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Ultraviolet Stability in Euclidean Scalar Field Theories

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Abstract. We develop a technique for reducing the problem of the ultraviolet divergences and their removal to a free field problem. This work is an example of a problem to which a rather general method can be applied. It can be thought as an attempt towards a rigorous version (in 2 or 3 space-time dimensions) of the analysis of the structure of the functional integrals developed in [9], the underlying mechanism being essentially the same as in [11, 3].

1. Introduction

The free euclidean field in $\mathbb{R}^d$ is the gaussian field with covariance operator:

$$C=(1-D)^{-1}, \tag{1.1}$$

where D is the Laplace operator in $\mathbb{R}^d$.

We call $(\varphi_\xi)_{\xi\in\mathbb{R}^d}$ the random field with the above covariance and we shall represent φ as a sum of independent, identically distributed up to scale factors, random fields $(\varphi_\xi^{(N)})_{\xi\in\mathbb{R}^d}$, $N=0,1,\ldots$. The field $\varphi^{(N)}$ has, by definition, the following covariance operator:

$$C^{(N)}=(\gamma^{2N}-D)^{-1}-(\gamma^{2N+2}-D)^{-1} \tag{1.2}$$

and γ will be appropriately chosen close to 1. From (1.2) it follows that if $d\leqq 3$ the kernel $C_{\xi\eta}^{(N)}$ of $C^{(N)}$ in $\mathbb{R}^d$ is finite when $\xi=\eta$:

$$C_{\xi\xi}^{(N)}\equiv C_{00}^{(N)}=\gamma^{(d-2)N}\frac{\gamma^2-1}{(2\pi)^d}\int\frac{d^dk}{(1+k^2)(\gamma^2+k^2)}\equiv c_\gamma\gamma^{(d-2)N}. \tag{1.3}$$

Hence it will be convenient to introduce the normalized field:

$$z_\xi^{(N)}=\frac{\varphi_\xi^{(N)}}{\sqrt{2C_{\xi\xi}^{(N)}}}=\frac{\varphi_\xi^{(N)}}{\sqrt{2c_\gamma\gamma^{(d-2)N}}} \tag{1.4}$$

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and to represent φ as:

$$\varphi_\xi = \sum_{N=0}^{\infty} \sqrt{2c_\gamma \gamma^{(d-2)N}}\, z_\xi^{(N)}. \tag{1.5}$$

We shall furthermore define the cut-off fields $\varphi^{[\leq N]}$ with length cut-off γ^{-N} as:

$$\varphi_\xi^{[\leq N]} = \sum_{k=0}^{N} \sqrt{2c_\gamma \gamma^{(d-2)k}}\, z_\xi^{(k)}. \tag{1.6}$$

Such fields are, also, normalizable: their normalized representatives can be defined as:

$$X_\xi^{(N)} = \frac{\varphi_\xi^{[\leq N]}}{\sqrt{2c_\gamma \sum_{k=0}^{N} \gamma^{(d-2)k}}} \tag{1.7}$$

which obey the recursion relation:

$$X_\xi^{(N)} = \frac{z_\xi^{(N)} + \sqrt{\Gamma_N} X_\xi^{(N-1)}}{\sqrt{1+\Gamma_N}}, \tag{1.8}$$

where:

$$\Gamma_N = \sum_{k=0}^{N-1} \frac{\gamma^{(d-2)k}}{\gamma^{(d-2)N}} = \frac{1-\gamma^{-(d-2)N}}{\gamma^{d-2}-1} = \begin{cases} \dfrac{1-\gamma^{-N}}{\gamma-1} & d=3 \\ N & d=2. \end{cases} \tag{1.9}$$

We shall call $\hat{P}_i$ the probability distribution of $z^{(i)}$ and $\hat{\mathscr{E}}_i$ shall be the expectation operation (i.e. integration) with respect to $\hat{P}_i$. We define also $P_N = \prod_{i=0}^{N} \hat{P}_i$.

If I denotes a cube centred at the origin the "bare interaction" is defined as[1]

$$V_{0,I}^{(N)} = -\lambda \int_I :(\varphi_\xi^{[\leq N]})^4: d\xi. \tag{1.10}$$

The "third order renormalized interaction" is defined as:

$$V_I^{(N)} = V_{0,I}^{(N)} - \frac{1}{2!}\langle (V_{0,I}^{(N)})^2\rangle_{(2)} - \frac{1}{2!}\langle (V_{0,I}^{(N)})^2\rangle_{(0)} - \frac{1}{3!}\langle (V_{0,I}^{(N)})^3\rangle_{(0)}, \tag{1.11}$$

1 As usual

$$:(\varphi_\xi^{[\leq N]})^n: = \left(2c_\gamma \sum_{k=0}^{N} \gamma^{(d-2)k}\right)^{n/2} \mathrm{H}_n(X_\xi^{(N)}) \equiv (2c_\gamma \gamma^{(d-2)N}(1+\Gamma_N))^{n/2} H_n(X_\xi^{(n)}),$$

where H_n is the n-th Hermite polynomial, e.g. $H_4(x) = x^4 - 3x^2 + \frac{3}{4}$.

where

$$\frac{1}{2}\langle (V_{0,I}^{(N)})^2\rangle_{(2)} = \frac{\lambda^2 4^2\cdot 3!}{2}\int_{I^2} d\xi\, d\eta (C_{\xi\eta}^{[\leqq N]})^3 : (\varphi_\xi^{[\leqq N]})^2 :, \tag{1.12}$$

$$\frac{1}{2}\langle (V_{0,I}^{(N)})^2\rangle_{(0)} = \frac{\lambda^2 4!}{2}\int_{I^2} d\xi\, d\eta (C_{\xi\eta}^{[\leqq N]})^4, \tag{1.13}$$

$$\frac{1}{3!}\langle (V_{0,I}^{(N)})^3\rangle_{(0)} = -\frac{\lambda^3}{3!}\binom{4}{2}^3 (2!)^3 \int_{I^3} d\xi\, d\eta\, d\zeta (C_{\xi\eta}^{[\leqq N]})^2 (C_{\eta\xi}^{[\leqq N)2} (C_{\xi}^{[\leqq N]})^2, \tag{1.14}$$

and

$$C_{\xi\eta}^{[\leqq N]} \equiv \int \varphi_\xi^{[\leqq N]}\varphi_\eta^{[\leqq N]} P_N(dz). \tag{1.15}$$

The "ultraviolet problem" that we study in this paper is the following: prove the existence of $E_+(\lambda)$, $E_-(\lambda)$ such that:

i) $$\exp - E_-(\lambda)|I| \leqq \int \exp V_I^{(N)} P_N(dz) \leqq \exp E_+(\lambda)|I| \tag{1.16}$$

if $|I|$ is the volume of I;

ii) $$\lim_{\lambda\to 0} \lambda^{-3} E_\pm(\lambda) = 0. \tag{1.17}$$

The technique that we use would allow to treat more general problems and does not distinguish between the $d=2$ and the $d=3$ cases.

Our technique is inductive: we shall obtain the estimates by successively integrating over $z^{(N)}, z^{(N-1)}, \ldots$, and it will be necessary to really do only one step in this process. The structure of the integrals does not change after each integration because of our scale invariant choice of the regularization. The fields $z^{(N)}$, $z^{(N-1)}, \ldots$, are, in fact, essentially identical in a probabilistic sense and are independent: the distribution of $z^{(N)}$ is the same as that of the field $z^{(0)}$ regarded on a scale γ^{-N}. In probability one has $z_\xi^{(N)} = z_{\gamma^N\xi}^{(0)}$ and this makes it convenient to introduce the random field

$$Z_\xi^{(N)} = z_{\gamma^{-N}\xi}^{(N)} \tag{1.18}$$

which will be useful later. All the fields $Z^{(N)}$ are identically and independently distributed.

We have tried to make a self contained exposition, of our results. However we think it essential that the reader gets, before looking at the details of this work, some non-superficial familiarity with [2]. In the theory in [2], first reference, we think, the deceiving "simplicity" of the superrenormalizable scalar field theories becomes transparent and does not disappear in the mist of some heavy technical, and uninteresting, details.

2. Reduction to Perturbation Theory

To evaluate the integral in (1.16) we shall use a technique which relies on the estimates in Lemma 1 below and which is described in a slightly more general setting.

We shall, for definiteness, only consider the $d=3$ case.

Let S, S', D, D' be non negative integers and let $\kappa>0$, $\beta\in(0,\frac{1}{2})$. Let J, I be two sets in $\mathbb{R}^3$ such that $J\subset I$ and I is a cube centred at the origin. Define, for

$$Z\in C^{(\beta)}(\mathbb{R}^3),\left[C^{(\beta)}(\Lambda)\equiv\left\{f\,\middle|\,\sup_{\substack{|\xi-\eta|\leqq 1\\ \xi,\eta\in\Lambda}}\frac{|f(\xi)-f(\eta)|}{|\xi-\eta|}+\sup_{\xi\in\Lambda}|f(\xi)|<+\infty\right\}\right]:$$

$$\begin{aligned}H_J(Z)=&\sum_{p=1}^{S}\sum_{\substack{\sum_{i=1}^{p}n_i\leqq D\\ n_i>0}}\int_{J^p}A^{n_1\dots n_p}_{\xi_1\dots\xi_p}e^{-\kappa d(\xi_1,\dots,\xi_p)}Z^{n_1}_{\xi_1}Z^{n_2}_{\xi_2}\dots Z^{n_p}_{\xi_p}d\xi_1\\ &\cdot d\xi_2\dots d\xi_p+\sum_{q=1}^{S'}\sum_{\substack{\sum_{i=1}^{q}m_i\leqq D'\\ m_i>0}}A^{m_1\dots m_q}_{\xi_1\xi'_1\dots\xi_q\xi'_q}e^{-\kappa d(\xi_1\xi'_1\dots\xi_q\xi'_q)}\\ &\cdot\prod_{j=1}^{q}\left(\frac{|Z_{\xi_j}-Z_{\xi'_j}|}{|\xi_j-\xi'_j|^{\beta}}\right)^{m_j}d\xi_1\dots d\xi'_q,\end{aligned}\tag{2.1}$$

where

1) $d(\xi_1,\dots,\xi_p)$ or, more generally, if $E_1, E_2,\dots,E_p$ are p sets in $\mathbb{R}^3$, $d(E_1,E_2,\dots,E_p)$ denotes the length of the shortest connected graph linking $E_1,\dots,E_p$ ("graph distance of $E_1,E_2,\dots,E_p$").

2) The functions $A^{\cdots}_{\cdots}$, which will be denoted $\underline{A}$, will be supposed to have some boundedness properties which can be most conveniently described in terms of a pavement Q_1 of $\mathbb{R}^d$ with cubic tesserae with side size 1 and with sides parallel to the coordinate axes. If the generic tessera of Q_1 is denoted by Δ the boundedness condition is

$$\|\underline{A}\|=\sup\int_{\Delta_1\times\dots\times\Delta_p}|A^{n_1\dots n_p}_{\xi_1\dots\xi_p}|\,d\xi_1\dots d\xi_p+\sup\int_{\Delta'_1\times\Delta''_1\times\dots\times\Delta'_q\times\Delta''_q}|A^{m_1\dots m_q}_{\xi_1\xi'_2\dots\xi_q\xi'_q}|\,d\xi_1\dots d\xi'_q<+\infty,\tag{2.2}$$

where the supremums are over the possible choices of the tesserae $\Delta_1,\dots,\Delta_p$, Δ'_1, $\Delta''_1,\dots,\Delta'_q,\Delta''_q$ in Q_1 and over the possible choices of $p,q,n_1,\dots,n_p$, $m_1,\dots,m_q$.

We wish to estimate expressions like[2]

$$\int\exp\hat{V}_J P(dz),\tag{2.3}$$

where $\hat{V}_J$ is a function which is simply related to a function of the type (2.1), [cf. (2.9) below], and P is the gaussian process on $\mathscr{S}'(\mathbb{R}^d)$ whose covariance operator is the inverse of [cf. (1.2)]

$$A=(\gamma^2-1)^{-1}(1-D)(\gamma^2-D).\tag{2.4}$$

The technique to estimate (2.3) is to introduce, for some $\beta>0$, the P-measurable events:

$$E^B_\Delta=\{Z\,|\,Z\in\mathscr{S}'(\mathbb{R}^3),\ \|Z\|_{C^{(\beta)}(\Delta)}<B(1+d(I,\Delta))\},\tag{2.5}$$

where Δ is a tessera of Q_1, and their complements. We shall denote χ^B_Δ the characteristic function of E^B_Δ and $\mathring{\chi}^B_\Delta=1-\chi^B_\Delta$ the characteristic function of the

2 Cf. Sect. 4.

complements of E_Δ^B: obviously

$$1 \equiv \sum_{R \subset Q_1} \left(\prod_{\Delta \in R} \mathring{\chi}_\Delta^B\right)\left(\prod_{\Delta \notin R} \chi_\Delta^B\right), \tag{2.6}$$

where $R = (\Delta_1, \Delta_2, \ldots)$ denotes a subset of Q_1 which will often be identified with the set $R = \bigcup_{i \geqq 1} \Delta_i$. Then (2.6) implies

$$\int \exp \hat{V}_J P(dZ) = \sum_{R \subset Q_1} \int \mathring{\chi}_R^B \chi_{R^c}^B \exp \hat{V}_J P(dZ), \tag{2.7}$$

where R^c = complement of R in Q_1

$$\mathring{\chi}_R^B = \prod_{\Delta \in R} \mathring{\chi}_\Delta^B \qquad \chi_{R^c}^B = \prod_{\Delta \in R^c} \chi_\Delta^B . \tag{2.8}$$

We now suppose that $\exists c_1, \varrho_1 > 0$ such that

$$\begin{aligned} &1)\ \hat{V}_J(Z) \equiv H_J(Z) \quad \text{if} \quad \chi_{Q_1}^B(Z) = 1, \\ &2)\ \hat{V}_J(Z) \leqq c_1 B^{\varrho_1} \|\Delta\| \, |\hat{R} \cap J| + H_{J \setminus \hat{R}}(Z) \quad \text{if} \quad \mathring{\chi}_R^B(Z) \chi_{Q_1 \setminus R}^B(Z) = 1 \end{aligned} \tag{2.9}$$

for all $\hat{R} \supset R$, $\hat{R} \subset Q_1$. In the sequel we shall only be interested in

$$\hat{R} = \{\Delta \mid \Delta \in Q_1, d(\Delta, R) \leqq B^3\}. \tag{2.10}$$

Then, obviously

$$\int \exp \hat{V}_J P(dZ) \geqq \int \chi_{Q_1}^B \exp H_J P(dZ), \tag{2.11}$$

$$\begin{aligned} \int \exp \hat{V}_J P(dZ) \leqq \sum_{R \subset Q_1} &\exp c_1 B^{\varrho_1} \|\Delta\| \, |\hat{R} \cap J| \\ &\cdot \int \exp H_{J \setminus \hat{R}} \mathring{\chi}_R^B \chi_{Q_1 \setminus R}^B P(dZ). \end{aligned} \tag{2.12}$$

Therefore concrete estimates can be obtained as soon as one finds a way of estimating integrals like:

$$\int \mathring{\chi}_R^B \chi_{Q_1 \setminus R}^B \exp H_{J \setminus \hat{R}} P(dZ) \qquad R \subset Q_1 . \tag{2.13}$$

In [2] we developed a method of analysis of integrals like (2.13): the analogue in the present case, of the results of [2] would be the following lemma:

Lemma 1. *Let γ be fixed close enough to 1. Given $t \geqq 0$, integer, there exist functions $B^*, G, G', \varrho, \varrho', \varrho'' > 0$ depending only on t, D, D', κ such that, if $B > B^*$ and if R is any subset of Q_1:*

$$\begin{aligned} \int \mathring{\chi}_R^B \chi_{Q_1 \setminus R}^B e^{H_{J \setminus \hat{R}}} P(dZ) \leqq &\exp(|I| \delta(B, \|\Delta\|) + |\hat{R} \cap J| \delta'(B, \|\Delta\|)) \\ &\cdot \exp\left\{\sum_{k=1}^{t} \frac{\mathscr{E}^T(H_J; k)}{k!}\right\} \left(\int \mathring{\chi}_R^B P(dZ)\right)^{1/2}, \end{aligned} \tag{2.14}$$

where $\mathscr{E}^T(\cdot\,; k)$ denotes the truncated P-expectation of order k[3] *and*

$$\begin{aligned} &\delta(B, \|\Delta\|) = G[(\|\Delta\| B^\varrho e^{\varrho'(\|\Delta\| B^\varrho)})^{t+1} + e^{-\varrho'' B^2 + \varrho' \|\Delta\| B^\varrho}], \\ &\delta'(B, \|\Delta\|) = G' \|\Delta\| B^\varrho e^{\varrho' \|\Delta\| B^\varrho}. \end{aligned} \tag{2.15}$$

3 See Appendix A for a precise definition of the truncated expectations $\mathscr{E}^T$, $\hat{\mathscr{E}}_i^T$

Furthermore, if $B > B^*$:

$$\int \chi^B_{Q_1} e^{H_J} P(dZ) \geqq \exp - |I| \delta(B, \|\underline{A}\|) \cdot \exp\left\{ \sum_{k=1}^{t} \frac{\mathscr{E}^T(H_J; k)}{k!} \right\}. \tag{2.16}$$

The above lemma, which we prove in Sect. 5 and 6, allows, as it will be shown in Sects. 3 and 4, to reduce the problem of estimating the integral in (1.16) to a "perturbation theory" of order 3 problem.

The remarkable fact, in our opinion, is that the "reduction to perturbation theory" is achieved via some laborious but completely straightforward algebra: the real difficulty in the whole proof is that of proving Lemma 1.

