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UNAMBIGUOUS $\pi\pi$ S WAVES FROM GENERAL PRINCIPLES AND THE EXISTENCE OF THE ρ MESON

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A B S T R A C T

We present a calculation of S wave $\pi\pi$ amplitudes based upon known rigorous results due to analyticity, unitarity and crossing symmetry : inequalities and crossing sum rules. The experimental P wave phase shifts between 500 and 1100 MeV are used as an input, and it is shown that the method leads to sharp S waves phase shifts predictions for a given very low energy P wave amplitude. Uncertainties in the extrapolation of the experimental P wave phase shifts downwards are mainly responsible for a reasonable dispersion of S wave phase shifts. Much care is taken in order to introduce correctly long range forces into the game. The results, in very good agreement with experiment, are physically interpreted, and compared with those of other models, in particular current algebra and field theory calculations. S wave scattering lengths and zeroes are found to be in agreement with current algebra models. A well defined σ meson is predicted under the form of a second sheet pole in $I=0$ S wave amplitude. Its mass, m_σ = 420 MeV, is remarkably independent of any variation of all parameters, in particular of the P wave scattering length, and it exists even if the phase shift does not pass through 90°. This phase shift is found to be of the "down" type above the ρ mass in the absence of inelasticity. Our conclusion is that, if there are no strong $I=2$ D wave forces, the relationship between S and P waves is well determined by analyticity, crossing symmetry and unitarity.

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I. INTRODUCTION

A lot of experimental [1),2)] and theoretical studies has been devoted to low energy $\pi\pi$ scattering during the recent past years. Among theoretical approaches, we encounter current algebra methods [3)], dispersion relation techniques [4),5)], bootstraps [6)], Lagrangian models [7),8),9)], dual models [10)], etc. Another kind of theory, initiated by Wanders and Piguet [11)] and continued by various people [12),13),14)] consists in constructing low energy amplitudes by some kind of parametrization which is constrained to fulfil all theoretical requirements imposed by analyticity (and thus crossing symmetry) and unitarity. The reason for such an approach is due to the existence of a considerable amount of exact results based upon axiomatic field theory and upon positivity conditions imposed by unitarity. A big part of these results has been derived by Martin [15)]. A quite complete review of what is known and what has been done about $\pi\pi$ scattering has been given by Basdevant and Reignier [16)]. Specific references will be quoted all along this paper.

Our work falls into the second category of theoretical approaches of the problem of $\pi\pi$ scattering. The purpose of it is twofold. First we notice that experimental information suffers from two kinds of uncertainties : one of them is the uncertainty about very low energy phase shifts (scattering lengths), and the other one concerns the shape of the S_0 phase shift above the ρ mass (is there a σ meson ?). So we first want to make predictions about S waves. Our second purpose is more theoretical : the various models or theories which have been, or will be, quoted here, often lead to different predictions; so we have to try to understand what is, and what is not model dependent. In particular, we want to investigate whether or not the known exact constraints are sufficient in order to fix sharply the low energy phase shifts even if we are dealing with a model with free parameters. Of course, we shall have to accept some qualitative assumptions (like smoothness, for example), which will be discussed, and also we obviously need some input physical information in order to prevent us from finding zero as a solution of the scattering problem. Here we have chosen to take as an input the P wave phase shifts in the region where they have been measured. In this respect, the Lagrangian model which looks the most like our approach is the so-called ρ model of Basdevant and Zinn-Justin [9].

In this paper we start from a 7 parameter unitary form of the isospin 0 and 2 S waves. We then construct a P wave amplitude which reproduces the known phase shifts and contains the long range forces generated by itself and by S waves. All known exact constraints are then imposed in order to determine an allowed domain of parameters. The main results of our work are the following.

i) Though we have more parameters than equations to be fulfilled, we find a rather small range of possible physical phase shifts. Furthermore, the major part of the range is due to uncertainty in the knowledge of low energy P wave phase shifts.

ii) Positions of the zeros of the S wave amplitudes, and values of the scattering lengths are compatible with current algebra conditions, though no such constraint has been directly put in. A physical interpretation of this result is given.

iii) The S_0 phase shift becomes large around the ρ mass, but we have no solution corresponding to a <u>narrow</u> σ meson. In all our solutions, we do find a second sheet pole in the S_0 amplitude, even when the phase shift does not pass through 90 degrees. This pole is always far away from the real axis. The mass of the σ meson is 420 MeV, independent of the free parameter variations.

iv) Our particular way of constructing the P wave allows a complete discussion and a good physical interpretation of the preceding results.

In the next section, we briefly recall what are the tools which are at our disposal to restrict the domain of possible low energy scattering amplitudes : inequalities and crossing sum rules. The third section is devoted to a detailed quantitative presentation of one of our solutions and to qualitative predictions based upon simple considerations on the physical content of the relations of Section II. In

Section IV, we completely specify the parametrization of the S wave amplitudes, the construction of the P wave amplitude, and give some indication about the computational problems. All the results are given in Section V, together with a discussion of them; in particular, we compare our approach with that of Bonnier and Gauron [14] and discuss why we do not find the same results. In our conclusion, Section VI, we discuss the problem of the D waves, not constructed in this paper, and give some indications about what can be done along the same lines if one tries to include inelasticities in the problem. We also investigate to what extent the low energy determination is precise enough in order to make predictions about high energy behaviour.

II. FORMALISM - CROSSING SUM RULES AND INEQUALITIES FOR PARTIAL WAVES

The content of this section is well known. We just recall the results which we are to use, for the sake of clarity and completeness.

II.1 Kinematics - Notations - Amplitudes

For isospin I_s in the s channel, the partial wave expansion of the full amplitude $A^{I_s}(s,t)$ is defined by

$$A^{I_s}(s,t) = \sum_\ell (2\ell+1) f_\ell^{I_s}(s) \, P_\ell(\cos\theta_s) \qquad \text{(II.1)}$$

with $\cos\theta_s = 1+(2t/(s-4\mu^2))$. In the following, the pion mass is taken to be unity. Normalizations are such that the differential cross-section is

$$\frac{d\sigma}{dt} = \frac{2\pi}{sq^2} |A|^2 \qquad \text{(II.2)}$$

and the total cross-section

$$\sigma_{TOT} = \frac{4\pi}{q\sqrt{s}} \, \mathcal{I}m \, A(s,t=0) \qquad \text{(II.3)}$$

with $q = \frac{1}{2}\sqrt{s-4}$. According to these conventions, the partial waves are related to the elastic phase shifts $\delta_\ell^{I_s}$ through

$$f_\ell^{I_s}(s) = \frac{\sqrt{s}}{q} \, e^{i\delta_\ell^{I_s}} \sin\delta_\ell^{I_s} \qquad \text{(II.4)}$$

Scattering lengths are defined by

$$a_\ell^{I_s} = \lim_{q^2\to 0} \delta_\ell^{I_s}/q^{2\ell+1} = \lim_{q^2\to 0} \frac{1}{2} f_\ell^{I_s}(s)/q^{2\ell}$$

Since in the major part of this paper we shall be essentially interested in S and P waves, we shall often use as a short-hand notation

$f^0(s)$, $f^2(s)$ and $f^1(s)$ for the isospin 0 and 2 S waves, and the P wave, respectively and a_0, a_2, a_1 for the corresponding scattering lengths. Also $f^{00}(s)$ will denote the combination $(f^0+2f^2)/3$, the S wave for $\pi_0\pi_0$ elastic scattering.

The isospin crossing matrix between s and t channel amplitudes reads

$$A^{I_s}(s,t) = \sum_{I_t} C_{I_s I_t} A^{I_t}(t,s) \tag{II.5}$$

where indices I_s and I_t refer to the rows and columns respectively of the matrix

$$C = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix} \tag{II.6}$$

The crossing matrix C verifies $C^2 = \mathbb{1}$.

Analyticity and crossing [15] allow to write the Froissart-Gribov representation of the partial waves $f_\ell^{I_s}(s)$ for $\ell \geq 2$, and $s < 4$, namely

$$f_\ell^{I_s}(s) = 4 \sum_{I_t} \frac{C_{I_s I_t}}{\pi(s-4)} \int_4^\infty dt\, A_s^{I_t}(t,s)\, Q_\ell\left(1+\frac{2t}{s-4}\right) \tag{II.7}$$

where $A_s^{I_t}(t,s)$, the isospin I_t absorptive part in the t channel, is positive for $0 \leq s \leq 4$; Q_ℓ is the usual Legendre function of the second kind. Under certain conditions, Eq. (II.7) may happen to be valid also for $\ell = 1$. We shall come back to this question when discussing our P wave construction.

The discontinuity of $f_\ell^{I_s}(s)$ across the left-hand cut $s \leq 0$, $|s|$ not too large [17], is given by

$$\operatorname{Im} f_\ell^{I_s}(s) = \frac{2\sum_{I_t} C_{I_s I_t}}{s-4} \int_4^{4-s} A_s^{I_t}(t,s)\, P_\ell\left(1+\frac{2t}{s-4}\right) dt \qquad \text{(II.8)}$$

This formula holds for all partial waves (finite integration range). The behaviour of $\operatorname{Im} f_\ell^{I_s}(s)$ near $s=0$ can be studied on this formula [11),14)]. The first two terms $(-s)^{3/2}$ and $(-s)^{5/2}$ are given for the S waves in Section IV and Appendix 1 in terms of scattering lengths and effective ranges.

Let us finally recall that for $0 \leq s \leq 4$ (and also in a certain negative interval), $A_s^{I_t}$ can be represented as a convergent partial wave expansion in the t channel. So, the contributions of particular partial waves (low energy phase shifts in the t channel) to $f_\ell^{I_s}(s)$ can be easily evaluated using Eq. (II.7). This will be especially useful in our construction of the P wave.

II.2 Rigorous results for partial waves : crossing sum rules

Since the first work by Balachandran and Nuyts [18)], the implications of crossing symmetry upon partial waves have been extensively studied in the recent past years [19),20)]. Applications for the use of parametrizing the $\pi\pi$ partial waves have been particularly studied by Roskies [21)]. Here, we shall use the five equations between S and P wave moments which have been written down in this reference, more explicitly

$$\int_0^4 (s-4)\, R_0^0\, (2f^0 - 5f^2)\, ds = 0 \qquad \text{(E1)}$$

$$\int_0^4 (s-4)\, R_1^0\, (f^0 + 2f^2)\, ds = 0 \qquad \text{(E2)}$$

(II.9)

$$\int_0^4 (s-4)\, R_1^0 (2 f^0 - 5 f^2)\, ds \;-\; 9 \int_0^4 (s-4)^2 R_0^1 f^1 ds = 0 \quad \text{(E3)}$$

$$\frac{1}{4}\int_0^4 (s-4)\, R_2^0 (2 f^0 - 5 f^2)\, ds + \frac{3}{2}\int_0^4 (s-4)^2 R_1^1 f^1 ds = 0 \quad \text{(E4)}$$

$$\frac{1}{16}\int_0^4 (s-4)\, R_3^0 (2 f^0 - 5 f^2)\, ds - \frac{15}{16}\int_0^4 (s-4)^2 R_2^1 f^1 ds = 0 \quad \text{(E5)}$$

(II.9)

Coefficients and normalizations have been chosen for practical computational convenience. In particular, R_i^ℓ are the following polynomials in S, proportional to the Jacobi polynomials $P_i^{(2\ell+1,0)}((2s-4)/4)$ (and thus orthogonal on the interval $0 \leq s \leq 4$ with the corresponding weights) :

$$R_0^0 = 1 \qquad\qquad R_0^1 = 1$$
$$R_1^0 = 3s-4 \qquad\qquad R_1^1 = 5s-4$$
$$R_2^0 = 10s^2 - 32s + 16 \qquad\qquad R_2^1 = 21s^2 - 48s + 16 \quad \text{(II.10)}$$
$$R_3^0 = 35s^3 - 180s^2 + 240s - 64$$

II.3 Rigorous results for partial waves : inequalities

A huge number of inequalities between the values of the partial waves in the unphysical region $0 \leq s \leq 4$ have been written down. They are consequences of crossing symmetry and positivity. Some of them concern the $\pi_0\pi_0$ amplitudes, either S waves only [22),23)], or S and D waves simultaneously [24)]. Other inequalities involve S and P waves [25)]. Finally, a new class of inequalities has been introduced [26] which involves the values of the partial wave amplitudes and of their derivatives with respect to S. In this section, we shall only recall those inequalities which are either instructive for qualitative arguments, or found to be particularly restrictive for the allowed values of our set of parameters [27].

Inequalities for $\pi^0\pi^0$ S waves

The most interesting inequalities are [22),23)]

$$f^{00}(4) - f^{00}(0) \geq 0 \qquad \text{(S1)}$$

$$f^{00}(3.205) \geq f^{00}(0.2134) \geq f^{00}(2.9863) \qquad \text{(S2)}$$

$$f^{00}(0) \geq f^{00}(3.19) \qquad \text{(S3)} \qquad \text{(II.11)}$$

$$f'^{00}(0) \leq 0 \qquad \text{(S4)}$$

$$f''^{00}(s) \geq 0 \quad \text{for} \quad 0 \leq s \leq 1.7 \qquad \text{(S5)}$$

Furthermore, it is known that $f^{00}(s)$ has a minimum for

$$1.127 \leq s \leq 1.697 \qquad \text{(S6)}$$

Refs. 22),23),26), and this minimum is the only one in the unphysical interval $0 \leq s \leq 4$.

Among all these conditions, we found that, in our particular model, inequality (S3) is the most restrictive one. The consequences of these results for the behaviour of $f^{00}(s)$ between 0 and 4 are conveniently visualized on the tentative drawing of Fig. 1.

We see from this drawing that we are invited to consider the function $f^{00}(s)$ roughly as a parabola with a minimum somewhere between 1.2 and 1.7. However, we expect at least two important deviations from this parabolic shape. The first one, near $s=0$, comes out from the existence of a $(-s)^{\frac{3}{2}}$ term in the imaginary part on the left-hand cut. Indeed, if $f^{00}(s)$ satisfies a subtracted dispersion relation, it means that the second derivative $f''^{00}(s) \to +\infty$ for $s \to +0$. The second deviation expected is due to the unitarity cut at $s=4$: for any finite scattering length the derivative $f'^{00}(s)$ is infinite at $s=4$. The corresponding effect is more or less important according to the fact that the combination a_0+2a_2 of the scattering lengths is large or not.

