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Ultrafast Consensus in Small-World Networks

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Abstract— **In this paper, we demonstrate a phase transition phenomenon in algebraic connectivity of small-world networks. Algebraic connectivity of a graph is the second smallest eigenvalue of its Laplacian matrix and a measure of speed of solving consensus problems in networks. We demonstrate that it is possible to dramatically increase the algebraic connectivity of a regular complex network by 1000 times or more without adding new links or nodes to the network. This implies that a consensus problem can be solved incredibly fast on certain small-world networks giving rise to a network design algorithm for ultrafast information networks. Our study relies on a procedure called "random rewiring" due to Watts & Strogatz (Nature, 1998). Extensive numerical results are provided to support our claims and conjectures. We prove that the mean of the bulk Laplacian spectrum of a complex network remains invariant under random rewiring. The same property only asymptotically holds for scale-free networks. A relationship between increasing the algebraic connectivity of complex networks and robustness to link and node failures is also shown. This is an alternative approach to the use of percolation theory for analysis of network robustness. We also show some connections between our conjectures and certain open problems in the theory of random matrices.**

Keywords: small-world networks, networked systems, consensus algorithms, phase transition, graph Laplacians, algebraic connectivity, network robustness, random matrices

I. Introduction

Complex networks are abundant in large-scale engineering, biological, and social systems. Some examples include power networks, metabolic and gene networks [17], co-authorship network of scientists [27], biological network of oscillators [44], [19], [18], [24], [39], economic networks [15], sensor networks [10], [30], [7], [38], swarms of networked unmanned autonomous vehicles (UAVs) [32], [36], [6], and self-organizing biological swarms [25], [41]. For recent surveys on complex networks, the reader can refer to [40], [28].

In 1998, Watts & Strogatz [42] introduced a network model called *small-world network* that was capable of interpolating between a regular network and a random network using a single parameter. A small-world is a network with a relatively small *characteristic length*[1]. In a small-world, any two nodes can be linked using a few steps despite the large size of the network. For example, the world-wide web (www) with $n = 8 \times 10^8$ nodes has a characteristic length of 18.5 [3].

The small-world model of Watts & Strogatz initiated a tremendous amount of interest among researchers from multiple fields to study *topological properties* of complex networks. These properties include degree distribution, characteristic length, clustering coefficient (see [42], [28]), robustness to node failure, and search issues. The researchers who have most contributed to this effort came from fields such as statistical physics, computer science, economics, mathematical biology, communication networks, and power networks.

In most engineering and biological complex systems, the nodes have a dynamics—they are not labels or names of actors. In other words, "real-life" engineering networks are interconnection of dynamic systems. The same applies to broad examples of biological networks including gene networks and coupled neural oscillators. From the perspective of systems & control theory, the stability properties of *collective dynamics* of networks of *dynamic agents* is of interest. This motivates exploration of *spectral properties* of complex networks. In the past, the study of spectral properties of random networks has been given little attention. This paper is a first step towards understanding the behavior of Laplacian spectra of complex networks and its application in design of ultrafast information networks.

We use *consensus problems* [35], [33] as a framework to convey our ideas regarding the connections between spectral properties of complex networks and ultrafast solution to distributed decision-making problems for interacting groups of agents. Distributed computation based on solving consensus problems has direct implications on *sensor networks & data fusion* [38], *load balancing* [21], and *swarms/flocks* [32], [36]. Moreover, *synchronization of coupled oscillators* [44], [19], [18], [24] that has received tremendous attention over the past 35 years is a special case of a nonlinear consensus problem in networks [35] regarding the frequency of oscillation of all nodes[2]. For this case, algebraic connectivity is (locally) a measure of speed of synchronization.

One of the first class of complex networks are *random graphs* introduced by Erdös & Rényi (ER) [9] nearly 50 years ago. A random graph can be constructed by connecting any pairs of vertices of the graph with probability p. Random graphs are perhaps the most well-studied model of random networks. The unique feature of the ER model

[1] The average distance between two nodes in the network over all pairs of distinct nodes. The distance is the length of the shortest path connecting two nodes.

[2] Further details regarding the connection between the two problems in the subject of an upcoming article.

is that beyond a critical value $p > p_c \approx 1/n$, a giant connected component forms in the network [9], [40]. This phenomenon is the first known example of a *combinatorial phase transition*. Other forms of combinatorial and algorithmic phase transition phenomena were later discovered in discrete mathematics and computer science.

The next generation of random networks consist of three models: i) *small-world networks* by Watts & Strogatz (WS) [42], ii) semiregular small-world networks by Newman, Moore, & Watts (NMW) [29], and iii) scale-free networks (e.g. www) by Barabási & Albert (BA) [1]. We will describe the WS and NMW network models in details and leave the discussion of the BA model to a future occasion. The ER model is inadequate for our study since it is not guaranteed to be connected unless the graph is relatively dense. That is why we focus on spectral properties of WS and NMW small-world models.