3. The Lower Bound

The idea is to obtain the lower bound in (1.16) by recursively applying Lemma 1 to analyze:

$$\int \left(\prod_{k=0}^{N} \chi^B_k(X^{(k)}) \right) \exp V_I^{(N)} \left(\prod_{k=0}^{N} P_k(dz) \right), \tag{3.1}$$

where χ^B_k is the characteristic function of the event[4]

$$\left\{ X^{(k)} \mid \|X^{(k)}\|_\Delta = \left(\sup_{\xi \in \Delta} |X^{(k)}_\xi| + \sup_{\substack{\xi \in \Delta, \eta \in \mathbb{R}^d \\ \gamma^k |\xi - \eta| \leqq 1}} \frac{|X^{(k)}_\xi - X^{(k)}_\eta|}{|\xi - \eta|^\beta} \right) < B_k(1 + \gamma^k d(I, \Delta)) \quad \forall \Delta \in Q_k \right\}, \tag{3.2}$$

where Q_k is a pavement of $\mathbb{R}^d$ with cubic tesserae with side size γ^{-k} and sides parallel to the coordinate axes of $\mathbb{R}^d$; β will be chosen once and for all equal to 1/4, say.

The first remark is that $V_I^{(N)}$, defined in (1.11), thought as a function of $Z^{(N)}$, cf. (1.18), at $X^{(N-1)}$ fixed and such that $\chi^B_{N-1}(X^{(N-1)}) = 1$, has the form (2.1) with I replaced by $\gamma^N I$ and $\underline{A}$ such that:

$$\|\underline{A}\| \leqq \lambda \gamma^{-N} \bar{A} B^{\bar{\varrho}}_{N-1} (1 + \lambda^2), \tag{3.3}$$

where $\bar{A}$ is a fixed constant and in general:

$$B_k = B(1+k)^4 \log(e + \lambda^{-1}) \qquad k = 0, 1, \ldots. \tag{3.4}$$

The estimate (3.3) easily follows from the explicit expression (1.12) ÷ (1.15) and (1.10) and some patience. Therefore Lemma 1 can be immediately applied to (3.1)[5]: taking into account the necessary change of scale and the associated scale

4 The power $(1+k)^4$ and the choice $\beta = \frac{1}{4}$ are not optimal but just arbitrary and convenient. The $\log(e + \lambda^{-1})$ factor is also quite arbitrary and has been included to improve the small λ estimates

5 Notice that $|X^{(N-1)}_\xi| < B'$, $|z^{(N)}_\xi| < \sqrt{1 + \Gamma_N} B - \sqrt{\Gamma_N} B' = \tilde{B}$ imply, for $\tilde{\tilde{B}} > 0$, that $|X^{(N)}_\xi| < B$; also $|X^{(N-1)}_\xi - X^{(N-1)}_\eta| \leqq B'(\gamma^{N-1}|\xi - \eta|)^\beta$, $|z^{(N)}_\xi - z^{(N)}_\eta| \leqq \tilde{B}(|\xi - \eta| \gamma^N)^\beta$ imply $|X^{(N)}_\xi - X^{(N)}_\eta| \leqq B(|\xi - \eta| \gamma^N)^\beta$

factors one finds from (3.3), (2.16), (2.15):

$$\int\left(\prod_{k=0}^{N}\chi_k^B(X^{(k)})\right)\cdot\exp V_I^{(N)}\hat{P}_N(dz^{(N)})$$
$$\geqq(\exp-\varepsilon(N)|I|)\cdot\left(\prod_{k=0}^{N-1}\chi_k^B(X^{(k)})\right)\cdot\exp\left\{\sum_{k=1}^{3}\frac{\hat{\mathscr{E}}_N^T(V_I^{(N)};k)}{k!}\right\},\tag{3.5}$$

where, for suitable $\bar{g}$, $\bar{G}$, $\bar{\varrho}>0$[6]:

$$\varepsilon(k)=\bar{G}(\lambda\gamma^{-k}k^{\bar{\varrho}})^{4-1/2}\gamma^{3k}e^{\bar{g}\lambda^{\bar{\varrho}}}\qquad k=0,1,\ldots.\tag{3.6}$$

Formulae (1.11)÷(1.15) show that the expression in curly brackets in (3.5) is a polynomial in λ of degree 6. To give (3.5) a form suited for further analysis denote, given a polynomial in λ, $p(\lambda)=\sum_{n\geqq 0}c_n\lambda^n$:

$$[p(\lambda)]_{(t)}=\sum_{n=0}^{t}c_n\lambda^n.\tag{3.7}$$

Then another algebraic analysis of the form (1.11)÷(1.15) of $V_I^{(N)}$ shows that, possibly changing $\bar{G}$, $\bar{\varrho}$, $\bar{g}$ in (3.6):

$$\left|\sum_{k=1}^{3}\frac{\hat{\mathscr{E}}_N^T(V_I^{(N)};k)}{k!}-\left[\sum_{k=1}^{3}\frac{\hat{\mathscr{E}}_N^T(V_I^{(N)};k)}{k!}\right]_{(3)}\right|\leqq\varepsilon(N)|I|\tag{3.8}$$

if $\chi_{N-1}^B(X^{(N-1)})=1$. Hence

$$\int\left(\prod_{k=0}^{N}\chi_k^B(X^{(k)})\right)\exp V_I^{(N)}\hat{P}_N(dz)$$
$$\geqq\left(\prod_{k=0}^{N-1}\chi_k^B(X^{(k)})\right)\left(\exp\left[\sum_{k=1}^{3}\frac{\hat{\mathscr{E}}_N^T(V_I^{(N)};k)}{k!}\right]_{(3)}\right)\exp-2\varepsilon(N)|I|.\tag{3.9}$$

The very remarkable fact is that the function in square brackets in (3.9) has still the same structure (2.1), if thought as a function of $Z^{(N-1)}$ at fixed $X^{(N-2)}$, and the new structure functions $\underline{A}'$ are such that $\|\underline{A}'\|$ verifies a bound like (3.3), with $N-1$ replacing N, if $\chi_{N-2}^B(X^{(N-2)})=1$.

This property is not obvious and is an important aspect of the theory to check it by an explicit, quite long but straightforward, calculation of the various gaussian expectations appearing in the r.h.s. of (3.9), see Appendix B.

Then it appears natural to define, inductively, for $J\subset I$:

$$\begin{aligned}&\tilde{V}_J^{(N)}=V_J^{(N)}\\&\tilde{V}_I^{(N-k)}=\left[\sum_{j=1}^{3}\frac{\hat{\mathscr{E}}_{N-k+1}^T(\tilde{V}_J^{(N-k+1)};j)}{j!}\right]_{(3)}\end{aligned}\tag{3.10}$$

6 $4-1/2$ is not optimal and in (3.6) we could replace $\lambda^{4-1/2}$ by $\lambda^4(\log(e+\lambda^{-1}))^{D+D'}$. The constant $\bar{\varrho}$ is some maximal constant introduced to avoid using too many constants and, finally, $\bar{g}$, $\bar{G}$, $\bar{\varrho}$ are B-dependent (but finite for $B>B^*$)

for $k=1,2,\ldots,N+1$. It turns out that $\tilde{V}_J^{(-1)}\equiv 0$.

In Appendix B we give the explicit expressions for $\tilde{V}_J^{(N-k)}$ cast in the form (2.1): it can be verified on such expressions, with some patience, that the structure functions $\underline{A}^{(N-k)}$ of $V_I^{(N-k)}$, thought as a function of $Z^{(N-k)}$, verify, if $\chi^B_{N-k-1}(X^{(N-k-1)})\equiv 1$, the bound:

$$\|\underline{A}^{(N-k)}\| \leqq \lambda \tilde{A} \gamma^{-(N-k)} B^{\tilde{\varrho}}_{N-k}(1+\lambda^2), \tag{3.11}$$

where $\tilde{A}$, $\tilde{\varrho}$ are suitable constants.

Furthermore, if $|X_\xi^{(N-k)}| \leqq B_{N-k}$ for all $\xi \in J$:

$$\left|\sum_{j=1}^{3} \frac{\hat{\mathscr{E}}^T_{N-k+1}(\tilde{V}_J^{(N-k+1)};j)}{j!} - \tilde{V}_J^{(N-k)}\right| < \varepsilon(N-k+1)\,|I|, \tag{3.12}$$

where ε can be taken that defined in (3.6), possibly readjusting there the $\bar{G},\bar{\varrho}$ parameters.

Therefore Lemma 1 of Sect. 2 immediately implies, by the already remarked scale invariance properties of the gaussian measures $\hat{P}^{(i)}$ that:

$$\begin{aligned}&\int (\exp \tilde{V}_I^{(N-k+1)}) \left(\prod_{i=0}^{N-k+1} \chi_i^B(X^{(i)})\right) P_{N-k+1}(dz)\\ &\quad \geqq (\exp -\varepsilon(N-k+1)\,|I|) \int (\exp \tilde{V}_I^{(N-k)}) \left(\prod_{i=0}^{N-k} \chi_i^B(X^{(i)})\right) P_{N-k}(dz).\end{aligned} \tag{3.13}$$

Hence, by iteration, (3.8) and (3.13) imply:

$$\int (\exp \tilde{V}_I^{(N)}) P_N(dz) \geqq \exp - \sum_{k=0}^{\infty} \varepsilon(k)\,|I| \tag{3.14}$$

i.e. one can take

$$E_-(\lambda) = G_-(\lambda e^{\varrho_- \lambda^{\varrho_-}})^{4-1/2} \tag{3.15}$$

for suitably chosen G_-,ϱ_-. This is the desired lower bound for the integral in (1.16).

4. The Structure of $\tilde{V}_I^{(h)}$ and the Upper Bound

To show that the upper bound can be deduced from the first estimate (2.14) in Lemma 1 we have to go into a somewhat more detailed analysis of the structure of $\tilde{V}_J^{(h)}$.

This function is, in general, a polynomial of degree not larger than 8 in the field $\varphi^{[\leqq h]}$ which we simply denote φ, calling $X=X^{(h)}$, $z=z_{(h)}$, $X'=X_{(h-1)}$, $\varphi'=\varphi^{[\leqq h-1]}$. The monomials composing the polynomial will have the following structure: either they have the form, $(\kappa>0)$:

$$\begin{aligned}&(\lambda\gamma^{2h})^p(\lambda^2 h\gamma^h)^q(\lambda^3 h)^r \int d\xi_1\ldots d\xi_p d\eta_1\ldots d\eta_q d\eta'_1\ldots d\eta'_r\\ &\quad\cdot(\exp -\kappa\gamma^h d(\xi_1,\ldots,\eta_1,\ldots,\eta'_r))(:X^{n_1}_{\xi_1}\ldots X^{m'_r}_{\eta'_r}:)\\ &\quad\cdot \underline{A}^{n_1\ldots n_p m_1\ldots m_q m'_1\ldots m'_r}_{\xi_1\ldots\xi_p\eta_1\ldots\eta_q\eta'_1\ldots\eta'_r}\end{aligned} \tag{4.1}$$

with $p+2q+3r\leqq 3$, $\sum_{i=1}^{q} m_i+\sum_{i=1}^{p} n_i+\sum_{i=1}^{r} m'_i\leqq 8$, $m_i\geqq 0$, $n_i\geqq 0$, $m'_i\geqq 0$, or they have the form, $(\kappa>0)$:

$$\lambda^2\gamma^{4h}\int_{J^2} d\xi d\eta \underline{A}_{\xi\eta} e^{-\kappa\gamma^h d(\xi,\eta)}\frac{:(X_\xi-X_\eta)^2:}{(\gamma^h|\xi-\eta|)^{1/2}} \tag{4.2}$$

here $\kappa>0$ is I,J,N-independent.

To describe the properties of the functions $\underline{A}$, which depend also on N,h, we use the previously introduced pavements Q_h of $\mathbb{R}^3$ with cubic tesserae with side size γ^{-h}. Then there exists $\bar{A}>0$, I,J,N,h-independent, such that

$$\begin{aligned}&(\gamma^{3h})^{p+q+r}\int_{\Delta_1\times\ldots\times\Delta_{p+q+r}}|\underline{A}^{n_1\ldots n_p m_1\ldots m_q m'_1\ldots m'_r}_{\xi_1\ldots\xi_p\eta_1\ldots\eta_q\eta'_1\ldots\eta'_r}|d\xi_1\ldots d\eta'_r\leqq\bar{A}\,,\\ &\gamma^{6h}\int_{\Delta_1\times\Delta_2}|\underline{A}_{\xi\eta}|d\xi d\eta\leqq\bar{A}\end{aligned} \tag{4.3}$$

$\forall\Delta_1,\ldots,\Delta_{p+q+r}\in Q_h$. Here $\gamma^{-3h}=$ volume of $\Delta\in Q_h$.

To help the intuition it is important to realize that if

$$\begin{aligned}&|X_\xi|\leqq b &&\forall\xi\in J\,,\\ &|X_\xi-X_\eta|\leqq b(\gamma^h|\xi-\eta|)^{1/4} &&\forall\xi,\eta\in J\end{aligned} \tag{4.4}$$

then in (4.1), (4.2) the terms which contain a power λ^p have a magnitude $\lesssim b^8(\gamma^{-h})^p$, if $b>1$, i.e., despite the appearances, the larger the value of p is the smaller the corresponding terms are. This claim can be easily checked by writing the integrals (4.1), (4.2) as sums of integrals over the tesserae of Q_h and, then, applying the estimates (4.3). This essentially means that, when (4.4) holds the counterterms and the higher order terms are negligible compared to the "bare interaction" (linear in λ).

We shall write the sum of the terms in (4.1), (4.2) collecting them in groups which appear naturally if one computes explicitly $\tilde{V}_I^{(h)}$, cf. Appendix B.

Namely:

$$\begin{aligned}\tilde{V}_I^{(h)}=V_I^{(h)}&+\sum_{i=h}^{N-1}\mathscr{E}_{>h}(\bar{W}_{I^2}^{(i)})+\Delta_I^{(h)}\\ &+\sum_{i=h+1}^{N-1}\mathscr{E}_{>h}(\bar{W}_{I^2}^{(i)}\cdot(V_{0,I}^{(i)}-V_{0,I}^{(h)}))\,,\end{aligned} \tag{4.5}$$

where $V_I^{(h)}$, $V_{0,I}^{(i)}$ have already been defined in Sect. 2 (we recall that the superscript denotes the high frequency cut off of the field), $\mathscr{E}_{>h}$ denotes the integration with respect to the variables $z^{(h+1)},\ldots,z^{(N)}$ and $\Delta_I^{(h)}$ is a sum of functions of the type (4.1), while $\bar{W}_G^{(i)}$ is, if $C_i(\xi,\eta)\equiv C_{\xi\eta}^{[\leqq i]}$:

$$-\frac{\lambda^2 4^2 3!}{4}\int_G d\xi d\eta(C_{i+1}^3(\xi,\eta)-C_i^3(\xi,\eta))(:(\varphi_\xi^{[\leqq i]}-\varphi_\eta^{[\leqq i]})^2:) \tag{4.6}$$

i.e. (cf. p. 2, Footnote 1)

$$W_G^{(h)} = \sum_{i=h}^{N-1} \mathscr{E}_{>h}(\bar{W}_G^{(i)}) = -\frac{\lambda^2 4^2 3!}{4} \int_G d\xi d\eta (C_N(\xi,\eta)^3 - C_h(\xi,\eta)^3)$$

$$\cdot :(\varphi_\xi - \varphi_\eta)^2: = \frac{-\lambda^2 4^2 3!}{4} 2c_\gamma (1+\Gamma_h)\gamma^{4h} \int_G d\xi d\eta$$

$$\cdot \{\gamma^{-3h}(C_N(\xi,\eta)^3 - C_h(\xi,\eta)^3)\cdot(|\xi-\eta|\gamma^h)^{1/2}\} \frac{:(X_\xi - X_\eta)^2:}{(|\xi-\eta|\gamma^h)^{1/2}}. \tag{4.7}$$

So we shall rewrite (4.5) as

$$\tilde{V}_I^{(h)} = (V_I^{(h)} + \Delta_I^{(h)}) + \left(W_{I^2}^{(h)} + \sum_{i=h+1}^{N-1} \mathscr{E}_{>h}(\bar{W}_{I^2}^{(i)} \cdot (V_{0,I}^{(i)} - V_{0,I}^{(h)}))\right) \tag{4.8}$$

and we shall also need the fact that $W_G^{(h)}$ is the only term in (4.8) which will be thought of the form (4.2).

For the analysis of the upper bound it is convenient to introduce a new interaction $\hat{V}_J^{(h)}$ obtained from $\tilde{V}_J^{(h)}$ by eliminating from J points where the field X is large or rough.

For this purpose, given $X^{(h)}, z^{(h)}$ we define in general:

$$D_h^g(X^{(h)}) = \{\xi \mid \xi \in \mathbb{R}^3, |X_\xi^{(h)}| \geqq B_h(1+\gamma^h d(\xi, I))\}, \tag{4.10}$$

$$D_h^r(X^{(h)}) = \{\xi,\eta \mid \xi,\eta \in \mathbb{R}^3 \times \mathbb{R}^3, |\gamma^h(\xi-\eta)| \leqq 1, |X_\xi^{(h)} - X_\eta^{(h)}| \geqq B_h(\gamma^h|\xi-\eta|)^{1/4}(1+\gamma^h d(\xi,I))\}. \tag{4.11}$$

$$\begin{aligned} R_h(z^{(h)}) = \{\Delta \mid \Delta \in Q_h \text{ such that } \exists \xi \in \Delta \text{ where} \\ |z_\xi^{(h)}| > B_h'(1+\gamma^h d(\Delta, I)) \text{ or } \exists \xi,\eta \in \Delta \text{ where} \\ |z_\xi^{(h)} - z_\eta^{(h)}| > B_h'(\gamma^h|\xi-\eta|)^{1/4}(1+\gamma^h d(\Delta,I)) \text{ and } |\xi-\eta|\gamma^h \leqq 1\}, \end{aligned} \tag{4.12}$$

where B_h was introduced in (3.4) and B_h' will be chosen as[7]

$$B_h' = B_h/50(1+h^2). \tag{4.13}$$

So $R_h = (\Delta_1, \Delta_2, \ldots)$ is a sequence of Δ's in Q_h: however in the following we shall identify R_h with $\bigcup_{\Delta \subset R_h} \Delta \subset \mathbb{R}^3$.

