Ulyanov-type inequalities between Lorentz-Zygmund spaces

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Abstract. We establish inequalities of Ul'yanov-type for moduli of smoothness relating the source Lorentz-Zygmund space $L^{p,r}(\log L)^{\alpha-\gamma}$, $\gamma>0$, and the target space $L^{p^*,s}(\log L)^{\alpha}$ over $\mathbb{R}^n$ if $1<p<p^*<\infty$ and over $\mathbb{T}^n$ if $1<p\leqslant p^*<\infty$. The stronger logarithmic integrability (corresponding to $L^{p^*,s}(\log L)^{\alpha}$) is balanced by an additional logarithmic smoothness.

1. Introduction

In [**31**, (3.6)'] Ul'yanov has shown, that, for functions $f\in L^p(\mathbb{T})$, $1\leqslant p<\infty$,

$$\omega_k(f,\delta)_{p^*}\lesssim\left(\int\limits_0^\delta [t^{-\sigma}\omega_k(f,t)_p]^{p^*}\frac{dt}{t}\right)^{1/p^*},\qquad \frac{1}{p^*}=\frac{1}{p}-\sigma,\quad 0<\sigma<1/p,\quad k\in\mathbb{N},$$

where the k-th order modulus of smoothness $\omega_k(f,\delta)_p$ is defined in the standard way by

$$\omega_k(f,\delta)_p=\sup_{|h|\leqslant\delta}\|\Delta_h^k f\|_p\,,\quad \Delta_h f(x)=f(x+h)-f(x),\ \ \Delta_h^k=\Delta_h\Delta_h^{k-1}.$$

The importance of Ulyanov-type inequalities follows from its relation to problems in the theory of function spaces, approximation theory, and interpolation theory – see, e.g., [**17**], [**23**], [**28**]. In recent years numerous contributions – see, e.g., [**10**], [**11**], [**30**], [**23**], [**28**], [**26**] – extended and improved this result in various directions. To obtain sharp Ulyanov-type inequalities, it turned out necessary to introduce moduli of fractional orders. The modulus of smoothness $\omega_\kappa(f,\delta)_p$ of fractional order $\kappa>0$ of a function $f\in L^p(\mathbb{R}^n)$ (or $f\in L^p(\mathbb{T}^n)$), $1\leqslant p<\infty$, is given by (cf. [**5**, p. 788])

$$\omega_\kappa(f,\delta)_p=\sup_{|h|\leqslant\delta}\|\Delta_h^\kappa f(x)\|_{L^p(\mathbb{R}^n)}\,,\qquad \Delta_h^\kappa f(x)=\sum_{\nu=0}^\infty(-1)^\nu\binom{\kappa}{\nu}f\left(x+\nu h\right).$$

Then a typical sharp Ulyanov-type inequality for $f\in L^p(\mathbb{R}^n)$, $1<p<\infty$, reads as follows ([**23**], [**28**])

$$\omega_\kappa(f,\delta)_{p^*}\lesssim\left(\int_0^\delta [t^{-\sigma}\omega_{\kappa+\sigma}(f,t)_p]^{p^*}\frac{dt}{t}\right)^{1/p^*},\qquad \frac{1}{p^*}=\frac{1}{p}-\frac{\sigma}{n}\,,\quad 0<\sigma<n/p. \tag{1.1}$$

Here we use the notation $A\lesssim B$, with $A,B\geqslant 0$, for the estimate $A\leqslant c\,B$, where c is a positive constant, independent of the appropriate variables in A and B. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ (and say that A is equivalent to B). For two normed spaces X and Y, we will use the notation $Y\hookrightarrow X$ if $Y\subset X$ and $\|f\|_X\lesssim\|f\|_Y$ for all $f\in Y$.

In this paper we replace the space $L^p(\mathbb{T})$ by the Lorentz-Zygmund space $L^{p,r}(\log L)^\alpha$ over $\mathbb{R}^n$ or $\mathbb{T}^n$. In particular, in the case of the torus $\mathbb{T}^n$ we can consider the limit case $\sigma=0$.

To define the Lorentz-Zygmund spaces $L^{p,r}(\log L)^\alpha(\mathbb{R}^n)$, $1\leqslant p,r\leqslant\infty$, $\alpha\in\mathbb{R}$, we introduce the logarithmic function $\ell(t)=(1+|\log t|)$, $t>0$. A measurable function f belongs to the space $L^{p,r;\alpha}\equiv$

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$L^{p,r}(\log L)^\alpha(\mathbb{R}^n)$ if

$$\|f\|_{p,r;\alpha} := \begin{cases} \left(\int_0^\infty [t^{1/p}\,\ell^\alpha(t) f^*(t)]^r \frac{dt}{t}\right)^{1/r} < \infty \;, & r<\infty \\ \sup_{t>0} t^{1/p}\,\ell^\alpha(t) f^*(t) < \infty \;, & r=\infty, \end{cases}$$

where f^* denotes the non-increasing rearrangement of f. Thus $L^p = L^{p,p;0}$ and $\|f\|_p = \|f\|_{p,p;0}$. In the case of the torus, the integration extends over the interval $(0,1)$ – see [**2**, p. 253]; the Lorentz-Zygmund spaces are rearrangement invariant Banach function spaces if $p>1$. For all these concepts see, e.g., [**2**, Chap. 2], [**19**].

Let us first formulate and comment our two main results. The former concerns functions defined on the Euclidean space $\mathbb{R}^n$, the latter functions on the torus $\mathbb{T}^n$.

Theorem 1. *Let* $\kappa>0$, $1<p<\infty$, $0<\sigma<n/p$, *and* $\alpha\in\mathbb{R}$.
(a) *If* $\gamma\geqslant 0$ *and* $1\leqslant r\leqslant s\leqslant\infty$, *then*

$$\omega_\kappa(f,\delta)_{p^*,s;\,\alpha} \lesssim \Big(\int_0^\delta [t^{-\sigma}\ell^\gamma(t)\,\omega_{\kappa+\sigma}(f,t)_{p,r;\,\alpha-\gamma}]^s\frac{dt}{t}\Big)^{1/s}, \qquad \delta\to 0+, \quad \frac{1}{p^*}=\frac{1}{p}-\frac{\sigma}{n}, \tag{1.2}$$

for all $f\in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{R}^n)$.
(b) *If* $\gamma<0$ *and* $1\leqslant r\leqslant s\leqslant\infty$, *then inequality* (1.2) *holds only if* $f=0$. *If* $s<r$, *then inequality* (1.2) *with* $\alpha=\gamma=0$ *is not true for all* $f\in L^{p,r}(\mathbb{R}^n)$.

Remark 1.1. Theorem 1 *shows how the logarithmic component in smoothness on the right-hand side of* (1.2) *leads to an additional logarithmic integrability on its left-hand side. In contrast to the Riesz fractional integration* $\|f\|_{p^*}\lesssim\|(-\Delta)^{\sigma/2}f\|_p$, $1/p^*=1/p-\sigma/n$, *we note that, for a fixed* p, *the logarithmic Riesz integration still leads to a type of* L^p*-space (see* [**20**]*); cf. also the embedding*

$$L^{p,r}(\log L)^\alpha(\mathbb{R}^n)\hookrightarrow L^{p,s}(\log L)^\beta(\mathbb{R}^n), \quad 1<p<\infty,\;\; r\leqslant s,\;\; \beta\leqslant\alpha\,.$$

Theorem 2. (a) *Let* $\kappa>0$, $1<p<\infty$, $0<\sigma<n/p$, $1\leqslant r\leqslant s\leqslant\infty$, $\alpha\in\mathbb{R}$, *and* $\gamma\geqslant 0$. *Then*

$$\omega_\kappa(f,\delta)_{p^*,s;\,\alpha} \lesssim \Big(\int_0^\delta [t^{-\sigma}\ell^\gamma(t)\,\omega_{\kappa+\sigma}(f,t)_{p,r;\,\alpha-\gamma}]^s\frac{dt}{t}\Big)^{1/s}, \qquad \delta\to 0+, \quad \frac{1}{p^*}=\frac{1}{p}-\frac{\sigma}{n}, \tag{1.3}$$

for all $f\in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$. *Inequality* (1.3) *holds for* $\gamma<0$ *only if* f *is constant.*
(b) *Let* $\kappa>0$, $1<p<\infty$, *and* $\alpha\in\mathbb{R}$.

(i) *If* $1<r\leqslant s<\infty$ *then, for all* $f\in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$ *when* $\delta\to 0+$,

$$\omega_\kappa(f,\delta)_{p,s;\,\alpha} \lesssim \Big(\int_0^\delta [\ell^{\gamma-1/s}(t)\,\omega_\kappa(f,t)_{p,r;\,\alpha-\gamma}]^s\frac{dt}{t}\Big)^{1/s} + \ell^\gamma(\delta)\,\omega_\kappa(f,\delta)_{p,r;\,\alpha-\gamma}, \quad \gamma>0. \tag{1.4}$$

(ii) *If* $1\leqslant s<r<\infty$ *then, for all* $f\in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$ *when* $\delta\to 0+$,

$$\omega_\kappa(f,\delta)_{p,s;\,\alpha} \lesssim \Big(\int_0^\delta [\ell^{\gamma-1/r}(t)\,\omega_\kappa(f,t)_{p,r;\,\alpha-\gamma}]^s\frac{dt}{t}\Big)^{1/s} + \ell^{\gamma+1/s-1/r}(\delta)\,\omega_\kappa(f,\delta)_{p,r;\,\alpha-\gamma}, \quad \gamma>1/r-1/s. \tag{1.5}$$

Remark 1.2. (a) *The two terms on the right-hand side of* (1.5) *are independent of each other: Consider the case* $p=r$, $\kappa>0$, $\alpha=\gamma>0$. *We can choose (see* [**21**, Thm. 2] *for* $\kappa\in\mathbb{N}$ *and* [**25**, Thm. 2.5] *for* $\kappa>0$*)* f *sufficiently regular such that* $\omega_\kappa(f,t)_p\approx\ell^{-1/s+1/r-\gamma}(t)\big(\ell(\ell(t))\big)^{-\beta}$, *where* $\beta>1/s$, *to obtain that the first integral term is equivalent to* $\big(\ell(\ell(\delta))\big)^{1/s-\beta}$, *while the second behaves like* $\big(\ell(\ell(\delta))\big)^{-\beta}$. *Next, if* $\omega_\kappa(f,t)_p\approx t^\kappa$, *then the first term leads to* $\ell^{\gamma-1/r}(\delta)\,\delta^\kappa$, *the second one to* $\ell^{1/s-1/r+\gamma}(\delta)\,\delta^\kappa$. *Analogously, the independence of the two terms on the right-hand side of* (1.4) *can be shown: Consider* $\omega_\kappa(f,t)_p\approx\ell^{-\gamma}(t)\big(\ell(\ell(t))\big)^{-\beta}$, $\beta>1/s$, *and* $\omega_\kappa(f,t)_p\approx t^\kappa$.

