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Ultimate Precision of Adaptive Noise Estimation

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We consider the estimation of noise parameters in a quantum channel, assuming the most general strategy allowed by quantum mechanics. This is based on the exploitation of unlimited entanglement and arbitrary quantum operations, so that the channel inputs may be interactively updated. In this general scenario, we draw a novel connection between quantum metrology and teleportation. In fact, for any teleportation-covariant channel (e.g., Pauli, erasure, or Gaussian channel), we find that adaptive noise estimation cannot beat the standard quantum limit, with the quantum Fisher information being determined by the channel's Choi matrix. As an example, we establish the ultimate precision for estimating excess noise in a thermal-loss channel, which is crucial for quantum cryptography. Because our general methodology applies to any functional that is monotonic under trace-preserving maps, it can be applied to simplify other adaptive protocols, including those for quantum channel discrimination. Setting the ultimate limits for noise estimation and discrimination paves the way for exploring the boundaries of quantum sensing, imaging, and tomography.

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Quantum metrology [1–5] deals with the optimal estimation of classical parameters encoded in quantum transformations. Its applications are many, from enhancing gravitational wave detectors [6,7], to improving frequency standards [8], clock synchronization [9], and optical resolution [10–12], just to name a few. Understanding its ultimate limits is therefore of paramount importance. However, it is also challenging, because the most general strategies for quantum parameter estimation exploit adaptive, i.e., feedback-assisted, quantum operations (QOs) involving an arbitrary number of ancillas.

Adaptive protocols are difficult to study [13–18] but a powerful tool can now be borrowed from the field of quantum communication. In this context, Ref. [19] has recently designed a general and dimension-independent technique which reduces adaptive protocols into a block form. This technique of "teleportation stretching" is particularly powerful when the protocols are implemented over suitable teleportation-covariant channels [19], which are those channels commuting with the random unitaries induced by teleportation. This is a broad class, including Pauli, erasure [20], and bosonic Gaussian channels [21].

In this work, we exploit the tool of teleportation stretching to simplify adaptive protocols of quantum metrology. We discover that the adaptive estimation of noise in a teleportation-covariant channel cannot beat the standard quantum limit (SQL). Our no-go theorem also establishes that this limit is achievable by using entanglement without adaptiveness, so that the quantum Fisher information (QFI) [1] assumes a remarkably simple expression in terms of the channel's Choi matrix. As an application, we set the ultimate adaptive limit for estimating thermal noise in Gaussian channels, which has implications for continuous-variable quantum key distribution (QKD) and, more generally, for measurements of temperature in quasimonochromatic bosonic baths.

Because our methodology applies to any functional of quantum states which is monotonic under completely positive trace-preserving (CPTP) maps, we may simplify other types of adaptive protocols, including those for quantum hypothesis testing [22–26]. Here, we find that the ultimate error probability for discriminating two teleportation-covariant channels is reached without adaptiveness and determined by their Choi matrices. Applications are for protocols of quantum sensing, such as quantum reading [27–34] and illumination [35–38], and for the resolution of extremely close temperatures [39,40].

Adaptive protocols for quantum parameter estimation.— The most general adaptive protocol for quantum parameter estimation can be formulated as follows. Let us consider a box containing a quantum channel $\mathcal{E}_\theta$ characterized by an unknown classical parameter θ. We then pass this box to Alice and Bob, whose task is to retrieve the best estimate of θ. Alice prepares the input to probe the box, while Bob gets the corresponding output. The parties may exploit unlimited entanglement and apply joint QOs before and after each probing. These QOs may distribute entanglement and contain measurements that can always be postponed at the end of the protocol (thanks to the principle of deferred measurement [20]).