Here is an outline of the paper: Some background on consensus problems from systems & control point of view is presented in Section II. Small-world networks and their semiregular version are described in Section III. Random rewiring procedure for general networks is given in Section IV. Our main numerical results and conjectures on Laplacian spectral properties of small-world networks and network resilience are presented in Section V. The connections between Laplacian of random networks and random matrix theory is discussed in Section VI. Finally, concluding remarks are made in Section VII.

II. Consensus Problems in Networks

Consider a network of integrator agents $\dot{x}_i = u_i$ with topology $G = (V, E)$ in which each agent only communicates with its neighboring agents $N_i = \{j \in V : \{i, j\} \in E\}$ on $G = (V, E)$. Here, $V = \{1, 2, \dots, n\}$ and $E \subset [V]^2$ (the set of 2-element subsets of V) denote the set of nodes and edges/links of the network, respectively. In [35], [33], Olfati-Saber & Murray show that the following linear dynamic system

$$\dot{x}_i(t) = \sum_{j \in N_i} (x_j(t) - x_i(t)) \qquad (1)$$

solves a consensus problem. More precisely, let $a_1, \dots, a_n \in \mathbb{R}$ be n constants, then with the set of initial states $x_i(0) = a_i$, the state of all agents asymptotically converges to the average value $\bar{a} = 1/n \sum_i a_i$ provided that the network is connected. The *collective dynamics* of the agents in (1) can be expressed as

$$\dot{x}(t) = -Lx(t) \qquad (2)$$

where $L = L(G)$ is the *Laplacian matrix* of graph G. The Laplacian is defined as $L = D - A$ where D is a diagonal degree matrix of G with an ith element that is the degree $d_i = |N_i|$ of node i. Let us denote the eigenvalues of Laplacian L by

$$\lambda_1 \leq \lambda_2 \leq \cdots \lambda_n.$$

Note that the Laplacian matrix always has a zero eigenvalue $\lambda_1 = 0$ corresponding to the aligned state $x = (1, 1, \dots, 1)^T$. In addition, if G is connected, then $\lambda_2 > 0$ [13]. Apparently, the analysis of consensus problems in networks reduces to *spectral analysis of Laplacian of the network topology*. Particularly, λ_2 is the measure of *speed of convergence* (or *performance*) of the consensus algorithm in (1) [33]. λ_2 is named the *algebraic connectivity* of the graph by Fiedler [12] due to the following inequality:

$$\lambda_2(G) \leq \nu(G) \leq \eta(G) \qquad (3)$$

where $\nu(G)$ and $\eta(G)$ are node-connectivity and edge connectivity of a graph, respectively (see [4] for definitions). According to this inequality, *a network with a relatively high algebraic connectivity is necessarily robust to both node-failures and edge-failures*. A lower bound on this degree of robustness is $\lceil \lambda_2 \rceil$.

For a network with communication time-delays, the consensus algorithm takes the following form [35], [33]:

$$\dot{x}_i(t) = \sum_{j \in N_i} \{x_j(t - \tau) - x_i(t - \tau)\} \qquad (4)$$

with a collective dynamics

$$\dot{x} = -Lx(t - \tau). \qquad (5)$$

We assume that the time-delay in all links is equal to τ (see [35] for the general case). A necessary and sufficient condition for stability of system (5) is given in [33] as

$$\tau < \tau_{max} = \frac{\pi}{2\lambda_n}. \qquad (6)$$

Thus, λ_n is a measure of *robustness to delay* for reaching a consensus in a network.

The main result of this paper is that λ_2 for a regular network can be increased multiple orders of magnitude via changing the *inter-agent information flow G* without increasing the total number of the links of the network. Moreover, this change has a negligible effect on λ_n (the system remains robustness to delay).

Some variations of consensus problems on graphs include the following areas: networks with switching topology [16], [33], [26], [34], consensus on digraphs [33], [26], and asynchronous consensus [14].

III. Small-World Networks

Small-world phenomenon is a feature of certain complex networks in which any two arbitrary nodes can be connected using a few links [23]. This means that the average distance between two nodes (i.e. characteristic length) is relatively small in small-worlds.

In 1998, Watts & Strogatz (WS) [42] introduced a model called *small-world network* with the capability to interpolate between a regular lattice and a random graph using a single parameter p (see Fig. 1). They demonstrated that broad examples of social, biological and physical complex networks fall within the category of small-world networks including power networks in the western US, the neural network of a worm called C. Elegans, a certain co-authorship network of scientists, and the network of actors who played in the same movies [42], [40].