(b) Theorem 2 (b) *in the case* $1<p<\infty$, $1\leqslant s<r<\infty$, $n=1$, $\kappa=1$, *and* $\alpha=\gamma=0$ *is contained in* [**22**, p. 336] *by Sherstneva.*

(c) *Estimate* (1.4) *in the case* $s = r = p$, *and* $n = 1$ *is an improvement of*

$$\omega_k(f,\delta)_{p,p;\gamma} \lesssim \int_0^\delta \ell^\gamma(u)\,\omega_k(f,u)_p\,\frac{du}{u}\,, \qquad \delta \to 0+,\;\; k\in\mathbb{N},\;\; \gamma>0,\;\; f\in L^p(\mathbb{T}),\;\; 1<p<\infty\,,$$

(*see* [**30**]) *which follows as a specification of an abstract Ulyanov-type inequality for semigroups in Banach spaces. Indeed,*

$$\begin{aligned}\Big(\int_0^\delta [\ell^{\gamma-1/r}(u)\omega_k(f,u)_p]^p\frac{du}{u}\Big)^{1/p} &\lesssim \Big(\sum_{j=-\infty}^{0}\int_{2^{j-1}\delta}^{2^j\delta}[\ell^\gamma(u)\omega_k(f,u)_p]^p\,\frac{du}{u}\Big)^{1/p}\\ \lesssim \Big(\sum_{j=-\infty}^{0}\ell^{\gamma p}(2^{j-1}\delta)\omega_k^p(f,2^j\delta)_p\Big)^{1/p} &\lesssim \sum_{j=-\infty}^{0}\ell^\gamma(2^{j-1}\delta)\omega_k(f,2^j\delta)_p\int_{2^{j-1}\delta}^{2^j\delta}\frac{du}{u}\,,\end{aligned}$$

by the monotonicity properties of the modulus of smoothness. Here the last term is approximately $\int_0^\delta \ell^\gamma(u)\,\omega_k(f,u)_p\,\frac{du}{u}$. *Moreover,*

$$\ell^\gamma(\delta)\omega_k(f,\delta)_p \approx \ell^\gamma(\delta)\frac{\omega_k(f,\delta)_p}{\delta^k}\int_0^\delta u^{k-1}\,du \lesssim \int_0^\delta \ell^\gamma(u)\omega_k(f,u)_p\,\frac{du}{u}\,,$$

since $\ell^\gamma(\delta)$ *is decreasing and* $\omega_k(f,\delta)_p/\delta^k$ *is almost decreasing on* $(0,1)$.

In what follows appropriate (modified) K-functionals play an essential role since they can be identified with the occurring moduli of smoothness. To make this more precise, introduce the Riesz potential space

$$H_\lambda^{p,r;\,\alpha}(\mathbb{R}^n) := \{g\in L^{p,r}(\log L)^\alpha(\mathbb{R}^n):\; |g|_{H_\lambda^{p,r;\,\alpha}} := \|(-\Delta)^{\lambda/2}g\|_{p,r;\alpha}<\infty\}\,, \qquad \lambda>0,$$

where $(-\Delta)^{\lambda/2}$ is to be understood in the standard way (cf. [**3**, p. 147]). As K-functional on the couple $(L^{p,r}(\log L)^\alpha(\mathbb{R}^n), H_\lambda^{p,r;\alpha}(\mathbb{R}^n))$, we will mainly use the expression

$$K(f,t;L^{p,r}(\log L)^\alpha, H_\lambda^{p,r;\,\alpha}) := \inf_{g\in H_\lambda^{p,r;\,\alpha}}\Big(\|f-g\|_{p,r;\alpha}+t|g|_{H_\lambda^{p,r;\,\alpha}}\Big).$$

The following lemma contains some characterizations of this K-functional; here we use the notation $\mathcal{F}$ for the Fourier transformation and $\mathcal{F}^{-1}$ for its inverse.

Lemma 1.1. *Let* $1<p<\infty$, $1\leqslant r\leqslant\infty$, $\alpha\in\mathbb{R}$, *and* $\lambda>0$. *Define on* $L^{p,r}(\log L)^\alpha(\mathbb{R}^n)$ *the generalized Weierstrass means* W_t^λ *and de la Vallée-Poussin means* V_t *by*

$$W_t^\lambda f := \mathcal{F}^{-1}[e^{-(t|\xi|)^\lambda}]*f, \qquad V_tf := \mathcal{F}^{-1}[\chi(t|\xi|)]*f, \qquad t>0,$$

where $\chi\in C^\infty[0,\infty)$ *is such that* $\chi(u)=1$ *for* $0\leqslant u\leqslant 1$ *and* $\chi(u)=0$ *for* $u\geqslant 2$. *Then*

$$K(f,t^\lambda;L^{p,r}(\log L)^\alpha,H_\lambda^{p,r;\,\alpha}) \approx \|f-W_t^\lambda f\|_{p,r;\alpha}\,, \tag{1.6}$$

$$K(f,t^\lambda;L^{p,r}(\log L)^\alpha,H_\lambda^{p,r;\,\alpha}) \approx \|f-V_tf\|_{p,r;\alpha}+t^\lambda|V_tf|_{H_\lambda^{p,r;\,\alpha}}\,, \tag{1.7}$$

$$\omega_\lambda(f,t)_{p,r;\alpha} \approx K(f,t^\lambda;L^{p,r}(\log L)^\alpha,H_\lambda^{p,r;\alpha})\,. \tag{1.8}$$

On $L^p(\mathbb{R}^n)$, $1<p<\infty$, the first two characterizations are folklore, (1.8) has been shown by Wilmes [**33**]. For the sake of completeness, we give a proof of (1.6) and (1.7) in Subsection 2.1. Since in the derivation of the three characterizations only Fourier multiplier arguments are used, i.e., one works with bounded linear operators, the extension to $L^{p,r}(\log L)^\alpha$-spaces is immediate by an interpolation argument given in [**12**, Cor. 3.15]. Namely, this corollary says that quasilinear bounded operators $T: L^p\to L^p$, $1<p<\infty$, are also bounded on the interpolation spaces $L^{p,r}(\log L)^\alpha$, $\alpha\in\mathbb{R}$, $1\leqslant r\leqslant\infty$.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. Since the Fourier multipliers with respect to $\mathbb{R}^n$ have periodic counterparts (cf. [**24**, Chap. VII]), the abstract arguments are independent of the underlying measure space, and the Wilmes' characterization also holds in the periodic situation [**34**], we obtain the sublimiting case $0<\sigma<n/p$ of Theorem 2 as that of Theorem 1 (a); details are left to the reader. Finally, in Section 3 we treat the limiting case $\sigma=0$ for Lorentz-Zygmund spaces over $\mathbb{T}^n$.

2. The sublimiting case $\mathbf{p < p^*}$ for Lorentz-Zygmund spaces over $\mathbb{R}^\mathbf{n}$

The proof of Theorem 1 (a) essentially runs as follows: Replace the modulus of smoothness on the left-hand side of (1.2) by an appropriate (modified) K-functional, estimate the latter by a K-functional with respect to $L^{p,r}(\log L)^{\alpha-\gamma}$-spaces, apply a Holmstedt-type formula (cf. [**15**, Thm. 3.1 (c)]) and go back to the associated modulus of smoothness on $L^{p,r}(\log L)^{\alpha-\gamma}$. For this purpose, we have to prove a series of results, e.g., embedding of a homogeneous Besov-type space into some Lorentz-Zygmund space, etc.

2.1. Auxiliary means. As already observed, by [**12**, Cor. 3.15], we need to prove (1.6) and (1.7) only on $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$.

Proof of (1.6) and (1.7). We start with (1.6). By [**27**, Cor. 2.3])

$$\|\mathcal{F}^{-1}[e^{-(t|\xi|)^\lambda}]\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{1-e^{-(t|\xi|)^\lambda}}{(t|\xi|)^\lambda}\Big]\Big\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{(t|\xi|)^\lambda e^{-(t|\xi|)^\lambda}}{1-e^{-(t|\xi|)^\lambda}}\Big]\Big\|_1 \lesssim 1, \qquad t>0. \tag{2.1}$$

Therefore, using Minkowski's inequality and the boundedness of the first two terms in (2.1), we get for any $g \in H^p_\lambda$,

$$\|f - W^\lambda_t f\|_p \leqslant \|(f-g) - W^\lambda_t(f-g)\|_p + \|g - W^\lambda_t g\|_p \lesssim \|f-g\|_p + t^\lambda |g|_{H^p_\lambda}\,,$$

since $g - W^\lambda_t g = \mathcal{F}^{-1}[(1-e^{-(t|\xi|)^\lambda})(t|\xi|)^{-\lambda}] * t^\lambda(-\Delta)^{\lambda/2} g$. Taking the infimum over all g, we arrive at the part " $\gtrsim$ " of the estimate in (1.6). Similarly, using the boundedness of the third term in (2.1), we obtain the converse estimate

$$K(f, t^\lambda; L^p, H^p_\lambda) \leqslant \|f - W^\lambda_t f\|_p + |W^\lambda_t f|_{H^p_\lambda} \lesssim \|f - W^\lambda_t f\|_p\,,$$

completing the proof of (1.6).