It is also interesting to notice that, when expanding $f^{00}(s)$ into Jacobi polynomials

$$f^{00}(s) = \sum_{i=0}^{\infty} \alpha_i R_i^0 \tag{II.12}$$

the Roskies equation (E2), (II.9), means that $\alpha_1=0$, so that, up to a constant, the first non-trivial term in (II.12) is proportional to $R_2^0 = 10s^2-32s+16$, a parabola the minimum of which lies at $s=1.6$. Then we expect that if (II.12) is a rapidly convergent series, the minimum of $f^{00}(s)$ should be not too far from 1.6. Note that the coefficient of R_2 is given by the Roskies equation

$$2\int_0^4 (4-s) R_2^0 f^{00}(s)\,ds = 5\int_0^4 (4-s)^3 f_2^{00}(s)\,ds \tag{II.13}$$

which means that the larger the D waves are, the larger α_2 is. Of course, Eq. (II.13) also implies that α_2 is positive, since the combination f_2^{00} is, so that the extremum of $\alpha_2 R_2$ at $s=1.6$ is actually a minimum. Other considerations can be made using sum rules which involve D waves : they will not be made in this paper, where we limit ourselves to S and P waves. Note that the condition $\alpha_2 \geq 0$ is precisely the type of inequalities we refer to in Ref. 27). It is clear that if f^{00} has the shape of Fig. 1, implied by all inequalities, then α_2 will be > 0.

Inequalities involving S and P waves [25)]

We do not quote any of these inequalities because it happens that, in the course of our calculations, we never found any violation of them once S and P waves verify the five equations (E1) to (E5).

Inequalities involving derivatives of S and P waves [26]

We found that two of them were particularly interesting, namely

$$f'^{00}(0) < -\frac{1}{4}\left[2f^{00}(4) - f^{00}(2) - f^{00}(0)\right] \qquad \text{(S7)} \qquad \text{(II.14)}$$

$$2f'^{2}(0) - 2f'^{1}(0) < f^{2}(0) - f^{1}(0) - 2/3\left(f^{0}(4) - f^{2}(4)\right) \qquad \text{(II.15)}$$

Inequality (S7) is in fact an improvement of (S4) which states that $f'^{00}(0)$ is negative. Not only this derivative is negative, but it has to be smaller than a combination of values of $f^{00}(s)$ which is itself negative, due to inequalities (S1) to (S6).

II.4 Dispersion relation for the $I_t = 1$ amplitude at threshold

We just recall the result [28] which we want to use as an a posteriori check of our solutions :

$$\frac{2a_0 - 5a_2}{6} = \frac{\mu}{8\pi^2}\int_{4\mu^2}^{\infty} \frac{ds}{\left[s(s-4\mu^2)\right]^{1/2}} \left\{\sigma^{+-}(s) - \sigma^{++}(s)\right\} \qquad \text{(II.16)}$$

where σ^{+-} and σ^{++} are the $\pi^+\pi^-$ and $\pi^+\pi^+$ total cross-sections, respectively *).

*) This relation is valid if both the Pomeranchuk theorem and asymptotic dominance of the imaginary part of the amplitude hold.

III. QUALITATIVE PREDICTIONS AND PRESENTATION OF A TYPICAL SOLUTION

Before explaining S and P wave parametrizations and giving all our results and discussions, here we present a typical solution for the S wave amplitudes, fulfilling all requirements of Section II. We do that in order to illustrate to what extent most of the obtained results can be easily understood, and even qualitatively predicted, through very simple considerations about the crossing and unitarity constraints.

This solution will be referred to as "central" solution throughout this paper. Here "central" just means that it corresponds to values of the parameters which lie somewhere in the middle of the domains allowed by constraints of Section II.

III.1 Phase shifts

In Figure 2, we have plotted the central solution we obtain for δ_0^0 and δ_0^2 phase shifts, together with experimental data of Refs. 1c) and 2). On these results, the following features clearly appear :

i) the $I=2$ S wave phase shift δ_0^2 is small, negative and slowly decreasing;

ii) the $I=0$ S wave phase shift δ_0^0 is large, positive, rapidly increasing and of the "down" type above the ρ mass.

In the present solution, it approaches 90°. Though this value is not reached, we nevertheless interpret this phase shift as a broad σ resonance, since in $f^0(s)$ we actually obtain a second sheet complex pole at

$$s \simeq 7.5 - i\,8.2 \qquad \text{(III.1)}$$

which is indeed far away from the right-hand cut, and corresponds for the mass and width of the σ to

$$m_\sigma \simeq 420 \text{ MeV} \qquad \Gamma_\sigma \simeq 380 \text{ MeV} \tag{III.2}$$

Let us show that the main features of the S wave amplitudes can be easily inferred from the hypothesis that the D waves are small in the region $0 \leq s \leq 4$. Assuming that for $s < 4$, $t < 4$ and $u < 4$, the amplitudes are well approximated by S and P waves only, then crossing clearly implies that these partial wave amplitudes are linear. Furthermore, they cannot be independent of each other. Their relations can be found for example by writing that if $A(s,t,u)$ is the usual t-u symmetric amplitude then

$$A(s,t,u) = \alpha + \beta s$$

and :

$$\begin{aligned} B(s,t,u) &\equiv A(t,u,s) = \alpha + \beta t \\ C(s,t,u) &\equiv A(u,s,t) = \alpha + \beta u \end{aligned} \tag{III.3}$$

so that S_0, S_2 and P_1 waves only depend on two parameters in the linear approximation. It is in fact more instructive for us to derive the corresponding relations between S and P waves from the Roskies sum rules which we are to use extensively all along this paper. Assuming that

$$\begin{aligned} g^0(s) &= \alpha_0 + \beta_0 s \\ g^2(s) &= \alpha_2 + \beta_2 s \\ g^1(s) &= \beta_1 (s-4) \end{aligned} \tag{III.4}$$

we observe that Eqs. (II.9) - (E4) and (E5) - are identities and that the three first ones give

$$\begin{aligned} \beta_0 &= 2\beta_1/3 \\ \beta_2 &= -\beta_1/3 \\ 2\alpha_0 - 5\alpha_2 &= -36\beta_1 \ , \end{aligned} \tag{III.5}$$

We actually find that the three considered waves only depend on two parameters, for example β_1 and $\alpha_0+2\alpha_2$. The interesting consequence for the phase shift behaviour is the following : if the P wave is increasing ($\beta_1 > 0$), then f^0 is <u>increasing</u> like $\frac{2}{3}f^1(s)$ and f^2 is <u>decreasing</u>, but <u>more slowly</u>, like $-\frac{1}{2}f^0(s)$. This property is related, via crossing symmetry, to the usual statement that P wave exchange is more attractive for $I=0$ than it is repulsive for $I=2$. Conversely, as can be seen in the Froissart-Gribov formula for $f^1(s)$,

$$f^1(s) = \frac{4}{\pi(s-4)} \cdot \int_4^\infty \left[\frac{1}{3}A_s^0(t,s) + \frac{1}{2}A_s^1(t,s) - \frac{5}{6}A_s^2(t,s)\right] Q_1\left(1+\frac{2t}{s-4}\right) dt \quad \text{(III.6)}$$

the only <u>positive</u> contribution to $f^1(0)$ (<u>negative</u> contribution to β_1) comes out from the isospin 2 term in the t channel. In particular, Eq. (III.6) contains the well-known fact which is at the origin of the bootstrap idea, that $I_t=1$ forces are attractive for the $I_s=1$ amplitude. What we want to emphasize is that, furthermore, once we know that these forces do exist (the ρ meson exists), then β_1 tends to be positive, which in turn implies that β_0 also does, leading to appreciable δ_0^0 phase shifts and thus enhancing the tendency of β_1 to be positive. In the bootstrap language, we would say that the ρ meson supports itself at the same time directly, via $A_s^{I_t=1}$, and through its effects in the S waves. In terms of contributions to $f^1(0)$, we have found that both effects have the same orders of magnitude [see Section IV.2 and especially Eq. (IV.15)]. We also see how the assumption that the D_2 wave is small enters into the game : a low energy $I=2$ D wave resonance, for example, could destroy the simple picture described above. This point will be discussed in the last section.

We can even go further in the discussion of simple approximations of the amplitudes between 0 and 4 and look at quadratic terms [13a,c]. Using here the Balachandran-Nuyts expansion in terms of Jacobi polynomials we obtain

$$g_0^I(s) = 2\left[a_{00}^I P_0^{(1,0)} + 2a_{01}^I P_1^{(1,0)} + 3a_{02}^I P_2^{(1,0)}\right]$$

$$g^1(s) = 2\left(\frac{4-s}{4}\right)\left[2a_0^1 P_0^{(3,0)} + 3a_1^1 P_1^{(3,0)}\right] \qquad \text{(III.7)}$$

The sum rules (II.9) give

$$2a_{00}^0 - 5a_{00}^2 = 0 \qquad \text{(III.8a)}$$

$$a_{01}^0 + 2a_{01}^2 = 0 \qquad \text{(III.8b)}$$

$$2a_{01}^0 - 5a_{01}^2 = -9a_0^1 \qquad \text{(III.8c)}$$

$$2a_{02}^0 - 5a_{02}^2 = 6a_1^1 \qquad \text{(III.8d)}$$

[the fifth Eq. (II.9) is identically satisfied]. The implication of (III.8b) has already been mentioned in Section II in the discussion of the shape of $f^{00}(s)$. This shape implies in particular

$$a_{02}^0 + 2a_{02}^2 > 0 \qquad \text{(III.9)}$$

[positive curvature of $f^{00}(s)$].

Now what do we learn about the curvatures of individual partial waves ? Following the same lines as in the linear case, we want some information about the curvature of $f^1(s)$, namely about a_1^1. This can be done at least near $s=0$, because of the $(-s)^{\frac{3}{2}}$ behaviour of the discontinuity near $s=0-$. From Eq. (AI.8) of Appendix AI, we see that, if $2(a_0)^2-5(a_2)^2$ is positive (which happens to be the case), then $f''^1(0)$ is infinite and negative, so that with the definitions of Eq. (III.7), we have

$$a_1^1 > 0 \qquad \text{(III.10)}$$

So, if we are in the same general situation as described in the linear case (no substantial $I = 2$ forces), from Eqs. (III.8d), (III.9) and (III.10), we obtain

$$a_{02}^{0} > 0 \tag{III.11}$$

$$\frac{2}{5} a_{02}^{0} > a_{02}^{2} \tag{III.12}$$

Using the result of Piguet and Wanders [27] that a_{02}^{2} is positive, we have the following qualitative prediction : the first correction to the linear approximation tends to enhance the already growing $I = 0$ S wave, whereas, though in a much weaker way, it tends to prevent the already decreasing $I = 2$ S wave from getting eventually too large in absolute value.

This is exactly what we obtain in our final results. The reason for such an agreement between qualitative predictions and actual computations can be understood as follows. If we assume that partial waves verify dispersion relations, their Jacobi moments between 0 and 4 appear as integrals on the left and right-hand cuts. In these integrals, the discontinuities are weighted by functions which, due to the orthogonality of Jacobi polynomials, are more and more decreasing at large s as soon as the order of the moment is increased. So, the highest are the orders of the considered moments, the best are the predictions based upon nearest singularity arguments.

III.2 Scattering lengths and zeros of the amplitudes

For our so-called central solution, we have obtained the following scattering lengths

$$\begin{aligned} a_0 &= 0.206 \\ a_2 &= -0.073 \\ a_1 &= 0.045 \end{aligned} \tag{III.13}$$

so that the ratio a_0/a_2 is -2.82, to be compared to the Weinberg prediction $-7/2$.

Both $I=0$ and 2 S wave amplitudes have a zero between 0 and 4. Their values are

$$z_0 = 1.4$$
$$z_2 = 1.27 \qquad \text{(III.14)}$$

It will be seen later that z_0 is a parameter which may be varied between 1.1 and 1.7, $z_0 = 1.4$ is chosen in the middle of this domain, and z_2 is the corresponding <u>computed</u> zero of $f^2(s)$. The existence of nearby zeros in the S waves is a well-known result of current algebra and we have to understand why we find them here, since no kind of Adler self-consistency condition has been apparently *) put in.

Let us go back to the linear approximation of Eq. (III.4) with constraints (III.5). In this approximation $z_0 = -\alpha_0/\beta_0$ and $z_2 = -\alpha_2/\beta_2$, and Eq. (III.5) leads to the relation

$$4z_0 + 5z_2 = 12 \qquad \text{(III.15)}$$

This law will be verified on the set of all our results in Section V. For the moment, we just want to point out that Eq. (III.15) is verified by $z_0 = z_2 = 4/3$ **), which is not surprising in the linear approximation since it just represents the fact that $A^{I_t=1}(4/3,4/3,4/3) = 0$ by s,u antisymmetry. The value 4/3 for z_0 and z_2 corresponds to a ratio a_0/a_2 equal to −2, a value which can be obtained by assuming that the only forces in the t channel are isospin 1 forces. So, we understand the values finally obtained for z_0 z_2, linked by Eq. (III.15), and a_0/a_2 as reflecting the two following facts :

i) the amplitudes are quasi-linear (small D waves);

ii) isospin 1 forces in the t channel are dominant.

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*) These words will be commented in Section V.4 ii) where we compare our approach to another one where the Adler condition is explicitly imposed.

**) If we take $z_0 = 4/3$ in its allowed domain of variation, we obtain $z_2 = 1.32$ as a result of our calculations.

Unlike the current algebra approach, we see that all these results are based upon on-mass shell considerations. Deviations from Eq. (III.15) and from $a_0/a_2 = -2$ then do not come out from off-mass shell extrapolation, but from the two following facts :

i) unitarity tends to bend the amplitudes;

ii) isospin 1 forces generate non-negligible $I = 0$ forces.

Let us also mention that the minimum of $f^{00}(s)$ is found to be at $s = 1.67$, which is not far from the value 1.6 predicted by the quadratic approximation.

Finally we can compare the values of partial wave amplitudes at $s = 0$ to the "linear" law

$$2 f^0(0) - 5 f^2(0) = 9 f^1(0)$$

given by the last Eq. (III.5). In the present solution, we find

$$f^0(0) = -0.158$$
$$f^2(0) = 0.091$$
$$f^1(0) = -0.084$$

so that the left-hand side of the above equation is equal to -0.771, to be compared to $9f^1(0) = -0.756$. This agreement confirms that the linear approximation between 0 and 4 is a good one.

If one is interested in reconstructing the solution described here above, one can look at the definite parametrization of S and P waves in Section IV, where numerical values of all necessary parameters are given.