We can now define $\hat{V}_J^{(h)}$ in terms of the functions V, W, Δ describing $\tilde{V}_I^{(h)}$:

$$\hat{V}_I^{(h)} = V_{I\setminus D_h^g}^{(h)} + \Delta_{I\setminus D_h^g}^{(h)} + W_{(I\setminus D_h^g)^2\setminus D_h^r}^{(h)} + \sum_{i=h+1}^{N-1} \mathscr{E}_{>h}(\bar{W}_{(I\setminus D_h^g)^2\setminus D_h^r}^{(i)} \cdot (V_{0,I\setminus D_h^g}^{(i)} - V_{0,I\setminus D_h^g}^{(h)})). \tag{4.14}$$

This function does not have the structure (2.1) because it is not "polynomial" in X: the X's enter also in the $\bar{A}$-functions and in the domains of integration. It is

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7 Of course 50 is not optimal: it is however a simple convenient choice. Neither the $(1+h^2)$ factor is optimal

therefore useful to introduce yet another function $H_J^{(h)}$ which, if thought as a function of $Z^{(h)}$, has the structure (2.1):

$$H_J^{(h)} = V^{(h)}_{J\backslash D^g_{h-1}} + \Delta^{(h)}_{J\backslash D^g_{h-1}} + W^{(h)}_{(J\backslash D^g_{h-1})^2\backslash D^r_{h-1}} + \sum_{i=h+1}^{N-1} \mathscr{E}^g_{>h}(\tilde{W}_{(J\backslash D^g_{h-1})^2\backslash D^r_{h-1}} \cdot (V^{(i)}_{0,J\backslash D^g_{h-1}} - V^{(h)}_{0,J\backslash D^g_{h-1}})). \tag{4.15}$$

Clearly the function $H_J^{(h)}(X^{(h)})$ has, $\forall h$, the structure (2.1) if thought as a function of $Z^{(h)}$ [since $D^g_{h-1}(X^{(h-1)})$, $D^r_{h-1}(X^{(h-1)})$ are $z^{(h)}$-independent], with J replaced by $\gamma^h J$.

The coefficient functions of $H_J^{(h)}$ denoted $\underline{H}^{(h)}$, verify[8]:

$$\|\underline{H}^{(h)}\| \leqq \lambda \bar{H} \gamma^{-h} B^{\bar{\sigma}}_{h-1} e^{\bar{\sigma}\lambda^3} \tag{4.16}$$

if $\bar{H}, \bar{\sigma}$ are suitably chosen constants.

There are some simple algebraic relations among $V_J^{(h)}$, $\tilde{V}_J^{(h)}$, $\hat{V}_J^{(h)}$, $H_J^{(h)}$ which either immediately follow from (4.2), (4.9), (4.14), (4.15) or from (3.10), (4.14), (4.15) after some simple gaussian integrations.

Such properties are the aspect that we need of the formal positivity of $\lambda:\varphi^4:$ for φ large. Technically they hold because of the positivity and the bounds (4.3) on $\underline{A}_\xi, \underline{A}_{\xi\eta}$ and because of the very special recursive form of $\tilde{V}_J^{(h)}$. A crucial role is played also by the superrinormalizability which allows to say that if h is large the terms in the $\hat{V}^{(h)}$'s are of different orders of magnitude (according to the powers of λ that they contain): the terms with λ^p have order γ^{-hp}, cf. (4.8).

Lemma 2. *$\exists$ an integer valued function $\bar{h}(B,\lambda)$ and two constants $\bar{C}, \bar{B} > 0$ such that for all $h \geqq \bar{h}(B,\lambda)$, $\forall J \subset I$, $\forall B \geqq \bar{B}$:*

$$\text{i)}\quad V_J^{(N)} \leqq \hat{V}_J^{(N)} \tag{4.17}$$

$$\text{ii)}\quad \hat{V}_J^{(h)} \leqq (\bar{c}\lambda e^{\bar{c}\lambda^3} \gamma^{2h} B_h^{\bar{c}}) \gamma^{-3h} \#(\hat{R}_h \cap J)) + H^{(h)}_{J\backslash \hat{R}_h}, \tag{4.18}$$

where $\hat{R}_h = \{\Delta \mid \Delta \in Q_h, d(\Delta, R_h(z^{(h)})) \leqq (B_h')^3 \gamma^{-h}\}$, (see (2.10), (4.12), (4.13))[9]*; $\#(R_h \cap J)$ = number of elements of R_h intersecting J.*

$$\text{iii)}\quad \sum_{k=1}^{3} \frac{\hat{\mathscr{E}}_h^T(H_J^{(h)}; k)}{k!} - \hat{V}_J^{(h-1)} \leqq |I| \cdot (\bar{c}\lambda e^{\bar{c}\lambda^3} B^{\bar{c}}_{h-1} \gamma^{-h})^4 \gamma^{3h} \tag{4.19}$$

iv) *$\bar{h}(B,\lambda) \equiv 0$ if λ is small enough (but this property does not hold uniformly in $B \geqq \bar{B}$).*

In Appendix C we illustrate the proof of Lemma 2.

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8 The natural estimate would have $(\lambda + \bar{\sigma}\lambda^3)$ instead of $\lambda e^{\bar{\sigma}\lambda^3}$ in (4.16). It is however notationally convenient to use the bounds $\lambda e^{\bar{\sigma}\lambda^3}$

9 Again $\bar{c}$ is a maximal constant introduced to avoid using too many symbols

It is now possible to complete the derivation of the upper bound. Choose once and for all $B=\bar{B}+B^*$, cf. Lemmas 1 and 2.

The inequality (4.17) shows that the integral:

$$\int \exp \hat{V}_I^{(N)} P_N(dz) \tag{4.20}$$

provides an upper bound to our integral (1.16).

To estimate (4.20) via Lemma 1 we apply the identities (2.6), (2.7):

$$\int \exp \hat{V}_I^{(h)} \hat{P}_h(dz^{(h)}) = \sum_{R \subset Q_1} \int \mathring{\chi}_R^{B'_h}(Z^{(h)}) \chi_{Q_1 \setminus R}^{B'_h}(Z^{(h)}) \cdot (\exp \hat{V}_I^{(h)}) \hat{P}_h(dz^{(h)}), \tag{4.21}$$

where $\chi_R^B, \mathring{\chi}_R^B$ are defined by (2.8), (2.5) with, naturally, I replaced by $\gamma^h I$; $\hat{V}_I^{(h)}$ is now to be thought as a function of the field $Z^{(h)}$. It is immediately realized that, the scale factors are such that $\mathring{\chi}_R^{B'_h}(Z^{(h)}) \cdot \chi_{Q_1 \setminus R}^{B'_h}(Z^{(h)}) = 1$, if and only if $R_h(z^{(h)}) = \gamma^{-h} R$: this gives to (4.21) a simple meaning.

After the appropriate scalings we can try to apply the method of Sect. 2 to estimate (4.21): therefore we apply (4.18) to (4.21) to obtain:

$$\int (\exp \hat{V}_I^{(h)}) \hat{P}_h(dz^{(h)}) \leqq \sum_{R \subset Q_1} \{ \int \mathring{\chi}_R^{B'_h}(Z^{(h)}) \chi_{Q_1 \setminus R}^{B'_h}(Z^{(h)}) \cdot (\exp H_{\gamma^h I \setminus \hat{R}}^{(h)}) \hat{P}_h(dz^{(h)}) \} \exp \bar{c} \lambda e^{\bar{c}\lambda^3} \gamma^{2h} B_h'^{\bar{c}} \gamma^{-3h} \# (R \cap \gamma^h I), \tag{4.22}$$

where $\hat{R}$ is defined here by (2.10) with B'_h replacing B.

We can now apply Lemma 1 because, $H^{(h)}$, as a function of $Z^{(h)}$, has the form (2.1), cf. (4.15): we obtain, using (4.19) to elaborate the curly bracket term arising from the application of (2.14) to our case:

$$\begin{aligned} \int (\exp \hat{V}_I^{(h)}) \hat{P}_h(dz^{(h)}) &\leqq (\exp \hat{V}_I^{(h-1)}) (\exp |I| \gamma^{3h} \\ &\cdot \delta(B'_h, \|\underline{H}^{(h)}\|)) \cdot \sum_{R \subset Q_1} (\exp [\bar{c} \lambda e^{\bar{c}\lambda^3} B_h'^{\bar{c}} \gamma^{-h} \\ &+ \delta'(B'_h, \|\underline{H}^{(h)}\|)] \# (\hat{R} \cap \gamma^h I)) \cdot (\int \mathring{\chi}_R^{B'_h}(Z^{(h)}) \hat{P}_h(dz^{(h)}))^{1/2} \end{aligned} \tag{4.23}$$

provided $h \geqq \bar{h}(B\lambda)$.

The dangerous-looking sum in (4.23) can be easily bounded by the use of the following tail lemma (cf. [8], Proposition 4):

Lemma 3. *$\exists c_1, c_2 > 0$ such that if $R = (\Delta_1, \Delta_2, \ldots) \subset Q_1$ and I is a cube centred at the origin:*

$$\int \mathring{\chi}_R^B(Z) \hat{P}_0(dZ) \leqq \prod_i \exp(c_1 - c_2 B^2 (1 + d(\Delta_i, I))). \tag{4.24}$$

After the obvious change of scale we deduce from Lemma 3 that the integral in the r.h.s. of (4.23) can be bounded by:

$$\left(\int \mathring{\chi}_R^{B'_h}(Z^{(h)}) \hat{P}_h(dz^{(h)})\right)^{1/2} \leqq \prod_i \exp \tfrac{1}{2}(c_1 - c_2 B_h'^2 (1 + d(\Delta_i, \gamma^h I))) \tag{4.25}$$

which permits to bound the sum over R in (4.23) by calling μ_h the term in square brackets in (4.23):

$$\prod_{\Delta\in Q_I}\left(1+\exp\left(\mu_h+\frac{c_1}{2}-\frac{c_2}{2}(B_h')^2(1+d(\Delta,\gamma^h I))\right)\right)$$

$$\leqq \exp[c_3 e^{\mu_h+\frac{c_1}{2}-\frac{c_2}{2}(B_h')^2}]\gamma^{3h}|I| \tag{4.26}$$

for a suitable $c_3>0$, $\forall h$.

Recalling (2.15) and (4.16) the error terms in (4.23) give rise to the following expression

$$|I|\gamma^{3h}\Big\{G((\|\underline{H}^{(h)}\|(B_h')^{\varrho}e^{\varrho'\|\underline{H}\|(B_h')^{\varrho}})^4$$

$$+e^{-\varrho''(B_h')^2+\varrho'\|\underline{H}^{(h)}\|(B_h')^{\varrho}})+c_3\exp\Big[G'(\|\underline{H}^{(h)}\|(B_h')^{\varrho}$$

$$\cdot e^{\varrho'\|H^{(h)}\|(B_h')^{\varrho}})+\frac{c_1}{2}-\frac{c_2}{2}(B_h')^{\varrho}\Big]\Big\}\leqq |I|\varepsilon_h(\lambda), \tag{4.27}$$

where $\varepsilon_h(\lambda)$ can be thought as given by (3.6), possibly readjusting there the parameters $\bar{g},\bar{G}$.

Hence (4.23) becomes, for $h\geqq\bar{h}(B,\lambda)$:

$$\int(\exp\hat{V}_I^{(h)})\hat{P}_h(dz)\leqq(\exp\hat{V}_I^{(h-1)})(\exp|I|\varepsilon_h(\lambda)) \tag{4.28}$$

which by iteration implies, if $h\geqq\bar{h}(B,\lambda)$:

$$\int\exp\hat{V}_I^{(N)}P_N(dz)\leqq\exp\left(|I|\sum_{h=\bar{h}}^{\infty}\varepsilon_h(\lambda)\right)$$

$$\cdot\int(\exp\hat{V}_I^{(\bar{h}-1)})P_{\bar{h}-1}(dz) \tag{4.29}$$

and, if λ is so small that $\bar{h}\equiv 0$, we can take in (1.16), cf. (3.15):

$$E_+(\lambda)=\sum_{h=0}^{\infty}\varepsilon_h(\lambda)=E_-(\lambda) \tag{4.30}$$

or generally if:

$$E^r(\lambda)=\sup_I\sup_z(|I|^{-1}\hat{V}_I^{(\bar{h}-1)}(z)) \tag{4.31}$$

we can take

$$E_+(\lambda)=E_-(\lambda)+E^r(\lambda). \tag{4.32}$$

This concludes the theory of the ultraviolet bound to third order.

The following sections will be devoted to the proof of the Lemma 1.

5. Geometric Preliminaries to the Proof of Lemma 1. The Integration Grid

The proof of the main lemma is identical in spirit to that of [2]: this can be fully realized after acquiring the basic properties of the gaussian processes associated to the fourth order differential operator (2.4), of strong elliptic type, and a good understanding of the theory of the Dirichlet problem associated to (2.4).

Before formulating the basic propositions on the above questions we must do a somewhat complicated geometric construction whose motivation becomes transparent only after seeing its use in the proof of Lemma 1.

The construction which follows is identical to the one of [8] where we called it the construction of an "integration grid" for the measure P associated to (2.4).

Let $Q_1, Q_2, \ldots, Q_4$ be four pavements of $\mathbb{R}^3$ with cubic tesserae with side size 1.

We suppose that the centers of the tesserae of such pavements are on the step-1 lattices of R^3 with origin, respectively, at the points

$$\xi_k = \frac{2^{k-1}-1}{2^{k-1}}(1,1,1) \qquad k=1,2,3,4 \tag{5.1}$$

and with edges parallel to the coordinate axes of $\mathbb{R}^3$.

After constructing such pavements we turn each tessera into a smooth region by some deformations.

Let δ be smaller than 1/80 (cf. Sect. 4 of [8]).

i) Shrink each tessera about its center by a homothety factor $(1-\delta)$: after this first operation the four deformed pavements are no longer such because they leave unpaved corridors of width $<1/40$ between them.

ii) Turn every corner into a smooth corner and, also, every edge, of any order, into a smooth edge.

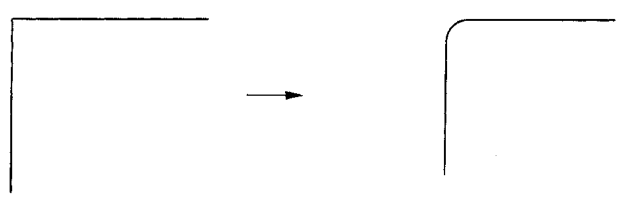

Fig. 1. Case $d=2$

iii) We now wish to modify more the boundaries of Q_2, Q_3, Q_4 in such a way that they intersect in a smooth way between each other and with the boundaries of Q_1. We also want that the modified regions are conically regular for cones with opening zero at least (cf. Appendix A, definitions). We add the further condition that if two points belong to the same boundary Σ' which has been deformed in order to cross smoothly another boundary Σ and if they lie on opposite sides of Σ then their distance is larger than say, $\delta/100$: in other words, upon crossing, the surfaces must stay adherent for a while.

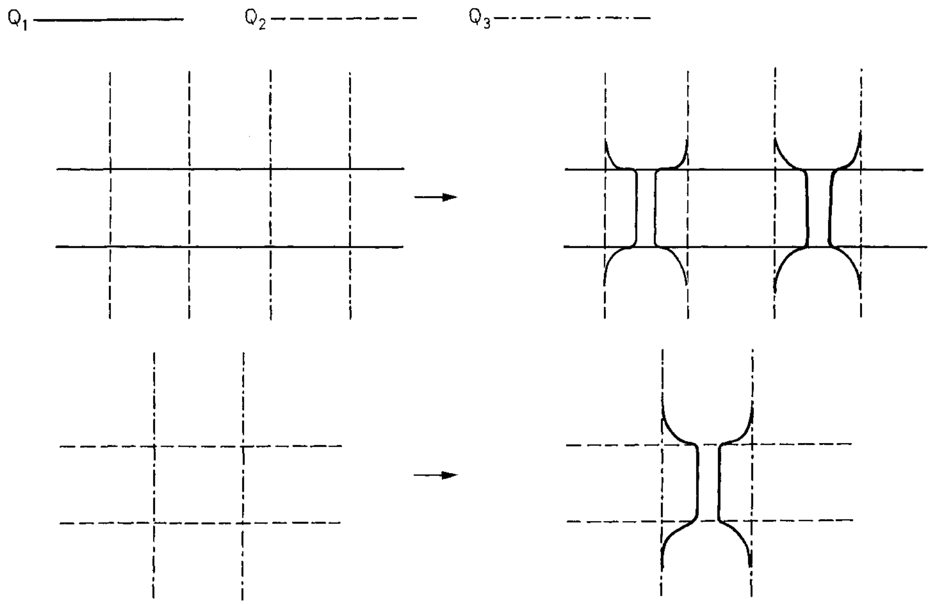

Fig. 2

The 2-dimensional situation is easily described by pictures and the last condition means that if a crossing is enlarged in scale it looks like:

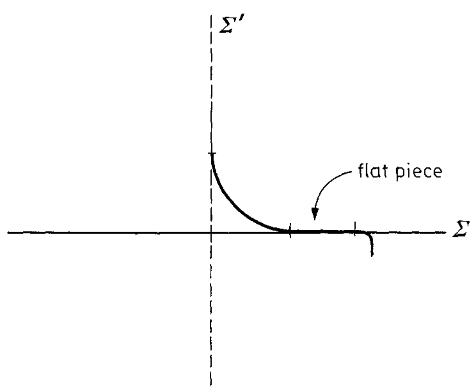

Fig. 3

We also require that in the deformations we only allow displacements of at most $\delta/2$.

iv) Finally we suppose that the contacts between different surfaces are of infinite order in the following sense. Let Σ, Σ' be two boundaries which cross as in

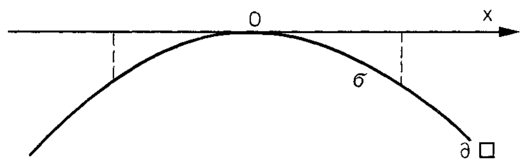

Fig. 4

iii); let $\xi \in \Sigma \cap \Sigma'$. Set up a cartesian reference system with origin in ξ and with plane $x_3 = 0$ coinciding with the tangent plane to Σ in ξ. Then the surface Σ' has a contact of infinite order with Σ if

1) The surface Σ' can be described in the above reference system and in a neighborhood of $\Sigma \cap \Sigma'$ by a function

$$x_3 = v(\underline{x}) \qquad \underline{x} = (x_1, x_2) \in U, \tag{5.2}$$

where U is a suitable neighborhood of $\underline{0} \in \mathbb{R}^2$.