In our formulation, we assume that Alice has a local register with an ensemble of systems $\mathbf{a} = \{a_1, a_2, \ldots\}$. Similarly, Bob has another local register $\mathbf{b} = \{b_1, b_2, \ldots\}$. These registers are intended to be dynamic, so that they can be depleted or augmented with quantum systems. Thus, when Alice picks an input system $a \in \mathbf{a}$, we update her

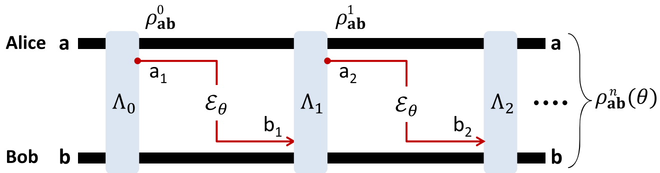

FIG. 1. Arbitrary adaptive protocol for quantum parameter estimation. After preparation of the register state $\rho^0_{\mathbf{ab}}$ by means of an initial QO Λ_0, Alice starts probing the box $\{\mathcal{E}_\theta\}$ by sending a system a_1 from her register, with Bob getting the output b_1. This is repeated n times with each transmission $a_i \to b_i$ interleaved by two QOs Λ_{i-1} and Λ_i. The output state $\rho^n_{\mathbf{ab}}(\theta)$ is finally subject to an optimal measurement.

register as $\mathbf{a} \to \mathbf{a}a$. Then, suppose that system a is transmitted to Bob, who receives the output system b. The latter is stored in his register, updated as $\mathbf{b}b \to \mathbf{b}$.

The first part of the protocol is the preparation of the initial register state $\rho^0_{\mathbf{ab}}$ by applying the first QO Λ_0 to some fundamental state. After this preparation, the parties start the adaptive probings. Alice picks a system $a_1 \in \mathbf{a}$ and send it through the box $\{\mathcal{E}_\theta\}$. At the output, Bob receives a system b_1, which is stored in his register $\mathbf{b}$. At the end of the first probing, the two parties applies a joint QO Λ_1, which updates and optimizes their registers for the next uses. In the second probing, Alice picks another system $a_2 \in \mathbf{a}$, sends it through the box, with Bob receiving b_2 and so on. After n probings, we have a sequence of QOs $\mathcal{P} = \{\Lambda_0, \ldots, \Lambda_n\}$ generating an output state $\rho^n_{\mathbf{ab}}(\theta)$ for Alice and Bob [41]. See Fig. 1.

The final step consists of measuring the output state. The outcome is processed into an unbiased estimator of θ, with an associated protocol-dependent QFI

$$I^n_\theta(\mathcal{P}) = \frac{8\{1 - F[\rho^n_{\mathbf{ab}}(\theta), \rho^n_{\mathbf{ab}}(\theta + d\theta)]\}}{d\theta^2}, \qquad (1)$$

with $F(\rho, \sigma) \coloneqq \mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}$ being the fidelity [42]. By optimizing over all adaptive protocols, we define the adaptive QFI $\bar{I}^n_\theta \coloneqq \sup_{\mathcal{P}} I^n_\theta(\mathcal{P})$, so that the minimum error variance in the estimation of θ satisfies the quantum Cramer-Rao bound (QCRB) [1,2] $\mathrm{Var}(\theta) \geq 1/\bar{I}^n_\theta$.

Teleportation stretching for quantum metrology.—We now compute the adaptive QFI. Consider the class of teleportation-covariant channels in arbitrary dimension as generally defined in Ref. [19]. They correspond to those quantum channels commuting with the random unitaries induced by teleportation, which are Pauli operators at finite dimension and displacement operators at infinite dimension [45–47]. By definition, a quantum channel $\mathcal{E}$ is called "teleportation covariant" if, for any teleportation unitary U, we may write [19]

$$\mathcal{E}(U\rho U^\dagger) = V\mathcal{E}(\rho)V^\dagger, \qquad (2)$$

for some other unitary V. This is a common property, owned by Pauli, erasure, and bosonic Gaussian channels.

Because of Eq. (2), we can simulate the channel $\mathcal{E}$ via local operations and classical communication (LOCC) applied to a suitable resource state. In fact, as explained in Figs. 2(a)–2(b), channel $\mathcal{E}$ can be simulated by a teleportation LOCC $\mathcal{T}$ performed over the channel's Choi matrix $\rho_{\mathcal{E}}$; i.e., we may write [19]

$$\mathcal{E}(\rho) = \mathcal{T}(\rho \otimes \rho_{\mathcal{E}}). \qquad (3)$$

This simulation is intended to be asymptotic for bosonic channels [19]. We consider $\mathcal{E}(\rho) = \lim_\mu \mathcal{T}_\mu(\rho \otimes \rho^\mu_{\mathcal{E}})$, where $\mathcal{T}_\mu$ is a sequence of teleportation LOCCs and $\rho^\mu_{\mathcal{E}} \coloneqq \mathcal{I} \otimes \mathcal{E}(\Phi^\mu)$ is a sequence computed on two-mode squeezed vacuum (TMSV) states Φ^μ [21], so that $\Phi \coloneqq \lim_\mu \Phi^\mu$ defines the asymptotic Einstein-Podolsky-Rosen (EPR)