III.3 Check by dispersion relation

As a consistency check in the physical region, we use dispersion relations (II.16), where the left-hand side is directly given by the scattering lengths, whereas the right-hand side is computed by using

- at low energy, the contributions of the obtained S and P waves, and of the f_o meson;
- above the f_o meson mass, a Regge-like term corresponding to ρ exchange.

Detailed parametrization of these various terms will be given in the next section. We just want here to give the numerical results for the "central" solution. We write Eq. (II.16) in the following form

$$L \equiv \frac{2a_0 - 5a_2}{6} = S_0 + S_2 + P_1 + F_0 + As \equiv R \qquad \text{(III.16)}$$

where the numbers on the right-hand side R correspond respectively to the contributions of δ_0^0, δ_0^2, P wave, f_o meson and ρ Regge exchange. We find

$$L = 0.1297 \qquad R = 0.1300 \qquad \text{(III.17)}$$

so that

$$\frac{|L-R|}{|R|} \simeq 2.3 \text{‰} \qquad \text{(III.18)}$$

with

$$S_0 = 6.844\ 10^{-2} \qquad S_2 = -1.534\ 10^{-2}$$
$$F_0 = 0.889\ 10^{-2} \qquad As = 2.939\ 10^{-2}$$
$$P1 = 3.859\ 10^{-2}\ .$$

We see that the agreement is nearly perfect. In particular, we note that As is about 3/4 of the direct ρ contribution P1, and then far from being negligible.

IV. DESCRIPTION OF THE S AND P WAVE AMPLITUDES

Since we are interested in low energy properties of S and P waves, we are faced with the problem of getting a description of nearest cuts as good as possible. Unitarity gives the answer for the right-hand cut (we assume elastic unitarity) and the near left-hand cut is related to the imaginary part of the same amplitudes in the physical region (for small D waves in the low energy region). Of course, the difficulty is to put both information at the same time in a closed parametrization. In order to partially solve the problem, we notice that, due to centrifugal barrier, the effect of the unitarity cut upon the behaviours in the unphysical region (where all constraints are put on) is smaller for a P wave than for S waves. This is the reason why we choose to have exact (elastic) unitarity for S waves, and only approximate unitarity for the P wave. For this last one, conversely, we can describe correctly the near left-hand cut (long range forces) for any given S waves in the physical region. Thus, all the free parameters are put in a unitary parametrization of the S waves, and the Roskies equations (II.9) are then used to determine them, so carrying the effect of long range forces into S waves in an "effective" way *). In fact, as will be seen very soon, the beginning of the S wave left-hand cut can also be exactly described.

IV.1 S wave parametrization

For each amplitude f^0 and f^2, we take a three-parameter form, corresponding to its value at some point, and to the possibility of getting at least one pole and one zero. Practically, convenient parameters are $f^I(0)$, the scattering lengths $a_I = f^I(4)/2$ and zeros z_I $(f^I(z_I)=0)$. We also introduce one additional parameter s_o, common to both S waves, the meaning of which will be soon cleared up. Using a K matrix formalism, we write

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*) In the Bonnier and Gauron model [14], all three waves are parametrized in the same unitary way. The meaning and the consequences of this different approach will be discussed in Section V.

$$f^I(s) = 1/\left[K^I(s) + \frac{2q}{\pi\sqrt{s}} \mathrm{Log}\left[(\sqrt{s-4}+\sqrt{s})/2\right] - i\,q/\sqrt{s} \right] \qquad \text{(IV.1)}$$

a choice which ensures elastic unitarity whenever K^I is real analytic on the physical real axis, and which leaves all left-hand cut effects in K^I. The logarithmic term is the usual Chew-Mandelstam function. Our explicit form for K^I is

$$K^I(s) = \frac{A^I + B^I s + h^I J(s) + g^I W(s)}{(1 + C^I s)} \qquad \text{(IV.2)}$$

where

- A^I, B^I, C^I are constants. In some sense, they are the basic parameters such as to reproduce the main L.H.C. effects. For instance, $A+Bs$ allows any kind of pole or resonance, whereas $1+Cs$ allows the existence of a zero at $s=-1/C$.

- $J(s) = \phi_{\frac{3}{2}}(s)$; $W(s) = \phi_{5/2}(s)$ with

$$\phi_i(s) = \frac{s_0^i}{\pi} \int_{-\infty}^{0} \frac{(-x)^i\, dx}{(1+x^4)(x - s/s_0)} \qquad \text{(IV.3)}$$

  h^I and g^I are constants. The added terms $hJ+gW$ give the correct behaviour of the L.H.C. near $s=0$, without introducing any new parameter.

Some comments are in order :

i) As is well known [11),14)], and recalled in Appendix AI, $\mathrm{Im}\, f^I(s)$, for small negative s, can be expanded as

$$\mathrm{Im}\, f^I(s) = \alpha(-s)^{3/2} + \beta(-s)^{5/2} + \dots \qquad \text{(IV.4)}$$

and α, β only depend on the scattering lengths and effective ranges of the S waves [see Eqs. (AI.4) and (AI.7)]. So in $K^I(s)$ we need functions which account for such a behaviour. Now, of course, we do not know on which range of the L.H.C. an expansion of the type (IV.4) is valid. The two functions $J(s)$ and $W(s)$ are precisely functions, the imaginary part of which behaves like $(-s)^{3/2}$ and $(-s)^{5/2}$ respectively for for $-s_o \lesssim s \leq 0$, and go to zero rapidly when s is below $-s_o$. This is our justification for the cut-off factor $1/(1+x^4)$ in integral (IV.3), and for taking s_o as a free parameter *).

ii) due to the relationship between the coefficients α, β and the effective range expansion parameters, only 2x3 out of 2x5 parameters A,B,C,h and g are in fact independent. Their relations with the 2x3 physical parameters $f^I(0)$, a_I and z_I are given in Appendix AII.

iii) it can be seen by looking at the behaviour of various parts of $1/f^I(s)$ that $f^I(s) \to 0$ as $1/\log s$ as $s \to \infty$. Of course this property has no particular significance since here we keep elastic unitarity at all energy. The question of inelasticity will be discussed in Section V. Note that the function $hJ(s)+gW(s)$, which accounts for the local behaviour of f near $s=0$ goes to zero like $1/s$ as $s \to \infty$, to be compared to $A+Bs$, a linear form. This ensures that this function only plays a local role, as it should.

IV.2 P wave representation

i) the ρ meson

Since we want to use P wave phase shifts as an input, as a first component of $f^1(s)$ we consider the analytic function

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*) A discussion of this point will be given in Section V.4 i), when comparing our work with that of Ref. 14).

$$B^1(s) = \frac{(s-4)}{\pi} \int_4^\infty \frac{\mathrm{Im}\, f^1(s')\, ds'}{(s'-s)(s'-4)} \tag{IV.5}$$

where $\mathrm{Im}\, f^1(s)$ is obtained by fitting to the P wave phase shift data [2]. Practically we used

$$\mathrm{Im}\, f^1(s) = \frac{\sqrt{s}}{9} \frac{\left(\frac{s-4}{s+s_R}\right)^3 Q_R^2}{(m_\rho^2 - s)^2 + \left(\frac{s-4}{s+s_R}\right)^3 Q_R^2} \tag{IV.6}$$

with

$$Q_R^2 = \left(\frac{m_\rho^2 + SR}{m_\rho^2 - 4}\right)^3 m_\rho^2 \Gamma_\rho^2 \ . \tag{IV.7}$$

m_ρ and Γ_ρ are the mass and width of the ρ meson. SR is introduced in such a way to realize conveniently the threshold behaviour; the corresponding P wave scattering length

$$a_1 = \frac{8 m_\rho \Gamma_\rho (m_\rho^2 + SR)^{3/2}}{(s_R + 4)^{3/2} (m_\rho^2 - 4)^{5/2}} \tag{IV.8}$$

is very sensitive to this parameter. Since the low energy behaviour, and then a_1, is very important for the determination of the S waves (see qualitative discussion in Section III), we have been led in our fit to experimental data to give the lowest energy points a larger weight *).

However, different sets of parameters (SR, m_ρ, Γ_ρ) give equally good fits, so that, since a_1 strongly depends on SR, we have kept SR as a parameter which is free inside a domain of values compatible with a good fit to P wave of Ref. 2). SR may be varied

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*) This is also justified by the fact that the integral in Eq. (IV.4) is rapidly convergent and not very sensitive to the value of $\delta^1(s)$ above the ρ peak.

between approximatively 5 and 50. For each SR, m_ρ and Γ_ρ are adapted by best fit to the data. We give in Fig. 2a this best fit for SR around 15.

ii) The near left-hand cut

With $B^1(s)$, we then describe correctly the imaginary part on the right-hand cut; keeping SR free ensures a sufficient freedom in the region where phase shifts are not measured. Now we have to introduce a function which gives the correct discontinuity on the left-hand cut. As long as s is not too large and negative, this discontinuity only depends on the absorptive part in the crossed channels, Eq. (II.8). Neglecting waves higher than S and P waves is certainly a good approximation for $(4-s) < m^2_{f_0}$, and, on the other hand, $(4-s) \simeq m^2_{f_0}$ is a region where the third double spectral function contributes to $\mathrm{Im}\, f^1(s)$. Since for the moment we are only interested in getting a function which have the correct imaginary part on the near L.H.C., we then construct a second component $f^1_{S,P}(s)$ to $f^1(s)$, obtained by retaining only S and P waves in the Froissart-Gribov formula (II.7), namely

$$f^1_{S,P}(s) = \frac{4}{\pi(s-4)} \int_4^{TC} \left[\frac{2\,\mathrm{Im}\, f^0(t) - 5\,\mathrm{Im}\, f^2(t)}{6} + 3\,\mathrm{Im}\, f^1(t) \cos\theta_t \right] \cdot Q_1\left(1 + \frac{2t}{s-4}\right) dt \qquad \text{(IV.9)}$$

In this formula, f^0 and f^2 are the S waves as given by Eqs. (IV.1), (IV.2) and (IV.3) for a given set of parameters. $\mathrm{Im}\, f^1(s)$ is given by Eq. (IV.6). Integration in the range (4,TC) instead of $(4,\infty)$ as in a true Froissart-Gribov formula accounts for the fact that we have no reason to believe in our parametrization of $\mathrm{Im}\, f^{0,1,2}$ for too large values of the energy (inelasticity, etc.). The actual value of TC which we chose was $TC = 115\,\mu^2$, a convenient choice which will be explained later, but is in fact unimportant here since the large values of t in Eq. (IV.9) only contribute to the far away left-hand cut.

iii) Evaluation of $f^1(0)$

If we now consider $f^1(s)-B^1(s)-f^1_{S,P}(s)$, we see that this function has no important cut at small distance $(-40 \lesssim s \lesssim 40$, say) so that what we really need is essentially a smooth real function, representing effectively the short range forces, and which has to realize unitarity together with B^1 and $f^1_{S,P}$. Unitarity so yields constraints for $s \geq 4$, but since our input information for computing S waves is the P wave between 0 and 4 via Roskies rules and inequalities, we think it safer to also find some constraint for $f^1(s)$ around $s=0$. This is the reason why we now assume that the Froissart-Gribov formula actually converges (cf., footnote on p.10) and compute $f^1(0)$ via Eq. (II.7). We divide the interval $(4,+\infty)$ into two parts $(4,TC)$ and $(TC,+\infty)$. Between 4 and TC we use the partial wave expansion of $A^{I_t}_s(t,s)$, and above TC a Regge like form corresponding to the ρ trajectory in the s channel. Due to the occurrence of resonances, ρ, f_0, g, etc., in the t channel, we tried several values of TC successively, namely $TC=((m_\rho^2+m_{f_0}^2)/2)$, including in $\int_4^{TC}$ S waves and the ρ meson, $TC=((m_{f_0}^2+m_g^2)/2)$, including also the f_0 meson contribution, and TC above the g meson, then taking its contribution also into account. We have verified that the variation of $\int_4^{TC}+\int_{TC}^{\infty}$ for such values of TC is very small, and anyway smaller than the uncertainty due to the high energy part $\int_{TC}^{\infty}$. We finally used

$$T_C = (m_{f_0}^2 + m_g^2)/2 = 115\,\mu^2$$

and find for $f^1(0)$

$$f^1(0) = f^1_{S,P}(0) + f^1_{D_0}(0) + f^1_{as}(0) \qquad \text{(IV.10)}$$

where $f^1_{S,P}$ is given in Eq. (IV.9),

$$f^1_{D_0}(s) = \frac{4}{\pi(s-4)}\int_4^{T_C}\left[\frac{\mathrm{Im}\, f_2^0(t)}{3}\; 5P_2(\cos\theta_t)\right] Q_1\left(1+\frac{2t}{s-4}\right) dt \qquad \text{(IV.11)}$$

with

$$\left\{\begin{array}{l} \mathrm{Im}\, f^{0}_{S_e}(t) = \frac{1}{2}\sqrt{\frac{t-4}{t}}\; \frac{[(t-4)/(t+S_F)]^5 Q_F^2}{(m_f^2-t)^2+[(t-4)/(t+S_F)]^5 Q_F^2} \\ Q_F^2 = [(m_{f_0}^2+S_F)/(m_{f_0}^2-4)]^3 m_{f_0}^2 \Gamma_{f_0}^2 \\ m_{f_0} = 1.264 \text{ MeV},\ \Gamma_{f_0} = 145 \text{ MeV},\ S_F = 20\mu^2 , \end{array}\right. \tag{IV.12}$$

and

$$f^{1}_{S_{as.}}(0) = -\frac{1}{\pi}\int_{TC}^{\infty} A_t^{I_s=1}(0,t)\, Q_1\left(1-\frac{t}{2}\right) dt \tag{IV.13}$$

with

$$\left\{\begin{array}{l} A_t^{I_s=1} = \beta \sin\pi\alpha_\rho(0) \left(\frac{t}{T_0}\right)^{\alpha_\rho(0)} \Gamma(1-\alpha_\rho(0)) \\ \alpha_\rho(0) = 0.5 \quad ; \quad 0.6\, \text{GeV}^2 \leq T_0 \leq 1.\ \text{GeV}^2 \\ \beta = \frac{6 m_\rho^2 \Gamma_\rho \alpha'_\rho T_0}{(m_\rho^2-4)^{3/2}} \quad ; \quad \alpha'_\rho = \frac{1-\alpha_\rho(0)}{m_\rho^2} \quad . \end{array}\right. \tag{IV.14}$$