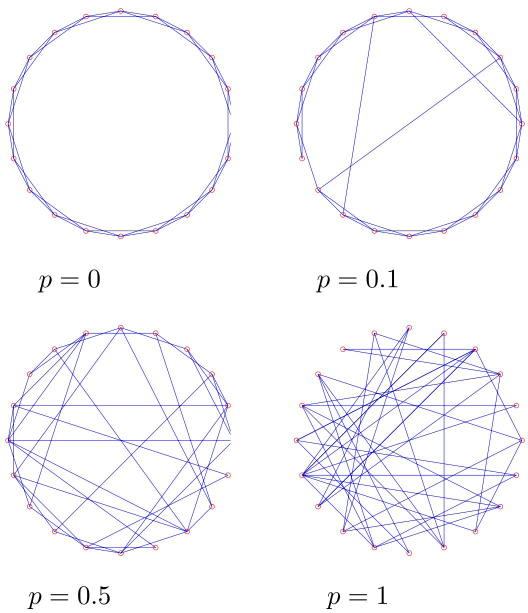


Fig. 1. Small-world networks $G(p)$ with $(n, k) = (20, 2)$ for $p = 0, 0.1, 0.5, 1$.

To construct a small-world network, one starts with a one-dimensional lattice on a ring with n nodes in which every node is connected to its nearest neighbors up to the range k. Let us denote this lattice by $G_0 = C(n, k)$. Then, one rewires every link with probability p by changing one of the endpoints of a link uniformly at random. No self-loops or repeated links are allowed. (Note: a mathematically precise rewiring algorithm will be presented later for an arbitrary network). Fig. 1 demonstrates small-world networks obtained via random rewiring for various values of parameter p.

In [42], Watts & Strogatz numerically showed that the characteristic length l of a small-world network considerably reduces over the range $0.0001 \leq p \leq 0.01$ and remains almost unchanged for $p > 0.01$. This indicates that random rewiring with a small value of p creates a small-world out of a regular network that originally has a large diameter.

Newman, Moore, & Watts (NMW) [29] introduced a modified form of the small-world network model that is the addition of a regular lattice plus a random graph. No rewiring is needed. Starting with a lattice every two nodes are connected with a probability of ϕ per number of links of the initial lattice. The NMW model has $nk + nk\phi$ links on average. Using mean-field theory from statistical physics, Newman *et al.* analytically derived a formula for the characteristic length of the network as $l = \frac{n}{k} f(nk\phi)$ with

$$f(x) = \frac{1}{2\sqrt{x^2 + 2x}} \tanh^{-1} \frac{x}{\sqrt{x^2 + 2x}}. \tag{7}$$

We refer to the Newman-Moore-Watts model as a *semiregular small-world network*. Semiregular small-world model can be effectively used as an approximate model of the WS model. The degree distribution of both models is known and can be found in [28].

Our main contribution is to show that the *algebraic connectivity of a small-world network can be made more than 1000 times greater than a regular network*. This means that small-world networks go through a *spectral phase transition phenomenon* that was unknown before. This spectral phase transition allows achieving *ultrafast consensus in small-world networks*.

Remark 1. By spectral properties of a graph, we mean spectral properties of the Laplacian matrix which fundamentally differs from the spectral properties of the adjacency matrix of a graph as considered in [11]. The spectrum of the adjacency matrix has no relevance to stability properties of system (2).

IV. Random Rewiring Algorithm: Network Evolution

The random rewiring procedure in [42] can be generalized to networks with arbitrary topologies in a straightforward manner. For future applications, we formally describe this algorithm in details. The byproduct of this formalism is the fact that small-world networks can be obtained as the limit of a finite-time evolution of a *dynamic graph* [22] with three key elements: a non-deterministic *graphical dynamics* specified by random rewiring algorithm, an initial state that is a regular lattice, and a terminal state that is a small-world network.

Let $J_i = N_i \cup \{i\}$ denote the set of inclusive neighbors of node i and $J_i^c = V \setminus J_i$ denote the set of non-neighbors. Let $e = \{i, j\}$ be an edge of the graph G and $e_d = (i, j)$ be its oriented form. The rewired edge is a new edge $e'_d = (i, j')$ in which j' is a node in J_i^c that is chosen uniformly at random with probability $q_i = 1/|J_i^c|$. Let $e' = \{i, j'\}$ be the unoriented form of e'_d. The randomly rewired graph $G' = R_p(G, e)$ is obtained from G by replacing the edge $e = \{i, j\}$ by its rewired version $e' = \{i, j'\}$ with probability p. If a node is connected to every other node, there are no non-neighbors (or $J_i^c = \emptyset$) and we set $j' = j$. In case $J_i = V$, no rewiring occurs. Here, R_p is a random edge rewiring operation that rewires a single edge of the graph with probability p. The following diagram clarifies the edge rewiring procedure:

$$\{i, j\} \xrightarrow[\text{orient}]{} (i, j) \xrightarrow[\text{rand. rewire}]{} (i, j') \xrightarrow[\text{unorient}]{} \{i, j'\}$$