2) If $\underline{j} = (j_1, j_2)$ are two integers and if $\partial^{(\underline{j})}$ denotes $\frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdot \frac{\partial^{j_2}}{\partial x_2^{j_2}}$ then $\forall a > 0$, $\forall \underline{j}$:

$$\sup_{\underline{x} \in U} \left| \frac{\partial^{\underline{j}} v(\underline{x})}{v(\underline{x})^{1-a}} \right| < +\infty . \tag{5.3}$$

We call $\tilde{Q}_{1,1}, \tilde{Q}_{2,1}, \tilde{Q}_{3,1}, \tilde{Q}_{4,1}$ the sets of deformed tesserae. If we scale by a homothety factor ℓ such assembly of boxes we obtain new families $\tilde{Q}_{1,\ell}, \ldots, \tilde{Q}_{4,\ell}$. The factor ℓ will be chosen later.

As usual in the theory of partial differential equations we introduce on each $\square \in \tilde{Q}_{i,\ell}$, $i = 1, \ldots, 4$, a covering of $\partial \square$ with regular surface elements, regularly spaced as $\ell \to \infty$ (see Appendix A, Definitions 3), which will be denoted $\sigma_1, \sigma_2, \ldots$. To each surface element we associate its local system of coordinates, cf. Appendix A, Definition 3, see also Fig. 4. If f is a distribution in $\mathscr{D}'(\partial \square)$ and if $\alpha_{\sigma_1}, \alpha_{\sigma_2}, \ldots$ is a partition of unity on $\partial \square$ associated with the regular regularly spaced covering (see Appendix A, Definition 3), we consider the distribution $\alpha_\sigma f$ with support on σ and call $\overline{\alpha_\sigma f}$ its representative in the local system of coordinates associated with $\sigma : \overline{\alpha_\sigma f} \in \mathscr{D}'(\mathbb{R}^2)$.

The following norms will be used to measure the magnitude of f:

$$\| f \|_{C_s^{(\varepsilon)}(\sigma)} = \| \overline{\alpha_\sigma f} \|_{C_s^{(\varepsilon)}(\mathbb{R}^2)}, \tag{5.4}$$

where for all $s \in \mathbb{R}$, $\forall \varepsilon \in (0, 1)$:

$$\| g \|_{C_s^{(\varepsilon)}(\mathbb{R}^2)} = \| (1 - \underline{D})^{\frac{s-\varepsilon}{2}} g \|_{\bar{C}^{(\varepsilon)}(\mathbb{R}^2)}, \tag{5.5}$$

$$\| h \|_{\bar{C}^{(\varepsilon)}(\mathbb{R}^2)} = \sup_{\underline{x} \in \mathbb{R}^2} e^{\sqrt{|\underline{x}|}} |h(\underline{x})| + \sup_{\substack{\underline{x}, \underline{y} \in \mathbb{R}^2 \\ |\underline{x} - \underline{y}| \leq 1}} e^{\sqrt{|\underline{x}|} + \sqrt{|\underline{y}|}} \frac{|h(\underline{x}) - h(\underline{y})|}{|\underline{x} - \underline{y}|^\varepsilon} \tag{5.6}$$

and $\underline{D} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

In the theory of the elliptic operator A appears a "conical regularity" parameter, associated with the theory of the double layer potentials, $\Theta(\gamma)$ which tends to 0 as $\gamma \to 1$. We shall therefore choose the value of γ, so far arbitrary, so close to 1 that all the tesserae $\square \in \tilde{Q}_{i,\ell}$, $i=1, \dots, 4$ are conically regular with respect to the cones with opening $\Theta(\gamma)$. This is possible, [cf. requirement iii) in the above construction], since the conical regularity is a homothety invariant notion.

We are now in a position to formulate the results on the theory of the elliptic operator A and of the associated gaussian process on $\mathscr{S}'(\mathbb{R}^3)$.

Proposition 1. $\exists \ell_0 > 0$, $\kappa > 0$ *such that* $\forall \ell \geqq \ell_0$, $\forall \varepsilon \in (0,1)$, $\forall s \in \mathbb{R}$ *the equation:*

$$\begin{aligned} Au &= 0 \quad \text{in} \quad \square \\ \partial^j u &= z^{(j)} \quad \text{on} \quad \partial \square, \quad j=0,1 \end{aligned} \tag{5.7}$$

for $\square \in \bigcup_{i=1}^{4} \tilde{Q}_{i,\ell}$ *and for* $\underline{z} = (z^{(0)}, z^{(1)}) \in \prod_{j=0}^{1} \cdot C^{(\varepsilon)}_{s-j}(\partial\square)$[10] *has a unique solution,* $u(\underline{z})$, *taking the boundary value in the sense of the traces on surfaces parallel to the boundary (see Appendix A, Definition 4).*

Furthermore if $s > \varepsilon$, $\ell \geqq \ell_0$, $\exists c^{s,\varepsilon}$ *such that*

$$\|u(\alpha_\sigma \underline{z})\|_{C^{(\varepsilon)}(\Delta \cap \square)} \leqq C^{s,\varepsilon} e^{-\kappa d(\sigma,\Delta)} \sum_{j=0}^{1} \|z^{(j)}\|_{C^{(\varepsilon)}_{s-j}(\sigma)} \,. \tag{5.8}$$

The following result on the theory of the gaussian measure P on $\mathscr{S}'(\mathbb{R}^d)$ associated with the operator A will play a major role in the proof of Lemma 1. It concerns essentially a support property for P.

Call $\Sigma_{1,\ell}, \Sigma_{2,\ell}, \Sigma_{3,\ell}, \Sigma_{4,\ell}$ the families of surface elements of $\partial\tilde{Q}_{1,\ell}, \dots, \partial\tilde{Q}_{4,\ell}$ considered before and call $\Sigma_\ell = \Sigma_{1,\ell} \cup \dots \cup \Sigma_{4,\ell}$. We shall refer to Σ_ℓ as to a "complete integration grid" for P, [8].

Given a cube I centred at the origin, let:

$$\bar{E}^{B,s,\varepsilon,\ell}_\sigma = \left\{ z \,\middle|\, z \in \mathscr{S}'(\mathbb{R}^3), \sum_{j=0}^{1} \left| \partial^j z \right|_{C^{(\varepsilon)}_{s-j}(\sigma)} < B_\sigma \right\}, \tag{5.9}$$

where $\sigma \in \Sigma_\ell$, $B_\sigma = B(1 + d(\sigma, I))$ and ∂^j denotes the j-th normal derivative of z on σ (cf. [8], Proposition 2).

Let $\bar{\chi}^{B,s,\varepsilon,\ell}_\sigma$ be the characteristic function of $\bar{E}^{B,s,\varepsilon,\ell}$ which we shall abbreviate $\bar{\chi}^{B,s,\ell}_\sigma$ or $\bar{\chi}^{B,s}_\sigma$ or $\bar{\chi}^{B}_\sigma$ when ε appears clear or when ε, ℓ appear clear or when ε, ℓ, s appear clear:

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10 f is said to be in $C^{(\varepsilon)}_s(\partial\square)$ if

$$\|f\|_{C^{(\varepsilon)}_s(\partial\square)} = \sup_{i=1,\dots,n} \|f\|_{C^{(\varepsilon)}_s(\sigma_i)} < +\infty$$

where $\sigma_1, \dots, \sigma_n$ is the covering associated with $\partial\square$

Proposition 2. *Fix* $\varepsilon\in(0,\frac{1}{2})$ *and* $s<\frac{1}{2}$, $\varepsilon<s$. *There exist constants* $c_1, c_2, \ldots, c_6, \ell_1$ *such that* $\forall \ell \geqq \ell_1$:

i) $\forall B \geqq c_3 + c_4 \log \ell$

$$\int P(dZ) \prod_{\sigma\in\Sigma_\ell} \bar{\chi}_\sigma^{B,s} \geqq \exp - c_1 e^{-c_2 B^2} |I| . \tag{5.10}$$

ii) *Let* $S \subset \Sigma_\ell$, $\forall B_\sigma \geqq c_3 + c_4 \log \ell$, $\sigma \in S$:

$$\int P(dZ) \prod_{\sigma\in S} (1-\bar{\chi}_\sigma^{B,s}) \leqq \prod_{\sigma\in S} \exp(c_5 - c_6 B_\sigma^2) . \tag{5.11}$$

iii) *Let* χ_Δ^B *be the characteristic function of the event*

$$E_\Delta^B = \{Z | Z \in \mathscr{S}'(\mathbb{R}^3), \|Z\|_{C^{(\varepsilon)}(\Delta)} < B(1+d(\Delta, I)\} ,$$

where $\Delta \in Q_1$ *(cf.* (2.5)*), then if* $B_\Delta = B(1+d(\Delta, I))$:

$$\int P(dZ) \Big(\prod_{\Delta\in Q_1} \chi_\Delta^B \Big) \Big(\prod_{\sigma\in\Sigma} \bar{\chi}_\sigma^B \Big) \geqq \exp - c_1 e^{-c_2 B^2} |I| . \tag{5.12}$$

$\forall B \geqq c_3 + c_4 \log \ell$, *and if* $R \subset Q_1$:

$$\int P(dZ) \Big(\prod_{\Delta\in R} (1-\chi_\Delta^B) \Big) \leqq \prod_{\Delta\in R} \exp(c_5 - c_6 B_\Delta^2) . \tag{5.13}$$

iv) *The estimates* (5.11), (5.13) *hold also if P is replaced by a probability measure* P^0 *which is a gaussian measure associated with the operator A considered with Dirichlet (i.e. null) boundary conditions on some set (cf. Proposition 3 below).*

Finally we shall need a representation for the P-distribution of a random field Z conditioned to taking a given value $\tilde{Z}$ outside a given regular region Λ.

Given an open set $O \subset \mathbb{R}^3$ we call $\mathscr{B}_O$ the σ-algebra of P-measurable sets of $\mathscr{S}'(\mathbb{R}^3)$ associated with the functions on $\mathscr{S}'(\mathbb{R}^3)$ having the form

$$Z \to Z(f) = \int_{\mathbb{R}^3} f(\xi) Z_\xi d\xi \qquad Z \in \mathscr{S}'(\mathbb{R}^3) \tag{5.14}$$

with $f \in \mathscr{D}(\mathbb{R}^3)$, $\operatorname{supp} f \subset O$. Define

$$\mathscr{B}_{\bar{\Lambda}} = \bigcap_{O \supset \bar{\Lambda}} \mathscr{B}_O \qquad \mathscr{B}_{\bar{\Lambda}_c} = \bigcap_{O \supset \bar{\Lambda}_c} \mathscr{B}_O , \tag{5.15}$$

where Λ_c = complement of Λ in $\mathbb{R}^3$ and the bar denotes closure, ($\Lambda_c \equiv \bar{\Lambda}_c$).

Proposition 3. i) *P-almost surely a randomly chosen distribution* $\tilde{Z}$ *is locally Hölder continuous with exponent* $\varepsilon<\frac{1}{2}$. *Furthermore* $\tilde{Z}$ *has, P-almost surely, a trace on the boundary of a given regular region* Λ *together with its first normal derivative* $\partial\tilde{Z}$. *The pair* $(\tilde{Z}, \partial\tilde{Z})$ *of distributions on* $\partial\Lambda$ *is in*

$$C_s^{(\varepsilon)}(\partial\Lambda) \times C_{s-1}^{(\varepsilon)}(\partial\Lambda), \qquad \forall \varepsilon\in(0,\tfrac{1}{2}), \qquad \forall s<\tfrac{1}{2}$$

[8]. *The trace of* $\tilde{Z}$ *on* $\partial\Lambda$ *is the same (except for the sign) both if* $\partial\Lambda$ *is considered as the boundary of* Λ *or of the complement of* Λ.

ii) *The solution* $u(\tilde{Z})$ *of the equation*

$$\begin{aligned} Au&=0 \quad \text{in} \quad \Lambda \\ u&=\tilde{Z}, \quad \partial u=\partial\tilde{Z} \quad \text{on} \quad \partial\Lambda \\ u&=\tilde{Z} \quad \text{in} \quad \Lambda^c \end{aligned} \tag{5.16}$$

with $u\in C^\infty(\Lambda)$ *and taking the boundary value in the sense of the traces on parallel surfaces in the spaces*

$$C_s^{(\varepsilon)}(\partial\Lambda)\times C_{s-1}^{(\varepsilon)}(\partial\Lambda), \qquad \varepsilon\in(0,\tfrac{1}{2}), \qquad s<\tfrac{1}{2},$$

is a random field in $\mathscr{S}'(\mathbb{R}^d)$ *with samples almost surely Hölder continuous with exponent* $<\frac{1}{2}$. *We shall denote* $(\mathscr{S}'(\mathbb{R}^3), \mathscr{B}, \bar{P})$ *or simply* $\bar{P}$ *the measure image of* P *under the map* $\tilde{Z}\to u(\tilde{Z})$.

iii) *The random variables* $\tilde{Z}-u(\tilde{Z})$ *and* $u(\tilde{Z})$ *are* P*-independent and the variable* $\tilde{\zeta}=\tilde{Z}-u(\tilde{Z})$ *has the same distribution of a gaussian random variable on* $\mathscr{S}'(\mathbb{R}^3)$ *whose covariance is the Green's function of the operator* A *with null boundary conditions on* $\partial\Lambda$. *We shall call* $(\mathscr{S}'(\mathbb{R}^3), \mathscr{B}, \bar{P}^0)$ *the measure on* $\mathscr{S}'(\mathbb{R}^3)$ *image of* P *under the map* $\tilde{Z}\to\tilde{\zeta}=\tilde{Z}-u(\tilde{Z})$.

iv) *The map* $\tilde{Z}\to(\tilde{\zeta}, u(\tilde{Z}))$ *is an isomorphism (mod. 0) between the measure spaces*

$$(\mathscr{S}'(\mathbb{R}^3), \mathscr{B}, P) \quad \text{and} \quad (\mathscr{S}'(\mathbb{R}^3)\times\mathscr{S}'(\mathbb{R}^3), \mathscr{B}\times\mathscr{B}, \bar{P}^0\times\bar{P}).$$

v) *If* F *is* $\mathscr{B}_{\bar{\Lambda}}$*-measurable and* G *is* $\mathscr{B}_{\Lambda_c}$*-measurable:*

$$\int P(d\tilde{Z})F(\tilde{Z})G(\tilde{Z})=\int P(d\tilde{Z})G(\tilde{Z})\int P^0(d\zeta)\cdot F(\zeta+u(\tilde{Z})). \tag{5.17}$$

The above proposition is essentially due to Pitt [10]; the parts i) and ii) are discussed in [8].

We shall use Proposition 3 via the "Markov property" (5.17). The random field $u(\tilde{Z})$ is called the "center" of the conditional distribution in Λ of Z given its value $\tilde{Z}$ outside Λ.

A corollary of Propositions 1 and 3 is

Corollary 4. *Let* $0<\varepsilon<s<\frac{1}{2}$ *and let* $\tilde{Z}\in\mathscr{S}'(\mathbb{R}^3)$ *be such that*

$$\prod_{\sigma\in\partial\square}\bar{\chi}_\sigma^{B,s}(\tilde{Z})=1 \quad \text{for some} \quad \square\in\bigcup_{i=1}^{4}\tilde{Q}_{i,\ell}$$

with ℓ*-large enough (depending on* ε, s*). Then the center of the conditional* P*-distribution in* $\square$ *of* Z *given its value* $\tilde{Z}$ *outside* $\square$ *verifies the bound*

$$\|u(\tilde{Z})\|_{C^{(\varepsilon)}(\Delta\cap\square)}\leqq K(\varepsilon,s)Be^{-\frac{\kappa}{2}d(\Delta,\partial\square)}(\|\tilde{Z}\|_{C_s^{(\varepsilon)}(\partial\square)}+\|\partial\tilde{Z}\|_{C_{s-1}^{(\varepsilon)}(\partial\square)}). \tag{5.18}$$

$\forall\Delta\in Q_1$, $\forall B\geqq 0$ *and for suitably chosen constants* $K(\varepsilon, s)$ *and* κ.

Combining Corollary 4 and Proposition 2, iv) and (5.8) and some geometrical considerations we obtain also the following corollary (see Sect. 5 in [9]):

Corollary 5. *Under the same assumptions of Corollary 4, there exists $\varrho>0$ such that if*

$$\tilde{Z}\in\mathscr{S}'(\mathbb{R}^3), \qquad \prod_{\sigma\in\partial\square}\chi_\sigma^{\varrho B}(\tilde{Z})=1 \quad \textit{for some} \quad \square\in\bigcup_{i=1}^{4}\tilde{Q}_{i,\ell}$$

then, $\forall B>c_3+c_4\log\ell$:

$$\int P^0(d\zeta)\Big(\prod_{\Delta\cap\square\neq\emptyset}\chi^{\frac{B}{2}}_{\Delta\cap\square}(\zeta+u(\tilde{Z}))\Big)\Big(\prod_{\substack{\sigma\subset\square,\\ \sigma\in\Sigma_\ell}}\chi_\sigma^B(\zeta+u(\tilde{Z}))\Big)\geqq\exp-c_1e^{-c_2B^2}|\square|, \tag{5.19}$$

where c_1,c_2,c_3,c_4 can be taken the same as those in Proposition 2[11].