Comments about the evaluation of $f^1(0)$

a) We use in Eq. (IV.12) a form inspired by the ρ contribution parametrization. We verified that the value of $f^1_{D_0}(0)$ is not very sensitive to the value of S_F which determines the D_0 scattering length. The value $S_F = 20\,\mu^2$ has been retained by analogy with the final values of SR but we do not claim that the corresponding D_0 scattering length is the right one; we should at least obtain a correct order of magnitude. The problem of D waves will be discussed elsewhere (see Section VI).

b) When computing $f^1_{as}(0)$, we have made simple assumptions concerning the high t behaviour of the $I_s = 1$ amplitude. The trajectory $\alpha_\rho(0) + \alpha'_\rho s$ passes through 1 at the ρ mass, and the value $\alpha_\rho(0) = 0.5$ was taken as a commonly adopted value in phenomenological analysis. Variations around 0.5 do not change anything substantially. The residue variation between $s = 0$ and $s = m^2_\rho$ has been put in through the scale parameter T_0, and the reduced residue β has been determined by extrapolation to the ρ mass [Eq. (IV.14)].

c) Typical orders of magnitude for the various components of $f^1(0)$ are given by

$$f^1(0) \simeq -8.4 \; 10^{-2}$$

$$f^1_S(0) : f^1_P(0) : f^1_{D_0}(0) : f^1_{as.}(0) \simeq -3.3 : -2.6 : -0.6 : -1.9 \qquad \text{(IV.15)}$$

iii) Final representation of $f^1(s)$

We now achieve the P wave parametrization by adding to $f^1_{S,P}(s) + B^1(s)$ a two pole function $L^1(s)$:

$$f^1(s) = f^1_{S,P}(s) + B^1(s) + L^1(s) \qquad \text{(IV.16)}$$

$$L^1(s) = \tau_1 \frac{(s-4)(s-\tau_2)}{(s-s_1)(s-s_2)} \qquad \text{(IV.17)}$$

where the two pole positions s_1 and s_2 have been a priori fixed at $-30\,\mu^2$ and $-50\,\mu^2$, in order to simulate short range forces, whereas τ_1 and τ_2 are fixed by the two following conditions :

a)
$$L^1(0) = f^1_{D_0}(0) + f^1_{as.}(0) \qquad \text{(IV.18)}$$

obtained by comparison of (IV.16) and (IV.10). Here we see why it was convenient to choose the same value of T_C in (IV.9) and in computing $f^1(0)$.

b) $f^1(s)$ is exactly elastic unitary at one energy $s = \sigma$, a condition which fixes the value of $\mathrm{Re}\, f^1(\sigma)$ once $\mathrm{Im}\, f^1(\sigma)$ is given by Eq. (IV.6). Practically we took $\sigma = 14\mu^2$, a value between $\pi\pi$ threshold and m_ρ^2.

A detailed discussion of the P wave representation will be given after the general results in Section V.

IV.3 Computational considerations

We just sketch the way we used successively all constraints due to analyticity, crossing and unitarity. We had essentially three computational steps.

i) Roskies sum rules

They allow to determine five out of the nine parameters SR, T_0, z_0, z_2, $f^0(0)$, $f^2(0)$, a_0, a_2, s_0. Practically, we first take SR and T_0 fixed in their allowed range.

Equation (IV.9) permits us to compute the part of the P wave which depends on S waves for any set of the seven S waves parameters. We then explore the (s_0, z_0) plane, and for each point of this plane, we look for solutions of the five Roskies equations in the space of the five S wave parameters left free.

These equations are solved using the CERN Minuits Program, with the help of which we minimize the function

$$M = \sum_i \alpha_i \, (E_i)^2 \qquad \text{(IV.19)}$$

where E_i are the left-hand sides of Eq. (II.9). Coefficients α_i have been empirically optimized in such a way that, for some random set of parameters, contributions of the various terms $\alpha_i(E_i)^2$ are of the same order of magnitude : the point here is that the successive moments of the partial waves, which appear in (II.9), are decreasing very fast with the index i (which means that the functions are smooth);

so, in general, when we are far from a solution, E_5 is very small as compared to E_1. Now, a solution is obtained when $M=0$. Of course, we never reach exactly zero, but numbers of the order of 10^{-8} to 10^{-14}, depending on actual specifications of the programme. In order to verify that we actually have a solution, we computed at the end of the minimization quantities of the type $c_i = |E_i|/|e_i|$, where e_i is the part of the E_i concerning one of the partial waves. A domain of solutions is a domain where all c_i are currently of the order of 10^{-7} to 10^{-10}, as compared to regions where $c_i \gtrsim 10^{-2}$ to 10^{-1}. Due to numerical precision problems, we are not able to define an exact boundary between these two kinds of domains, but accuracy is in fact quite sufficient in order to get a good determination of the physical parameters.

Our last remark, which we consider as a very important one is that the orthogonality of the polynomials which appear in the crossing sum rules is indispensable : if, for example, we replace Eq. (II.9), $E_1 = E_3 = E_4 = E_5 = 0$, by a new set

$$\left\{ E'_i = \sum_j \alpha_{ij} E_j = 0 \right\} \quad \begin{matrix} i = 1,3,4,5 \\ j = 1,3,4,5 \end{matrix}$$

obtained by using the lowest order equations in order to simplify the highest order ones, we find that small errors in solving $E'_i = 0$ can imply substantially larger errors for $E_i = 0$ (especially for E_5). We have actually observed that doing this leads to instabilities, which manifest themselves by difficulties in finding a true minimum for M, or by giving not small enough a minimum [29]. Using instead orthogonal polynomials, like in Eq. (II.9), is safer and considerably increases the convergence speed in the search for solutions. Time needed for finding one solution on the CERN 6600 Computer is typically of the order of 6 to 12 seconds, corresponding to 400 to 800 trial sets of S wave parameters [for each set, one has to compute the S waves, the part $f_S^1(s)$ of the P wave which depends on the S waves, and the moments of S and P wave amplitudes which enter into the five Roskies equations].

ii) Inequalities

Inequalities are checked for sets retained in step i). Since at this stage a lot of positivity and crossing symmetry has already been put in, only a few inequalities are still constraining. They are used to reduce the domain of allowed solutions. At this stage, inequalities (S3) [Eq. (II.11)] and (S7) [Eq. (II.14)] appear to be the most stringent *).

iii) P wave unitarity

We finally measure the degree at which P wave unitarity is violated. Recalling that by construction $f^1(s)$ is exactly unitary at $s = 14\mu^2$, we consider

$$\Delta R/R = \frac{|\mathrm{Re}\, f^1(s) - \mathrm{Re}\, f^1_u(s)|}{|\mathrm{Re}\, f^1_u(s)|} \tag{IV.20}$$

with $\mathrm{Re}\, f^1(s)$ as given by Eq. (IV.16), and

$$|\mathrm{Re}\, f^1_u(s)| = \left[\mathrm{Im}\, f^1(s)\left(1 - \frac{q}{\sqrt{s}}\,\mathrm{Im}\, f^1(s)\right)\right]^{1/2} \tag{IV.21}$$

the real part that $f^1(s)$ should have in order to be exactly unitary. Deviations of $\Delta R/R$ from 0 measure the degree of violation, except near $s = m_\rho^2$ where $\mathrm{Re}\, f^1_u$ and $\mathrm{Re}\, f^1(s)$ do not vanish exactly at the same energy. The selection procedure of solutions for which the P wave is practically unitary will be given later.

iv) Numerical values

Numerical values of the constants needed for the reconstruction of S and P waves in the case of our so-called central solution of Section III are :

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*) Here we start with sum rules and observe that a lot of inequalities are then verified. The converse is true : if one satisfies all inequalities, then in general, Roskies sum rules are not too badly violated [30].

P wave : SR = 15 [for which we obtain by fit $m_\rho = 767$ MeV, $\Gamma_\rho = 135$ MeV],

$T_o = 0.8\ \text{GeV}^2$.

S waves : $A_o = -6.602$, $B_o = 0.5224$, $C_o = -0.7143$,

$h_o = 0.3093$, $g_o = -0.1277$;

$A_2 = 10.57$, $B_2 = 1.012$, $C_2 = -0.7886$,

$h_2 = 0.6002$, $g_2 = -0.4496$,

$s_o = 1.6$.

V. GENERAL RESULTS AND DISCUSSION

V.1 Presentation of the results

In Section III, we have presented a solution for S and P waves which fulfils all known theoretical constraints. We recall that since in fact we have more parameters than equations, this solution was chosen out of a band of possible solutions verifying all requirements. To be more specific, let us recall that we have a model in which, once the P wave phase shifts between 500 and 1100 MeV, and the mass and width of the f_o meson are given, we are left with

7 parameters for the S waves : $f^{0,2}(0)$, $f^{0,2}(4)$, $z^{0,2}$ and s_o;

2 parameters for the P wave : namely SR and T_o.

The first very important result we want to mention is that, though we are left with 4 free parameters SR, T_o, s_o and z_o once the five Roskies equations are verified, other constraints (namely inequalities and unitarity of the P wave) lead to a relatively small domain of possible solutions. Things go as follows :

a) For SR and T_o fixed, we have looked for solutions in a very large domain in the (s_o, z_o) plane. But we find already above $z_o \sim 4$ and below $z_o \sim -6$ S waves which violate more and more Roskies sum rules as $|z_o|$ increases, this being also true for lower and lower $|z_o|$ as s_o increases. As a result, we find two domains in the (s_o, z_o) plane, $\mathfrak{D}_1$ and $\mathfrak{D}_2$, in which the five Roskies equations are verified.

$\mathfrak{D}_1$ is characterized by the fact that z_o is negative (in fact it can reach 0 for small s_o). The corresponding value of z_2 is in general larger than 4 [cf., Eq. (III.15)] and this means that $\mathfrak{D}_1$ contains a domain of solutions of the Castoldi type [4] (positive $I = 2$ S wave scattering length). But $\mathfrak{D}_1$ is ruled out by inequalities, essentially by inequalities (S3) [Eq. (II.11)] and (S7) [Eq. (II.14)]. This is a general result, independent of SR and T_o. So we shall not comment further on this domain.

$\mathfrak{D}_2$ is instead characterized by z_o somewhere between 0 and 4. Correspondingly, z_2 is also between 0 and 4 (see qualitative discussion in Section III), so that the $I = 2$ scattering length is always negative in $\mathcal{D}_2$. Once all inequalities are imposed, and mainly (S3) and (S7), $\mathcal{D}_2$ reduces to a small domain $\mathcal{D}'_2$ in s_o and z_o. This domain is given in Fig. 3a for $SR = 15$ and $T_o = 0.8$ GeV^2.

b) When SR and T_o are varied, SR in a domain allowed by experimental P wave phase shifts and T_o between 0.6 and 1 GeV^2, the domain $\mathcal{D}'_2$ moves a little bit, without changing very much the qualitative results (z_o and z_2 in particular are always severely constrained to lie in a small region of the interval 0 - 4).

c) The violation of unitarity in the P wave at low energy is measured by the quantity $\Delta R/R$ [Eq. (IV.20)-(IV.21)]. In order not to have a bias due to bad unitarity in the low energy P wave amplitude, we want to retain only sets of solutions of Eq. (II.9) and inequalities which agree the best with the unitarity condition. This can be done because we have found that, as a function of the free parameters, the maximum deviation M of $(\Delta R/R)$ from 0, for $4 \leq s \leq 18$, has a well defined minimum. For instance, in Fig. 3b, we give the variation of M as a function of SR for a fixed point of $\mathcal{D}'_2$ and for $T_o = 0.8$ GeV^2. From the order of magnitudes which we have found for the lowest minima, we may finally retain as a unitarity criterion

$$\Delta R/R \leq 2.5\ \%$$

for $4 \leq s \leq 18$.

It is very remarkable that once such a restriction has been imposed, unitarity is still very well verified up to rather high energies : on Fig. 4 we compare $\mathrm{Re}\, f^1(s)$ and $\mathrm{Re}\, f^1_u(s)$ for s between 4 and 42. The agreement obtained confirms that our understanding of P wave dynamics, including the ρ meson region is fairly good. Final solutions for SR and T_o fixed correspond to a domain $\mathcal{D}''_2(s_o, z_o)$. $\mathcal{D}''_2$ exists only for $14 \leq SR \leq 18$ and then moves from lower to upper part of $\mathcal{D}'_2$. It is for $SR = 15$ that $\mathcal{D}''_2$ has a maximum extension and a good central position inside $\mathcal{D}'_2$, as shown in Fig. 3a.

V.2 Quantitative results

In Fig. 5, for $SR = 15$ and $T_0 = 0.8$ GeV^2, we give the band of solutions corresponding to the domain $\mathcal{D}''_2$ described in Section V.1 a) and c). Note that the corresponding band of solutions is very narrow. For this range of solutions, in Fig. 6a and 6b, we also plot z_2 against z_0 (the variation appearing practically independent of s_0) and a_2 against a_0 when s_0 and z_0 vary in the allowed domain, a plot which is of interest for comparison with current algebra, or other models predictions. We see that a_0 and a_2 not so sharply restricted as the phase shifts $\delta^0_0(s)$ and $\delta^2_0(s)$ are. However, we find that the quantity $(2a_0-5a_2)$ varies in $\mathcal{D}''_2$ only between 0.764 and 0.779.

In Fig. 7 we present our phase shifts for T_0 fixed at 0.8 GeV^2, and the central point of $\mathcal{D}''_2$ when SR varies between 14 and 18. The corresponding band is appreciably wider than in Fig. 5 : here we want to emphasize that this part of the uncertainty on S wave phase shifts is due to the uncertainty about very low energy P wave phase shifts (or scattering length a_1). What we predict is the relation between $\delta^0_0(s)$, $\delta^2_0(s)$ and a_1, a_1 varying between 0.04 and 0.047 when SR varies from 18 to 14. Note that since a_1 is a decreasing function of SR, according to our qualitative discussion of Section III, the larger SR is, the smaller is δ^0_0.

In Fig. 8, the phase shifts correspond to T_0 varying between 0.6 and 1 GeV^2, for $SR = 15$ and the point $s_0 = 1.6$, $z_0 = 1.5$ of $\mathcal{D}''_2$. Again, the corresponding band is narrow. This is a satisfactory result since the limitation $[0.6, 1]$ for T_0 has no deep justification. Here, we have no simple interpretation of the sense of variation of phase shifts : on the one hand, the contribution to $f^1(0)$ increases with T_0, but, on the other, we find that S waves decrease, leading as a final result to a smaller total $f^1(0)$ for larger T_0.