Let $e_1, e_2, \ldots, e_m$ be the sequence of the edges of graph G. Set $G_0 = G$ and define the following graphical dynamics

$$G_{t+1} = R_p(G_t, e_{t+1}), \quad t = 0, 1, 2, \ldots, (m - 1) \tag{8}$$

Then, the *randomly rewired network* is defined as $G(p) := G_m$ with $m = |E|$. Clearly, (8) can be viewed as a discrete-time dynamic system with a state that is a dynamic graph of order n driven by an input $v_t = e_{t+1}$. The special choice of $G_0 = C(n,k)$ and $m = nk$ gives the small-world networks in [42]. The benefit of our formalism is that any other type of lattice or regular network can be used as an initial state of system (8). This includes 2-D and 3-D *toroidal grids* and networks induced by α*-lattices* [32].

Here, our main focus will be devoted to random rewiring of the ring lattice $G_0 = C(n,k)$ with purely local links. In comparison, a small-world network contains *nonlocal interconnections* (or shortcuts) as well as local links. We will demonstrate that the existence of nonlocal links are crucial in dramatic improvement of algebraic connectivity of small-world networks.

Considering that $G(p)$ is a random network, its eigenvalues are random variables. This means that $\lambda_i(p) = \lambda_i(G(p)) := \lambda_i(L(G(p)))$ are random variables for $i = 1,\ldots,n$. Thus, by saying "algebraic connectivity" of a small-world network $G(p)$, we mean the expected value $\bar{\lambda}_2(G(p))$ of the random variable $\lambda_2(G(p))$. Throughout this paper, with a slight abuse of notation, we refer to "expected value of algebraic connectivity" of a random network as "algebraic connectivity" and denote it by $\lambda_2(p)$.

Even for a highly structured graph such as $C(n,k)$, the probability space of $G(p)$ is very large for $n \gg 1$ with $k = O(log(n))$. One approximate way to calculate $\lambda_2(p)$ is to generate a number of instances of $G(p)$ for a given $p > 0$ using the random rewiring algorithm. Then, use the average value of their algebraic connectivity as an estimate of $\lambda_2(L(G(p)))$.

V. Spectral Phase Transition in Small-World Networks (main results)

In this section, we characterize the behavior of algebraic connectivity and λ_n of small-world networks based on a set of systematic numerical experiments on complex networks. This allows us to formulate formal conjectures regarding the behavior of λ_2 and λ_n of small-world networks that can be later tackled by various theories. Here are a number of motivating questions that guide our study:

1) When does $\lambda_2(p)$ increase as a function of p?
2) Is $\lambda_2(p)$ a monotonic function of p?
3) Does random rewiring increase $\lambda_2(G_t(p))$ (on average for a fixed p) as a function of the evolution time-step t?
4) Is there an interval for p such that $\lambda_2(p)$ increases by a factor of 100, 1000, or 10,000?
5) What happens to $\lambda_n(p)$ upon rewiring?

So far, there have been no analytic or experimental studies on spectral properties of small-world networks that address any of the above questions. Answering these questions is crucial in further understanding of properties of complex networks of interconnected dynamic systems that arise in engineering and biological systems. This knowledge is particularly beneficial for *design of ultrafast information networks*.

TABLE I
Test Graphs

$C(n,k)$	n	k	$m = \|E\|$
g_1	100	2	200
g_2	200	3	600
g_3	500	3	1500
g_4	1000	5	5000

We generate multiple (10 to 20) samples of small-world networks $G(p)$ for each p with initial states specified in Table I. The parameter p is chosen on a logarithmic scale between 0.01 to 1 (25 data points). In fact, for $0 < p < 0.01$, no significant change in $\lambda_2(p)$ could be observed. The last entry of Table I (or g_4) corresponds to the graph parameters used to create the data in Fig. 2 of [42] which is added for comparison purposes. All entries of the table satisfy $k \approx \log(n)/2$ meaning that $m = |E| = O(n\log(n))$.

Definition 1. (algebraic connectivity gain & robustness to delay) Let $\lambda_i(p) = \lambda_i(G(p))$ and note that $G(0) = G_0$. We refer to $\gamma_2(p) = \lambda_2(p)/\lambda_2(0)$ as the *algebraic connectivity gain* of $G(p)$. In addition, $\gamma_n(p) = \lambda_n(p)/\lambda_n(0)$ is called the measure of *robustness to delay* of the network $G(p)$ (as $\gamma_n(p)$ increases, the network can tolerate smaller delays).

