{x}(t_0)=y(t_0)<0$, $w(t)$ crosses $\overline{Q_3Q_4}$ from below to above at $t=t_0$.

By arguments similar to those above, we can show (24) for $w(t_0)=Q_4$ and that $w(t)$ crosses $C_3(\alpha)-\{Q_4\}$, $C_4(\alpha)$ and $\overline{Q_6Q_7}$ transversely from outside to inside. Thus the positive invariance of D_α is proved.

Next we show our remaining assertion to the effect that if $w(t)\in D_\alpha-D_A$, then $w(T)\in J_A$ for some $T\geq t_0$. Let P_i be the point on J_A corresponding to Q_i on J_α for $1\leq i\leq 8$ (see Figure 1, where the shadowed region denotes D_A). To the contrary, suppose that $w(t)\notin J_A$ for all $t\geq t_0$, that is, $w(t)\in D_\alpha-D_A$ for all $t\geq t_0$. For $w(t)=(x(t),y(t))$, since $x(t)$ is bounded for $t\geq t_0$, there is a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k\to\infty$ as $k\to\infty$, such that

$$\dot{x}(t_k)=y(t_k)\to 0 \qquad \text{as} \quad k\to\infty ,$$

which implies that $w(t_k)$ approaches $\overline{P_4Q_4}$ or $\overline{P_6Q_7}$ as $k\to\infty$. In some neighbourhoods of $\overline{P_4Q_4}$ and $\overline{P_7Q_7}$, where $|x|\geq 1$ and $|y|$ is sufficiently small, we obtain by the same argument as in Step 12 that $|\dot{y}(t)|\geq b$, which guarantees that $w(t)$ crosses $\overline{P_4Q_4}$ or $\overline{P_7Q_7}$ at some t close to t_k. Therefore $w(t)$ makes a complete clockwise revolution around D_A. This is a contradiction, because $w(t)$ cannot cross Γ from above to below. The proof of Theorem 1 is complete.

REMARK 3. In the above argument it is seen that $w(t)$ starting outside D_A never makes a complete revolution around D_A. Such a behavior was already observed for the equation (3) with $g(x)=x$ by setting a critical point at infinity (cf. [20, p. 316]).

4. Unbounded solutions. We show that the condition (v.1) in Theorem1 cannot by dropped, by constructing a continuous function $a(t)$, $1\leq a(t)\leq 6$, such that the equation

$$\ddot{x}+|\dot{x}|\dot{x}+e^{a(t)|x|}x=0 \tag{27}$$

has a solution $x(t)$ with

$$\limsup_{t\to\omega}|x(t)|=\infty \qquad \text{for some } \omega\in R . \tag{28}$$

For the system

$$\dot{x}=y, \quad \dot{y}=-|y|y-e^{a(t)|x|}x , \tag{29}$$

which is equivalent to (27), we shall show that there exists a solution $y(t)$ satisfying

$$\limsup_{t\to\omega}|y(t)|=\infty \tag{30}$$

and that (30) implies (28).

In order to prove the assertion, it is sufficient to construct a continuous function $a(t)$ and τ_2 for any y_0 and τ_1, where $|y_0|>N$ for some constant $N>0$, such that

$$\tau_2>\tau_1 , \qquad 1\leq a(t)\leq 6 \text{ on } [\tau_1,\tau_2] ,$$

$$a(\tau_1)=a(\tau_2)=1$$

and that the solution $(x(t), y(t))$ with $x(\tau_1)=0$ and $y(\tau_1)=y_0$ satisfies

$$|y(\tau_2)|\geq|y_0|+1 \quad \text{and} \quad x(\tau_2)=0 . \tag{31}$$

Let N be a large positive number such that if $|y|\geq N$, then the solution $x=x(y)$ of the equation

$$y^2=\frac{2}{9}+\left(\frac{2}{3}x-\frac{2}{9}\right)e^{3x}$$

satisfies

$$\frac{1}{8}(1-e^{4x})+\frac{x}{2}e^{4x}>(|y|+2)^2,$$

and in the following, take y_0 such that $|y_0|\geq N$. Since the right hand side of (29) is symmetric with respect to the origin, we may assume that $y_0>0$ and set $\tau_1=t_1$.

Step 1. First of all we consider the solution of the system

(32) $$\dot{x}=y, \qquad \dot{y}=-|y|y-e^{|x|}x$$

with $x(t_1)=0$ and $y(t_1)=y_0$. In the first quadrant of the phase plane (32) yields that

(33) $$y\frac{dy}{dx}=-y^2-e^x x,$$

and by setting $z=y^2$, we obtain the linear equation

$$\frac{dz}{dx}=-2z-2e^x x,$$

and hence

$$z=e^{-2x}\left\{y_0^2-2\int_0^x e^{3u}u\,du\right\},$$

that is,

(34) $$y^2=e^{-2x}\left\{y_0^2-\frac{2}{9}-\left(\frac{2}{3}x-\frac{2}{9}\right)e^{3x}\right\}.$$

Setting t_2 to be the first $t>t_1$ such that $y(t)=0$ and putting $x_0=x(t_2)$, we have

(35) $$y_0^2=\frac{2}{9}+\left(\frac{2}{3}x_0-\frac{2}{9}\right)e^{3x_0},$$

which implies that $x_0\to\infty$ as $y_0\to\infty$.