For all the solutions which have been finally retained, the $I = 0$ S wave amplitude has always a pole in the second sheet of the s plane. The very interesting result is that, though we do not get a unique solution for the phase shifts, but rather a band of solutions, the <u>mass</u> of the predicted σ meson is found to be practically independent of all the free parameters, and equal to 420 MeV. The deviation from this value is never larger than 3 MeV for all solutions. At the same time, on the contrary, the width Γ_σ varies between 270 MeV and 520 MeV, when we follow the solutions from the upper to the lower one in terms of δ_0^0 phase shifts. One might argue that the precise location of this pole is not significant in the sense that it is far away in the imaginary direction : in fact, we are dealing with analytic functions and our predictions should be valid in some circle centered at threshold. The distance from threshold of the poles obtained is at most $12\,\mu^2$, which is not a large distance.

In order to allow easy reconstruction of some of our solutions, we give the values of the constants needed for reproducing some of the curves which have been drawn (Table I).

Finally, let us give the result of the dispersion relation [Eq. (II.16)], considered as a consistency check. We have shown the relative importance of the various terms of the R.H.S. for our so-called central result in Section III. Here we shall just mention that for all the retained solutions we have

$$\left| \frac{RHS - LHS}{RHS} \right| \lesssim 3\,‰$$

This excellent agreement of our results with dispersion relations is a further confirmation of a good crossing symmetry. This can be especially seen by noting that when we vary T_0, everything else being kept fixed, we have the following chain of reactions :

i) if T_0 increases, as we said, S waves (and especially the S_0 wave) and $f^1(0)$ decrease;

ii) correspondingly, the L.H.S. <u>and</u> the contribution of the S waves in the R.H.S. both decrease in the dispersion relation;

iii) but at the same time, the contribution of the asymptotic part in the dispersion relation increases.

Crossing is then checked in the following sense : the variation of the asymptotic part just cancels <u>the difference between the variations due to S waves</u> in the R.H.S. and the L.H.S. of the dispersion relation. Numbers are given in Table II.

We shall come back to this interesting point in our conclusion.

V.3 Discussion

Our results should be compared both to experimental data and other theoretical approaches.

As far as the experiment is concerned, we have compared our results to the data of Refs. 1c) and 2). These are the data which are quoted in all our figures. We see that our solutions are clearly favourable to the so-called "down" solution above the ρ mass. Taking into account the dispersion of experimental results [1a)], and keeping in mind that extraction of $\pi\pi$ phase shifts from experiment is difficult, we may consider that our solutions are in agreement with experiment.

Now, the two important questions from a theoretical point of view are :

a) what are the true scattering lengths ? There is no clear experimental answer for the moment. Most of the models predict in particular $a_2 < 0$, and as we said, we find that, <u>in our model</u>, a_2 positive is ruled out by inequality S_7. As can be seen in Ref. 4) using dispersion relations, where a_2 is found < 0, the sign of a_2 has some connection with the sign of the D_2 wave. It is stated in particular that, when D_2 is increased, a_2 tends to be larger. Presumably, our solution should be associated with a <u>negative</u> D_2 wave. This will be elucidated in a forthcoming work.

b) is there a well-defined σ meson ? It is worth while noticing that the experimental phase shift analysis still allows two possible results, the so-called "up" and "down" solutions above the ρ mass, the first one leading to a rather narrow $I = 0$ resonance. The interesting point here is that theoretical models also fall either in one category or in the other one. This invites us to discuss the features of our model which seem to impose the down solution.

Discussion of our S wave parametrization

Let us first recall that we have for each S wave a three parameter form which allows (but does not impose) to have essentially one pole, one zero and a given value for some s. This is realized by the homographic function $(A+Bs)/(1+Cs)$. From this point of view, we can think that this parametrization is sufficiently flexible in order to reproduce any a priori given S wave. However, let us remark that if we manage for example to have a pole at $s=m^2$, a zero somewhere, and a given value at $s=0$, say, then the residue of the pole is fixed, and so is the width of the resonance. Thus one could argue that our "down" solution just emerges from this lack of independence between the "normalization" of the amplitude at some point and the pole residue. We believe that this argument is not valid. Indeed, let us assume for a moment that we add to our homographic function some term $\varepsilon V(s)$, with ε sufficiently small in order not to modify substantially the amplitude between 0 and 4 where all constraints are imposed, and $V(s)$ sufficiently increasing with s in order to make the amplitude rapidly small after the resonance. This would possibly lead to a narrow σ meson without changing the S wave in the unphysical region. Our point is that the obtained solution would be no more crossing symmetric because $f^1(0)$, which is fairly sensitive to the crossed S waves [see Eq. (IV.15)], would be no more adapted to this new situation in the physical region. Our conclusion is : if a narrow σ meson actually exists, this means either that we do not understand the forces which are responsible for the observed P wave amplitude, or that we have to include important inelastic effects in all waves. This question is connected to the next point we want to discuss.

Up to what energy should we believe in our solution ?

If we look at our solutions, we see that the S_0 wave phase shift becomes rather flat for $s > 20$ or $30\mu^2$. Roughly speaking, that means that our homographic function has reached its asymptotic value. Again one can think that this feature is due to the absence of flexibility of our parametrization. From the above discussion, we would rather say the following : crossing symmetry requires a large imaginary part in the S_0 wave on a wide energy range, so that if this wave is elastic unitary, the phase shift has to stay far from 0 or 180° for a long time (leading to a broad "resonance"). But, if a substantial amount of inelasticity is allowed, then we may have the situation in which the phase shift goes rapidly to 0 or 180° (or in fact, any other value), the imaginary part being produced by absorption in 4π and $K\bar{K}$ channels. The case $\delta_0^0 \to 180^0$ would be the so-called up solution, which, in our opinion, should be associated with important inelasticity. In such a case, we could say that we start with a narrow resonance which is broadened by absorption *). We think it interesting to study this question from both experimental and theoretical points of view.

Discussion of the P wave

In our construction of the P wave, we first had to use as an input the P wave phase shifts in the region where they have been measured, but also to make additional assumptions in order to be able to evaluate its value in the unphysical region, in practice the value of $f^1(0)$. The first assumption we made is that the Froissart-Gribov formula for $f^1(0)$ actually converges. This allows to compute the contribution of any partial wave in the crossed channel, and in particular of the considered S waves and of the given P wave itself. We do not think that this assumption is a strong one. Once it is made, we still have to be more specific about other waves, and practically to assume :

i) that the D waves are essentially $I=0$ (f_0 meson used as an input, no substantial D_2 wave). We shall come back to this point in the last section.

ii) that the asymptotic behaviour of $A^{I_s=1}(0,t)$ is well described by the existence of a ρ Regge trajectory in the s channel. As we have said (Section IV.2), such a behaviour introduces the new parameter T_0 which we varied in order to obtain an effective representation of possibly neglected short range forces in the P wave. The chosen variation range for T_0, namely 0.6 to 1 GeV^2, corresponds first to what is commonly used in phenomenological analysis of high energy data in other reactions, but also

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*) We thank J. Zinn-Justin for an interesting discussion about the question of the existence, or not, of narrow S wave resonances.

to a region in which we actually verified that the same $f^1(0)$ is obtained when making the asymptotic behaviour start either from $TC = (m_{f_0}^2 + m_g^2)/2$ (between the f_0 and g meson masses), or from above the g meson (taking then the g contribution explicitly into account as a 3^- resonance).

Maybe these assumptions are not all valid, but we can make at least two comments :

- under these assumptions, we obtain a very good internal consistency between input and output information, as can be seen from the discussion of the dispersion relation;
- the fact that P wave unitarity is verified within an extremely good approximation (see Fig. 4) by using only a smooth function $L^1(s)$ [Eq. (IV.17)] in order to simulate all effects which are not due to crossed S and P waves means that presumably we have no trouble with the description of long range forces. Unitarity of the P wave could even be improved by varying the parameters s_1, s_2, and σ which appear in $L^1(s)$, but we do not think this would be very significant since anyway the structure of this function is not too well adapted to a detailed description of short range forces.

V.4 Further comparison with other models

From the point of view of the method, our approach is comparable to that of Bonnier and Gauron [14]. From a physical point of view, we already said that our model has some similarities with the Lagrangian model of Basdevant and Zinn-Justin [9] where the ρ meson is also used as an input. Let us comment further upon these two points.

i) Comparison with the model of Ref. 14)

The main idea of both models is the same : given some input physical information, analyticity, crossing and unitarity constraints should be strong enough in order to allow the prediction of new things. And, of course, these constraints are the same for all people. However, we have already said that some care must be taken when handling Roskies sum rules and in this respect, we refer to Section IV.3 i) to mention a difficulty concerning numerical accuracy.

Apart from this technical point, a more interesting point is the comparison of parametrization. It is clear that if two different parametrizations do not give the same results when all constraints are fulfilled, there is some doubt about the actual usefulness of them. What are the differences in the parametrizations ? The first one is about the number of free parameters : we have nine of them instead of five, which leads to more flexibility in the shapes of the partial waves, and we already emphasized that the four parameters which are left free after verifying Eq. (II.9) are constrained to lie in a very small domain. More important from a physical point of view is the difference between both ways of handling left-hand cuts. In Ref. 14), all left-hand discontinuities are represented by functions, the power expansion of which

$$\mathrm{Im}\, f_\ell(s) = s\sqrt{-s}\ (a + bs + \cdots) \qquad (V.1)$$

verifies crossing at the order $(-s)^{7/2}$. This is also what de did for S waves (but not for the P wave for reasons which have been explained), but in a very different way. In the final form, our S waves have substantial discontinuities only in an interval $\{-s_0, 0\}$ *), which turns out to be indeed very small : s_0 has been found to lie always in the interval

$$1 \lesssim s_0 \lesssim 3 .$$

Let us point out that this feature is extremely satisfactory from the point of view of internal consistency. Expansion (V.1) is based upon the effective range expansion (AI.4) of the K matrix, the convergence of which is limited in all cases by the existence of the left-hand cut. So, the radius of convergence R in expansion (AI.4) is at most equal to 4. Furthermore, if partial waves have zeros between 0 and 8, this radius is even limited by $4-z_I$, where z_I is the zero which is the nearest to threshold. Recalling the approximate law :

*) The effect of the rest of the left-hand cuts is described by the two three-parameter forms $(A_I+B_Is)/(1+C_Is)$.

$$4z_0 + 5z_2 = 12 \tag{V.2}$$

derived in the linear approximation, and well verified a posteriori (see Fig. 6a), we see that the radius of convergence is bounded by

$$\begin{cases} R = \mathrm{Min}\left(4 - z_0, \frac{8}{5} + \frac{4z_0}{5}\right) \\ \text{for} \quad 0 \leq z_0 \leq 4 \end{cases} \tag{V.3}$$

The maximum allowed value of R is obtained for $z_0 = z_2 = 4/3$, for which $R_{max} = 8/3$. The maximum values of our parameter s_0, which cuts off expansion (V.1), agree surprisingly well with this estimate as can be seen for example on Fig. 3.

We thus believe that generalizing the method of Refs. 11) and 14) in order to calculate higher order terms in expansion (V.1) would not in fact improve the description of the left-hand discontinuity. By the way, it can be noted that the important part of the (near) left-hand cut is not concentrated in general near $s = 0$, but rather at $s \simeq 4 - m^2$, if m is the mass of any exchanged object : in the narrow resonance approximation, for example, ρ exchange leads to a cut which begins at $4 - m_\rho^2$, and it is then hopeless to try and describe it by an expansion at $s = 0$. In our opinion, the way out of this problem is either to use true N/D method [with all inherent problems, see for example Ref. 31)], or to increase the number of parameters which represent the long range forces, or to abandon exact unitarity and follow the same lines as we did for the P wave. From this point of view, the accuracy within which we were able to approach unitarity with very simple functions seems to be encouraging.

ii) Comparison with the Lagrangian models [9),8)]

In the ρ model of Ref. 9), the ρ meson is used as an input in a Lagrangian theory in which the (presumably) divergent perturbation series is treated with the Padé approximant method. The Adler self-consistency condition [32] is here imposed in the following way : this condition is satisfied by construction by the first order term, and used in the second order term calculation in order to fix the renormalization constant Z. Now, it can be seen [33] that choosing a substantially different value for Z would be equivalent to introducing a substantial amount of ϕ^4 interaction in the first order term, so destroying the initial physical idea that the ρ meson contribution is responsible for the main part of the amplitude in the middle of the Mandelstam triangle. On the other hand, it is known [7] that in a $\lambda \phi^4$ theory, one obtains a high degree of degeneracy between $I = 0$ and $I = 2$ states, and in particular that the f_0 meson is accompanied by a quasi-degenerate $I = 2$, D wave meson. Then, the fact that we obtain results comparable to current algebra results, and to those of Ref. 9), though we do not impose the Adler condition a priori, can be traced back to our hypothesis of no important D_2 wave contribution in the calculation of $f^1(0)$.

Finally, we would also like to comment about one of our results : in our model, by the very parametrization, the σ meson appears as elementary, as does a C.D.D. pole in an N/D calculation. So, if mass and coupling were used as basic parameters, of course, they would be independent of each other. What we find here is that, although m_σ and Γ_σ are calculated in terms of other free parameters (essentially SR and T_0), m_σ happens to be a constant when these parameters are varied. We have no clear explanation of this fact, but we want at least to point out that a similar situation arises in the so-called σ model of Ref. 8) and ρ model of Ref. 9) [*].

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*) We thank Dr. J.L. Basdevant for bringing this point to our attention.

VI. CONCLUSIONS

We have shown to what extent the knowledge of P wave phase shifts in a sufficient energy domain, together with a systematic and accurate use of all rigorous properties which emerge from the general principles of analyticity, crossing and unitarity, can yield quite definite predictions about other partial wave phase shifts (here for S waves). The main characteristics of this work are essentially :

i) from the point of view of the parametrization, a new way of handling with the left-hand cuts : the near L.H.C. of the P wave is carefully described, and the corresponding information is transmitted to S waves via crossing symmetry; crossing symmetry thus fixes the parameters of a form of the S waves which simulates the effects of the L.H.C., apart from the local behaviour near $s=0$ which is exactly built in;

ii) from the point of view of the method, a very accurate way of dealing with the crossing sum rules, which are shown to be extremely powerful in determining unknown parameters;

iii) from the point of view of physics, a detailed interpretation of the results in terms of a simple qualitative discussion of the content of the sum rules and inequalities. In particular, we have shown how the existence of the ρ meson, and the hypothesis that there is no large D_2 phase shifts at low energy, imply that the main characteristics of the S waves (scattering lengths and zeros especially) cannot be very different from what is predicted in special models, as the Weinberg current algebra model, or models in which the Adler condition has been a priori imposed. The predicted δ_0^2 phase shift is found to be remarkably stable against any variation of the free parameters in their allowed domain. For the δ_0^0 phase shift, we obtain a larger band of possible solutions than for δ_0^2. Recall that the parameter in which S wave phase shifts depend the most strongly is SR (tightly connected to the P wave scattering length a_1); so that we can say that <u>once the P wave is given</u> in the very low and low energy region, our method leads to a <u>sharp</u> determination of the S wave phase shifts.