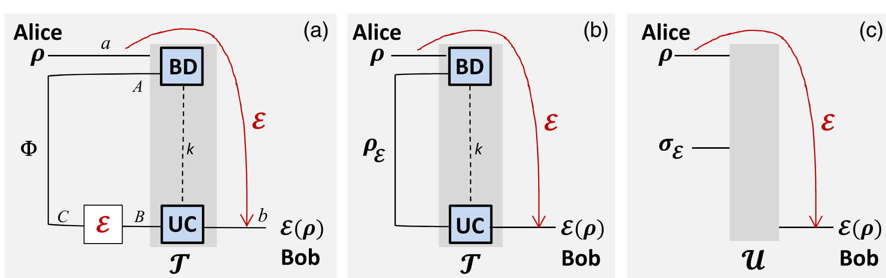

FIG. 2. Teleportation covariance and channel simulation. In panel (a), we consider a teleportation-covariant channel $\mathcal{E}$ (red curvy line) from Alice's system a to Bob's system b. This can be simulated by teleporting system a to system C, by means of a maximally entangled state Φ_{AC} and a Bell detection (BD) on systems a and A, with outcome k. System C is projected onto a state ρ_C which is equal to ρ_a up to a teleportation unitary U_k. Because of Eq. (2), we now have $\rho_B = \mathcal{E}(\rho_C) = \mathcal{E}(U_k\rho_a U_k^\dagger) = V_k\mathcal{E}(\rho_a)V_k^\dagger$ for some other unitary V_k. Upon receiving k from Alice, Bob may undo V_k on system B by applying a unitary correction (UC) V_k^{-1}. Thus, he retrieves the output state $\rho_b = \mathcal{E}(\rho_a)$. Overall, Alice's BD and Bob's UC represent a teleportation LOCC $\mathcal{T}$. As shown in panel (b), this is equivalent to simulate the channel by teleporting the state over the channel's Choi matrix $\rho_{\mathcal{E}} \coloneqq \mathcal{I} \otimes \mathcal{E}(\Phi)$, so that we may write Eq. (3). The teleportation simulation $(\mathcal{T}, \rho_{\mathcal{E}})$ becomes asymptotic $(\mathcal{T}_\mu, \rho^\mu_{\mathcal{E}})$ for bosonic channels. By comparing with panel (c), we see that we have provided a computable design for the tool of quantum simulation [48–50], reducing the quantum operation $\mathcal{U}$ to a teleportation LOCC $\mathcal{T}$, and the (difficult-to-find) program state $\sigma_{\mathcal{E}}$ to the channel's Choi matrix $\rho_{\mathcal{E}}$.

state and $\rho_{\mathcal{E}} := \lim_\mu \rho_{\mathcal{E}}^\mu$ defines the asymptotic Choi matrix [19]. In the following, for any pair of asymptotic states $\rho_{0,1} := \lim_\mu \rho_{0,1}^\mu$, we correspondingly extend a functional f to the limit as $f(\rho_0, \rho_1) := \lim_\mu f(\rho_0^\mu, \rho_1^\mu)$.

The teleportation-based simulation provides a powerful design to the generic tool of quantum simulation [48–50] which is described by

$$\mathcal{E}(\rho) = \mathcal{U}(\rho \otimes \sigma_{\mathcal{E}}), \tag{4}$$

where $\mathcal{U}$ is a trace-preserving QO [51] and $\sigma_{\mathcal{E}}$ is some program state, as in Fig. 2(c). First of all, we establish a simple criterion (teleportation covariance) that allows us to identify channels $\mathcal{E}$ that are simulable as in Eq. (3) and, therefore, programmable as in Eq. (4). Then, we give an explicit solution to Eq. (4), so that $\mathcal{U}$ reduces to teleportation and the program state $\sigma_{\mathcal{E}}$ is found to be the channel's Choi matrix (see Fig. 2). As we will see below, this insight drastically simplifies computations.