A. Behavior of λ_2

The algebraic connectivity gain of small-world networks $G(p)$ evolved from regular networks g_2, g_3, g_4 (defined in Table I) is shown in Fig. 2. The curve of $\gamma_2(p) = \lambda_2(p)/\lambda_2(0)$ has a S-shape that essentially remains the same for various network parameters. Each data point in this figure is obtained by averaging over 10 randomly rewired networks. The actual data from the simulation runs for g_4 is shown in Fig 3. Here are some observations and remarks:

i) A *phase transition in algebraic connectivity gain* γ_2 can be observed that starts around the critical value $p_c \approx 0.1$ regardless of the network size n and continues to $p_f = 0.68$ (the 23rd data point). This critical value is 10^3 times larger than the parameter value for the onset of the small-world phenomenon in [42] for $G_0 = C(1000,5)$ (Fig. 2 (d)).

ii) The maximum value of γ_2 is at the same order as the network size or greater, i.e. $\gamma_2^* = \max_p \gamma_2(p) \approx O(n)$. If this happens to be true for larger networks (e.g. $n = 10^4$ and larger), the implication is that λ_2 *of a complex regular network can be increased by multiple orders of magnitude (e.g. 4 or more) via random rewiring*. Apparently, for a network with $n = 1000$ nodes, λ_2 can be increased more than 1500 times over an interval of p.

iii) Random rewiring with high p does not necessarily increase λ_2 for complex regular networks. In fact, according to Fig. 2 (b), rewiring with $p = 1$ might lead to a gain that is smaller than γ_2^* for that network.

iv) We observed that in 100% of the 250 rewiring procedures performed on four regular networks preserved the connectivity of the network during the rewiring iterations for all $p \in (0, 1]$. This property does not hold for networks that are too sparse (e.g. $n \geq 100, k = 1$).

v) Random rewiring of a regular network that is not sufficiently complex does not necessarily increase λ_2! (see [31] for details). It turns out that the *dramatic increase in* λ_2 *requires both random rewiring with a high* p *and high complexity of the network.*

vi) For networks with $n \geq 100$, the algebraic connectivity gain $\gamma_2(p)$ is monotonically increasing in p regardless of the network parameters n, k over a large sub-interval $[p_c, p_f]$ of $[0, 1]$ (e.g. $p_c = 0.1$, $p_f = 0.68$).

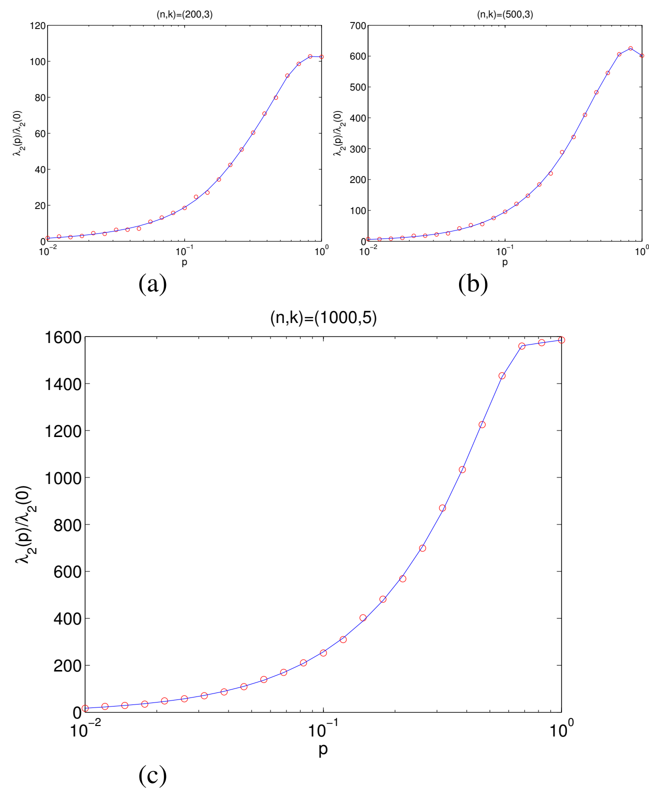


Fig. 2. The *S-shape curve* of algebraic connectivity gain $\lambda_2(p)/\lambda_2(0)$ for small-world networks starting from lattices a) g_2, b) g_3, and c) g_4.

From the above discussion, one can conclude that the complexity of small-world networks serves a great purpose: one can increase algebraic connectivity of a regular network via random rewiring by a factor of $O(n)$. The larger the size of the network, the greater the gain $\gamma_2(p)$. We summarize these assertions in the form of the following conjecture that captures the essential aspects of the behavior of $\lambda_2(p)$ for small-world networks.

Conjecture 1. *Let* $G(p)$ *with* $p > 0$ *be a small-world network that is evolved from an initial regular lattice* $G_0 = C(n, k)$ *with* $k = O(\log(n))$. *Then, for a network* $G(p)$ *that is sufficiently large, the following statements hold:*

i) *The exists a sub-interval* $[p_c, p_f]$ *of* $[0, 1]$ *of length* $\geq 1/2$ *such that* $\lambda_2(p)$ *of* $L(G(p))$ *is on average monotonically increasing in* p.