6. Proof of Lemma 1

Given $R\subset Q_1$ we have to estimate, cf. (2.10), (2.13):

$$\mathscr{G}_R=\int(\exp H_{J\backslash\hat{R}})\mathring{\chi}_R^B\chi^B_{Q_1\backslash R}dP. \tag{6.1}$$

Using the notations of Sect. 5 consider the four smoothed pavements $\tilde{Q}_{1,\ell}$, $\tilde{Q}_{2,\ell}$, $\tilde{Q}_{3,\ell}$, $\tilde{Q}_{4,\ell}$ with $\ell=B^2$ and choose ε,s in $(0,1/2)$, $\varepsilon<s$. We shall then choose $B>B_1^*$ with B_1^* such that $(B_1^*)^2\geqq\ell_0+\ell_1(\varepsilon,s)$, (cf. Propositions 1 and 2 and Sect. 5).

Recalling the definition, preceding Proposition 2, of $\bar{\chi}_\sigma^B$ for σ belonging to the integration grid $\Sigma_\ell=\Sigma_{1,\ell}\cup\ldots\cup\Sigma_{4,\ell}$, we see that:

$$1\equiv\sum_{S\subset\Sigma_\ell}\Big(\prod_{\sigma\in S}(1-\bar{\chi}_\sigma^B)\Big)\Big(\prod_{\sigma\notin S}\bar{\chi}_\sigma^B\Big). \tag{6.2}$$

Let:

$$\begin{aligned}&\mathscr{R}(S)=\{\text{set of the }\square\text{'s which touch some }\sigma\in S\}\\ &\hat{\mathscr{R}}(S)=\{\Delta\,|\,\Delta\in Q_1, d(\Delta,\mathscr{R}(S))\leqq B^3\}.\end{aligned} \tag{6.3}$$

We shall need the following properties of H_J which follow easily from its definition (2.1): given any two sets paved by Q_1 (R and $\hat{\mathscr{R}}$) there are constants g_1, g_2, ϱ such that:

$$\mathring{\chi}_R^B\chi^B_{Q_1\backslash R}H_{J\backslash\hat{R}}\leqq\mathring{\chi}_R^B\chi^B_{Q_1\backslash R}(H_{J\backslash\hat{R}\cup\hat{\mathscr{R}}}+g_1\cdot\|\Delta\|B^\varrho|\hat{\mathscr{R}}\cap J|) \tag{6.4}$$

and

$$\left|\sum_{k=1}^{t}\frac{\mathscr{E}^T(H_{J\backslash\hat{R}\cup\hat{\mathscr{R}}};k)}{k!}-\sum_{k=1}^{t}\frac{\mathscr{E}^T(H_J;k)}{k!}\right|\leqq g_2\,\|\Delta\|\,e^{g_2\|\Delta\|}\cdot(|\hat{R}\cap J|+|\hat{\mathscr{R}}\cap J|). \tag{6.5}$$

It is perhaps worth stressing that in (6.5) the sets $R,\hat{\mathscr{R}}$ are arbitrary sets paved by Q_1 (and have nothing to do with regions where the field z is not well behaved). Also it should be noted that (6.5) is a simple consequence of the polynomial nature of (2.1) and of its locality properties [expressed by the exponentials in (2.1)]. Therefore (6.5) should not be confused with the (4.19) which though quite similar, rests on the much more detailed positivity and structure properties of $V^{(N)}$.

11 In (5.19) we use the choice $B/2$ and B in the two characteristic functions because this is the choice that we shall need later: of course this is a quite arbitrary choice

To apply (6.5) to the analysis of $\mathcal{G}_R$ notice that if $\hat{\mathcal{R}} = \hat{\mathcal{R}}(S)$ then $|\hat{\mathcal{R}}| \leqq (2B)^9 |\mathcal{R}(S)|$ and, also, $|\hat{R}| \leqq (2B)^9 |R|$: therefore (6.2)–(6.4) imply:

$$\mathcal{G}_\emptyset \geqq \int \chi_Q^{B_1} \left(\prod_{\sigma \in \Sigma} \bar{\chi}_\sigma^B \right) (\exp H_J) P(dz), \tag{6.6}$$

$$\begin{aligned} \mathcal{G}_R \leqq \sum_{S \subset \Sigma_\ell} & (\exp g_1 \|\underline{A}\| (2B)^{9+\varrho} |\mathcal{R}(S) \cap J|) \\ & \cdot \int \mathring{\chi}_R^B \chi_{Q_1 \backslash R}^B \left(\prod_{\sigma \in S} (1 - \bar{\chi}_\sigma^B) \right) \cdot \left(\prod_{\sigma \notin S} \bar{\chi}_\sigma^B \right) (\exp H_{J \backslash \hat{R} \cup \hat{\mathcal{R}}(S)}) \cdot P(dz). \end{aligned} \tag{6.7}$$

We shall first see that (6.6), (6.7) allow to "reduce" our problem to the proof of the following, apparently even more difficult, lemma

Lemma 4. *With the above notations, given* $t \geqq 0$, $\exists M, \sigma_1, \sigma_2, \sigma_3$ *such that* $\forall B$ *large enough,* $\forall R \subset Q_1$, $\forall S \subset \Sigma_\ell$:

$$\begin{aligned} & \int \mathring{\chi}_R^R \chi_{Q_1 \backslash R}^B \left(\prod_{\sigma \in S} (1 - \bar{\chi}_\sigma^B) \right) \left(\prod_{\sigma \notin S} \bar{\chi}_\sigma^B \right) (\exp H_{J \backslash \hat{R} \cup \hat{\mathcal{R}}(S)}) P(dz) \\ & \leqq \left(\exp \left\{ \sum_{k=1}^t \frac{\mathscr{E}^T(H_{J \backslash \hat{R} \cup \hat{\mathcal{R}}(S)}; k)}{k!} \right\} \right) (\exp \bar{\delta}(B, \|\underline{A}\|) |I|) \\ & \cdot \left(\int \mathring{\chi}_R^B \left(\prod_{\sigma \in S} (1 - \bar{\chi}_\sigma^B) \right) P(dz) \right) \end{aligned} \tag{6.8}$$

with

$$\bar{\delta}(B, \|\underline{A}\|) = M[(\|\underline{A}\| B^{\sigma_1} e^{\sigma_1 \|\underline{A}\| B^{\sigma_1}})^{t+1} + e^{\sigma_2 \|\underline{A}\| B^{\sigma_1} - \sigma_3 B^2}]. \tag{6.9}$$

Furthermore:

$$\begin{aligned} \int \chi_{Q_1}^B \left(\prod_{\sigma \in \Sigma_\ell} \bar{\chi}_\sigma^B \right) (\exp H_J) P(dz) \geqq & (\exp - \bar{\delta}(B, \|\underline{A}\|) |I|) \\ & \cdot \left(\exp \left\{ \sum_{k=1}^t \frac{\mathscr{E}^T(H_J; k)}{k!} \right\} \right). \end{aligned} \tag{6.10}$$

While it is obvious that Lemma 4 solves the lower bound problem i.e. proves (2.16), some more work is necessary to obtain (2.14) from (6.8).

In fact combining (6.7), (6.8), (6.5) we see that

$$\begin{aligned} \mathcal{G}_R \leqq & \left(\exp \left\{ \sum_{k=1}^t \frac{\mathscr{E}^T(H_J; k)}{k!} \right\} \right) \cdot \left(\sum_S \left(\int \prod_{\sigma \in S} (1 - \bar{\chi}_\sigma^B) P(dz) \right)^{1/2} \right. \\ & \left. \cdot \exp[(g_1 + g_2) \|\underline{A}\| e^{g_2 \|\underline{A}\|} \cdot (2B)^{9+\varrho} (|R \cap J| + |\mathcal{R}(S) \cap J|)] \right) \\ & \cdot \left(\int \mathring{\chi}_R^B dP \right)^{1/2} \cdot \exp \bar{\delta}(B, \|\underline{A}\|) |I| \end{aligned} \tag{6.11}$$

using (5.11) the sum over S can be easily estimated [as in the similar case dealt with in the formulae (4.23)–(4.25)] and the result is just (2.14).

Hence Lemma 1 will be proven once we shall have proven Lemma 4.

We now prove Lemma 4. The proof that follows is an obvious adaptation of the proof in Sect. 5 of [2]. We shall try to make it using notations which underline the similarity of the hierarchical and the euclidean fields.

Denote $J_0 = J \backslash \hat{R} \cup \hat{\mathscr{R}}(S)$ and let $C = R \cup \mathscr{R}(S)$, C_1 = smallest set paved by Q_1 and such that $C_1 \supset C$, $d(\partial C_1, C) > 3B^2$.

Call $\bar{P}$ the probability measure $P(dZ|\bar{Z}_{C_1})$ which is the P measure conditioned to a given value $\bar{Z}$ of the field in the region C_1.

Consider the $\underline{A}$-functions in (2.1) and set:

$$\begin{aligned} \tilde{A}^{n_1 \dots n_p}_{\xi_1 \dots \xi_p} &= \left(\prod_{i=1}^{p} \chi_{J_0}(\xi_i) \right) A^{n_1 \dots n_p}_{\xi_1 \dots \xi_p} \\ \tilde{A}^{m_1 \dots m_q}_{\xi_1 \xi'_1 \dots \xi_q \xi'_q} &= \left(\prod_{i=1}^{q} \chi_{J_0}(\xi_i) \chi_{J_0}(\xi'_i) \right) A^{m_1 \dots m_q}_{\xi_1 \xi'_1 \dots \xi_q \xi'_q}, \end{aligned} \tag{6.12}$$

where χ_{J_0} is the characteristic function of the set J_0, and introduce the following very symbolical notations:

$$\begin{aligned} &A^{n_1, \dots, n_p}_{\Delta_1, \dots, \Delta_p} e^{-\kappa d(\Delta_1, \dots, \Delta_p)} Z^{n_1}_{\Delta_1} \dots Z^{n_p}_{\Delta_p} \\ &\equiv \int_{\Delta_1 \times \dots \times \Delta_p} \tilde{A}^{n_1 \dots n_p}_{\xi_1 \dots \xi_p} e^{-\kappa d(\xi_1, \dots, \xi_p)} Z^{n_1}_{\xi_1} \dots Z^{n_p}_{\xi_p} d\xi_1 \dots d\xi_p \end{aligned} \tag{6.13}$$

and similarly:

$$\begin{aligned} &A^{m_1, \dots m_q}_{\Delta_1, \dots, \Delta'_q} e^{-\kappa d(\Delta_1, \Delta'_1, \dots, \Delta_q, \Delta'_q)} \prod_{i=1}^{q} \left(\frac{(Z_{\Delta_i} - Z_{\Delta'_i})}{|\Delta_i - \Delta'_i|^{1/4}} \right)^{m_i} \\ &\equiv \int_{\Delta_1 \times \Delta'_1 \times \dots \times \Delta_q \times \Delta'_q} d\xi_1 d\xi'_1 \dots d\xi_q d\xi'_q \tilde{A}^{m_1 \dots m_q}_{\xi_1 \xi'_1, \dots \xi_q \xi'_q} e^{-\kappa d(\xi_1, \dots, \xi'_q)} \\ &\cdot \prod_{i=1}^{q} \left(\frac{(Z_{\xi_i} - Z_{\xi'_i})}{|\xi_i - \xi'_i|^{1/4}} \right)^{m_i}, \end{aligned} \tag{6.14}$$

where $\Delta_1, \Delta'_1, \dots, \Delta'_q$ are tesserae of the pavement Q_1.

The above notation is not really necessary, of course, however we like it because it suggests explicitly that Z is "constant and smooth" on the tesserae Δ of Q_1: this is not strictly true but the reason why the proof works is that it is "essentially" true.

Given any set F paved by Q_1 we set:

$$\begin{aligned} H(F) = &\sum_{p=1}^{s} \sum_{\substack{n_1, \dots, n_p \\ \Sigma_i n_i \leqq D}} \sum_{\Delta_1, \dots, \Delta_p \subset F} A^{n_1 \dots n_p}_{\Delta_1 \dots \Delta_p} e^{-\kappa d(\Delta_1, \dots, \Delta_p)} Z^{n_1}_{\Delta_1} \dots Z^{n_p}_{\Delta_p} \\ &+ \sum_{q=1}^{s'} \sum_{\substack{m_1, \dots, m_q \\ \Sigma_i m_i \leqq D'}} \sum_{\Delta_1, \Delta'_1, \dots, \Delta_q, \Delta'_q \subset F} A^{m_1 \dots m_q}_{\Delta_1 \dots \Delta'_q} e^{-\kappa d(\Delta_1, \dots, \Delta'_q)} \prod_{i=1}^{q} \left(\frac{(Z_{\Delta_i} - Z_{\Delta'_i})}{|\Delta_i - \Delta'_i|^{1/4}} \right)^{m_i}. \end{aligned} \tag{6.15}$$

If $\bar{J}_0$ is the smallest set paved by Q_1 and containing J_0 we see that

$$H_{J_0} \equiv H(\bar{J}_0) \tag{6.16}$$

and, therefore, we are led to the analysis of integrals like:

$$\int \left(\prod_{\Delta \in Q_1 \backslash R} \chi^B_\Delta(Z) \right) \left(\prod_{\sigma \in \Sigma_\ell \backslash S} \chi^B_\sigma(Z) \right) (\exp H(\bar{J}_0) \bar{P}(dZ). \tag{6.17}$$

By the definition of $\bar{P}$, cf. lines before (6.12), it is immediate to realize that the l.h.s. of (6.8) is just the integral of (6.17), thought as a function of the conditioning field $\bar{Z}$ which appears in $\bar{P}$, with respect to the measure

$$\left(\prod_{\Delta\in R} \mathring{\chi}_\Delta^B(\bar{Z})\right)\left(\prod_{\sigma\in S}(1-\bar{\chi}_\sigma^B)\right) P(d\bar{Z}). \tag{6.18}$$

So, if we can prove that (6.17) is uniformly bounded from above by the r.h.s. of (6.8) without the last factor, we shall have proved (6.8) itself.

Similarly to prove (6.10) we have to prove that (6.17) is bounded below by the r.h.s. of (6.10) when $R=S=\emptyset$ (in this case $C=\emptyset=C_1$ and $\bar{P}(d\bar{Z})\equiv P(d\bar{Z})$).

Therefore we shall prove:

$$\begin{aligned}&\int \bar{P}(dZ)\left(\prod_{\Delta\in Q_1\backslash R}\chi_\Delta^B(Z)\right)\left(\prod_{\sigma\in\Sigma_\ell\backslash S}\chi_\sigma^B(Z)\right)(\exp H(\bar{J}_0))\\&\quad\leqq\left(\exp\left\{\sum_{k=1}^{t}\frac{\mathscr{E}^T(H(\bar{J}_0);k)}{k!}\right\}\right)(\exp+\bar{\delta}(B,\|\underline{A}\|)|I|)\end{aligned} \tag{6.19}$$

and for $R=S=\emptyset$, i.e. for $C=C_1=\emptyset$:

$$\begin{aligned}&\int P(dZ)\left(\prod_{\Delta\in Q_1}\chi_\Delta^B(Z)\right)\left(\prod_{\sigma\in\Sigma_\ell}\chi_\sigma^B(Z)\right)(\exp H(\bar{J}))\\&\quad\geqq\left(\exp\left\{\sum_{k=1}^{t}\frac{\mathscr{E}^T(H(\bar{J});k)}{k!}\right\}\right)(\exp-\bar{\delta}(B,\|\underline{A}\|)|I|).\end{aligned} \tag{6.20}$$

The estimates (6.19), (6.20) can be estimated as follows (cf. [2]).

Given two different regions T,S, paved by Q_1, define the "interaction between T,S" as

$$H(T,S)=H(T\cup S)-H(T)-H(S). \tag{6.21}$$

Call (1), (2), (3), (4) the closed unpaved corridors of $\tilde{Q}_{1,\ell}$, $\tilde{Q}_{2,\ell}$, $\tilde{Q}_{3,\ell}$, $\tilde{Q}_{4,\ell}$ and $(12)=(1)\cap(2)$, $(123)=(1)\cap(2)\cap(3)$. By the construction $d((4),(123))\geqq\delta\cdot\ell$.

Consider the quasi-pavement $\tilde{Q}_{1,\ell}$ and the tesserae $\square\in\tilde{Q}_{1,\ell}$.

Call $\Gamma_1=$largest set paved by Q_1 contained in (1). Let $\square'$ be a cube inside $\square\in\tilde{Q}_{1,\ell}$ whose boundary is at a distance $\delta\cdot\ell$ from $\partial\square$:

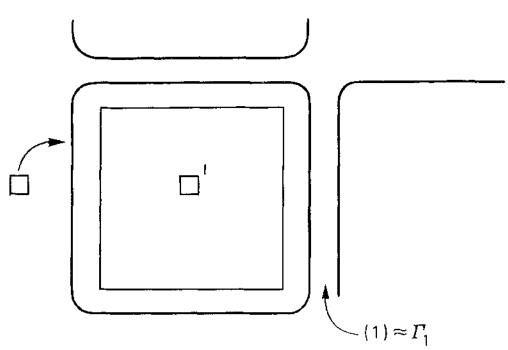


Fig. 5

Recalling that $\ell = B^2$ and that the only restriction on B was to be larger than B_1^* (cf. beginning of Sect. 6), so far, we may suppose that B is so large that $\frac{\delta \cdot \ell}{2} > 1$.