For a channel which is "Choi stretchable" as in Eq. (3), we may apply teleportation stretching [19,52]. After stretching, the output $\rho_{\mathbf{ab}}^n$ of an adaptive protocol for quantum or private communication takes the form

$$\rho_{\mathbf{ab}}^n = \bar{\Lambda}(\rho_{\mathcal{E}}^{\otimes n}), \tag{5}$$

where $\bar{\Lambda}$ is trace-preserving LOCC [53]. Here, to simplify quantum metrology, we do not need to enforce the LOCC structure, so that $\bar{\Lambda}$ may be an arbitrary CPTP map. In this sense, the following lemma provides a full adaptation of the tool for the task of parameter estimation [55].

Lemma 1 (stretching of adaptive metrology) Consider the adaptive estimation of the parameter θ of a teleportation-covariant channel $\mathcal{E}_\theta$. After n probings, the output of the adaptive protocol can be written as

$$\rho_{\mathbf{ab}}^n(\theta) = \bar{\Lambda}(\rho_{\mathcal{E}_\theta}^{\otimes n}) = \lim_\mu \bar{\Lambda}_\mu(\rho_{\mathcal{E}_\theta}^{\mu\otimes n}), \tag{6}$$

where $\bar{\Lambda}$ is a θ-independent CPTP map and $\rho_{\mathcal{E}_\theta}$ is the channel's Choi matrix. If channel $\mathcal{E}_\theta$ is bosonic, then the decomposition is asymptotic $(\bar{\Lambda}_\mu, \rho_{\mathcal{E}_\theta}^\mu)$ with a sequence of CPTP maps $\bar{\Lambda}_\mu$ and Choi-approximating states $\rho_{\mathcal{E}_\theta}^\mu$.

By exploiting Lemma 1, we now show that the adaptive estimation of noise in teleportation-covariant channels cannot exceed the SQL and can always be reduced to nonadaptive strategies. In fact, we have the following no-go theorem from teleportation [55].

Theorem 2 No-go: telecovariance implies SQL The adaptive estimation of the noise parameter θ of a teleportation-covariant channel $\mathcal{E}_\theta$ satisfies the QCRB $\mathrm{Var}(\theta) \geq 1/\bar{I}_\theta^n$, where the adaptive QFI takes the form

$$\bar{I}_\theta^n = nB(\rho_{\mathcal{E}_\theta}), \qquad B(\rho_{\mathcal{E}_\theta}) := \frac{8[1 - F(\rho_{\mathcal{E}_\theta}, \rho_{\mathcal{E}_{\theta+d\theta}})]}{d\theta^2}. \tag{7}$$

For large n, the QCRB is achievable by entanglement-based nonadaptive protocols. For bosonic channels, we implicitly assume $F(\rho_{\mathcal{E}_\theta}, \rho_{\mathcal{E}_{\theta+d\theta}}) := \lim_\mu F(\rho_{\mathcal{E}_\theta}^\mu, \rho_{\mathcal{E}_{\theta+d\theta}}^\mu)$.

There are two important aspects in this theorem. The first is the achievability of the bound [62]. The second is the extreme simplification of the adaptive QFI, which becomes a functional of the channel's Choi matrix, computable almost instantaneously for many channels. Because the QFI takes such a simple form, our results are easily extended to bosonic channels [63] and can also be generalized to multiparameter estimation [55]. The teleportation-based approach is so powerful that it is an open problem to find other channels (e.g., programmable) for which we may compute the adaptive QFI beyond the class of teleportation-covariant channels.

Analytical formulas.—Let us use Theorem 2 to study the adaptive estimation of error probabilities in qubit channels [20]. For a depolarizing channel with probability p, we find the asymptotically achievable bound [55]

$$\mathrm{Var}(p) \geq p(1-p)/n. \tag{8}$$

This result is also valid for the adaptive estimation of the probability p of a dephasing channel or an erasure channel [55]. Thus, we show that the bounds of Refs. [50,64] are adaptive in a straightforward way.