The agreement with experiment is rather good for both δ_0^0 and δ_0^2, and our predictions are clearly favourable to the <u>down</u> solution for δ_0^0 above the ρ mass. All solutions correspond to a second sheet complex pole in the $I=0$ S wave amplitude. The corresponding σ meson has a mass which is around 420 MeV, remarkably independent of the particular solutions considered, whereas the width varies between 270 MeV and 520 MeV. We have qualitatively discussed possible implications of non-zero inelasticity in this region. Anyway the sudden variations of the experimental phase shifts around 1 GeV clearly invite us to look at this problem in more details; this investigation, which is not allowed by our present unitary parametrization, is of great interest if one observes that in fact inelasticity seems to be non-zero even below the $K\bar{K}$ threshold, and thus to be presumably attributed to a 4π system.

All along this paper, we have emphasized the fact that all qualitative aspects of the solutions are easily understood once it is assumed that the D_2 wave is small : this point is particularly illustrated by the comparison with the Lagrangian ρ model and current algebra; in our model, the crucial point where this hypothesis enters is in the evaluation of $f^1(0)$ through the Froissart-Gribov formula. Isospin 2 contribution in $f^1(0)$ is <u>positive</u>, so that substantial D_2 wave makes $|f^1(0)|$ smaller, tending to decrease δ_0^0 and to increase δ_0^2. This can make a difference also for the locations of zeros and for the values of the scattering lengths. So, we have first to construct the D waves in a way coherent with our S and P waves, using the Froissart-Gribov formula in the unphysical region, unitarizing and again constraining the amplitudes to verify Martin inequalities [24)] and Roskies sum rules [21)]. Consistency will then be checked if we find the f_0 meson coming out in the D_0 wave and a small, presumably negative, D_2 wave. Now this situation which seems to be actually verified in nature, is usually referred to in the framework of duality as manifesting exchange degeneracy, which leads to the absence of exotic resonances ($I=2$ waves). So, the second question, directly connected to the construction of D waves is the following : can we build a consistent set of S, P and D waves, verifying all necessary conditions, but realizing a drastically different situation from the point of view of duality ? This is a particularly interesting problem if we remember the result of Atkinson [34], who has shown that one has a great arbitrariness in constructing functions which verify Mandelstam representations and unitarity in all channels (and then automatically fulfilling all constraints considered here). Keeping this in mind, we would like to investigate whether the existence of the ρ meson alone is sufficient in order to get rid of this arbitrariness for not too high energies.

Finally, let us recall that we have considered here one particular dispersion relation, and found that it constitutes a good consistency check of crossing symmetry for the full amplitudes $I_t = 1$. Considering the accuracy at which this dispersion relation is verified, we may now be more ambitious, and look at dispersion relations in order to make predictions about high energies. An example of such an application is to write a subtracted dispersion relation for $A^{00}(s,0,u)$. Using our S waves and a very rough estimate of the D^{00} wave *) in order to fix the substraction, and assuming a behaviour $i\,\mathrm{const} \times s$ for the asymptotic part of the forward amplitude, we have found that the dispersion relation is fulfilled for a $\pi\pi$ total cross-section at infinity of the order of 40 mb, a result which seems to be very encouraging if we remember that in such a calculation, this value comes out as the small difference between two large numbers. Further details and developments will be described elsewhere, together with our predictions for the D waves.

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*) According to this preliminary estimate, we find that the combination $(a_2^0+2a_2^2)/3$ of the D wave scattering lengths is of the order of $9 \cdot 10^{-4}$.

APPENDIX AI

BEHAVIOURS NEAR $s=0$

The subject of the behaviours of $\mathrm{Im}\, f_\ell(s)$ near $s=0$ has been studied many times in the literature, especially in Refs. 11), 14), and the result has been used extensively in the same kind of a context, for example in Ref. 14). So here we just recall how things go in our actual $\pi\pi$ case, for the sake of completeness. For simplicity, we consider S waves. We know that, for $s \leq 0$, $|s|$ not too large because of the third double spectral function,

$$\mathrm{Im}\, f_0^{I_s}(s) = \frac{2}{(s-4)} \int_4^{4-s} \sum_{I_t} C_{I_s I_t} A_s^{I_t}(t,s)\, dt \tag{AI.1}$$

Due to the integration range, for s close to zero, the values of t involved are close to 4 so that the threshold behaviour of t channel partial waves

$$\mathrm{Im}\, f_\ell^{I_t}(t) = q_t^{4\ell+1}\, a_\ell^{I_t}(t) \tag{AI.2}$$

determines the s behaviour of $\mathrm{Im}\, f_0^{I_s}(s)$. Putting $t-4=-sx$ we obtain

$$\mathrm{Im}\, f_0^{I_s}(s) = \frac{2(-s)^{3/2}}{(s-4)} \sum_{I_t} C_{I_s I_t} \sum_\ell \frac{(2\ell+1)}{2} \int_0^1 \sqrt{x}\, (-sx)^{2\ell}\, a_\ell^{I_t}(t) \cdot P_\ell\left(1-\frac{2}{x}\right) dx \tag{AI.3}$$

We see that, apart from the explicit $(-s)^{\frac{3}{2}}$ factor, the next term in s of the integral comes out of the next term of the expansion of $a_0(4-sx)$, and not from other waves than the S waves, due to the $(-sx)^{2\ell}$ factor. In particular the term $(-s)^{7/2}$ would receive a contribution from the P wave in the t channel. Now $\mathrm{Im}\, f_\ell^{I_t}(t)$ is not an analytic function of t, so that in order to give a sense to the expansion of $a_0(t)$, we use instead the K matrix expansion near threshold

$$q_t \cot g\, \delta_0^{I_t} = (1/a_{I_t}) + \frac{1}{2} r_{I_t} q_t^2 + \dots \qquad \text{(AI.4)}$$

with

$$\operatorname{Im} f_0^{I_t}(t) = q_t \sqrt{t}\, \left(q_t^2 \cot g^2 \delta_0^{I_t} + q_t^2\right)^{-1} \equiv q_t\, a_0^{I_t}(t) \qquad \text{(AI.5)}$$

Taking the first two terms of the expansion of $a_\ell^{I_t}(t)$ at $t = 4$, and integrating on x in (AI.3), we obtain

$$\operatorname{Im} f_0^{I_s}(s) = \alpha_{I_s} (-s)^{3/2} + \beta_{I_s} (-s)^{5/2} + O\left((-s)^{-7/2}\right) \qquad \text{(AI.6)}$$

with

$$\begin{cases} \alpha_0 = -\frac{1}{9}\left(a_0^2 + 5 a_2^2\right) \\ \alpha_2 = -\frac{1}{9}\left(a_0^2 + \frac{1}{2} a_2^2\right) \end{cases}$$

$$\begin{cases} \beta_0 = \frac{1}{360} \left\{ a_0^2 \left(7 + 6 a_0 (a_0 + r_0)\right) + 5 a_2^2 \left(7 + 6 a_2 (a_2 + r_2)\right) \right\} \\ \beta_2 = \frac{1}{360} \left\{ a_0^2 \left(7 + 6 a_0 (a_0 + r_0)\right) + \frac{1}{2} a_2^2 \left(7 + 6 a_2 (a_2 + r_2)\right) \right\} \end{cases} \qquad \text{(AI.7)}$$

It is easy to see that, for the P wave, the $(-s)^{\frac{3}{2}}$ term is given by the formula

$$\operatorname{Im} f_1^1(s) = \frac{1}{18} \left[2 a_0^2 - 5 a_2^2 \right] (-s)^{3/2} + O(-s)^{5/2} \qquad \text{(AI.8)}$$

APPENDIX AII

S WAVE PARAMETERS AS FUNCTIONS OF THE PHYSICAL PARAMETERS

Here we derive the equations which allow to compute the ten constants A_I, B_I, C_I, h_I and g_I which appear in Eq. (IV.2) as functions of the six more physical quantities a_I, z_I and $f^I(0)$. Using $K^I(s)$ as defined in Eq. (IV.1) and parametrized by Eq. (IV.2), and a_I and r_I as defined by Eq. (AI.4) we have from threshold behaviour

$$\frac{1}{2a_I} = K^I(4) \tag{AII.1}$$

$$r_I = \frac{1}{a_I} + \frac{4}{\pi} + 16\, K^{I\prime}(4) \tag{AII.2}$$

(In all the following, symbol ' means d/ds.) Now, near $s=0$, from Eqs. (IV.1)-(IV.3),

$$\mathrm{Im}\left[f^I(s)\right]^{-1} = -\frac{\mathrm{Im}\, f^I}{|f^I|^2} = h_I(-s)^{3/2} + (g_I + C_I h_I)(-s)^{5/2} + \dots$$

and by comparison with Eq. (IV.4) we obtain

$$h_I = -\alpha_I\, f^I(0)^{-2} \tag{AII.3}$$

which, due to (AI.7), gives h_I as a function of a_I and $f^I(0)$, and

$$g_I + C_I h_I = -\beta_I\, f^I(0)^{-2} + 2\alpha_I\, f^I(0)^{-1}\left(f_0^I(0)^{-1}\right)' \tag{AII.4}$$

Note that in this equation, both g_I's are involved in the right-hand side via β_I [Eq. (AI.7)] and $[f^{I-1}(0)]'$. Apart from the trivial relation

$$C_I = -1/z_I \tag{AII.5}$$

the last equation is given by the value at $s=0$, namely

$$A_I + h_I J(0) + g_I W(0) + \frac{1}{\pi} = f^I(0)^{-1} \tag{AII.6}$$

The system of 2×6 equations [(AII.1)-(AII.6)] is a linear system for the 2×6 constants A_I, B_I, C_I, h_I, g_I and r_I, the solution of which yields the needed result. Practically, Eq. (AII.3) directly gives h_o and h_2, Eq. (AII.5) gives C_o and C_2. A_I and B_I can then be computed as functions of g_o, g_2 and the physical parameters through equations (AII.1) and (AII.6), whereas r_I is given by Eq. (AII.2). The expressions are put in Eq. (AII.4), a coupled linear system of two equations which yields g_o and g_2. The final result is too lengthy to be given here. We prefer to give the numerical values of $A_I B_I C_I h_I g_I$ for the "central" solution given in Section III, in order to allow easier reconstruction of the corresponding S waves.

Solutions / Parameters	Curve (a) Fig.5	Curve (b) Fig.5	Curve (a) Fig.7	Curve (b) Fig.7	Curve (a) Fig.8	Curve (b) Fig.8
A_o	-8.1925	-5.2684	-6.7080	-5.9466	-6.9362	-7.0408
B_o	0.6251	0.42424	0.58715	0.32544	0.59134	0.50178
C_o	-0.86957	-0.57143	-0.75188	-0.57143	-0.75188	-0.75188
A_2	9.2729	13.212	9.5976	15.644	9.8505	10.515
B_2	0.90253	1.2564	0.94477	1.4322	0.99891	0.97098
C_2	-0.689	-0.9916	-0.75507	-1.0095	-0.75611	-0.7599
S_o	1.5	1.5	1.25	2.25	1.75	1.75
SR	15	15	14	18	15	15
T_o $(GeV)^2$	0.8	0.8	0.8	0.8	0.6	1
Associated constants						
m_ρ (MeV)	766.7	766.7	766.9	766.1	766.7	766.7
Γ_ρ (MeV)	135	135	134.5	136.4	135	135
h_o	0.48435	0.18079	0.37675	0.15485	0.37033	0.32825
g_o	-0.28185	-0.050267	-0.15621	-0.057881	-0.1557	-0.15849
h_2	0.53735	0.73517	0.60582	0.66381	0.59699	0.56394
g_2	-0.37228	-0.64735	-0.43792	-0.60858	-0.43262	-0.41625

TABLE I

Values of constants needed for reconstructing solutions labelled on the various figures. (Parametrization is given in Section IV.)

Solution	$10^2 S_0$	$10^2\ S_2$	$10^2 P_1$	$10^2 F_0$	$10^2 As$	$10^2\ L$	$10^2\ R$
I	7.593	-1.484	3.859	8.891	2.545	13.38	13.40
II	6.019	-1.489	3.859	8.891	3.286	12.53	12.56

TABLE II

Results of dispersion relation (II.16), with notations of Section III.3, for two solutions differing by T_0. For solution I, $T_0 =$ $= 0.6\ GeV^2$ and for solution II, $T_0 = 1\ GeV^2$. The variation of As just compensate the variation of $R-S_0-S_2$, so that $L = R$ in both cases within the same accuracy.

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FIGURE CAPTIONS

Figure 1 : Tentative drawing by hand of $f^{00}(s)$ due to inequalities. Scale and origin are arbitrary.

Figure 2 : a) Fit to P wave phase shifts from Eq. (IV.5) for $SR = 15$. Fits for SR between 14 and 18 are practically indistinguishable. Experimental data are elastic results of Ref. 2).

b) I=0 and I=2 S wave phase shifts for our central solution of Section III. Experimental data :

I elastic results from Ref. 2);

Ɨ Ref. 1c).

Figure 3 : a) Domains $\mathcal{D}'_2$ and $\mathcal{D}''_2$ defined in Section V.1 for $SR = 15$ and $T_0 = 0.8\ GeV^2$.

b) Variation of $M \equiv \max(\Delta R/R)(s)$ $(4 \leq s \leq 18)$ versus SR, for $T_0 = 0.8\ GeV^2$, $z_0 = 1.3$, $s_0 = 1.7$. The minimum around $SR = 15$ shows that in this region unitarity of the P wave is the best possible.

Figure 4 : Unitarity of the P wave : comparison of $\mathrm{Re}\, f^1_u(s)$ (continuous line), corresponding to exact elastic unitarity for P wave phase shifts of Fig. 2a, and of $\mathrm{Re}\, f^1(s)$ (dashed line) for our central solution of Section III. Below $s = 20$, both curves cannot be distinguished.

Figure 5 : Bands of solutions for the I=0 and I=2 S waves phase shifts. They correspond to s_0 and z_0 varying in the $\mathcal{D}''_2$ domain, with $T_0 = 0.8\ GeV^2$ and $SR = 15$ both fixed. Experimental data are the same as for Fig. 2b.