This means that B is thought to be larger than some constant B^{**}: to say that B is large enough will, from now on, mean that $B > B^{**}/(\varrho/2)^4$ where ϱ is the constant of Corollary 5, Sect. 5.

Then we notice that

$$H(\bar{J}_0) = \Big(H(\Gamma_1) + \sum_{\square} \psi_{\square}\Big) + H^{(c)} \equiv \hat{H}(\bar{J}_0) + H^{(c)} \tag{6.22}$$

if $H^{(c)} \equiv H(\bar{J}_0) - \hat{H}(\bar{J}_0)$ and, if $\bar{\square}$ is the smallest set paved by Q_1 and containing $\square$:

$$\psi_{\square} = H(\bar{\square}) + H(\bar{\square}, \Gamma_1). \tag{6.23}$$

The short range nature of $H(J)$, cf. the exponentials in (2.1), allows to say that $H^{(c)}$ which represents the "interaction between tesserae separated by a corridor of size $\sim 2\delta\ell$, can be bounded by (recalling that $\ell^2 = B^4$ is a measure of $|\partial\square|$):

$$\Big|H^{(c)} \cdot \Big(\prod_{\Delta \notin R} \chi_{\Delta}^{B}\Big)\Big| \leqq S_1 \|\underline{\Delta}\| B^{D+D'+4} e^{-\frac{\kappa}{2}\delta B^2} |I| \tag{6.24}$$

and S_1 is a suitable constant which can be easily found looking at (2.1).

Therefore our problem is to estimate

$$\int \bar{P}(dZ) \Big(\prod_{\Delta \notin R} \chi_{\Delta}^{B}(Z)\Big) \Big(\prod_{\sigma \in \Sigma_1 \setminus S} \chi_{\sigma}^{B}(Z)\Big) \exp \hat{H}(\bar{J}_0) \tag{6.25}$$

since neglecting H^c will cause, by (6.24), an error of the same form of the second contribution to $\bar{\delta}$ in (6.9) (with some other constants but with the same $\|\underline{\Delta}\|$ and B dependence).

We first prove a lower bound for (6.25) supposing $C = \emptyset$, i.e. $R = S = \emptyset$. We have, in this case:

$$\begin{aligned}[(6.25)] \geqq \int \bar{P}(d\tilde{Z}) &\Big(\prod_{\Delta \cap (1) \neq \emptyset} \chi_{\Delta \cap (1)}^{B/2}(\tilde{Z})\Big) \Big(\prod_{\sigma \subset (1)} \chi_{\sigma}^{B}(\tilde{Z})\Big) e^{H(\Gamma_1)} \\ &\cdot \prod_{\square} \Big\{ \int P^{(0)}(d\zeta_{\square}) \Big(\prod_{\Delta \cap \square \neq \emptyset} \chi_{\Delta \cap \square}^{B/2}(\zeta_{\square} + u(\tilde{Z}))\Big) \\ &\cdot \Big(\prod_{\sigma \subset \square} \chi_{\sigma}^{B}(\zeta_{\square} + u(\tilde{\zeta})\Big) (\exp \psi_{\square}) \Big\},\end{aligned} \tag{6.26}$$

where we have used the Markov property (5.17) denoting $\zeta_{\square}$ the Dirichlet field relative to $\square$ and to the operator A. We have also used the remark that if $\Delta_1 = \Delta \cap \square$ and $\Delta_2 = \Delta \cap \square_c$ and $f \in C^{(1/4)}(\Delta)$ then:

$$\|f\|_{C^{(1/4)}(\Delta)} \leqq \|f\|_{C^{(1/4)}(\Delta_1)} + \|f\|_{C^{(1/4)}(\Delta_2)}. \tag{6.27}$$

Let now $\varrho \in (0, 1)$ be the constant introduced in Corollary 5, Sect. 5, and set

$$\begin{aligned}\chi^{\Gamma_1, \varrho B}(Z) &= \Big(\prod_{\Delta \cap (1) \neq \emptyset} \chi_{\Delta \cap (1)}^{B\varrho/2}(Z)\Big) \Big(\prod_{\sigma \subset (1)} \chi_{\sigma}^{B\varrho}(Z)\Big) \\ \chi^{\square, B}(Z) &= \Big(\prod_{\Delta \cap \square = \emptyset} \chi_{\Delta \cap \square}^{B/2}(Z)\Big) \Big(\prod_{\sigma \subset \square} \chi_{\sigma}^{B}(Z)\Big),\end{aligned} \tag{6.28}$$

and denote $\bar{P}(dZ_\square/\tilde{Z})$ the P-conditional probability for the field Z in $\square$ given its value $\tilde{Z}$ in the corridor (1): by Proposition 3 we may identify $\zeta_\square + u(\tilde{Z})$ with $Z_\square$.

Then $\exists f_1, f_2, s_2 > 0$ such that:

$$\begin{aligned}
[(6.25)] \geqq & \int \bar{P}(d\tilde{Z})\chi^{\Gamma_1,\varrho B} e^{H(\Gamma_1)} \\
& \cdot \left[\prod_{\square \cap \bar{J}_0 \neq \emptyset} \left(\int \bar{P}(dZ_\square|\tilde{Z})\chi^{\square,B}(Z_\square) e^{\psi_\square \chi^{\square,B}}\right)\right] \\
& \cdot \left[\prod_{\square \cap \bar{J}_0 = \emptyset} \cdot \left(\int \bar{P}(dZ_\square|\tilde{Z})\chi^{\square,B}(Z_\square)\right)\right] \\
\geqq & \int \bar{P}(d\tilde{Z})\chi^{\Gamma_1,\varrho B} e^{H(\Gamma_1)} \\
& \cdot \left[\prod_{\square \cap \bar{J}_0 \neq \emptyset} \left(\int \bar{P}(dZ_\square|\tilde{Z}) e^{\psi_\square \chi^{\square,B}}\right)\right] \left[\prod_{\square \cap \bar{J}_0 = \emptyset} \left(\int \bar{P}(dZ_\square|\tilde{Z}) \cdot \chi^{\square,B}(Z_\square)\right)\right] \\
& \cdot \left(\exp - \mathcal{N}(\bar{J}_0) f_1 e^{-f_2 B^2} e^{2s_2 \|\Delta\| B^{D+D'} \ell^3}\right),
\end{aligned} \tag{6.29}$$

where $\mathcal{N}(\bar{J}_0)$ = number of $\square$'s in $\tilde{Q}_{1,\ell}$ such that $\square \cap \bar{J}_0 \neq \emptyset$: $\mathcal{N}(\bar{J}_0) \leqq |I|/\ell^3$. In (6.29) the error term arises because we replace in the intermediate integral in the r.h.s. of (6.29) one of the two $\chi^{\square,B}$ by 1. Such error can be estimated easily by using the inequality, valid for any random variable X with values in a Banach space and with distribution μ:

$$\begin{aligned}
& \left(\int \mu(dX)\chi(\|X\| < B) e^{V(X)}\right) \geqq \left(\int \mu(dX) e^{V(X)}\right) \\
& \quad \cdot \left(\int \mu(dX)\chi(\|X\| < B)\right)^{\theta \exp 2\|V\|_\infty} \qquad \theta \in [-1, 1],
\end{aligned} \tag{6.30}$$

where $\chi(\|X\| < B)$ is the characteristic function of the event $\{X \mid \|X\| < B\}$ and $\|V\|_\infty$ is the $L_\infty(\mu)$-norm of the, arbitrary, function $V \in L_\infty(\mu)$. The above relation applied to $\bar{P}(dZ_\square/\tilde{Z})$ and combined with Corollary 5, Sect. 5, to estimate

$$\int \bar{P}(dZ_\square|\tilde{Z})\chi^{\square,B}(Z_\square) \geqq \exp - c_1 e^{-c_2 B^2} \ell^3 \tag{6.31}$$

immediately yields (6.29).

The inequality (6.30) is an elementary inequality (for a proof see [2], p. 156).

Since $\mathcal{N}(\bar{J}_0) \leqq |I| \ell^{-3}$ the error term in (6.29) has the same form of the second contribution to $\bar{\delta}$ in (6.9) (with other constants) and therefore we do not have to be worried by it anymore.

We now compute the intermediate integral in the r.h.s. of (6.29) by a cumulant expansion, (see Appendix A, Definition 1), to order t:

$$\begin{aligned}
& \log \int \bar{P}(dZ_\square|\tilde{Z})(\exp \psi_\square \chi^{\square,B}) \\
& = \left\{\sum_{k=1}^{t} \frac{\mathscr{E}_{\tilde{Z}}^T(\psi_\square \chi^{\square,B}; k)}{k!}\right\} + (\text{error}),
\end{aligned} \tag{6.32}$$

where $\mathscr{E}_{\tilde{Z}}^T$ denotes the truncated expectation with respect to $\bar{P}(dZ_\square|\tilde{Z})$ and $\exists s_3 > 0$ such that:

$$\begin{aligned}
(\text{error}) = & \tau \frac{2^{(t+1)^2}(t+1)!}{(t+1)!} (s_3 B^{D+D'} \ell^3 \|\Delta\|)^{t+1} \\
& \cdot (\exp 2s_3 \|\Delta\| B^{D+D'} \ell^3)
\end{aligned} \tag{6.33}$$

with $\tau \in [-1, 1]$.

We see that the error (6.33) has, apart from the different values of the constants, the same form of the first contribution to $\bar{\delta}$ in (6.9).

Combining (6.29), (6.32), (6.33) we have obtained:

$$[(6.25)] \geqq \int \bar{P}(d\tilde{Z})\chi^{\Gamma_1, \varrho B} e^{H(\Gamma_1)} \cdot \Big[\prod_{\square \cap \bar{J}_0 = \emptyset} (\int \bar{P}(dZ_\square | \tilde{z})\chi^{\square, B}(Z_\square))\Big] \prod_{\square \cap \bar{J}_0 \neq \emptyset} \cdot \left(\exp\left\{\sum_{k=1}^{t} \frac{\mathscr{E}^T_{\tilde{Z}}(\psi_\square \chi^{\square, B}; k)}{k!}\right\}\right) \cdot (\exp(\text{error})), \tag{6.34}$$

where the (error) has the form $\delta'(B, \|\underline{A}\|)|I|$ with δ' given by (6.9) with, possibly, different values of the constants M, σ_1, σ_2, σ_3.

The next step consists in observing that the difference between $\chi^{\Gamma_1, \varrho B}\mathscr{E}^T_{\tilde{Z}}(\psi_\square \chi^{\square, B}; k)$ and $\chi^{\Gamma_1, \varrho B}\mathscr{E}^T_{\tilde{Z}}(\psi_\square; k)$ can be bounded, in absolute value, by an error term of the form $|I|\tilde{\delta}(B, \|\underline{A}\|)$ where $\tilde{\delta}$ has the same form as the second contribution to $\bar{\delta}$ in (6.9) with possibly different constants: so

$$[(6.25)] \geqq \int \bar{P}(d\tilde{Z})\chi^{\Gamma_1, \varrho B} e^{H(\Gamma_1)} \cdot \Big[\prod_{\square \cap \bar{J}_0 = \emptyset} (\int \bar{P}(dZ_\square | \tilde{Z})\chi^{\square, B}(Z_\square))\Big] \cdot \left(\exp \sum_{\square \cap \bar{J}_0 \neq \emptyset} \sum_{k=1}^{t} \frac{\mathscr{E}^T_{\tilde{Z}}(\psi_\square; k)}{k!}\right) \cdot (\exp - (\delta'(\|\underline{A}\|, B) + \tilde{\delta}(\|\underline{A}\|, B))|I|). \tag{6.35}$$

We now try to replace the conditional expectation $\mathscr{E}^T_{\tilde{Z}}$ by the unconditional one: of course this will be possible only "far from $\partial\square$". To explain what this means, let:

$$\psi_\square = (H(\square') + H(\square', \square_0)) + (H(\square_0, \Gamma_1) + H(\square_0)) + \tilde{H}^{(c)}$$

$$\equiv \psi^1_\square + \psi^2_\square + \tilde{H}^{(c)} \quad \text{where} \quad \square_0 = \bar{\square} \backslash \square' \tag{6.35'}$$

and $\tilde{H}^{(c)}$ can be bounded in the same way as (6.24) (because it contains interactions "across a corridor of width $\sim \delta \cdot \ell$): therefore, modulo an error $\tilde{\delta}'(\|\underline{A}\|, B)|I|$, with $\delta'(\|\underline{A}\|, B)$ having the same form of the second term in (6.9) with different constants we can replace in (6.35) the sum of expectations by:

$$\sum_{k=1}^{t} \frac{\mathscr{E}^T_{\tilde{Z}}(\psi^1_\square + \psi^2_\square; k)}{k!} \equiv \sum_{k=1}^{t} \sum_{k_1 + k_2 = k} \frac{\mathscr{E}^T_{\tilde{Z}}(\psi^1_\square, \psi^2_\square; k_1, k_2)}{k_1! k_2!}, \tag{6.36}$$

where we have used the Leibnitz formula for the cumulants (see Appendix A, Definition 1).

The gaussian integrals in (6.36) should now be grouped, before explicit computation, as:

$$\sum_{k=1}^{t} \sum_{\substack{k_1 > 0 \\ k_1 + k_2 = k}} \frac{\mathscr{E}^T_{\tilde{Z}}(\psi^1_\square, \psi^2_\square; k_1, k_2)}{k_1! k_2!} + \sum_{k=1}^{t} \frac{1}{k!} \mathscr{E}^T_{\tilde{Z}}(\psi^2_\square; k). \tag{6.37}$$

Use now the exponential decay of the free covariance and of the center field $u(\tilde{Z})$ from $\partial\square$ and the consequent "exponential similarity" of the free and of the Dirichlet covariances far from $\partial\square$. We shall combine this property with the exponential decay of the "interaction" in (2.1) and, by explicit computation of the gaussian integrals, it is easily seen that the first sum of (6.37) can be replaced by

$$\sum_{k=1}^{t}\sum_{\substack{k_1>0\\k_1+k_2=k}}\frac{\mathscr{E}^T(\psi_\square^1,\psi_\square^2;k_1,k_2)}{k_1!k_2!} \tag{6.38}$$

modulo an error $\tilde{\tilde{\delta}}''(\|\underline{A}\|,B)$ of the same form of $\tilde{\delta}'$, $\tilde{\delta}$ but with different constants.

Finally

$$\left(\exp\sum_{\square\cap\bar{J}_0\neq\emptyset}\sum_{k=1}^{t}\frac{\mathscr{E}_{\tilde{Z}}^T(\psi_\square^2;k)}{k!}\right)$$

$$=\left(\prod_{\square\cap\bar{J}_0\neq\emptyset}\int\bar{P}(dZ_\square|\tilde{Z})\chi^{\square,B}\cdot\exp\psi_\square^2\right)\cdot\exp(\text{error}) \tag{6.39}$$

with an error again of the by now usual form: this can be seen by doing backwards all the preceding steps. So

$$[(6.25)]\geqq\left\{\exp\sum_{\square\in\tilde{Q}_{1,\ell}}\sum_{k=1}^{t}\sum_{\substack{k_1>0\\k_1+k_2=k}}\frac{\mathscr{E}^T(\psi_\square^1,\psi_\square^2;k_1,k_2)}{k_1!k_2!}-\delta^{(1)}(\|\underline{A}\|,B)|I|\right\}$$

$$\cdot\left[\int\bar{P}(d\tilde{Z})\left(\prod_{\Delta\in Q_1}\chi_\Delta^{B\varrho/2}(\tilde{Z})\right)\left(\prod_{\sigma\in\Sigma_\ell}\chi_\sigma^{B\varrho/2}(Z)\right)e^{H(\Gamma_1^{(1)})}\right], \tag{6.40}$$

where the error $\delta^{(1)}(\|\underline{A}\|,B)$ has the same form (6.9) with other constants instead of M, σ_1, σ_2, σ_3 and

$$\Gamma_1^{(1)}=\bar{J}^{(0)}\Big\backslash\bigcup_{\square\in\tilde{Q}_{1,\ell}}\square' \tag{6.41}$$

i.e. we have "removed the interaction" except close to the corridor (1).

We can repeat four times the above argument [notice that the integral in (6.30)] has the same form as (6.25) with B replaced by $\frac{B\varrho}{2}$ and with $\bar{J}_0$ replaced by $\Gamma_1^{(1)}$.

Since $d((4),(123))$ is larger than 1 at the fourth step we obtain:

$$[(6.25)]\geqq\left(\exp\left\{\sum_{j=1}^{4}\sum_{\square\in\tilde{Q}_{j,\ell}}\sum_{k=1}^{t}\sum_{\substack{k_1>0\\k_1+k_2=k}}\frac{\mathscr{E}^T(\psi_\square^{1,j},\psi_\square^{2,j};k_1,k_2)}{k_1!k_2!}\right.\right.$$

$$\left.\left.+\delta^{(4)}(\|\underline{A}\|,B)|I|\right\}\right)\cdot\left[\int\bar{P}(d\tilde{Z})\left(\prod_{\Delta\in Q_1}\chi_\Delta^{B(\varrho/2)^4}(Z)\right)\right.$$

$$\left.\cdot\left(\prod_{\sigma\in\Sigma_\ell}\chi_\sigma^{B(\varrho/2)^4}(Z)\right)\right], \tag{6.42}$$

where $\delta^{(4)}$ has the "usual form" of (6.9) and $\psi_\square^{1,1}\equiv\psi_\square^1$, $\psi_\square^{2,1}\equiv\psi_\square^2$ and $\psi_\square^{1,j}$, $\psi_\square^{2,j}$ are defined in a way analogous to that used for $\psi_\square^1$ and $\psi_\square^2$: they are expressed in terms of the new "interactions" $H(\Gamma_1^{(1)})$, $H(\Gamma_2^{(12)})$, $H(\Gamma_3^{(123)})$ in the same way $\psi_\square^1$, $\psi_\square^2$ were expressed in terms of $H(\bar{J}_0)$.