Now consider (1.7). Since $V_t f \in C^\infty \cap L^p$ for any $f \in L^p$, the part " $\lesssim$ " is trivial. To verify the converse inequality, we note that, by [**27**, Cor. 2.3],

$$\|\mathcal{F}^{-1}[\chi(t|\xi|)]\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{1-\chi(t|\xi|)}{(t|\xi|)^\lambda}\Big]\Big\|_1 + \Big\|\mathcal{F}^{-1}\Big[\frac{(t|\xi|)^\lambda \chi(t|\xi|)}{1-e^{-(t|\xi|)^\lambda}}\Big]\Big\|_1 \lesssim 1, \qquad t>0. \tag{2.2}$$

The first two estimates show that $\|f - V_t f\|_p \lesssim K(f, t^\lambda; L^p, H^p_\lambda)$. Together with (1.6), the estimate of the third term in (2.2) finally implies that $t^\lambda |V_t f|_{H^p_\lambda} \lesssim K(f, t^\lambda; L^p, H^p_\lambda)$. □

Next we consider a theorem on *fractional integration*, a slight variant of [**20**, Thm. 2.1], which is based on a modified Bessel potential operator. We define the Riesz potential operator with logarithmic component by

$$I^{\sigma,\gamma} f := k_{\sigma,\gamma} * f\,, \qquad \mathcal{F}[k_{\sigma,\gamma}](\xi) = |\xi|^{-\sigma} \log^{-\gamma}(e + |\xi|^2), \quad 0<\sigma<n, \ \ \gamma>0.$$

Analogously to [**20**], we obtain that

$$k_{\sigma,\gamma}(x) \lesssim |x|^{\sigma-n} \ell^{-\gamma}(|x|), \quad k^*_{\sigma,\gamma}(t) \leqslant k^{**}_{\sigma,\gamma}(t) \lesssim t^{\sigma/n-1} \ell^{-\gamma}(t), \tag{2.3}$$

where $k^{**}_{\sigma,\gamma}(t) := t^{-1}\int_0^t k^*_{\sigma,\gamma}(u)\,du$ is the maximal function of k^* (cf. [**2**, p. 52]).

LEMMA 2.1. *Let* $1<p<\infty$, $1 \leqslant r \leqslant s \leqslant \infty$, $\alpha \in \mathbb{R}$, $\gamma \geqslant 0$, $0 < \sigma < n/p$, *and* $1/p^* = 1/p - \sigma/n$. *Then*

$$\|I^{\sigma,\gamma} f\|_{p^*,s;\alpha} \lesssim \|f\|_{p,r;\alpha-\gamma} \qquad \textit{for all } \ f \in L^{p,r}(\log L)^{\alpha-\gamma}.$$

The proof is analogous to that of [**20**, Thm. 2.1], since in [**20**] only estimates (2.3) were used to get the corresponding result for the Bessel-type potential operator.

The next lemma deals with a *Bernstein inequality* for logarithmic derivatives. Throughout the paper we put

$$B_R(0) := \{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}.$$

LEMMA 2.2. *Let* $1<p<\infty$, $1 \leqslant r \leqslant \infty$, $\alpha \in \mathbb{R}$, *and* $\gamma > 0$. *Then*

$$\|\mathcal{F}^{-1}[\log^\gamma(e+|\xi|^2)\widehat{g}\,]\,\|_{p,r;\alpha-\gamma} \lesssim \begin{cases} \ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}, & 1 \leqslant R, \\ \|g\|_{p,r;\alpha-\gamma} \qquad , & 0<R<1, \end{cases}$$

for all $g \in S'$ *with* $\operatorname{supp}\widehat{g} \subset B_R(0)$.

Proof. Let $\chi \in C^\infty[0,\infty)$ be as in Lemma 1.1. Again, in view of [**12**, Cor. 3.15], we only need to show that

$$\|\mathcal{F}^{-1}[\log^\gamma(e+|\xi|^2)\chi(|\xi|^2/R^2)]\,\|_1 \lesssim \ell^\gamma(R), \qquad R \geqslant 1,$$

which immediately follows by of [**27**, Cor. 2.3]. □

A combination of these two lemmas gives the following embedding.

Lemma 2.3. *Let* $1<p<\infty$, $1\leqslant r\leqslant s\leqslant\infty$, $\alpha\in\mathbb{R}$, $\gamma>0$, $0<\sigma<n/p$, *and* $1/p^*=1/p-\sigma/n$. *Then*

$$\|I^{\sigma,0}g\|_{p^*,s;\alpha} \lesssim \begin{cases} \ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}, & 1\leqslant R,\\ \|g\|_{p,r;\alpha-\gamma} \qquad\quad, & 0<R<1,\end{cases}$$

for all entire functions $g\in L^{p,r}(\log L)^{\alpha-\gamma}$ *with* $\operatorname{supp}\widehat{g}\subset B_R(0)$.

Proof. Note that $I^{\sigma,0}g = I^{\sigma,\gamma-\gamma}g = I^{\sigma,\gamma}\mathcal{F}^{-1}[\log^\gamma(e+|\xi|^2)\,\widehat{g}\,]$ and, therefore, by Lemmas 2.1 and 2.2,

$$\|I^{\sigma,0}g\|_{p^*,s;\alpha} \lesssim \|\mathcal{F}^{-1}[\log^\gamma(e+|\xi|^2)\,\widehat{g}\,]\,\|_{p,r;\alpha-\gamma} \lesssim \ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}\,, \qquad R\geqslant 1.$$

□

The following variant of a *Nikolskii inequality* will turn out to be useful.

Lemma 2.4. *Let* $1<p<p^*<\infty$, $1\leqslant r\leqslant s\leqslant\infty$, $\alpha\in\mathbb{R}$, *and* $\gamma\geqslant 0$. *Then*

$$\|g\|_{p^*,s;\alpha} \lesssim R^{n(1/p-1/p^*)}\ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}$$

for all $g\in L^{p,r}(\log L)^{\alpha-\gamma}$ *with* $\operatorname{supp}\widehat{g}\subset B_R(0)$, $R>0$.

Proof. Take χ from Lemma 1.1 and define $v_R(x) := \mathcal{F}^{-1}[\chi(|\xi|/R)](x)$. Then

$$|v_R(x)| \lesssim \frac{R^n}{(1+R|x|)^n}, \qquad v_R^*(t) \lesssim \frac{R^n}{(1+Rt^{1/n})^n}, \qquad v_R^{**}(t) \lesssim \min\Big\{R^n, \frac{1}{t}\Big\}.$$

By the assumption on the support of the Fourier transform of g, we have $v_R * g = g$. Therefore, by O'Neil's inequality,

$$g^*(t) = (v_R*g)^*(t) \lesssim t\,v_R^{**}(t)g^{**}(t) + \int_t^\infty v_R^*(u)g^*(u)\,du.$$

Hence,

$$\begin{aligned}\|g\|_{p^*,s;\alpha} \;&\lesssim\; \left(\int_0^\infty \Big[t^{1/p^*}\ell^\alpha(t)\min\Big\{R^n,\frac{1}{t}\Big\}\int_0^t g^*(u)\,du\Big]^s\frac{dt}{t}\right)^{1/s}\\ &\quad + R^n\left(\int_0^\infty\Big[t^{1/p^*}\ell^\alpha(t)\int_t^\infty \frac{g^*(u)}{(1+Ru^{1/n})^n}\,du\Big]^s\frac{dt}{t}\right)^{1/s} =: N_1+N_2.\end{aligned}$$

Observing that $t^\varepsilon\ell^\gamma(t)$, $\varepsilon>0$, is almost increasing and $t^{-\varepsilon}\ell^\gamma(t)$ is almost decreasing, elementary estimates lead to

$$\begin{aligned} N_1 \;&\leqslant\; R^n\left(\int_0^{R^{-n}}\Big[\{t^{1/p^*+1-1/p}\ell^\gamma(t)\}\,t^{1/p-1}\ell^{\alpha-\gamma}(t)\int_0^t g^*(u)\,du\Big]^s\frac{dt}{t}\right)^{1/s}\\ &\quad + \left(\int_{R^{-n}}^\infty\Big[\{t^{1/p^*-1/p}\ell^\gamma(t)\}\,t^{1/p-1}\ell^{\alpha-\gamma}(t)\int_0^t g^*(u)\,du\Big]^s\frac{dt}{t}\right)^{1/s}\\ &\lesssim\; R^{n(1/p-1/p^*)}\ell^\gamma(R)\left(\int_0^\infty\Big[t^{1/p-1}\ell^{\alpha-\gamma}(t)\int_0^t g^*(u)\,du\Big]^s\frac{dt}{t}\right)^{1/s}.\end{aligned}$$

Now apply a Hardy-type inequality [**12**, Lemma 3.1 (i)] to obtain (cf. the estimate [**20**, (2.5)])

$$N_1 \lesssim R^{n(1/p-1/p^*)}\ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}\,.$$

Similarly, handle the term N_2, use [**12**, Lemma 3.1 (ii)] (cf. the estimate [**20**, (2.6)]) to arrive at

$$N_2 \lesssim R^n\left(\int_0^\infty\Big[t^{1/p^*+1-1/r}\ell^\alpha(t)\frac{g^*(t)}{(1+Rt^{1/n})^n}\Big]^r dt\right)^{1/r} = R^n\left(\int_0^{R^{-n}}\ldots+\int_{R^{-n}}^\infty\ldots\right)^{1/r}.$$

Apply Minkowski's inequality, observe that

$$(1+Rt^{1/n})^n \approx \begin{cases} 1 , & 0<t<R^{-n}, \\ R^{-n}t^{-1}, & t \geqslant R^{-n}, \end{cases}$$

and use again the monotonicity properties of $t^{\pm\varepsilon}\ell^\gamma(t)$ to get $N_2 \lesssim R^{n(1/p-1/p^*)}\ell^\gamma(R)\,\|g\|_{p,r;\alpha-\gamma}$. □

Next we need an analog of Lemma 2.1 with Besov-type spaces involved instead of Riesz-type potential spaces. To this end, we define the *Besov-type space* $B^{(p,r;\beta),s}_{\sigma,\gamma}(\mathbb{R}^n)$, $\sigma>0$, $\beta,\gamma\in\mathbb{R}$, by

$$B^{(p,r;\beta),s}_{\sigma,\gamma} := \Big\{ f\in L^{p,r}(\log L)^\beta : |f|_{B^{(p,r;\beta),s}_{\sigma,\gamma}} := \Big(\int_0^\infty [u^{-\sigma}\ell^\gamma(u)\omega_{\kappa+\sigma}(f,u)_{p,r;\beta}]^s\frac{du}{u}\Big)^{1/s} < \infty\Big\}, \tag{2.4}$$

where $\kappa>0$. Note that the definition of $B^{(p,r;\beta),s}_{\sigma,\gamma}$ is independent of $\kappa>0$. This follows from the Marchaud inequality

$$\omega_\sigma(f,t)_{p,r;\beta} \lesssim t^\sigma \int_t^\infty u^{-\sigma-1}\omega_{\sigma+\kappa}(f,u)_{p,r;\beta}\,du \tag{2.5}$$

and a Hardy-type inequality [**12**, Lemma 3.1 (ii)]. To deduce (2.5), we refer to an abstract Marchaud inequality from [**29**] – see the next remark.