Figure 6 : a) Zeros of the $I=0$ and $I=2$ S waves. Plot of z_2 versus z_o for our solutions inside $\mathcal{D}''_2$ ($T_o = 0.8\ GeV^2$ and $SR = 15$) (continuous line) and for the linear approximation of Eq. (III.15) (dashed line).

b) Scattering lengths a_o and a_2. Plot of a_2 versus a_o : the three areas are the images of the three domains $\mathcal{D}''_2$ corresponding to $T_o = 0.8\ GeV^2$ and $SR = 14$, 15 and 18, respectively. The P wave scattering length a_1 is quoted.

Figure 7 : Bands of solutions for the $I=0$ and $I=2$ S waves phase shifts. Curves a) and b) are the solutions for the central point of $\mathcal{D}''_2$ corresponding to $SR = 14$ and to $SR = 18$, respectively. $T_o = 0.8\ GeV^2$ is fixed. Experimental data are the same as for Fig. 2b.

Figure 8 : Bands of solutions for the $I=0$ and $I=2$ S wave phase shifts. Curves a) and b) are the solutions for the same point $s_o = 1.75$, $z_o = 1.33$ of $\mathcal{D}''_2$ corresponding to $T_o = 0.6\ GeV^2$ and to $T_o = 1\ GeV^2$, respectively. $SR = 15$ is fixed. Experimental data are the same as for Fig. 2b.

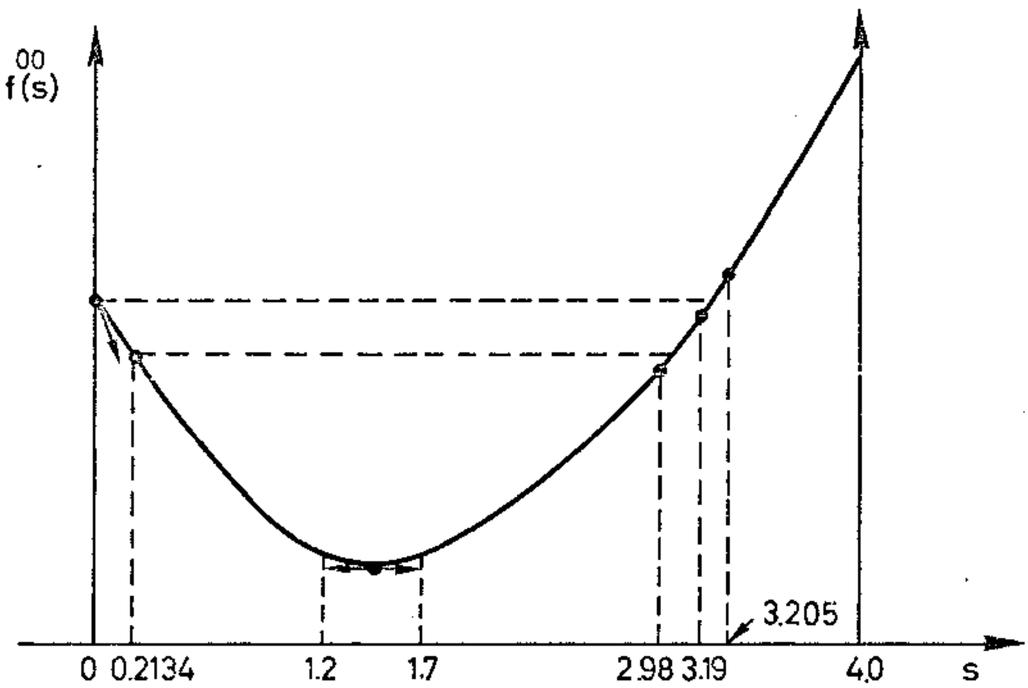

FIG.I

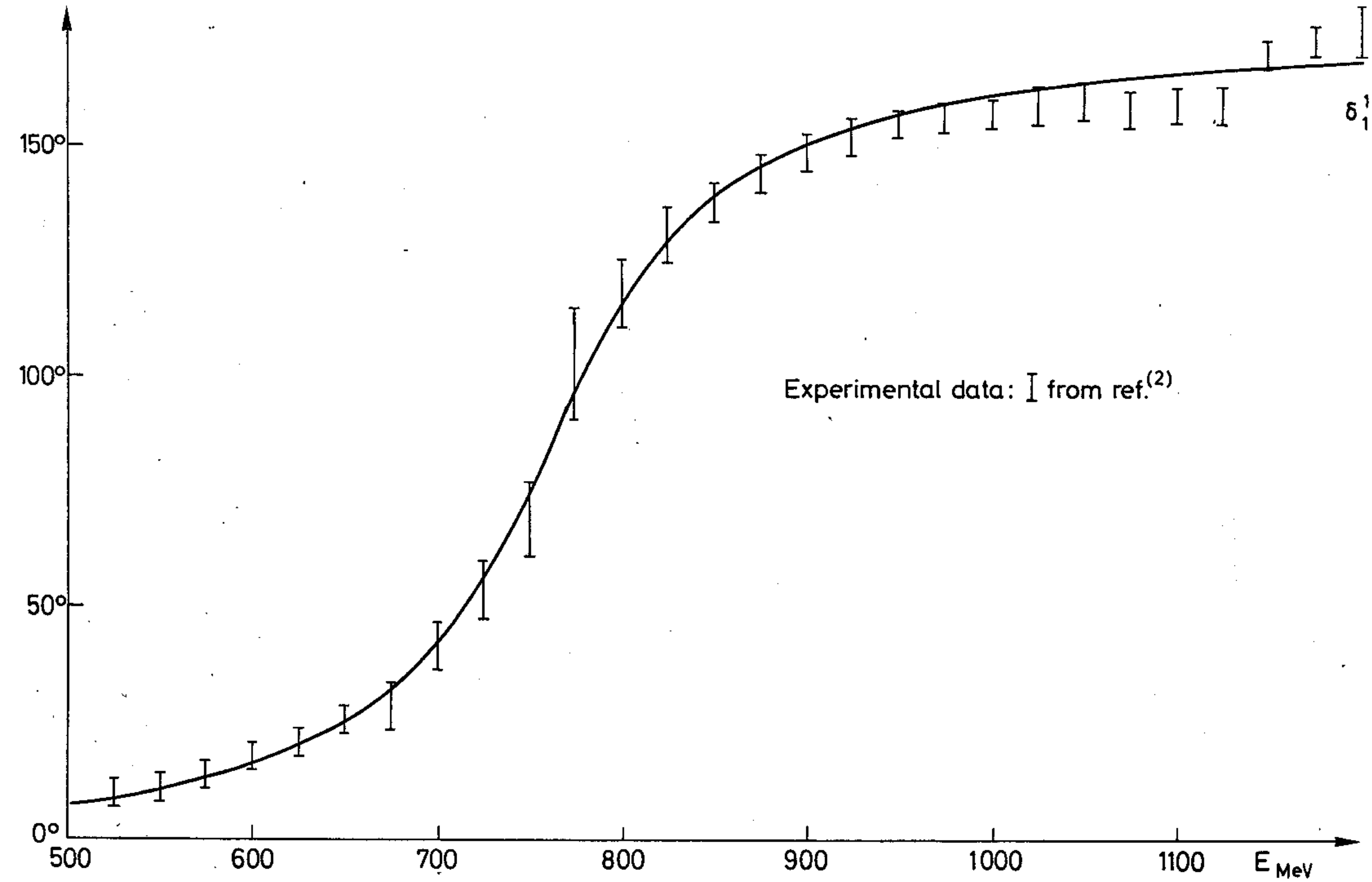

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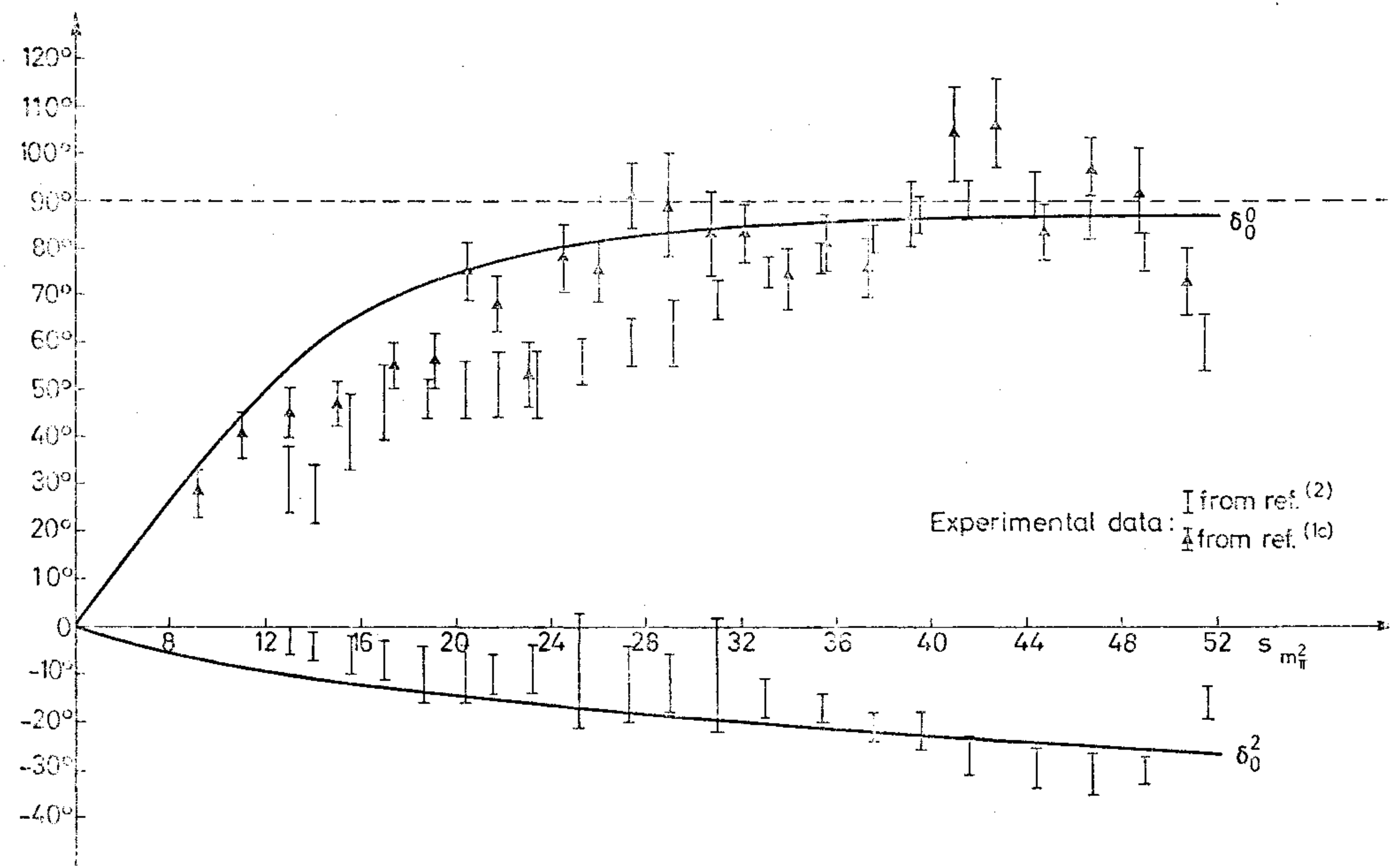
FIG II a)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Experimental data: I from ref. (2)
Experimental data: ⍋ from ref. (1c)

FIG. II b)

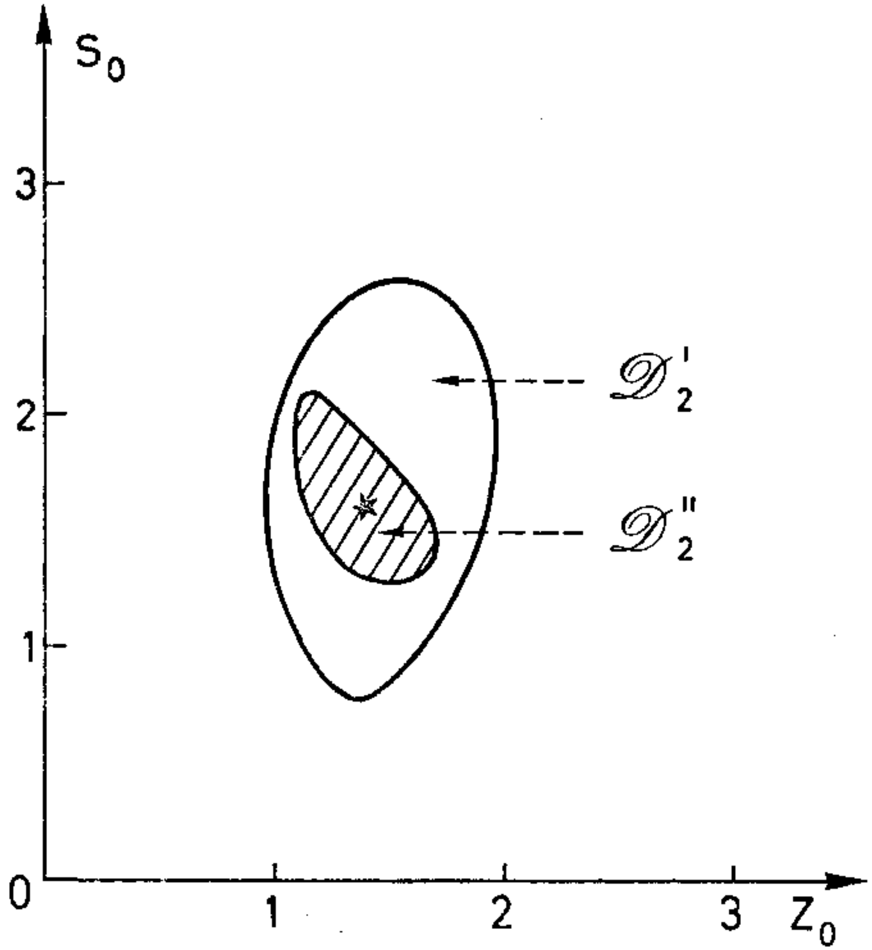


FIG. III a)