It is quite clear that, up to an error term of the form $\bar{\delta}(\|\underline{A}\|, B)|I|$ with $\bar{\delta}$ having the same form as the second term in (6.9), the sum of expectations in (6.42) is nothing else than

$$\sum_{k=1}^{t} \frac{\mathscr{E}^T(H(\bar{J}_0);k)}{k!} \tag{6.43}$$

hence (6.10) follows from (6.43), (6.42) and the above remarks by applying the "phase space" estimate of Proposition 2, (5.12), to bound from below the integral in the r.h.s. of (6.32).

The estimate (6.8) is obtained by some simple modifications of the preceding argument by being careful, every time, to choose upper rather than lower bounds: the procedure to eliminate the characterististic functions will, of course be slightly different. Instead of restricting the phase space by replacing B by ϱB in the corridors we shall have to increase the phase space by replacing B by B/ϱ inside the elements $\square$, of the integration grid, which are under consideration.

In the above proof we have avoided introducing all the corridors appearing in the analogous proof in [2]: they seem to make the whole proof clumsy and it turns out to be, hopefully, more clear if they are used only implicitly as in the above scheme.

7. Concluding Remarks

1. The technique used can be clearly extended to prove perturbation theory to "any order" in the following sense.

Define

$$V_I'^{(N)} = V_{0,I}^{(N)} - \frac{\langle (V_{0,I}^{(N)})^2 \rangle_{(2)}}{2!} \tag{7.1}$$

and, if $\mathscr{E}$ denotes the expectation with respect to P_N, cf. Sect. 1, let

$$V_I^{(N,t)} = \left[V_I^{(N)} - \sum_{k=1}^{t} \frac{\mathscr{E}^T(V_I'^{(N)};k)}{k!} \right]_{(t)}. \tag{7.2}$$

Then there exists $E(\lambda;t)$ such that if $t \geqq 3$

i) $\int (\exp V_I^{(N,t)}) P_N(dz) = \exp \theta E(\lambda;t)|I|$ (7.3)

for some $\theta \in [-1,1)$.

ii) $\lim_{\lambda \to 0} \lambda^{-t} E(\lambda;t) = 0$. (7.4)

The classical theory of renormalization implies that all the terms in the sum (7.2) with $k \geqq 4$ are finite; and, as $I \to \infty$, have a limit if divided by (I). In other words (7.3), (7.4) mean that the ground state energy can be bounded above and below by the result of formal perturbation theory to order t plus a remainder which is infinitesimal of order λ^{t+1} as $\lambda \to 0$.

2. If one treated the case $d=2$ to obtain the same results obtained in the $d=3$ case in this paper, then one would have to do exactly the same calculations and the

same steps presented here with only a few (minor) simplifications. Actually the difference would be the absence of the term $\frac{(\varphi_\xi - \varphi_\eta)^2}{|\xi-\eta|^{1/4}}$ in the effective hamiltonian: to say it better a term like $(\varphi_\xi - \varphi_\eta)^2$ would still be present but we would not need to multiply and divide it by $|\xi-\eta|^{1/4}$ and use the Hölder continuity of the fields to show that it does not cause problems.

3. The technique of this paper allows to study the integrals,

$$\frac{\int (\exp \varphi(f))(\exp V_I^{(N)}) P_N(d\varphi)}{\int (\exp V_I^{(N)}) P_N(d\varphi)}, \tag{7.4}$$

where $f \in \mathscr{D}(\mathbb{R}^d)$ and to derive for the Schwinger functions bounds of the type of the ones found by Feldman [3] before the "advent" of the cluster expansion.

4. The ultraviolet limit $N \to \infty$ of expressions like (7.4) should be possible using techniques similar to the ones involved in this work. However the infrared limit, $|I| \to \infty$, is a somewhat different problem and there seems to be no other possible approach apart from the cluster expansion [3].

5. In this paper we have tried to eliminate the "casistic" involved in the derivation of the upper bound in [2]: it is replaced in Sect. 4 by a "mechanical" algebraic discussion. The idea of using the functions $\hat{V}$ and H to avoid the casistics was inspired by the work in [6].

6. The final results of this work were all well known and has been obtained by other methods.

The idea of using conditionings and overlapping regions seems to be somewhat new, at least in Statistical Mechanics and Field Theory: it was inspired by the works [4, 5] and used already in [2]. Similar ideas are involved in the work [6], independent on ours, which unfortunately has not yet appeared.

It seems to us that our work goes in the direction of extending to field theory the work [5]: however this important point deserves further analysis.

A general background on the above problems can be obtained by [7].

Appendix A. Some Definitions

1. The "cumulants" or truncated expectations of a family of random variables $x_1, x_2, \ldots, x_s$ with respect to the probability measure P are

$$\begin{aligned} &\mathscr{E}^T(x_1, x_2, \ldots, x_s; k_1, k_2, \ldots, k_s) \\ &= \left[\frac{\partial^{k_1+\ldots+k_s}}{\partial\theta_1^{k_1}\ldots\partial\theta_s^{k_s}} \log \int \left(\exp \sum_{i=1}^{s} \theta_1 x_i\right) dP\right]_{\theta_1=\ldots=\theta_s=0}, \end{aligned} \tag{A.1}$$

where $k_1, \ldots, k_s = 0, 1, \ldots$.

The cumulants verify a summation property ("Leibnitz formula") [7]:

$$\mathscr{E}^T\left(\sum_{i=1}^{p} x_i; k\right) = \sum_{k_1+\ldots+k_p=k} \frac{k!}{k_1!\ldots k_p!} \mathscr{E}^T(x_1, \ldots, x_p; k_1, \ldots, k_p). \tag{A.2}$$

The cumulants make sense, in an obvious way, for any probability measure such that $\int |x_i|^\varrho dP > +\infty$, $\forall \varrho > 0$ and for $i = 1, \ldots, s$.

2. A region Λ of $\mathbb{R}^3$ is "regular" if there are a finite number of points $\xi_1, \ldots, \xi_s$ in $\partial\Lambda$ such that, when they are chosen as the origin of a coordinate system in which the plane $x_3=0$ is the tangent plane π_i to $\partial\Lambda$ in ξ_i, then $\exists v\in\mathscr{D}(\mathbb{R}^2)$ and the surface σ_v

$$x_3=v(x_1,x_2) \qquad x_1,x_2\in\mathbb{R}^2 \tag{A.3}$$

coincides with $\partial\Lambda$ in an open neighborhood of $\underline{0}=(0,0)$. Furthermore the "surface elements" thus described cover $\partial\Lambda$.

A region Λ is "conically regular" for cones with opening $\theta\in[0,\frac{\pi}{2})$ if the cones with apex on $\partial\Lambda$, and axis given by the outer normal (to $\partial\Lambda$) in their apex, intersect $\partial\Lambda$ in no point other than the apex if the opening of the cones is restricted to be $\leqq\theta$.

3. Given a regular region $\Lambda\subset\mathbb{R}^d$ a "regular regularly spaced" covering of the homothetic image $\partial\ell\Lambda$ of $\partial\Lambda$ is a family of coverings $\sigma_1, \ldots, \sigma_{n_\ell}$ of $\partial\ell\Lambda$ which, as ℓ varies in $[1,\infty]$, have the following properties:

i) Each σ_i can be described by an "equation"

$$x_d=v_\sigma(\underline{x}),\ \underline{x}\in\mathbb{R}^{d-1}, \qquad |\underline{x}|<1 \tag{A.4}$$

in a cartesian reference system with origin in some $\xi\in\sigma$ in which $x_d=0$ is the tangent plane to σ in ξ. Furthermore $v_\sigma\in\mathscr{D}(\mathbb{R}^{d-1})$.

ii) The sets σ_i^0 described in local coordinates by

$$x_d=v_\sigma(\underline{x}) \qquad |\underline{x}|\leqq\tfrac{1}{2} \quad \text{(say)} \tag{A.5}$$

also cover $\partial\ell\Lambda$.

iii) If $\sigma_i\cap\sigma_j=\emptyset$ then $d(\sigma_i,\sigma_j)\geqq\frac{1}{100}$ (say).

iv) Given i there are at most $C(\Lambda)d$ values of j such that $d(\sigma_i,\sigma_j)=0$.

A function f on $\partial\ell\Lambda$ with support on σ_i will be represented in the local system of coordinates associated with σ_i by a certain function $\bar{f}$ on $\mathbb{R}^{d-1}$.

Let $\alpha_1, \ldots, \alpha_{n_\ell}$ be a partition of unity on $\partial\ell\Lambda$ with functions such that $\operatorname{supp}\alpha_i\subset\sigma_i^0$. A family of such partitions will be called regular if $\exists$ a sequence A_p of real numbers such that

$$\|\bar{\alpha}_i\|_{C^{(p)}(\mathbb{R}^{d-1})}\leqq A_p \qquad \forall p\geqq 0, \qquad \forall\ell\geqq 1. \tag{A.6}$$

This notion is an asymptotic notion as well as the notion of regularly spaced covering of $\partial\ell\Lambda$.

In the course of this work we shall only meet four families of regions depending homothetically on a parameter $\ell\geqq 1$. They will be the tesserae $\square\in\tilde{Q}_{1,\ell}$, $\tilde{Q}_{2,\ell}$, $\tilde{Q}_{3,\ell}$, $\tilde{Q}_{4,\ell}$. We shall suppose to have associated once and for all a family of coverings which are regular and regularly spaced and a family of associated regular partitions of unity.

4. Let $u\in C^\infty(\square)$ and let $z^{(j)}\in C^{(\varepsilon)}_{s-j}(\partial\square)$, $j=0,1,\ldots,m-1$. We want to define the meaning of the statement that the j-th normal derivative of u takes the value $z^{(j)}$ on $\partial\square$ in the sense of the traces on surfaces parallel to the boundary. For every surface element σ of the covering of $\partial\square$ consider the functions on $\mathbb{R}^{d-1}$

$$\underline{x}\to\alpha_\sigma(\underline{x})\,(\partial^i u)\,(\underline{x},v_\sigma(\underline{x})+t)$$

defined as the j-th normal derivatives of u on the surface which, in the local system of coordinates associated with σ, has the equation

$$x_d = t + v_\sigma(\underline{x}).$$

Call $\alpha_\sigma(\partial^j u)_t$ such functions.

Then u will take the boundary value $z^{(0)}, \ldots, z^{(m-1)}$ in the sense of the traces on parallel surfaces if $\forall\sigma$ in the covering of $\partial\Box$:

$$\lim_{t\to 0^+} \|\alpha_\sigma(\partial^i u)_t - \alpha_\sigma z^{(j)}\|_{C^{(\varepsilon)}_{s'-j}(\mathbb{R}^{d-1})} = 0$$

$$j = 0, 1, \ldots, m-1, \forall s' < s.$$

Appendix B

The following expressions are easily proven by induction combined with simple algebra on Wick polynomials (see, for example [7])

$$\tilde{V}_I^{(N-1)} = V_I^{(N-1)} + W_{N-1} + (F_{N-1} + H_{N-1} + \tilde{W}_{N-1}) - \mathscr{E}(\tilde{W}_{N-1} V_{0,I}^{(N-1)}), \tag{B.1}$$

where, in general, if $C_i(\xi,\eta) \equiv C_{\xi\eta}^{[\leq i]}$ and $\varphi_\xi \equiv \varphi_\xi^{[\leq i]}$:

$$W_i = -3!4\lambda^2 \int_{I\times I} d\xi d\eta (C_{i+1}(\xi,\eta)^3 - C_i(\xi,\eta)^3) :(\varphi_\xi - \varphi_\eta)^2:, \tag{B.2}$$

$$\tilde{W}_i = \frac{\lambda^2}{2} \sum_{\ell=2}^{3} \binom{4}{\ell}^2 (4-\ell)! \int_{I^2} d\xi d\eta (C_{i+1}(\xi,\eta)^{4-\ell} - C_i(\xi,\eta))^{4-\ell}) \cdot :(\varphi_\xi^\ell \varphi_\eta^\ell):, \tag{B.3}$$

$$\begin{aligned} F_i = -\lambda^3 \sum_{r_1 r_2 r_3} d_{r_1 r_2 r_3} \int_{I^3} \{ & C_{i+1}(\xi,\eta)^{s_{12}} C_{i+1}(\eta,\zeta)^{s_{23}} C_{i+1}(\zeta,\xi)^{s_{31}} \\ & - 3 C_{i+1}(\xi,\eta)^{s_{12}} C_i(\eta,\zeta)^{s_{23}} C_i(\zeta,\xi)^{s_{31}} \\ & + 2 C_i(\xi,\eta)^{s_{12}} C_i(\eta,\zeta)^{s_{23}} C_i(\zeta,\xi)^{s_{31}} \} :(\varphi_\xi^{r_1} \varphi_\eta^{r_2} \varphi_\zeta^{r_3}): d\xi d\eta d\zeta \end{aligned} \tag{B.4}$$

with:

$$d_{r_1 r_2 r_3} = \begin{cases} \frac{1}{3!} \binom{4}{r_1} \binom{4}{r_2} \binom{4}{r_3} \binom{4-r_1}{s_{12}} \binom{4-r_2}{s_{23}} \binom{4-r_3}{s_{31}} s_{12}! s_{23}! s_{31}! \\ \text{if} \quad r_1 + r_2 + r_3 = 2r, 1 \leqq r \leqq 4, r_i \leqq 3, i = 1,2,3 \\ 0 \quad \text{otherwise} \end{cases}$$

$$s_{12} + s_{23} = 4 - r_2,\ s_{23} + s_{31} = 4 - r_3,\ s_{31} + s_{12} = 4 - r_1$$

$$\begin{aligned} H_i = \frac{\lambda^3}{2} \sum_{p=0}^{1} 4^2 3! \binom{2}{p} \binom{4}{p+2} (2-p)! \int_{I^3} d\xi d\eta d\zeta [& C_{i+1}(\xi,\eta)^{2-p} \\ & - C_i(\xi,\eta)^{2-p}] C_{i+1}(\eta,\zeta)^3 :(\varphi_\xi^{p+2} \varphi_\eta^p):. \end{aligned} \tag{B.5}$$

By induction:

$$\tilde{V}_I^{(h)}=V_I^{(h)}+W_{I^2}^{(h)}+\left\{\sum_{i=h+1}^{N-1}\mathscr{E}_{>h}(W_i(V_{0,I}^{(i)}-V_{0,I}^{(h)}))\right\}$$
$$+\tilde{W}_{I^2}^{(h)}+\left\{\sum_{i=h+1}^{N-1}[\mathscr{E}_{>h}(\tilde{W}_i(V_{0,I}^{(i)}-V_{0,I}^{(h)}))\right.$$
$$\left.-\mathscr{E}(\tilde{W}_i(V_{0,I}^{(i)}-V_{0,I}^{(h)}))]\right\}+F_{I^3}^{(h)}+H_{I^3}^{(h)}+C_{I^3}^{(h)}, \tag{B.6}$$

where $\mathscr{E}_{>h}$ denotes integration with respect to $z^{(h+1)},\dots,z^{(N)}$ and $\mathscr{E}$ is the full integration, and if we call $G_{I^3}^{(h)}$, $\tilde{G}_{I^3}^{(h)}$ the terms in the curly brackets, we have

$$W_{I^2}^{(h)}=-\lambda^2 4^2\cdot 3!\int_{I^2}(C_N(\xi,\eta)^3-C_h(\xi,\eta)^3):(\varphi_\xi-\varphi_\eta)^2:d\xi d\eta, \tag{B.7}$$

$$G_{I^3}^{(h)}=\lambda^3 4\cdot 3!\cdot 4\cdot 2\cdot 2\left\{\sum_{i=h+1}^{N-1}\int_{I^3}(C_{i+1}(\xi,\eta)^3-C_i(\xi,\eta)^3)\right.$$
$$\left.\cdot(C_i(\zeta,\xi)-C_i(\zeta,\eta)-C_h(\zeta,\xi)+C_h(\zeta,\eta)):(\varphi_\xi\varphi_\zeta^3):d\xi d\eta d\zeta\right\}$$
$$-\lambda^3 4^2\cdot 3!\cdot 6\left\{\sum_{i=h+1}^{N-1}\int_{I^3}(C_{i+1}(\xi,\eta)^3-C_i(\xi,\eta)^3)\cdot(C_i(\zeta,\xi)\right.$$
$$\left.\cdot C_i(\zeta,\eta)-C_i(\zeta,\xi)^2-C_h(\zeta,\xi)\,C_h(\xi,\eta)+C_h(\zeta,\xi)^2):(\varphi_\zeta)^2:d\xi d\eta d\zeta\right\}, \tag{B.8}$$

$$\tilde{W}_{I^2}^{(h)}=\frac{\lambda^2}{2}\sum_{\ell=2}^{3}\binom{4}{\ell}^2(4-\ell)!\int_{I^2}(C_N(\xi,\eta)^{4-\ell}-C_h(\xi,\eta)^{4-\ell})\cdot:(\varphi_\xi^\ell\varphi_\eta^\ell):d\xi d\eta, \tag{B.9}$$