Remark 2.1. *Let $(X,\|\cdot\|)$ be a (complex) Banach space and $\{T(t)\}_{t\geqslant 0}$ be an equibounded $(\mathcal{C}_0)$-semigroup of linear operators from X into itself with infinitesimal generator A_T (cf. [**3**, § 6.7]), i.e.,*

$$\begin{aligned} T(t_1+t_2) &= T(t_1)+T(t_2) \ \textit{for all } t_1,\, t_2\geqslant 0, \quad T(0)=I, \\ \|T(t)\| &\leqslant C \ \textit{ with a constant } C \textit{ independent of } t\geqslant 0, \\ \lim_{t\to 0+}\|T(t)f-f\| &= 0 \ \textit{ for each } f\in X \quad ((\mathcal{C}_0)\textit{-property}), \\ \lim_{t\to 0+}\Big\|\frac{T(t)f-f}{t}-A_Tf\Big\| &= 0 \ \textit{ for all } f\in D(A_T) \quad (\textit{domain of } A_T). \end{aligned}$$

The operator A_T is closed, $D(A_T)$ is a Banach space under the graph norm $\|g\|+\|A_Tg\|$, and the associated K-functional is given by

$$K(f,t;X,D(A_T)) := \inf_{g\in D(A_T)}\Big\{\|f-g\|+t\,\|A_Tg\|\Big\}.$$

If one defines the fractional power $(-A_T)^\mu$, $\mu>0$, of $(-A_T)$ by the strong limit

$$(-A_T)^\mu f := s\text{–}\lim_{t\to 0+}\frac{[I-T(t)]^\mu}{t^\mu}f,$$

then $(-A_T)^\mu$ is closed and [**29**, (1.12) and (1.5)] *imply that*

$$K(f,t^\mu;X,D((-A_T)^\mu)) \lesssim t^\mu\int_t^\infty u^{-\mu-1}K(f,u^{\mu+\kappa};X,D((-A_T)^{\mu+\kappa}))\,du \ \textit{ for any } \kappa>0. \tag{2.6}$$

Now observe that for $X=L^{p,r}(\log L)^\beta(\mathbb{R}^n)$, $1<p<\infty$, the generalized Weierstrass means $\{\mathfrak{W}^\mu_t\}_{t>0}$,

$$\mathfrak{W}^\mu_t f := \begin{cases} \mathcal{F}^{-1}[e^{-t|\xi|^\mu}]*f , & t>0 \\ f , & t=0, \end{cases} \tag{2.7}$$

differing from the Weierstrass means of Lemma 1.1 in the normalization of the parameter $t>0$, form (cf. [**3**, § 6.7] and [**12**, Cor. 3.15]) an equibounded $(\mathcal{C}_0)$-semigroup of linear operators of the required type and

$$D(A_{\mathfrak{W}^\mu}) = D((-A_{\mathfrak{W}^1})^\mu) = H^{p,r;\beta}_\mu .$$

Thus, (2.6) in combination with (1.8) gives the Marchaud inequality (2.5).

An important role in the proof of Theorem 1 is played by the following embedding.

Lemma 2.5. *Let* $1 < p < \infty$, $0 < \sigma < n/p$, $1 \leqslant r \leqslant s \leqslant \infty$, $\alpha \in \mathbb{R}$, *and* $\gamma \geqslant 0$. *Then*

$$\|f\|_{p^*,s;\alpha} \lesssim |f|_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} \quad \text{for all } f \in B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{\sigma}{n}.$$

Proof. Consider the partition of unity on $\mathbb{R}^n$,

$$\sum_{j=-\infty}^{\infty} \varphi_j(|\xi|) = 1 \ \text{ for } \ \xi \neq 0, \qquad \varphi_j(t) = \varphi(2^{-j}t), \quad \varphi(t) := \chi(t) - \chi(2t),$$

with the cut-off function χ from Lemma 1.1. Set

$$f_j := \mathcal{F}^{-1}[\varphi_j(|\xi|)] * f, \quad j \in \mathbb{Z}. \tag{2.8}$$

Under the assumption that

$$\|f\|^s_{p^*,s;\alpha} \lesssim \sum_{j=-\infty}^{\infty} [\,\ell^\gamma(2^j)2^{j\sigma}\|f_j\|_{p,r;\alpha-\gamma}\,]^s \tag{2.9}$$

holds, we show that the assertion of Lemma 2.5 is true. To this end, we first note that

$$\begin{aligned}\|\mathcal{F}^{-1}[\varphi_j(|\xi|)] * f\|_{p,r;\alpha-\gamma} &\leqslant \|f - V_{2^{-j}}f\|_{p,r;\alpha-\gamma} + \|f - V_{2^{1-j}}f\|_{p,r;\alpha-\gamma} \\ &\lesssim K(f, 2^{-j(\kappa+\sigma)}; L^{p,r;\alpha-\gamma}, H^{p,r;\alpha-\gamma}_{\kappa+\sigma})\end{aligned} \tag{2.10}$$

by Lemma 1.1. Therefore,

$$\begin{aligned}\|f\|^s_{p^*,s;\alpha} &\lesssim \sum_{j=-\infty}^{\infty} [\,\ell^\gamma(2^j)2^{j\sigma}K(f, 2^{-j(\kappa+\sigma)}; L^{p,r;\alpha-\gamma}, H^{p,r;\alpha-\gamma}_{\kappa+\sigma})\,]^s \int_{2^{j-1}}^{2^j} \frac{dt}{t} \\ &\approx \int_0^\infty [\,\ell^\gamma(t)t^{-\sigma}K(f, t^{\kappa+\sigma}; L^{p,r;\alpha-\gamma}, H^{p,r;\alpha-\gamma}_{\kappa+\sigma})\,]^s \,\frac{dt}{t},\end{aligned}$$

and Lemma 2.5 is established in view of (1.8) and (2.4) provided that (2.9) is valid.

We prove (2.9) by an argument communicated to us by A. Seeger. Choose $\widetilde{\varphi} \in C^\infty(0,\infty)$ with $\operatorname{supp}\widetilde{\varphi} \subset (1/4, 4)$ and $\widetilde{\varphi} = 1$ on $\operatorname{supp}\varphi$; set $\widetilde{\varphi}_j = \widetilde{\varphi}(2^{-j}\cdot)$. Define $T_{\widetilde{\varphi}_j}f := \mathcal{F}^{-1}[\widetilde{\varphi}_j(|\xi|)] * f$. Then $T_{\widetilde{\varphi}_j}f_j = f_j$ for the f_j's from (2.8). Recall that $\operatorname{supp}\widehat{f_j} \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$ and, therefore, by the Nikolskii inequality from Lemma 2.4,

$$\|\ell^{-\gamma}(2^j)2^{-j\sigma}T_{\widetilde{\varphi}_j}f\|_{p^*,s;\alpha} \lesssim \|T_{\widetilde{\varphi}_j}f\|_{p,r;\alpha-\gamma} \lesssim \|f\|_{p,r;\alpha-\gamma}, \qquad \sigma = n(1/p - 1/p^*), \tag{2.11}$$

for all $f \in L^{p,r;\alpha-\gamma}$. Now fix p, $1 < p < \infty$; choose p^*_0, p^*_1 such that $p < p^*_0 < p^* < p^*_1 < \infty$. Set $\sigma_0 = n(1/p - 1/p^*_0)$, $\sigma_1 = n(1/p - 1/p^*_1)$, consequently $\sigma_0 \neq \sigma_1$. Then (2.11) also holds with this fixed p but with (p^*, σ) replaced by (p^*_i, σ_i), $i = 0, 1$. Hence, for an arbitrary sequence $(F_j)_{j\in\mathbb{Z}}$, $F_j \in L^{p,r;\alpha-\gamma}$, we have

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{Z}} T_{\widetilde{\varphi}_j}F_j\Big\|_{p^*_i,s;\alpha} &\lesssim \sum_{j\in\mathbb{Z}} \|T_{\widetilde{\varphi}_j}F_j\|_{p^*_i,s;\alpha} \lesssim \sum_{j\in\mathbb{Z}} \ell^\gamma(2^j)\,2^{j\sigma_i}\|T_{\widetilde{\varphi}_j}F_j\|_{p,r;\alpha-\gamma} \\ &\lesssim \sum_{j\in\mathbb{Z}} \ell^\gamma(2^j)\,2^{j\sigma_i}\|F_j\|_{p,r;\alpha-\gamma}.\end{aligned} \tag{2.12}$$

Now apply an interpolation argument: Define the sequence space $\ell^q_\sigma(X)$, X a normed space, as the space of X-valued sequences $(F_j)_{j\in\mathbb{Z}}$ with

$$\|\,(F_j)_j\|_{\ell^q_\sigma} := \Big(\sum_{j\in\mathbb{Z}} [\,2^{j\sigma}\|\ell^\gamma(2^j)\,F_j\|_X\,]^q\Big)^{1/q} < \infty,$$

and a linear operator S by

$$S\big((F_j)_j\big) := \sum_{j\in\mathbb{Z}} T_{\widetilde{\varphi}_j}F_j.$$

Then (2.12) means that

$$S : \ell^1_{\sigma_i}(L^{p,r;\alpha-\gamma}) \to L^{p^*_i,s;\alpha}, \qquad i = 0, 1.$$

Since $\sigma_0 \neq \sigma_1$, we obtain, by [**3**, Thm. 5.6.1 (dotted version)] that

$$(\ell^1_{\sigma_0}(X), \ell^1_{\sigma_1}(X))_{\theta,q} = \ell^q_\sigma(X), \qquad \sigma = (1-\theta)\sigma_0 + \theta\sigma_1,\ \ 0<\theta<1,\ \ 1 \leqslant q \leqslant \infty\,. \tag{2.13}$$