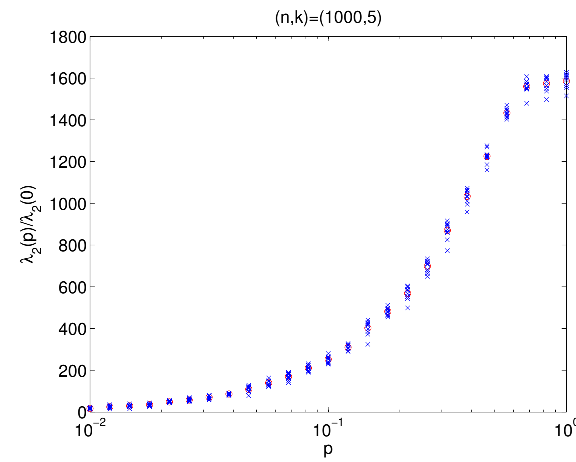


Fig. 3. Individual algebraic connectivity gains $\gamma_2(p)$ corresponding to the averaged data shown in Fig. 2 (c).

ii) *There exists a* $p^* > 0$ *such that* $\gamma_2(p) > n, \forall p > p^*$.

Now, let us explore whether the rewiring process on average increases $\lambda_2(p)$ during the evolution of small-world networks. For doing so, we have plotted the estimates of $\lambda_2(p)$ and the individual runs for 20 simulation runs in Fig. 4. During a large portion of the network evolution, $\lambda_2(p)$ is monotonically increasing as a function of the evolution time-step. For each individual simulation run, $\lambda_2(p)$ does not change monotonically. Though, in all cases, random rewiring increase λ_2 by 2-3 orders of magnitude. These observations can be also summarized as a conjecture:

Conjecture 2. *Let* $G_0 = C(n, k)$ *be a lattice with* $k = O(\log(n))$. *Let* $G_t(p)$ *denote the state of the network during its random rewiring evolution according to graphical dynamics* (8) *at time-step* t *with* $t = 1, 2, \ldots, m$ *and* $m = nk$. *Then, for a sufficiently large-scale network, the mean of* $\lambda_2(L(G_t(p)))$ *is monotonically increasing in* t *during the last* $\lceil m/2 \rceil$ *iterations of its evolution.*

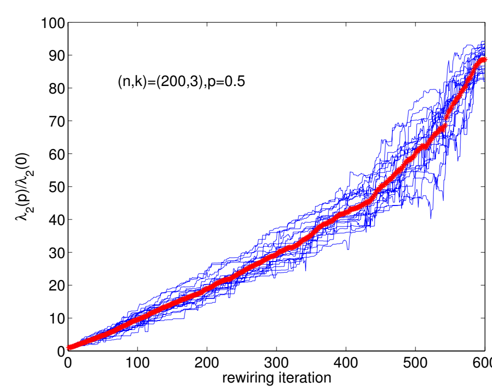
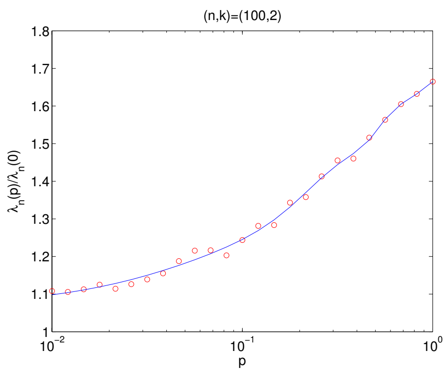


Fig. 4. Variations of $\gamma_2(p)$ during network evolution.

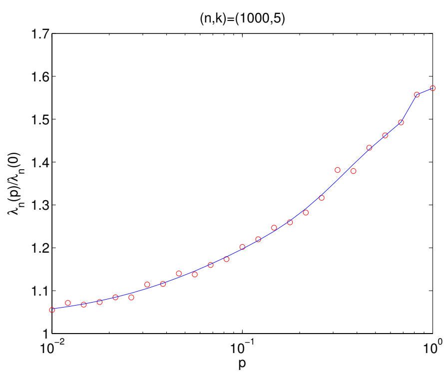
B. Behavior of λ_n

Given that λ_n is a measure of robustness to delays for a consensus algorithm, the main question that remains is whether a dramatic increase in $\gamma_2(p)$ leads to a considerable decrease in the robustness to delay gain $\gamma_n(p)$? To answer this question, we need to inspect the plots of $\gamma_n(p)$ for small-world networks with various parameters.

Fig. 5 shows the variation of the robustness to delay gain $\gamma_n(p)$ for various networks. Clearly, regardless of the size of the network, $\gamma_n(p)$ does not change significantly in any case. Therefore, a dramatic increase in $\gamma_2(p)$ only slightly increases $\gamma_n(p)$. In fact, $\gamma_n(p) < 2$ for all p for every case shown in Fig. 5.