Now consider a bosonic Gaussian channel which transforms input quadratures [21] $\hat{x} = (\hat{q}, \hat{p})^T$ as $\hat{x} \to \eta\hat{x} + |1-\eta|\hat{x}_T + \xi$, where η is a real gain parameter, $\hat{x}_T$ are the quadratures of a thermal environment with $\bar{n}_T$ mean number of photons, and ξ is an additive Gaussian noise variable with variance w. A specific case is the thermal-loss channel for which $0 \leq \eta < 1$ and $\xi = 0$. It is immediate to compute the ultimate (adaptive) limit for estimating thermal noise $\bar{n}_T > 0$ in such a channel. By using our Theorem 2 and the formula for the fidelity between multimode Gaussian states [65], we easily derive [55]

$$\mathrm{Var}(\bar{n}_T) \geq \bar{n}_T(\bar{n}_T + 1)/n, \tag{9}$$

which is achievable for large n.

The latter result sets the ultimate precision for estimating the excess (thermal) noise in a tapped communication line [66] or the temperature of a quasimonochromatic bosonic bath. Equation (9) is also valid for estimating thermal noise in an amplifier, defined by $\eta > 1$ and $\xi = 0$. Finally, for $\eta = 1$ and $\xi \neq 0$, we have an additive-noise Gaussian channel. The adaptive estimation of its variance $w > 0$ is limited by [55]

$$\mathrm{Var}(w) \geq w^2/n. \tag{10}$$

Adaptive quantum channel discrimination.—We can simplify other types of adaptive protocols whose performance is quantified by functionals which are monotonic under CPTP maps [67]. Thus, consider a box with two equiprobable channels $\{\mathcal{E}_k\} = \{\mathcal{E}_0, \mathcal{E}_1\}$. An adaptive discrimination protocol $\mathcal{P}$ consists of local registers prepared in a state $\rho^0_{\mathbf{ab}}$, which are then used to probe the box n times while being assisted by a sequence of QOs $\mathcal{P}$, similar to Fig. 1. The output state $\rho^n_{\mathbf{ab}}(k)$ is optimally measured [68] so that we may write the protocol-dependent error probability in terms of the trace distance D

$$p(k' \neq k|\mathcal{P}) = \frac{1 - D[\rho^n_{\mathbf{ab}}(0), \rho^n_{\mathbf{ab}}(1)]}{2}. \tag{11}$$

The ultimate error probability is given by optimizing over all adaptive protocols, i.e., $p_{\text{err}} := \inf_{\mathcal{P}} p(k' \neq k|\mathcal{P})$.

For the discrimination of teleportation-covariant channels, we may write the output state $\rho^n_{\mathbf{ab}}(k)$ using the same Choi decomposition of Eq. (6), proviso that we replace $\rho_{\mathcal{E}_\theta}$ with its discrete version $\rho_{\mathcal{E}_k}$, i.e.,

$$\rho^n_{\mathbf{ab}}(k) = \bar{\Lambda}(\rho^{\otimes n}_{\mathcal{E}_k}), \tag{12}$$

understood to be asymptotic for bosonic channels. We then prove [55] the following result which expresses p_{err} in terms of the trace distance between Choi matrices.

Theorem 3 Consider an adaptive protocol for discriminating two teleportation-covariant channels $\{\mathcal{E}_0, \mathcal{E}_1\}$. After n probings, the minimum error probability is

$$p_{\text{err}} = \frac{1 - D(\rho^{\otimes n}_{\mathcal{E}_0}, \rho^{\otimes n}_{\mathcal{E}_1})}{2}, \tag{13}$$

where $D = \lim_\mu D[\rho^{\mu \otimes n}_{\mathcal{E}_0}, \rho^{\mu \otimes n}_{\mathcal{E}_1}]$ for bosonic channels.

For programmable channels $\{\mathcal{E}_k\}$ with states $\{\sigma_{\mathcal{E}_k}\}$, we may only write the bound $p_{\text{err}} \geq [1 - D(\sigma^{\otimes n}_{\mathcal{E}_0}, \sigma^{\otimes n}_{\mathcal{E}_1})]/2$. In general, this is not achievable because we do not know if $\sigma_{\mathcal{E}_k}$ can be generated by transmission through $\mathcal{E}_k$. By contrast, for teleportation-covariant channels, the bound is always achievable and the optimal strategy is nonadaptive, based on sending parts of maximally entangled states and then measuring the output Choi matrices. Because of the equality in Eq. (13), we may write both lower and upper (single-letter) bounds. Using the Fuchs-van der Graaf relations [69], the quantum Pinsker's inequality [70,71], and the quantum Chernoff bound (QCB) [72–74], we find that the adaptive discrimination of teleportation-covariant channels must satisfy [55]