Moreover, $\big(L^{p_0^*,s;\alpha}, L^{p_1^*,s;\alpha}\big)_{\theta,q} = L^{p^*,q;\alpha}$, where $1/p^* = (1-\theta)/p_0^* + \theta/p_1^*$ and $0<\theta<1$. Thus, the real interpolation implies that

$$S : \ell^1_\sigma(L^{p,r;\alpha-\gamma}) \to L^{p^*,q;\alpha}. \tag{2.14}$$

Choose $F_j = f_j$ with f_j from (2.8). Then

$$S\big((F_j)_j\big) = S\big((f_j)_j\big) = \sum_{j\in\mathbb{Z}} T_{\widetilde{\varphi}_j} f_j = \sum_{j\in\mathbb{Z}} f_j = f$$

and, by (2.14),

$$\|f\|_{p^*,q;\alpha} \lesssim \left(\sum_{j\in\mathbb{Z}} [\, 2^{j\sigma} \|\ell^\gamma(2^j)\, f_j\|_{p,r;\alpha-\gamma}]^q \right)^{1/q},$$

which gives (2.9) on putting $q = s$. □

As already announced, we want to apply an appropriate *Holmstedt formula* for the proof of Theorem 1 (a). To this purpose, we introduce slowly varying functions. A measurable function $b : (0,\infty) \to (0,\infty)$ is said to be *slowly varying* on $(0,\infty)$, notation $b \in SV(0,\infty)$ if, for each $\varepsilon > 0$, there is an increasing function g_ε and a decreasing $g_{-\varepsilon}$ such that $t^\varepsilon b(t) \approx g_\varepsilon(t)$ and $t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t)$ for all $t \in (0,\infty)$. Clearly, one has that $\ell^\gamma \in SV(0,\infty)$, $\gamma \in \mathbb{R}$. For the sake of simplicity, in the following we assume that $t^{\pm\varepsilon} b(t)$ are already monotone. To describe the framework of the desired Holmstedt formula, let (X, Y) be a compatible couple of Banach spaces, where $Y \subset X$ has a seminorm $|\cdot|_Y$ such that $\|\cdot\|_Y := \|\cdot\|_X + |\cdot|_Y$ is a norm on Y. We will work with the (modified) K-functional

$$K(f, t; X, Y) := \inf_{g\in Y} (\|f-g\|_X + t|g|_Y)$$

and we will state a slight variant of the Holmstedt formula involving slowly varying functions given in [**15**, Thm. 3.1 (c)] without proof.

Lemma 2.6. *Let* $0 \leqslant \theta \leqslant 1$, $1 \leqslant s \leqslant \infty$, *and* $b \in SV(0,\infty)$. *Define the interpolation space* $(X,Y)_{\theta,s;b}$ *by*

$$(X,Y)_{\theta,s;b} := \Big\{ f \in X : |f|_{\theta,s;b} = \Big(\int_0^\infty [t^{-\theta} b(t) K(f,t;X,Y)]^s \frac{dt}{t} \Big)^{1/s} < \infty \Big\}.$$

If $0<\theta<1$, *then*

$$K(f, t^{1-\theta} b(t); (X,Y)_{\theta,s;b}, Y) \approx \Big(\int_0^t [u^{-\theta} b(u) K(f,u;X,Y)]^s \frac{du}{u} \Big)^{1/s}$$

for all $f \in X$ *and all* $t > 0$.

2.2. Proof of Theorem 1 (a). Using the characterization (1.8), we can reduce the problem to estimates between K-functionals. Thus,

$$\omega_\kappa(f,t)_{p^*,s;\alpha} \approx K(f, t^\kappa; L^{p^*,s;\alpha}, H_\kappa^{p^*,s;\alpha}) \leqslant \|f-g\|_{p^*,s;\alpha} + t^\kappa \|(-\Delta)^{\kappa/2} g\|_{p^*,s;\alpha} \ \text{ for all } \ g \in H_\kappa^{p^*,s;\alpha},$$

in particular, in view of Lemma 2.1, for all $g \in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}$. Consider $g_t = V_t g$, the de la Vallée-Poussin means of $g \in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}$ from Lemma 1.1. Note that

$$|g_t|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}} \lesssim |g|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}$$

since $\|\mathcal{F}^{-1}[\chi(t|\xi|)]\|_1 = O(1)$ by [**27**, Cor. 2.3]. As $\operatorname{supp} \widehat{g}_t \subset B_{2/t}(0)$, Lemmas 2.5 and 2.3 imply that

$$\omega_\kappa(f,t)_{p^*,s;\alpha} \lesssim |f-g_t|_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} + \ell^\gamma(t)\, t^\kappa \|(-\Delta)^{(\kappa+\sigma)/2} g_t\|_{p,r;\alpha-\gamma}\,. \tag{2.15}$$

We want to apply the Holmstedt formula from Lemma 2.6. To this end, we have to get rid of the de la Vallée-Poussin means, i.e., we have to estimate g_t by g. Clearly,

$$\omega_\kappa(f,t)_{p^*,s;\alpha} \lesssim |f-g|_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} + |g-g_t|_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} + \ell^\gamma(t)\, t^\kappa |g_t|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}$$

and, by the above argument, $|g_t|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}} \lesssim |g|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}$. Observe that

$$|g-g_t|_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} \approx \left(\left(\int_0^t+\int_t^\infty\right)[u^{-\sigma}\ell^\gamma(u)K(g-g_t,u^{\kappa+\sigma};L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})]^s\frac{du}{u}\right)^{1/s}.$$

Since $K(g-g_t,u^{\kappa+\sigma}) \leqslant u^{\kappa+\sigma}|g-g_t|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}$ and $K(g-g_t,u^{\kappa+\sigma}) \leqslant \|g-g_t\|_{p,r;\alpha-\gamma}$, we see that

$$|g-g_t|_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} \lesssim \ell^\gamma(t)\,t^\kappa|g_t|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}} + \ell^\gamma(t)\,t^{-\sigma}\|g-g_t\|_{p,r;\alpha-\gamma}.$$

The estimate

$$\|g-g_t\|_{p,r;\alpha-\gamma} \lesssim K(g,t^{\kappa+\sigma};L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma}) \lesssim t^{\kappa+\sigma}|g|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}$$

follows from Lemma 1.1, the definition of the K-functional and the fact that $g\in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}$. Thus, (2.15) holds for all $g\in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}$, which implies that

$$\omega_\kappa(f,t)_{p^*,s;\alpha} \lesssim \inf_{g\in H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}\big(|f-g|_{B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma}} + \ell^\gamma(t)\,t^\kappa|g|_{H^{p,r;\alpha-\gamma}_{\kappa+\sigma}}\big). \tag{2.16}$$

Now $B^{(p,r;\alpha-\gamma),s}_{\sigma,\gamma} = (L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})_{\theta,s;\ell^\gamma}$, $\theta=\sigma/(\kappa+\sigma)$, which directly follows from the definition of the interpolation space given in Lemma 2.6 and the characterization (1.8) of the involved K-functional. If we change the variable t^κ to $t^{1-\theta}$ and set $\rho(t)=t^{1-\theta}\ell^\gamma(t)$, we can interpret the right-hand side of (2.16) as $K\big(f,\rho(t);(L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})_{\theta,s;\ell^\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma}\big)$. By Lemma 2.6, the latter can be reformulated as follows

$$K\big(f,\rho(t);(L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})_{\theta,s;\ell^\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma}\big) \approx \left(\int_0^t[u^{-\theta}\ell^\gamma(u)K(f,u;L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})]^s\frac{du}{u}\right)^{1/s}.$$

Hence, using the change of variables and (1.8), we arrive at

$$\begin{aligned}\omega_\kappa(f,t)_{p^*,s;\alpha} &\lesssim \left(\int_0^t[u^{-\sigma}\ell^\gamma(u)K(f,u^{\kappa+\sigma};L^{p,r;\alpha-\gamma},H^{p,r;\alpha-\gamma}_{\kappa+\sigma})]^s\frac{du}{u}\right)^{1/s}\\ &\approx \left(\int_0^t[u^{-\sigma}\ell^\gamma(u)\,\omega_{\kappa+\sigma}(f,u)_{p,r;\alpha-\gamma}]^s\frac{du}{u}\right)^{1/s}.\end{aligned}$$

□

2.3. Proof of Theorem 1 (b). (i) First let $\gamma<0$. Then it follows from (1.8) that $f\in H^{p,r;\,\alpha-\gamma}_{\kappa+\sigma}$ implies that $\omega_{\kappa+\sigma}(f,t)_{p,r;\alpha-\gamma}=O(t^{\kappa+\sigma})$, $t\to 0+$. Together with (1.2) and the assumption $\gamma<0$, this gives $\omega_\kappa(f,\delta)_{p^*,s;\,\alpha}=o(\delta^\kappa)$, $\delta\to 0+$.
Therefore, by (1.8), it remains to show that

$$K(f,\delta^\kappa;L^{p^*,s}(\log L)^\alpha,H^{p^*,s;\alpha}_\kappa)=o(\delta^\kappa) \text{ as } \delta\to 0+ \qquad\Longrightarrow\qquad f=0.$$

¿From the proof of Lemma 1.1 it is clear that

$$\delta^\kappa\|(-\Delta)^{\kappa/2}W^\kappa_\delta f\|_{p^*,s;\alpha} \lesssim \|f-W^\kappa_\delta f\|_{p^*,s;\alpha} \lesssim K(f,\delta^\kappa;L^{p^*,s}(\log L)^\alpha,H^{p^*,s;\alpha}_\kappa).$$

Thus, by the Fatou property of the Lorentz-Zygmund spaces and the hypothesis, we have

$$\|(-\Delta)^{\kappa/2}f\|_{p^*,s;\alpha} \leqslant \liminf_{\delta\to 0+}\|(-\Delta)^{\kappa/2}W^\kappa_\delta f\|_{p^*,s;\alpha}=0.$$

But $(-\Delta)^{\kappa/2}f=0$ yields $f=0$, since $f\in L^{p^*,s}(\log L)^\alpha$. □

(ii) Concerning the case $s<r$, we show that, under this assumption, the Ulyanov-type inequality (1.2) implies a fractional integration theorem, which is false in Lorentz spaces. To this end, take $\alpha=\gamma=0$ and write $L^{p,r}:=L^{p,r;0}$. Consider the set of entire functions of exponential type

$$E_{p,r;R}:=\Big\{P_R\in L^{p,r}(\mathbb{R}^n):\ \operatorname{supp}\widehat{P}_R\subset B_R(0),\quad R>0\Big\}.$$