In the fourth quadrant (32) yields

(36) $$y\frac{dy}{dx}=y^2-e^x x,$$

and hence by the same argument as above we have

(37) $$y^2e^{-2x}-2(1+x)e^{-x}=\text{const}.$$

Substituting $x=x_0$ and $y=0$, we obtain

(38) $$y^2=2(1+x)e^x-2(1+x_0)e^{2x-x_0}.$$

Step 2. Secondly we consider the solution of the system

(39) $$\dot{x}=y, \qquad \dot{y}=-|y|y-e^{6|x|}x$$

with $x(t_2)=x_0$ and $y(t_2)=0$. Since (39) yields

(40) $$y\frac{dy}{dx}=y^2-e^{6x}x$$

in the fourth quadrant, by the same argument as in Step 1 we obtain

(41) $$y^2=e^{2x}\left(\frac{1}{8}e^{4x}-\frac{1}{2}xe^{4x}-\frac{1}{8}e^{4x_0}+\frac{1}{2}x_0e^{4x_0}\right).$$

Setting t_3 to be the first $t>t_2$ such that $x(t)=0$ and putting $y_1=y(t_3)$, we see that

$$y_1^2=\left\{\frac{1}{8}(1-e^{4x_0})+\frac{x_0}{2}e^{4x_0}\right\}.$$

Since $|y_0|>N$, (35) implies that $y_1^2>(y_0+2)^2$. Let δ be a small positive number such that

(42) $$\frac{(y_0+2)^2}{2\delta+(y_0+1)^2}\geq e^{2\delta}$$

and let t_4 be a number sufficiently close to t_3, $t_2<t_4<t_3$, such that

(43) $$y(t_4)<-(y_0+2)$$

and

(44) $$0<x(t_4)<\delta.$$

Step 3. Let $a(t, \varepsilon)$ be a function defined for $t\geq t_1$ and a small positive parameter ε such that

$$a(t,\varepsilon)=\begin{cases} 1 & (t_1\leq t\leq t_2) \\ 1+\dfrac{5}{\varepsilon}(t-t_2) & (t_2<t<t_2+\varepsilon) \\ 6 & (t_2+\varepsilon\leq t\leq t_4-\varepsilon) \\ 6-\dfrac{5}{\varepsilon}(t-t_4+\varepsilon) & (t_4-\varepsilon<t<t_4) \\ 1 & (t\geq t_4). \end{cases}$$

Clearly $a(t, \varepsilon)$ is continuous for t, $1\leq a(t, \varepsilon)\leq 6$ and

$$\int_{t_2}^{t_4} |a(t, \varepsilon)-6|\,dt=5\varepsilon . \tag{45}$$

We consider the solution $x(t)=X(t)$ and $y(t)=Y(t)$ of the system (29) with $a(t)=a(t, \varepsilon)$, which satisfies $X(t_1)=0$ and $Y(t_1)=y_0$. Since $a(t, \varepsilon)=1$ for $t_1 \le t \le t_2$, (29) coincides with (32), and hence $x=X(t)$ and $y=Y(t)$ satisfies (34) and $X(t_2)=x_0$ and $Y(t_2)=0$. We set G to be the region in the fourth quadrant bounded by the y-axis and the curve (38) and (41) (see Figure 2). For $t_2<t<t_4$, because $1 \le a(t, \varepsilon) \le 6$, we can see by the comparison of the equation $YdY/dX=Y^2-e^{a(t,\varepsilon)X}X$ with (36) and (40) that $(X(t), Y(t))$ remains in G as long as it belongs to the fourth quadrant.

We will show that if ε is sufficiently small, then

$$(X(t), Y(t)) \in G \quad \text{for} \quad t_2<t<t_4 , \tag{46}$$

$$X(t_4)<\delta \tag{47}$$

and

$$Y(t_4)<-(y_0+2) . \tag{48}$$

For the solution $(x(t), y(t))$ of (39), setting

$$\xi(t)=X(t)-x(t) \quad \text{and} \quad \eta(t)=Y(t)-y(t) ,$$

we see that $\xi(t_2)=\eta(t_2)=0$ because $X(t_2)=x(t_2)=x_0$ and $Y(t_2)=y(t_2)=0$, and that