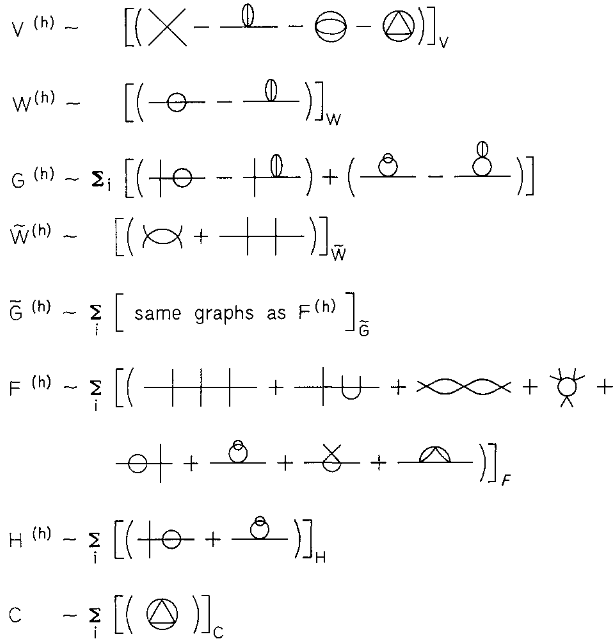
$$\tilde{G}_{I^3}^{(h)}=-3\lambda^3\sum_{\substack{r_1,r_2,r_3\\ s_{12}<3}}d_{r_1r_2r_3}\sum_{i=h+1}^{N-1}\int_{I^3}d\xi d\eta d\zeta$$
$$\cdot\{[C_{i+1}(\xi,\eta)^{s_{12}}-C_i(\xi,\eta)^{s_{12}}]\cdot[C_i(\eta,\zeta)^{s_{23}}C_i(\zeta,\xi)^{s_{12}}-C_h(\eta,\zeta)^{s_{23}}C_h(\zeta,\xi)^{s_{12}}]$$
$$\cdot[:(\varphi_\xi^{r_1}\varphi_\eta^{r_2}\varphi_\zeta^{r_3}):]\}, \tag{B.10}$$

$$F_{I^3}^{(h)}=-\lambda^3\sum_{r_1r_2r_3}d_{r_1r_2r_3}\sum_{i=h+1}^{N}\int_{I^3}d\xi d\eta d\zeta$$
$$\cdot[C_i(\xi,\eta)^{s_{12}}C_i(\eta,\zeta)^{s_{23}}C_i(\zeta,\xi)^{s_{31}}-3C_i(\xi,\eta)^{s_{12}}C_{i-1}(\eta,\zeta)^{s_{23}}C_{i-1}(\zeta,\xi)^{s_{31}}]$$
$$\cdot[:(\varphi_\xi^{r_1}\varphi_\eta^{r_2}\varphi_\zeta^{r_3}):], \tag{B.11}$$

$$H_{I^3}^{(h)}=\frac{\lambda^3}{2}\sum_{p=0}^{1}4^2\cdot 3!\binom{2}{p}\binom{4}{p}(2-p)!\sum_{i=h+1}^{N}\int_{I^3}d\xi d\eta d\zeta$$
$$\cdot[C_i(\xi,\eta)^{2-p}-C_{i-1}(\xi,\eta)^{2-p}]\,C_i(\eta,\zeta)^3:(\varphi_\xi^{p+2}\varphi_\eta^p):, \tag{B.12}$$

$$C_{I^3}^{(h)}=\frac{\lambda^3}{2}\binom{4}{2}^3 2^3\sum_{i=h+1}^{N}\int_{I^3}d\xi d\eta d\zeta$$
$$\cdot[C_i(\xi,\eta)^2-C_{i-1}(\xi,\eta)^2]\,C_h(\eta,\zeta)^2\,C_h(\zeta,\xi)^2. \tag{B.13}$$

It may be helpful for some readers to see the graphical representation of the above terms while checking the calculations:



The above representation is highly symbolical and the subscripts point out that the rules to associate a number to the diagrams are not the same for each class. They can be easily read from the explicit expressions given above.

It is convenient to rewrite $\tilde{V}^{(h)}$ as

$$\tilde{V}_I^{(h)} \equiv V_I^{(h)} + W_{I^2}^{(h)} + G_{I^3}^{(h)} + \Delta_I^{(h)}$$

and notice

$$\begin{aligned} G_{I^3}^{(h)} \equiv \lambda^3 4\cdot 3!\cdot 4\cdot 2 \Bigg\{ \sum_{i=h+1}^{N-1} \int_{I\times(I\times I)} & d\zeta d\xi d\eta (C_{i+1}(\xi,\eta)^3 - C_i(\xi,\eta)^3) \\ & [(C_i(\xi,\zeta) - C_i(\eta,\zeta)) - (C_h(\xi,\zeta) - C_h(\eta,\zeta))] : (\varphi_\xi - \varphi_\eta)\cdot \varphi_\zeta^3 : \Bigg\} \\ & - \lambda^3 4\cdot 3!\cdot 6 \Bigg\{ \sum_{i=h+1}^{N-1} \int_{I\times(I\times I)} d\zeta d\xi d\eta (C_{i+1}(\xi,\eta)^3 - C_i(\xi,\eta)^3) \\ & \cdot [(C_{i+1}(\zeta,\xi) - C_{i+1}(\zeta,\eta))^2 \\ & - (C_h(\zeta,\xi) - C_h(\zeta,\eta))^2] \cdot :(\varphi_\zeta)^2: \Bigg\} \end{aligned} \tag{B.14}$$

having used the symmetry of $C_{\zeta,\eta}$ in the first integral.

We now give an example of the method to get the bounds on the Λ-functions by explicitly estimating the contribution coming from $W_{I^2}^{(h)}$, for $d=3$.

We have to estimate

$$\begin{aligned} W_{\Delta,\Delta} = 4\cdot 3!\,\lambda^2 \gamma^h (1+\Gamma_h) & \Bigg\{ \Bigg[\int_{\Delta^2,\, \gamma^h|\xi-\eta| \leq 1} d\xi d\eta (\gamma^h |\xi-\eta|)^{1/2} \\ & \cdot (C_N(\xi,\eta)^3 - C_h(\xi,\eta)^3) \Bigg] \Bigg\} + \Bigg[\int_{\Delta^2,\, \gamma^h|\xi-\eta| \geq 1} d\xi d\eta (C_N(\xi,\eta)^3 - C_h(\xi,\eta)^3) \Bigg], \end{aligned} \tag{B.15}$$

where $\Delta \in Q_h$ and:

$$C_i(\xi,\eta) = \frac{e^{-|\xi-\eta|} - e^{-\gamma^{i+1}|\xi-\eta|}}{|\xi-\eta|} \tag{B.16}$$

hence, since $C_N^3 - C_h^3 \leqq 3C_N^2(C_N - C_h)$:

$$W_{\Delta,\Delta} \leqq \frac{4\cdot 3!\,\lambda^2\gamma^h}{\gamma-1}\left\{\int_{\Delta^2,\,\gamma^h|\xi-\eta|\leqq 1} d\xi d\eta \frac{(\gamma^h|\xi-\eta|)^{1/2}}{|\xi-\eta|^3} e^{-\gamma^h|\xi-\eta|} + \int_{\Delta^2,\,\gamma^h|\xi-\eta|>1} d\xi d\eta \frac{e^{-\gamma^h|\xi-\eta|}}{|\xi-\eta|^3}\right\} \tag{B.17}$$

changing scale by a factor γ^h the integrals become:

$$W_{\Delta,\Delta} \leqq \frac{4\cdot 3!\,\lambda^2\gamma^{-2h}}{\gamma-1}\left\{\int_{\mathbb{R}^3} d^3x \frac{e^{-|x|}}{|x|^{3-1/2}} + \int_{R^3,\,|x|\geqq 1} d^3x \frac{e^{-|x|}}{|x|^3}\right\}. \tag{B.18}$$

All the other estimates are very similar to this one.

Another example that we treat for purposes of illustration is, say, the first term in the r.h.s. of (B.8).

The A-function for this term is

$$\lambda^3 4\cdot 3!\cdot 4\cdot 2\cdot 2(1+\Gamma_h)^2 \sum_{i=h+1}^{N-1} (C_{i+1}(\xi,\eta)^3 - C_i(\xi,\eta)^3)(C_i(\zeta,\xi) - C_i(\zeta,\eta) - C_h(\zeta,\xi) + C_h(\zeta,\eta))\gamma^{2h} = A^{013}_{\xi\eta\zeta}(\lambda\gamma^{2h})^3. \tag{B.19}$$

Using the inequality

$$\int_\Delta |C_i(\zeta,\xi) - C_i(\zeta,\eta) - C_h(\zeta,\xi) + C_h(\zeta,\eta)| d\zeta \leqq \hat{c}\gamma^{-2h}(\gamma^h|\xi-\eta|)e^{-\frac{\gamma^h}{2}d(\Delta,\xi)}. \tag{B.20}$$

$\forall i \geqq h$, $\forall \gamma^h|\xi-\eta| \leqq 1$, $\forall \Delta \in Q$, *if* $\hat{c}$ is suitably chosen.

Therefore it easily follows:

$$\begin{aligned}(\lambda\gamma^{2h})^3 &\int_{\Delta\times\Delta_1\times\Delta_2} |A^{013}_{\xi\eta\zeta}| d\xi d\eta d\zeta \\ &\leqq (\lambda\gamma^{2h})^3 4\cdot 3!\cdot 4\cdot 2\cdot 2\cdot \hat{c}e^{-\frac{\gamma^h}{8}d(\Delta,\Delta_1,\Delta_2)} \\ &\quad \int_{\Delta\times\Delta_1\times\Delta_2} \left\{\frac{e^{-\frac{\gamma^h}{2}|\xi-\eta|}}{|\xi-\eta|^3}\hat{c}(\gamma^h|\xi-\eta|)\cdot\gamma^h e^{-\frac{\gamma^h}{2}|\xi-\eta|}\gamma^{2h}\right\} d\zeta d\eta d\xi \\ &\leqq \bar{c}\lambda^3\gamma^{-3h}.\end{aligned} \tag{B.21}$$

$\forall h \geqq 0$, $\forall \Delta, \Delta_1, \Delta_2 \in Q_h$, if $\hat{c}, \bar{c}$ are suitably chosen.

Appendix C

Proof of Lemma 2. By definition

$$\hat{V}_I^{(h)} = V^{(h)}_{I\backslash D_h^g} + W^{(h)}_{(I\backslash D_h^g)^2\backslash D_h^r} + \sum_{i=h+1}^{N-1} \mathscr{E}_{>h}((V^{(i)}_{0,I\backslash D_h^g} - V^{(h)}_{0,I\backslash D_h^g})\bar{W}^{(i)}_{(I\backslash D_h^g)^2\backslash D_h^r}) + \Delta^{(h)}_{I\backslash D_h^g}. \tag{C.1}$$

The results of Appendix B, i.e. the estimates on the A-coefficients cf. (4.3) show that there exist constants $\bar{g}, \bar{\varrho}, \bar{\varrho}'$ such that:

$$\begin{aligned}\hat{V}_I^{(h)} \leqq\ & V^{(h)}_{I\backslash D_h^g\cup\hat{R}_h} + W^{(h)}_{(I\backslash D_h^g\cup\hat{R}_h)^2\backslash D_h^r} \\ &+ \left\{\sum_{i=h+1}^{N-1} \mathscr{E}_{>h}((V^{(i)}_{0,I\backslash D_h^g\cup\hat{R}_h} - V^{(h)}_{0,I\backslash D_h^g\cup\hat{R}_h})\cdot \bar{W}^{(i)}_{(I\backslash D_h^g\cup\hat{R}_h)^2\backslash D_h^r})\right\} \\ &+ \Delta^{(h)}_{I\backslash D_h^g\cup\hat{R}_h} + \bar{g}\lambda e^{\varrho\lambda}B_h^{\varrho'}\gamma^{-h}\#(R_h\cap I).\end{aligned} \tag{C.2}$$

Then we notice that the definitions of R_h, D_h, D_{h-1} imply:

$$\hat{R}_h \supset D_h^g \backslash D_{h-1}^g \quad (I\backslash\hat{R}_h)^2 \backslash (D_h^r \cup D_{h-1}^r) = (I\backslash\hat{R}_h)^2 \backslash D_{h-1}^r \tag{C.3}$$

and, furthermore, in $D_{h-1}^g \backslash (D_h^g \cup \hat{R}_h)$ the field $X^{(h)}$ is still quite large

$$|X_\xi^{(h)}| \geqq \frac{\sqrt{\Gamma_h}\, B_{h-1} - B_h'}{\sqrt{1+\Gamma_h}} \geqq \frac{B_h}{(1+h)^2} \tag{C.4}$$

and in $I^2 \backslash [(D_h^g \cup \hat{R}_h)^2 \backslash (D_{h-1}^r \backslash D_h^r)]$:

$$\begin{aligned}|X_\xi^{(h)} - X_\eta^{(h)}| &\geqq \frac{\sqrt{\Gamma_h}\, B_{h-1}(\gamma^{h-1}|\xi-\eta|)^{1/4} - B_h'(\gamma^h|\xi-\eta|)^{1/4}}{\sqrt{1+\Gamma_h}} \\ &\geqq \frac{B_h}{(1+h)^2}(\gamma^h|\xi-\eta|)^{1/4}.\end{aligned} \tag{C.5}$$

Therefore we can use the positivity properties of V and W to conclude that if the constants $\tilde{g}, \tilde{\varrho}$ are suitably chosen

$$\begin{aligned}\hat{V}_I^{(h)} \leqq &\{V^{(h)}_{I\backslash(D^g_{h-1}\cup\hat{R}_h)} + W^{(h)}_{(I\backslash(D^g_{h-1}\cup\hat{R}_h))^2\backslash D^r_{h-1}}\} \\ &+ \left[\sum_{i=h+1}^{N-1} \mathscr{E}_{>h}((V^{(i)}_{0,I\backslash(D^g_{h-1}\cup\hat{R}_h)} - V^{(h)}_{0,I\backslash(D^g_{h-1}\cup\hat{R}_h)}) \cdot \bar{W}^{(i)}_{(I\backslash(D^g_{h-1}\cup\hat{R}_h))^2\backslash D^r_{h-1}})\right] \\ &+ \Delta^{(h)}_{I\backslash(D^g_{h-1}\cup\hat{R}_h)} + \tilde{g}(\lambda e^{\lambda\tilde{\varrho}} B_h^{\tilde{\varrho}})\gamma^{-h} \#(R_h \cap I)\end{aligned} \tag{C.6}$$

provided h is so large that the negative parts taken out of $V^{(h)}$ and $W^{(h)}$ overwhelm the higher order terms.

Since such negative terms can be bounded above, respectively, by

$$-c\lambda\gamma^{2h}(1+\Gamma_h)^2\left(\frac{B_h}{(1+h)^2}\right)^4 \int_{(D^g_{h-1}\backslash(D^g_h\cup\hat{R}_h))\cap I} d\xi \tag{C.7}$$

and

$$-\lambda^2 c \int_{(D^r_{h-1}\backslash(D^r_h\cup\hat{R}_h))^2\cap I^2} d\xi d\eta \left(\frac{B_h}{(1+h)^2}\right)^2 \gamma^h [C_N(\xi,\eta)^3 - C_h(\xi,\eta)^3](\gamma^h|\xi-\eta|)^{1/2} \tag{C.8}$$

it is easily seen from the estimates on the A-coefficients (4.3) that (C.6) always holds for h large and, if λ is small, it holds for all h's.

For instance we can consider the first of the two terms of third order containing W: to change it to the form in (C.6) we produce an error estimated, in absolute value, by:

$$c\lambda^3 4\cdot 3!\cdot 4\cdot 2 \int_{(D^r_{h-1}\backslash(D^g_h\cup\hat{R}_h)^2)\cap I^2} d\xi d\eta B_h^4 (\gamma^h|\xi-\eta|)^{1+1/4}(C_N(\xi,\eta)^3 - C_h(\xi,\eta)^3) \tag{C.9}$$

having used:

$$\begin{aligned}&\int |C_{i+1}(\xi,\zeta) - C_{i+1}(\eta,\zeta) - C_h(\xi,\zeta) + C_h(\xi,\eta)|\, d\zeta \\ &\quad \leqq c'\gamma^{-2h}(\gamma^h|\xi-\eta|)\end{aligned} \tag{C.10}$$

[cf. (B.20), also].

It remains to check the third statement of the lemma.

From the recursive structure seen in the proof on Appendix B it is easy to deduce the expression of $\sum_{k=1}^{3} \frac{\mathscr{E}_h^T(H_J^{(h)};k)}{k!}$ (which is a gaussian integral).

In general the sum of the first three cumulants of an expression like

$$\begin{aligned}&V^{(h)}_{I\backslash\mathscr{D}} + W^{(h)}_{(I\backslash\mathscr{D})^2\backslash\mathscr{D}'} + \Delta^{(h)}_{I\backslash\mathscr{D}} \\ &+ \left\{\sum_{i=h+1}^{N-1} \mathscr{E}_{>h}(\bar{W}^{(i)}_{(I\backslash\mathscr{D})^2\backslash\mathscr{D}'}(V^{(i)}_{0,I\backslash\mathscr{D}} - V^{(h)}_{0,I\backslash\mathscr{D}}))\right\}\end{aligned} \tag{C.11}$$

is exactly equal to

$$V_{I\backslash\mathcal{D}}^{(h-1)} + W_{(I\backslash\mathcal{D})^2\backslash\mathcal{D}'}^{(h-1)} + \Delta_{I\backslash\mathcal{D}}^{(h-1)} + \sum_{i=h}^{N-1} \mathcal{E}_{>(h-1)}(\bar{W}_{(I\backslash\mathcal{D})^2\backslash\mathcal{D}'}^{(i)} \cdot (V_{0,I\backslash\mathcal{D}}^{(i)} - V_{0,I\backslash\mathcal{D}}^{(h)})) + [\bar{W}_{(I\backslash\mathcal{D})^2\cap\mathcal{D}'}^{(h-1)}] + (\text{terms of order} \geqq 4 \text{ in } \lambda) \tag{C.12}$$

and the bound in the A-coefficients allow a naive bound on the higher order terms:

$$\tilde{c}\lambda^4 \tilde{B}_{h-1}^{\varrho'} e^{\lambda\tilde{\varrho}} \gamma^{-h}|I| \tag{C.13}$$

while the term in square brackets is not positive: so

$$\sum_{k=1}^{3} \frac{\mathcal{E}_h^T(H_I^{(h)};k)}{k!} - \hat{V}_I^{(h-1)} \leqq \tilde{c}\lambda^4 e^{\tilde{\varrho}\lambda} \tilde{B}_{h-1}^{\tilde{\varrho}'} \gamma^{-h}|I| \tag{C.14}$$

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