Then the following Riesz-type inequality holds:

$$|P_{1/\delta}|_{H^{p^*,s}_\kappa} \lesssim \delta^{-\kappa}\omega_\kappa(P_{1/\delta},\delta)_{p^*,s}, \quad P_{1/\delta}\in E_{p,r;1/\delta},\ \ 1<p^*<\infty,\ \ 1\leqslant s\leqslant\infty,\ \ \kappa>0. \tag{2.17}$$

Indeed, this is proved in [**33**] for $p^* = s$ and the argument following (1.8) shows that (2.17) is true. Formula (1.8) and the definition of the K-functional imply that

$$\omega_{\kappa+\sigma}(P_{1/\delta}, t)_{p,r} \approx K(P_{1/\delta}, t^{\kappa+\sigma}; L^{p,r}, H^{p,r}_{\kappa+\sigma}) \lesssim t^{\kappa+\sigma}|P_{1/\delta}|_{H^{p,r}_{\kappa+\sigma}}. \tag{2.18}$$

Estimates (2.17) and (2.18) applied to (1.2) lead to

$$|P_{1/\delta}|_{H^{p^*,s}_\kappa} \lesssim \delta^{-\kappa}\left(\int_0^\delta [t^{-\sigma}t^{\kappa+\sigma}|P_{1/\delta}|_{H^{p,r}_{\kappa+\sigma}}]^s \frac{dt}{t}\right)^{1/s} \approx |P_{1/\delta}|_{H^{p,r}_{\kappa+\sigma}}.$$

Since the estimates involved are independent of $\delta > 0$ and $\bigcup_{R>0} E_{p,r;R}$ is dense in $L^{p,r}$, we get

$$\|I^{\sigma,0}f\|_{p^*,s} \lesssim \|f\|_{p,r}, \qquad f \in L^{p,r},\ \ 1<p<\infty,\ \ 0<\sigma<n/p,\ \ \frac{1}{p^*} = \frac{1}{p} - \frac{\sigma}{n}. \tag{2.19}$$

However, this contradicts [**14**, Thm. 10.3]. □

3. The limiting case $\mathbf{p} = \mathbf{p}^*$ for Lorentz-Zygmund spaces over $\mathbb{T}^{\mathbf{n}}$

In this section we discuss the limiting case $\sigma = 0$, i.e., when $p = p^*$. When trying to follow the effective approach of Section 2, we encounter the difficulty that we cannot carry out the monotonicity arguments used in the proof of Lemma 2.4 on the whole half-line, but only on the interval $(0,1)$ or $(1,\infty)$ separately. There are two possibilities how to overcome this obstacle. One is to use the concept of broken indices for the log-function - see [**13**]. The other, which we make use of, is to restrict ourselves to the n-dimensional torus $\mathbb{T}^n$. In the following we use the standard Fourier series setting (cf. [**24**, Chap. VII]),

$$f(x) \sim \sum_{m\in\mathbb{Z}^n} \widehat{f}_m e^{2\pi imx}, \qquad \widehat{f}_m = \int_{\mathbb{T}^n} f(x)e^{-2\pi imx}\,dx, \qquad f \in L^1(\mathbb{T}^n),$$

and denote by $\mathcal{T}_N$ the set of all trigonometric polynomials of degree N, i.e.,

$$\mathcal{T}_N := \Big\{T_N = \sum_{|m|\leqslant N} c_m e^{2\pi imx} : \ c_m \in \mathbb{C},\ m \in \mathbb{Z}^n\Big\}, \qquad N \in \mathbb{N}_0\,.$$

Since in this section there will be no ambiguity, we use the notation of the previous sections though the underlying measure space is $\mathbb{T}^n$. Thus we write

$$L^{p,r;\alpha} := \{f \in L^1(\mathbb{T}^n) : \|f\|_{p,r;\alpha} := \left(\int_0^1 [t^{1/p}\ell^\alpha(t)f^*(t)]^r \frac{dt}{t}\right)^{1/r} < \infty\},$$

for the periodic Riesz-potential space $H^{p,r;\alpha}_\lambda(\mathbb{T}^n)$ of order $\lambda > 0$

$$H^{p,r;\alpha}_\lambda := \{g \in L^{p,r;\alpha} : |g|_{H^{p,r;\alpha}_\lambda} := \|(-\Delta)^{\lambda/2}g\|_{p,r;\alpha} < \infty\}, \quad (-\Delta)^{\lambda/2}g \sim \sum_{m\in\mathbb{Z}^n} |m|^\lambda\, \widehat{g}_m\, e^{2\pi imx},$$

and for the associated K-functional $K(f, t^\lambda; L^{p,r;\alpha}(\mathbb{T}^n), H^{p,r;\alpha}_\lambda(\mathbb{T}^n))$

$$K(f, t^\lambda; L^{p,r;\alpha}, H^{p,r;\alpha}_\lambda) := \inf_{g\in H^{p,r;\alpha}_\lambda}(\|f-g\|_{p,r;\alpha} + t^\lambda|g|_{H^{p,r;\alpha}_\lambda}), \quad \lambda > 0.$$

On account of the Poisson-summation formula (see [**32**, p. 37]) we note that the periodic analogs of (1.6) and (1.7) hold; the periodic analog of (1.8) is due to Wilmes [**34**]. Hence the following variant of Lemma 1.1 is true.

Lemma 3.1. *Let* $1<p<\infty$, $1\leqslant r\leqslant\infty$, $\alpha\in\mathbb{R}$, *and* $\lambda>0$. *Define on* $L^{p,r}(\log L)^\alpha(\mathbb{T}^n)$ *the generalized Weierstrass means* W^λ_t *and de la Vallée-Poussin means* V_t *by*

$$W^\lambda_t f := \sum_{m\in\mathbb{Z}^n} e^{-(t|m|)^\lambda}\widehat{f}_m e^{2\pi imx}, \qquad V_t f := \sum_{|m|\leqslant 2/t} \chi(t|m|)\widehat{f}_m e^{2\pi imx}, \qquad t>0,$$

where $\chi \in C^\infty[0,\infty)$ *is from* Lemma 1.1. *Then*

$$K(f, t^\lambda; L^{p,r}(\log L)^\alpha, H^{p,r;\,\alpha}_\lambda) \approx \|f - W^\lambda_t f\|_{p,r;\alpha}\,, \tag{3.1}$$

$$K(f,t^\lambda;L^{p,r}(\log L)^\alpha,H_\lambda^{p,r;\alpha})\approx\|f-V_tf\|_{p,r;\alpha}+t^\lambda|V_tf|_{H_\lambda^{p,r;\alpha}}\,, \tag{3.2}$$

$$\omega_\lambda(f,t)_{p,r;\alpha}\approx K(f,t^\lambda;L^{p,r}(\log L)^\alpha,H_\lambda^{p,r;\alpha})\,. \tag{3.3}$$

3.1. Proof of Theorem 2 (b). We start with deriving analogs of Lemmas 2.1, 2.2, and 2.4 in the limiting case $p=p^*$. These results will be used in the proof of Theorem 2 (b). Define a fractional integration $\tilde I^{0,\gamma}$ of logarithmic order $\gamma>0$ via $\tilde I^{0,\gamma}f:=\tilde k_{0,\gamma}*f$, where the Fourier series and the growth behavior (at the origin) of $\tilde k_{0,\gamma}$ – see [**32**, Thm. 7 (ii)] – are given by

$$\tilde k_{0,\gamma}(x)\sim\sum_{m\in\mathbb{Z}^n}\frac{e^{2\pi imx}}{\log^\gamma(e+|m|^2)},\qquad |\tilde k_{0,\gamma}(x)|\lesssim\frac{1}{|x|^n}\log^{-\gamma-1}\frac{1}{|x|}\,,\ \ x\to0. \tag{3.4}$$

As the next result is a slight variant of [**20**, Thm. 2.4], we state it without proof.

Lemma 3.2. *Let* $1<p<\infty$, $1\leqslant r\leqslant s\leqslant\infty$, $\alpha\in\mathbb{R}$ *and* $\gamma>0$. *Then*

$$\|\tilde I^{0,\gamma}f\|_{p,s;\alpha}\lesssim\|f\|_{p,r;\alpha-\gamma}\quad\textit{for all}\ \ f\in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n).$$

By the Poisson-summation formula (see [**32**, p. 37]), it is clear that the proof of Lemma 2.2 also works in the periodic situation. Hence, we obtain the following lemma.

Lemma 3.3. *Let* $1<p<\infty$, $1\leqslant r\leqslant\infty$, $\alpha\in\mathbb{R}$ *and* $\gamma>0$. *Then the Bernstein-type inequality*

$$\Big\|\sum_{|m|\leqslant N}\log^\gamma(e+|m|^2)\,c_ke^{2\pi imx}\Big\|_{p,r;\alpha-\gamma}\lesssim\ell^\gamma(N)\Big\|\sum_{|m|\leqslant N}c_me^{2\pi imx}\Big\|_{p,r;\alpha-\gamma}\,.$$

holds for all trigonometric polynomials of degree N.

A combination of these two lemmas yields a Nikolskii-type inequality for the limiting case.

Lemma 3.4. *Let* $1<p<\infty$, $1\leqslant r\leqslant s\leqslant\infty$, $\alpha\in\mathbb{R}$, *and* $\gamma>0$. *Then*

$$\Big\|\sum_{|m|\leqslant N}c_me^{2\pi imx}\Big\|_{p,s;\alpha}\lesssim\ell^\gamma(N)\,\Big\|\sum_{|m|\leqslant N}c_me^{2\pi imx}\Big\|_{p,r;\alpha-\gamma}\,.$$

for all trigonometric polynomials in $\mathcal{T}_N$, $N\in\mathbb{N}$.