(a)

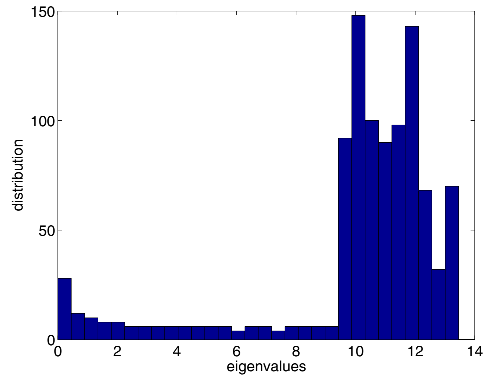


(b)

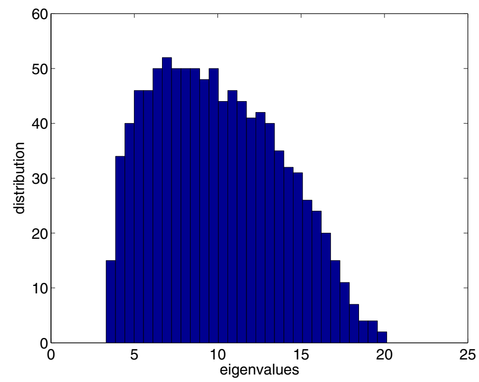
Fig. 5. $\gamma_n(p)$ of small-world networks for networks with different sizes: a) g_1 and b) g_4. One observes that $1 < \gamma_n(p) < 2$ regardless of the size.

C. Spectral Phase Transition

Fig. 6 shows the distribution of bulk eigenvalues (excluding $\lambda_1 = 0$) of a regular lattice $G_0 = C(1000, 5)$ and a small-world network $G(p)$ with $p = 0.68$ that is evolved from G_0. Numerically, it was determined that $\lambda_2(0) = 2.2 \times 10^{-3}$ and $\lambda_2(p) \approx 3.3$. Let $\kappa = \lambda_n/\lambda_2$ denote the (modified) *condition number* of L. Then, the condition number of Laplacian of the lattice in Fig. 6 (a) is approximately 10^3 times larger than the condition number of the small-world network $G(p)$ in Fig. 6 (b). Thus, upon random rewiring, the entire Laplacian spectrum of the network shifts towards λ_n. We refer to this dramatic shift of the bulk spectrum of a complex network as *spectral phase transition phenomenon*.



(a)



(b)

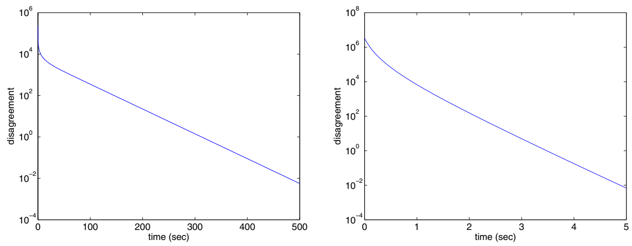
Fig. 6. Distribution of the bulk Laplacian spectrum of two complex networks with $n = 1000$ nodes: a) a regular lattice and b) a small-world network $G(p)$ with $p = 0.68$.

Let us define the *disagreement* of a group of agents with state x as $\varphi(x) = x^T L x$ [35], [33]. For a network with $n = 200$ and $k = 3$, the disagreement of a group of dynamic agents solving a consensus problem is shown in Fig. 7. The initial state of the agents is chosen as $x_i(0) = i$ for $i = 1, \ldots, n$. It takes 100 times longer to reach a consensus in a regular network compared to a small-world network $G(p)$ with $p = 0.68$. A communication network that implements such an ultrafast information flow via *routing* can be chosen to be a small-world model with $p = 0.01$. This way a network with few nonlocal physical links implements an information flow with numerous nonlocal information links.

D. Mean of the Bulk Eigenvalues of Complex Networks

The following theorem provides an invariance property of the bulk Laplacian spectrum of small-world models that are obtained via random rewired from *any* lattice:

Theorem 1. *(mean invariance of bulk eigenvalues) Let $G(p)$ be a small-world network of order n evolved from an arbitrary lattice G_0 with m links. Then, the mean $\bar{\lambda} = \frac{1}{n-1}\sum_{i\geq 2}\lambda_i$ of the bulk eigenvalues of Laplacian of $G_t(p)$ is an invariant quantity that does not depend on p or t (the evolution step-size). Moreover, $\bar{\lambda}(G(p)) = \bar{\lambda}(G_0)$.*



(a) (b)

Fig. 7. Comparison of time to reach a consensus between a regular network and a small-world model: a) $T_{\text{lattice}} = 500$ seconds and b) $T_{\text{small-world}} = 5$ seconds.