$$\dot{\xi}=\eta , \tag{49}$$

$$\dot{\eta}=Y^2-y^2-(e^{a(t,\varepsilon)X}X-e^{6x}x) . \tag{50}$$

Therefore, there is a positive constant L such that

$$|\dot{\eta}| \le L(|Y-y|+|X-x|+|a(t, \varepsilon)-6|)=L(|\xi|+|\eta|+|a(t, \varepsilon)-6|) \tag{51}$$

as long as $(X(t), Y(t)) \in G$. Setting $\rho(t)=|\xi(t)|+|\eta(t)|$, we see from [6] that $\rho(t)$ has the right-hand derivative $D^+\rho(t)$ and by (49) and (51) that

$$\begin{aligned} D^+\rho(t) &\le |\dot{\xi}|+|\dot{\eta}| \le (L+1)(|\xi|+|\eta|)+L|a(t, \varepsilon)-6| \\ &=(L+1)\rho(t)+L|a(t, \varepsilon)-6| . \end{aligned}$$

Therefore we obtain

$$\rho(t) \le L\int_{t_2}^{t} |a(s, \varepsilon)-6|\,e^{(L+1)(t-s)}ds \qquad \text{for} \quad t \ge t_2 ,$$

where we used $\rho(t_2)=0$, and hence by (45) $\rho(t)=O(\varepsilon)$ for $t_2 \le t \le t_4$, that is, $|X(t)-x(t)|+|Y(t)-y(t)| \to 0$ on $[t_2, t_4]$ as $\varepsilon \to 0$. Thus, (46), (47) and (48) are proved by (43) and (44).

Step 4. We consider (29) for $t \ge t_4$, and then since $a(t, \varepsilon)=1$ for $t \ge t_4$, it again

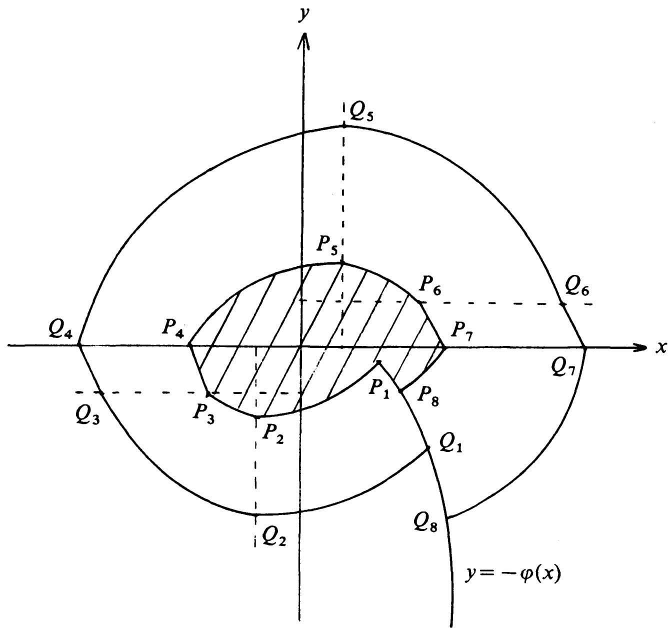

FIGURE 1.

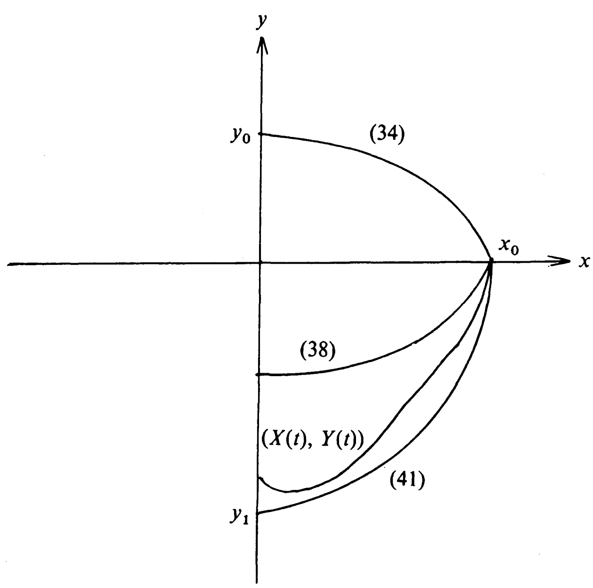

FIGURE 2.

coincides with (32), and hence $x=X(t)$ and $y=Y(t)$ satisfies (37) in the fourth quadrant. Setting t_5 to be the first $t>t_4$ such that $X(t)=0$ and putting $Y_1=Y(t_5)$, we have

$$Y_1^2=2+e^{-2X(t_4)}Y^2(t_4)-2e^{-X(t_4)}(1+X(t_4)),$$

and hence by (47), (48) and (42), we obtain

$$Y_1^2\geq 2+e^{-2\delta}(y_0+2)^2-2(1+\delta)\geq e^{-2\delta}(y_0+2)^2-2\delta\geq(y_0+1)^2.$$

Thus, setting $\tau_2=t_5$, we obtain (31), and (30) implies (28) by (35). The proof is complete.

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