Proof. By Lemma 3.2,

$$\Big\|\sum_{|m|\leqslant N}c_me^{2\pi imx}\Big\|_{p,s;\alpha}=\Big\|\sum_{|m|\leqslant N}\frac{\log^\gamma(e+|m|^2)}{\log^\gamma(e+|m|^2)}c_me^{2\pi imx}\Big\|_{p,s;\alpha}\lesssim\Big\|\sum_{|m|\leqslant N}\log^\gamma(e+|m|^2)\,c_me^{2\pi imx}\Big\|_{p,r;\alpha-\gamma}$$

and an application of Lemma 3.3 gives the assertion. □

To formulate an analog of Lemma 2.5 in our limiting case, we need the Besov-type space involving only the logarithmic smoothness ℓ^γ, $\gamma>0$, defined by

$$B_{0,\gamma}^{(p,r;\beta),s}(\mathbb{T}^n):=\Big\{f\in L^{p,r;\beta}:\ |f|_{B_{0,\gamma}^{(p,r;\beta),s}}:=\Big(\int_0^1[\,\ell^\gamma(u)\,\omega_\kappa(f,u)_{p,r;\beta}]^s\frac{du}{u}\Big)^{1/s}<\infty\Big\},$$

where $\kappa>0$. The notation $B_{0,\gamma}^{(p,r;\beta),s}$ is justified by the fact that the definition is independent of $\kappa>0$. To verify this, we make use of the notion of best approximation. Here $E_N(f)_{p,r;\beta}$ denotes the error of approximation of $f\in L^{p,r;\beta}$ by elements from $\mathcal{T}_N$, given by

$$E_N(f)_{p,r;\beta}=\inf\{\|f-T_N\|_{p,r;\beta}:T_N\in\mathcal{T}_N\}$$

and we call $T_N^{p,r;\beta}(f)\in\mathcal{T}_N$ the best approximation to $f\in L^{p,r;\beta}$ from $\mathcal{T}_N$. Next we observe that, for any $\kappa>0$,

$$E_j(f)_{p,r;\,\alpha-\gamma}\lesssim\omega_\kappa(f,1/j)_{p,r;\alpha-\gamma}\lesssim\frac{1}{(j+1)^\kappa}\sum_{i=0}^j(i+1)^{\kappa-1}E_i(f)_{p,r;\alpha-\gamma}.\qquad j\in\mathbb{N}_0\,. \tag{3.5}$$

Here the first estimate is the Jackson inequality which can be easily derived from the classical Jackson's theorem for the integer order moduli of smoothness (see [**8**, Thm. 2.1]):

$$E_j(f)_{p,r;\,\alpha-\gamma} \lesssim \omega_{[\kappa]+1}(f,1/j)_{p,r;\alpha-\gamma} \lesssim \omega_\kappa(f,1/j)_{p,r;\alpha-\gamma}\,.$$

The second estimate in (3.5) is the weak inverse inequality which is known (see [**7**, Thm. 2.3]) for the case $\kappa\in\mathbb{N}$. We can prove it, for any $\kappa>0$, as follows. By (3.2),

$$\omega_\kappa(f,1/2^m)_{p,r;\alpha-\gamma} \approx \|f-V_{2^{-m}}f\|_{p,r;\alpha-\gamma} + 2^{-m\kappa}|V_{2^{-m}}f|_{H_\kappa^{p,r;\,\alpha-\gamma}}\,.$$

Now we use the fact that the de la Vallée-Poussin sum satisfies $\|V_t f\|_{p,r;\alpha-\gamma}\leqslant C\|f\|_{p,r;\alpha-\gamma}$ and $V_{1/N}T_N = T_N$, $T_N\in\mathcal{T}_N$. Therefore, one has (see also [**6**, Sect. 4])

$$\|f-V_{2^{-m}}f\|_{p,r;\alpha-\gamma} \lesssim E_{2^m}(f)_{p,r;\alpha-\gamma}. \tag{3.6}$$

We now need the Bernstein inequality in $L^{p,r;\,\alpha-\gamma}(\mathbb{T}^n)$,

$$|T_N|_{H_\kappa^{p,r;\,\alpha-\gamma}} = |V_{1/N}T_N|_{H_\kappa^{p,r;\,\alpha-\gamma}} \lesssim N^\kappa K(T_N,t^\kappa;L^{p,r}(\log L)^\alpha,H_\kappa^{p,r;\,\alpha}) \lesssim N^\kappa\|T_N\|_{p,r;\alpha-\gamma}\,,$$

which follows from (3.2). This estimate and (3.6) yield

$$\begin{aligned}|V_{2^{-m}}f|_{H_\kappa^{p,r;\,\alpha-\gamma}} &= |\sum_{l=1}^m (V_{2^{-l}}f - V_{2^{-l+1}}f) + V_1 f|_{H_\kappa^{p,r;\,\alpha-\gamma}}\\ \lesssim \sum_{l=1}^m 2^{l\kappa}\|V_{2^{-l}}f - V_{2^{-l+1}}f\|_{p,r;\alpha-\gamma} &+ \|V_1f\|_{p,r;\alpha-\gamma} \lesssim E_0(f)_{p,r;\alpha-\gamma} + \sum_{l=0}^{m-1} 2^{l\kappa}E_{2^l}(f)_{p,r;\alpha-\gamma}.\end{aligned}$$

Thus, we get

$$\omega_\kappa(f,1/2^m)_{p,r;\alpha-\gamma} \lesssim 2^{-m\kappa}\Big(E_0(f)_{p,r;\alpha-\gamma} + \sum_{l=0}^{m-1}2^{l\kappa}E_{2^l}(f)_{p,r;\alpha-\gamma}\Big),$$

which is equivalent to the last estimate in (3.5). Using monotonicity properties of the modulus of smoothness, we get

$$\left(\int_0^1 [\,\ell^\gamma(u)\,\omega_\kappa(f,u)_{p,r;\beta}]^s\frac{du}{u}\right)^{1/s} \approx \left(\sum_{\nu=1}^\infty [\,\ell^\gamma(1/\nu)\,\omega_\kappa(f,1/\nu)_{p,r;\beta}]^s\frac{1}{\nu}\right)^{1/s}.$$

This estimate, (3.5), and Hardy's inequality imply that, for any γ, $s>0$,

$$\left(\int_0^1 [\,\ell^\gamma(u)\,\omega_\kappa(f,u)_{p,r;\beta}]^s\frac{du}{u}\right)^{1/s} \approx \left(\sum_{\nu=1}^\infty [\,\ell^\gamma(1/\nu)\,E_{\nu-1}(f)_{p,r;\beta}]^s\frac{1}{\nu}\right)^{1/s}. \tag{3.7}$$

Note that in the case $0<s<1$ we use the following Hardy-type inequality for monotonic sequences $\{\varepsilon_i\}$ (cf. [**4**]): $\sum_{\nu=1}^\infty \nu^{-1}\Big[\ell^\gamma(\nu)\nu^{-\kappa}\sum_{i=0}^\nu (i+1)^{\kappa-1}\varepsilon_i\Big]^s \lesssim \sum_{\nu=1}^\infty \nu^{-1}[\,\ell^\gamma(\nu)\,\varepsilon_{\nu-1}]^s$.

Finally, (3.7) immediately implies that the definition of $B_{0,\gamma}^{(p,r;\beta),s}$ is independent of $\kappa>0$.

Lemma 3.5. (a) *Let either $\Omega=(a,b)$ with $-\infty<a<b<\infty$ or let $\Omega\subset\mathbb{R}^n$, $n\geqslant 2$, be a bounded domain with a Lipschitz boundary. If $1<p,r<\infty$, $1\leqslant s<\infty$, $\alpha\in\mathbb{R}$, and $\beta>-1/s$, then*

$$B_{0,\beta}^{(p,r;\alpha),s}(\Omega) \hookrightarrow L^{p,s;\beta+\alpha+1/\max\{s,r\}}(\Omega)\,.$$

(b) *If $\Omega=\mathbb{T}^n$, $n\geqslant 1$, $1<p,r<\infty$, and $\alpha\in\mathbb{R}$, then*

$$B_{0,\gamma-1/s}^{(p,r;\alpha-\gamma),s}(\mathbb{T}^n) \hookrightarrow L^{p,s;\alpha}(\mathbb{T}^n)\,, \qquad \gamma>0, \quad 1<r\leqslant s<\infty, \tag{3.8}$$

$$B_{0,\gamma-1/r}^{(p,r;\alpha-\gamma),s}(\mathbb{T}^n) \hookrightarrow L^{p,s;\alpha}(\mathbb{T}^n)\,, \qquad \gamma>1/r-1/s, \quad 1\leqslant s<r<\infty, \tag{3.9}$$

Proof. (a) By [**19**, Thm. 4.6],

$$L^{p,r;\alpha}(\Omega) \hookrightarrow L^{p_1}(\Omega) \quad \text{for any } p_1 \in [1,p). \tag{3.10}$$

If $n \geqslant 2$, choose p_1 such that

$$\max\left\{1, \frac{np}{n+p}\right\} < p_1 < \min\{p,n\}. \tag{3.11}$$

Together with the (generalized) Sobolev embedding theorem (cf., e.g., [**12**, Thm. 4.8 and Thm. 4.2]), embedding (3.10) implies that

$$W^1L^{p,r;\alpha}(\Omega) \hookrightarrow W^1L^{p_1}(\Omega) \hookrightarrow L^{p_1^*}(\Omega), \qquad \frac{1}{p_1^*} = \frac{1}{p_1} - \frac{1}{n}.$$

If $n = 1$, then, cf. [**1**, Lemma 5.8, p. 100], $W^1L^1(\Omega) \hookrightarrow C(\overline{\Omega})$, which, together with (3.10) shows that the embedding

$$W^1L^{p,r;\alpha}(\Omega) \hookrightarrow W^1L^{p_1}(\Omega) \hookrightarrow L^{p_1^*}(\Omega)$$

remains true with any $p_1^* \in [1,\infty]$, and hence with p_1^* satisfying $1 < p < p_1^* < \infty$. Combining the embedding $W^1L^{p,r;\alpha}(\Omega) \hookrightarrow L^{p_1^*}(\Omega)$ with the trivial embedding $L^{p,r;\alpha}(\Omega) \hookrightarrow L^{p,r;\alpha}(\Omega)$ and using a limiting interpolation, we arrive at

$$X := \left(L^{p,r;\alpha}(\Omega), W^1L^{p,r;\alpha}(\Omega)\right)_{0,s;\beta} \hookrightarrow \Big(L^{p,r;\alpha}(\Omega), L^{p_1^*}(\Omega)\Big)_{0,s;\beta} =: Y$$

for any $s \in [1,\infty]$ and $\beta \in \mathbb{R}$. Since (cf. [**16**, (1.6)])

$$K(f,t; L^{p,r;\alpha}(\Omega), W^1L^{p,r;\alpha}(\Omega)) \approx \min\{1,t\}\|f\|_{p,r;\alpha} + \omega_1(f,t)_{p,r;\alpha}$$

for all $f \in L^{p,r;\alpha}(\Omega) + W^1L^{p,r;\alpha}(\Omega)$ and every $t > 0$, one can show that $X = B^{(p,r;\alpha),s}_{0,\beta}(\Omega)$. Note that

$$\|f\|_X := \|f\|_{p,r;\alpha} + \left(\int_0^1 [\ell^{\beta}(t)\,\omega_1(f,t)_{p,r;\alpha}]^s \,\frac{dt}{t}\right)^{1/s}.$$

Moreover, if $1 \leqslant s < \infty$ and $\beta > -1/s$, then, using [**13**, Thm. 5.9$^+$, Thm. 4.7$^+$ (ii), p. 952], we obtain that $Y \hookrightarrow L^{p,s;\beta+\alpha+1/\max\{s,r\}}(\Omega)$ and the result follows.