Proof. For a graph G with m edges, Laplacian L, and eigenvalues $\lambda_1 \leq \ldots, \lambda_n$, we have

$$(n-1)\bar{\lambda}(G) = \sum_{i\geq 2} \lambda_i = \sum_{i\geq 1} \lambda_i = \text{trace}(L) = \sum_i d_i = 2m.$$

Thus, $\bar{\lambda}(G) = 2m/(n-1)$. Since the number of edges and nodes of $G_t(p)$ remain invariant during random rewiring, we get $\bar{\lambda}(G_t(p)) = \bar{\lambda}(G_0)$. Furthermore, setting $t = m$ gives $\bar{\lambda}(G(p)) = \bar{\lambda}(G_0)$. □

Keep in mind that G_0 is not restricted to be a ring lattice in Theorem 1. For the two distributions of the bulk eigenvalues shown in Fig. 6, the mean $\bar{\lambda}$ of the bulk spectra of G_0 and $G(p)$ are the same and equal to $\bar{\lambda} = 10^4/999 \approx 10$ (or $\bar{\lambda} \approx 2k$).

A modified version of Theorem 1 for the NMW model can be stated as follows: Let $G(\phi)$ be a semiregular small-world network with parameters n, k, ϕ. Then, on-average $\bar{\lambda}(G(\phi)) = \bar{\lambda}(G_0) + 2nk\phi/(n-1) \approx 2k(1+\phi)$ for $n \gg 1$.

The invariance property of Theorem 1 does not hold for scale-free networks because the number of their nodes and links grows by time (see [1], [2] for more details). However, it is possible to calculate the limit of $\bar{\lambda}$ after a long evolution period. Let $G(t)$ be a scale-free network at the evolution time-step t and assume initially the network started with a finite (but unknown) number of nodes n_0 and no links. At every time-step t, one node is added and connected to $s \leq n_0$ existing nodes in any manner (including preferential growth [2]). Then, $m(t) = st$ and $n(t) = n_0 + t$. Thus, $\bar{\lambda}(G(t)) = 2m(t)/(n(t) - 1) \approx 2s$ for a well-developed network with $t \gg 1$.

For a random graph Γ with parameter p_0, $\bar{\lambda}(\Gamma) = np_0$. Given that a small-world model $G(p)$ with $p = 1$ behaves like a random graph with $p_0 = 2nk/n(n-1) \approx 2k/n$ for $n \gg 1$, it turns out that $\bar{\lambda}(G(p)|_{p=1}) = np_0 \approx 2k$. This value is consistent with our previous calculation of $\bar{\lambda}$ for the small-world model. Note that we have managed to determine $\bar{\lambda}$ for all four models of complex networks.

E. Algebraic Connectivity and Robustness to Failures

According to the Fiedler's inequality in (3), increasing the algebraic connectivity of a network renders the network robust to link and node failures. Since $\lceil \lambda_2 \rceil = 4$ for the spectrum of $G(p)$ in Fig. 6 (b), we have $\eta(G(p)) \geq \nu(G(p)) \geq 4$. Thus, any 4 links or nodes of a sample of the small-world network $G(p)$ with $\lambda_2 = 3.3$ can fail and still the network will remain connected. This is an alternative approach to the use of *percolation theory* for analysis of network resilience [5], [28].

VI. Connections to Random Matrices

The question is that how one proves any of the aforementioned conjectures? A look at the structure of the Laplacian matrix $L = D - A$ makes it clear that L is a random matrix, i.e. a matrix with entries that are random variables. The theory of random matrices [20] was originally motivated by the study of many-body particle systems in physics. The first major result in random matrices is the original work of Wigner [43]. Specifically, we are interested in distribution of individual eigenvalues of Laplacian and not how the entire spectrum is distributed.

The *diagonal elements of L are random variables with identical distributions to the degree distribution of a random network*. Hopefully, the degree distributions of all random network models are known (see [28]). However, the degree distribution and the distribution of all off-diagonal elements of L differ. The author is not aware of any results that can deal with *eigenvalue distribution of random matrices with entries that have heterogeneous distributions*. We assume this is an open problem that needs to be addressed by mathematical physicists who specialize in such distribution calculations [8], [37].

VII. Conclusions

We demonstrated that the algebraic connectivity of a regular network can be considerably increased by a factor of 1000 via random rewiring that turns a local link to a nonlocal link (no new links are added). This procedure was originally introduced by Watts & Strogatz [42]. This phase transition in algebraic connectivity of small-world networks makes them ideal candidates for design of ultrafast information networks.

We posed two conjectures based on a numerical analysis of the behavior of λ_2 and λ_n for small-world networks. We proved that the mean of the bulk Laplacian spectrum of small-world networks is invariant and does not change via rewiring. This property only holds asymptotically for scale-free networks. A relationship between increasing the algebraic connectivity of complex networks and network robustness to link and node failures was also demonstrated.

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