$$\frac{1 - \sqrt{\min\{1 - F^{2n}, nS\}}}{2} \leq p_{\text{err}} \leq \frac{Q^n}{2} \leq \frac{F^n}{2}, \tag{14}$$

where $F := F(\rho_{\mathcal{E}_0}, \rho_{\mathcal{E}_1})$, $Q := \inf_s \text{Tr}(\rho^s_{\mathcal{E}_0}, \rho^{1-s}_{\mathcal{E}_1})$, and $S := (\ln\sqrt{2}) \min\{S(\rho_{\mathcal{E}_0}||\rho_{\mathcal{E}_1}), S(\rho_{\mathcal{E}_1}||\rho_{\mathcal{E}_0})\}$, with $S(\rho||\sigma)$ being the relative entropy [75]. Here, recall that the QCB is tight for large n [72], so that $p_{\text{err}} \simeq Q^n/2$. All these functionals are asymptotic for bosonic channels.

In particular, for two thermal-loss channels with identical transmissivity but different thermal noise, $\bar{n}_0$ and $\bar{n}_1$, we may take the limit and compute [55]

$$Q = \inf_s [(\bar{n}_0 + 1)^s (\bar{n}_1 + 1)^{1-s} - \bar{n}^s_0 \bar{n}^{1-s}_1]^{-1}. \tag{15}$$

For these channels, it is interesting to study the infinitesimal discrimination $\bar{n}_0 = \bar{n}_T$ and $\bar{n}_1 = \bar{n}_T + d\bar{n}_T$. As we show in a lemma [55], when we consider the discrimination of two infinitesimally close states, ρ_θ and $\rho_{\theta+d\theta}$, the n-copy minimum error probability can be connected with the QCRB for estimating parameter θ. Applying this result to the asymptotic Choi matrices of the thermal-loss channels and taking the limit of large n, we get [55] $p_{\text{err}} \simeq e^{-n\Sigma}/2$, where $\Sigma = [8\bar{n}_T(\bar{n}_T + 1)]^{-1} d\bar{n}^2_T$ for $\bar{n}_T > 0$. For the specific case of $\bar{n}_T = 0$ (infinitesimal discrimination from vacuum noise), we have a discontinuity, and we may write $\Sigma = d\bar{n}_T$ [55]. These results represent the ultimate adaptive limits for resolving two temperatures, e.g., for testing the Unruh effect [39] or the Hawking radiation in analogue systems [40].

Conclusions.—In this Letter, we have established the ultimate limits of adaptive noise estimation and discrimination for the wide class of teleportation-covariant channels, which includes fundamental transformations for qubits, qudits, and bosonic systems. We have reduced the most general adaptive protocols for parameter estimation and channel discrimination into much simpler block versions, where the output states are simply expressed in terms of Choi matrices of the encoding channels. This allowed us to prove that the optimal noise estimation of teleportation-covariant channels scales as the SQL and is fully determined by their Choi matrices. Our work not only shows that teleportation is a primitive for quantum metrology but also provides remarkably simple and practical results, such as the precision limit for estimating the excess noise of a thermal-loss channel, which is a basic channel in continuous variable QKD. Setting the ultimate precision limits of noise estimation and discrimination has broad implications, e.g., in quantum tomography, imaging, sensing, and even for testing quantum field theories in noninertial frames.

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Note added.–Recently, related work [76] appeared on the arXiv.

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[63] In particular, this extension regards the estimation of noise parameters (thermal background) but not loss parameters. For the latter, we always obtain the trivial bound $\mathrm{Var}(\theta) \ge 0$, which comes from $\lim_\mu F(\rho^\mu_{\mathcal{E}_\theta}, \rho^\mu_{\mathcal{E}_{\theta+d\theta}}) = 0$. In fact, we can always perfectly distinguish and estimate two infinitesimally close transmission parameters in the limit of infinite input energy. Finding the optimal adaptive estimation of the loss parameter of a Gaussian channel with an input energy constraint is an open problem subject to investigation.

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