(b) Replace α by $\alpha - \gamma$, take $\beta = \gamma - 1/s$ and $\beta = \gamma - 1/r$, to obtain embeddings (3.8) and (3.9), respectively. $\square$

Proof of Theorem 2 (b). Unlike the proof of part (a), here we will use the technique based on estimates of the best approximations rather than a Holmstedt-type formula.
By (3.3) and (3.2), we have

$$\omega_\kappa(f,1/N)_{p,s;\alpha} \lesssim \|f - V_{1/N}f\|_{p,s;\alpha} + N^{-\kappa}|V_{1/N}f|_{H_\kappa^{p,s;\,\alpha}} =: I + II\,. \tag{3.12}$$

(i) Let us first handle the case $r \leqslant s$. Lemma 3.4 together with (3.2) and (3.3) gives

$$II \lesssim (\log N)^{\gamma}\omega_\kappa(f,1/N)_{p,r;\alpha-\gamma}\,. \tag{3.13}$$

Concerning I, we first observe that under our restriction on the parameters, by (3.8),

$$\begin{aligned} \|f\|_{p,s;\alpha} &\lesssim \|f\|_{p,r;\alpha-\gamma} + \Big(\int_0^1 [\ell^{\gamma-1/s}(t)\,\omega_1(f,t)_{p,r;\alpha-\gamma}]^s\,\frac{dt}{t}\Big)^{1/s} \\ &\lesssim \|f\|_{p,r;\,\alpha-\gamma} + \Big(\sum_{j=1}^{\infty}[\ell^{\gamma-1/s}(j)\,E_j(f)_{p,r;\alpha-\gamma}]^s\,\frac{1}{j}\Big)^{1/s} \end{aligned} \tag{3.14}$$

for all $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$, where the latter inequality follows by (3.7). For arbitrary $g \in L^{p,r;\alpha-\gamma}(\mathbb{T}^n)$ set $f := g - T_N^{p,r;\alpha-\gamma}(g)$. This implies that

$$E_j(f)_{p,r;\alpha-\gamma} \leqslant \|f\|_{p,r;\alpha-\gamma} = \|g - T_N^{p,r;\alpha-\gamma}(g)\|_{p,r;\alpha-\gamma} = E_N(g)_{p,r;\alpha-\gamma}\,, \qquad 0 \leqslant j \leqslant N$$

and

$$E_j(f)_{p,r;\alpha-\gamma} = E_j(g)_{p,r;\alpha-\gamma}\,, \qquad j \geqslant N.$$

Rewrite (3.14) for the above function $f = g - T_N^{p,r;\alpha-\gamma}(g)$ to get

$$\begin{aligned} E_N(g)_{p,s;\,\alpha} &\lesssim \|g - T_N^{p,r;\alpha-\gamma}(g)\|_{p,s;\,\alpha} \\ &\lesssim E_N(g)_{p,r;\,\alpha-\gamma} + \Big[\Big(\sum_{j=1}^{N} + \sum_{j=N+1}^{\infty}\Big)[\ell^{\gamma-1/s}(j)\,E_j(f)_{p,r;\,\alpha-\gamma}]^s\,\frac{1}{j}\Big]^{1/s} \\ &\lesssim E_N(g)_{p,r;\,\alpha-\gamma} + E_N(g)_{p,r;\,\alpha-\gamma}\Big(\sum_{j=1}^{N}[\ell^{\gamma-1/s}(j)]^s\frac{1}{j}\Big)^{1/s} \\ &\qquad\qquad + \Big(\sum_{j=N+1}^{\infty}[\ell^{\gamma-1/s}(j)\,E_j(g)_{p,r;\,\alpha-\gamma}]^s\,\frac{1}{j}\Big)^{1/s} \\ &\lesssim \ell^{\gamma}(N)\,E_N(g)_{p,r;\,\alpha-\gamma} + \Big(\sum_{j=N+1}^{\infty}[\ell^{\gamma-1/s}(j)\,E_j(g)_{p,r;\,\alpha-\gamma}]^s\frac{1}{j}\Big)^{1/s}. \end{aligned}$$

Observe that, by [**8**, Thm. 2.1],

$$E_j(g)_{p,r;\,\alpha-\gamma} \lesssim \omega_{[\kappa]+1}(g,1/j)_{p,r;\,\alpha-\gamma} \lesssim \omega_\kappa(g,1/j)_{p,r;\,\alpha-\gamma}\,,$$

and that, by (3.6), $\|g - V_{1/N}g\|_{p,s;\alpha} \lesssim E_N(g)_{p,s;\alpha}$ to get the desired estimate for I. Together with (3.13), this establishes (1.4).

(ii) Let us now consider the case $1 \leqslant s < r$. Concerning I, we first observe that under our restriction on the parameters, by (3.9),

$$\|f\|_{p,s;\alpha} \lesssim \|f\|_{p,r;\alpha-\gamma} + \Big(\int_0^1 [\ell^{\gamma-1/r}(t)\,\omega_1(f,t)_{p,r;\alpha-\gamma}]^s\,\frac{dt}{t}\Big)^{1/s}.$$

Now follow straightforward the proof in (i) to obtain

$$E_N(g)_{p,s;\,\alpha} \lesssim \ell^{\gamma+1/s-1/r}(N)\,E_N(g)_{p,r;\,\alpha-\gamma} + \Big(\sum_{j=N+1}^{\infty}[\ell^{\gamma-1/r}(j)\,E_j(g)_{p,r;\,\alpha-\gamma}]^s\frac{1}{j}\Big)^{1/s}$$

for any $g \in L^{p,r;\alpha-\gamma}(\mathbb{T}^n)$. This implies that

$$I \lesssim \ell^{\gamma+1/s-1/r}(\delta)\,\omega_\kappa(f,\delta)_{p,r;\,\alpha-\gamma} + \Big(\int_0^\delta [\ell^{\gamma-1/r}(t)\,\omega_\kappa(f,t)_{p,r;\,\alpha-\gamma}]^s\frac{dt}{t}\Big)^{1/s}. \tag{3.15}$$

With regard to II, we need the following variant of Nikolskii's inequality for trigonometric polynomials $T_N \in \mathcal{T}_N(\mathbb{T}^n)$ which states that

$$\|T_N\|_{p,s;\alpha} \lesssim (\log N)^{\gamma+1/s-1/r}\|T_N\|_{p,r;\alpha-\gamma}\,, \qquad \gamma > 1/r - 1/s, \quad s < r, \tag{3.16}$$

and which will be proved below. Suppose (3.16) is true. Then

$$II \lesssim N^{-\kappa}(\log N)^{\gamma+1/s-1/r}|V_{1/N}f|_{H_\kappa^{p,r;\,\alpha-\gamma}} \lesssim (\log N)^{\gamma+1/s-1/r}\omega_\kappa(f,1/N)_{p,r;\alpha-\gamma}\,,$$

by (3.2) and (3.3). In view of (3.15), this proves assertion (1.5).

To prove (3.16), we need the following Remez inequality (see [**9**] and also [**18**])

$$T_N^*(0) \leqslant C(n)T_N^*(N^{-n}), \qquad N \in \mathbb{N}, \tag{3.17}$$

where T_N^* is the non-increasing rearrangement of T_N. Then

$$\|T_N\|_{p,s;\alpha}^s \lesssim \int_0^{N^{-n}} [\,t^{1/p}\ell^\alpha(t)T_N^*(t)\,]^s\frac{dt}{t} + \int_{N^{-n}}^1 [\,t^{1/p}\ell^\alpha(t)T_N^*(t)\,]^s\frac{dt}{t} =: I_1 + I_2\,.$$

By (3.17),

$$\begin{aligned} I_1 &\lesssim T_N^*(0)^s \int_0^{N^{-n}} [\, t^{1/p}\ell^{\alpha}(t)\,]^s \frac{dt}{t} \lesssim T_N^*(0)^s N^{-ns/p}\ell^{\alpha s}(N) \\ &\lesssim T_N^*(0)^s \ell^{\gamma s}(N) \left(\int_0^{N^{-n}} [\, t^{1/p}\ell^{\alpha-\gamma}(t)\,]^r \,\frac{dt}{t} \right)^{s/r} \\ &\lesssim \ell^{\gamma s}(N) \left(\int_0^{N^{-n}} [\, t^{1/p}\ell^{\alpha-\gamma}(t) T_N^*(t)\,]^r \,\frac{dt}{t} \right)^{s/r} \lesssim \ell^{\gamma s}(N) \|T_N\|^s_{p,r;\alpha-\gamma} \,. \end{aligned}$$

Finally, by Hölder's inequality,

$$\begin{aligned} I_2 &\lesssim \left(\int_{N^{-n}}^1 [\, t^{1/p}\ell^{\alpha-\gamma}(t) T_N^*(t)\,]^r \frac{dt}{t} \right)^{s/r} \left(\int_{N^{-n}}^1 \ell^{\gamma sr/(r-s)}(t) \,\frac{dt}{t} \right)^{(r-s)/r} \\ &\lesssim \ell^{\gamma s+1-s/r}(N) \left(\int_{N^{-n}}^1 [\, t^{1/p}\ell^{\alpha-\gamma}(t) T_N^*(t)\,]^r \frac{dt}{t} \right)^{s/r} \lesssim \ell^{\gamma s+1-s/r}(N) \|T_N\|^s_{p,r;\alpha-\gamma} \,. \end{aligned}$$

Note that the power of the $\ell(N)$-factor is positive since $\gamma > 1/r - 1/s$. □

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