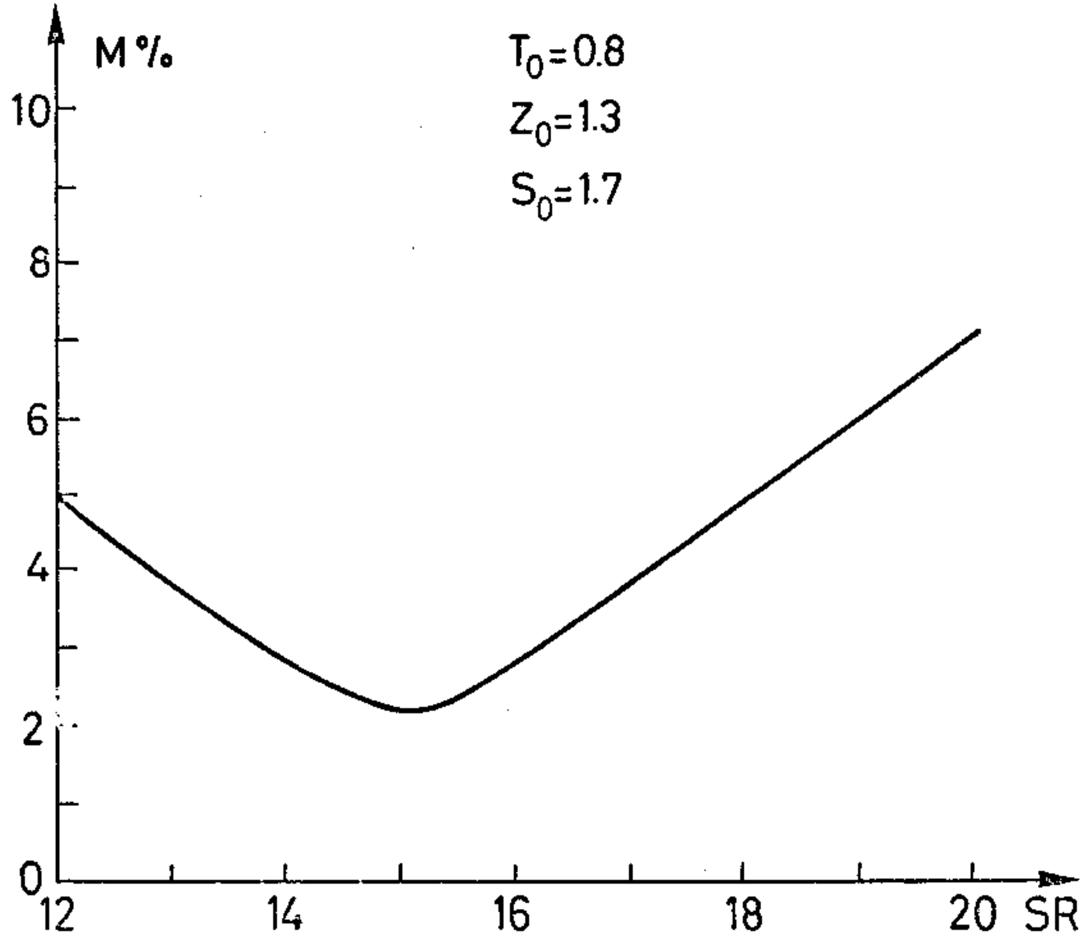


FIG. III b)

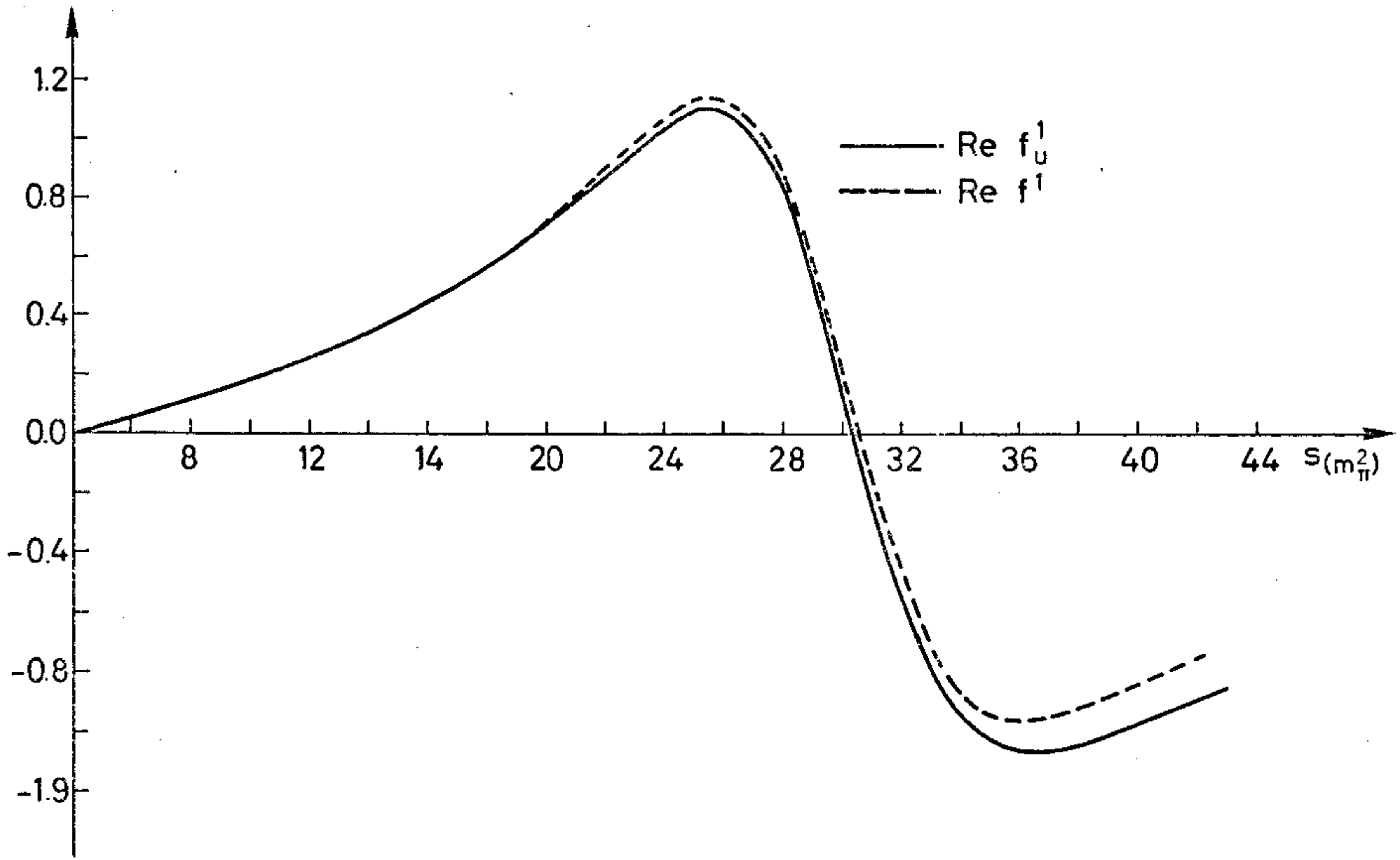
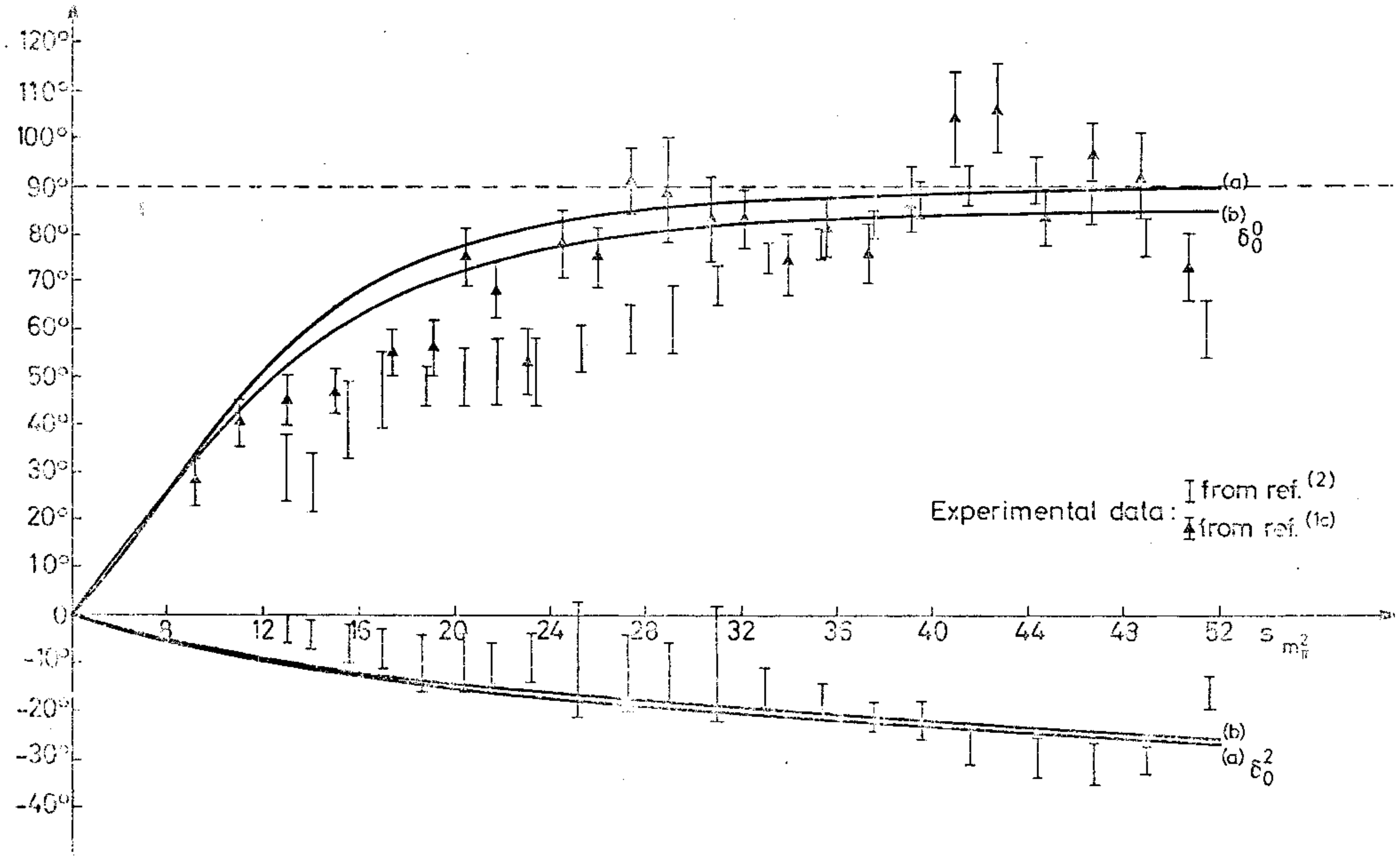


FIG. IV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$s \; m_\pi^2$
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Experimental data: from ref. (2)
from ref. (1c)

FIG. V

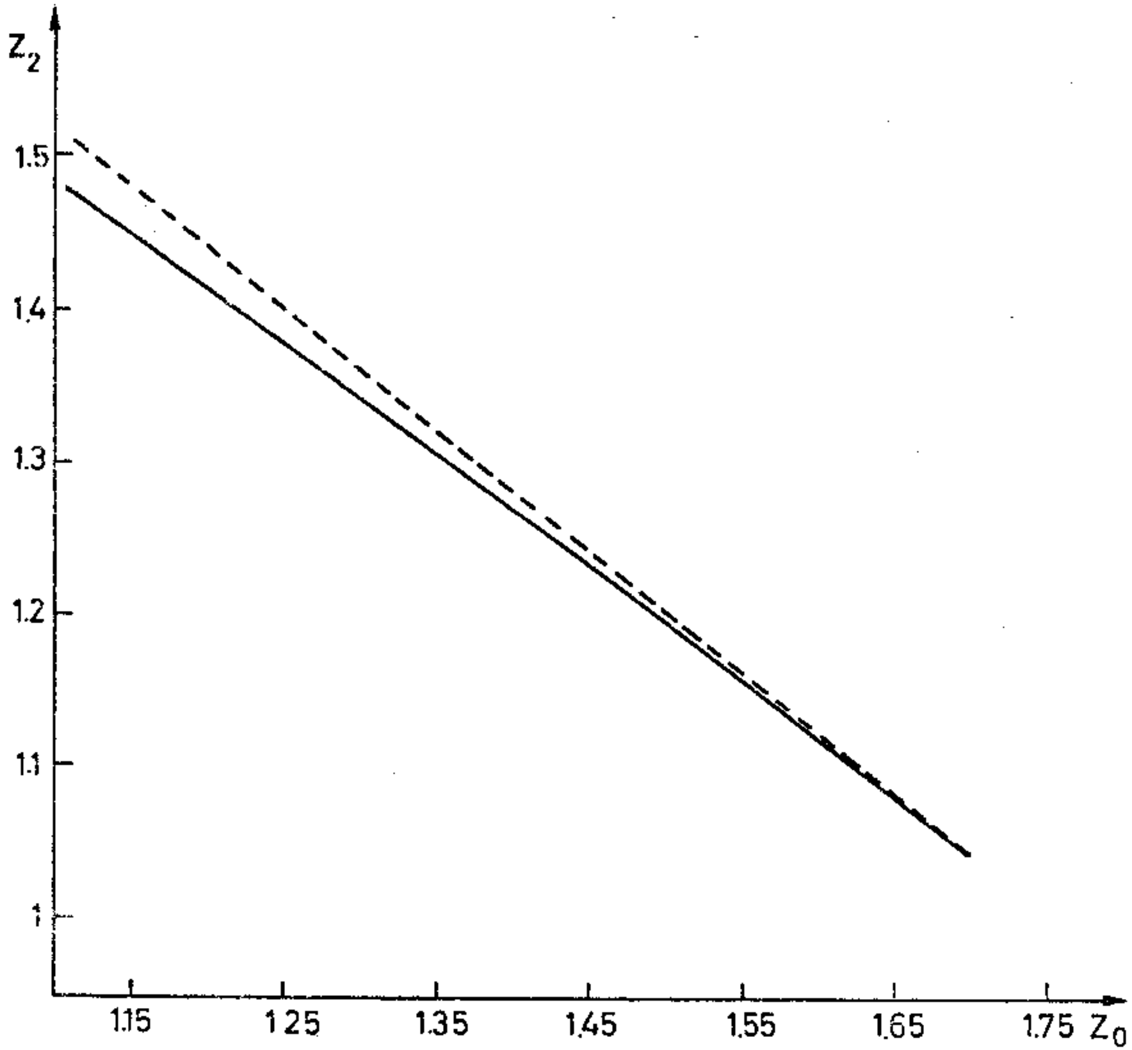


FIG. VI a)

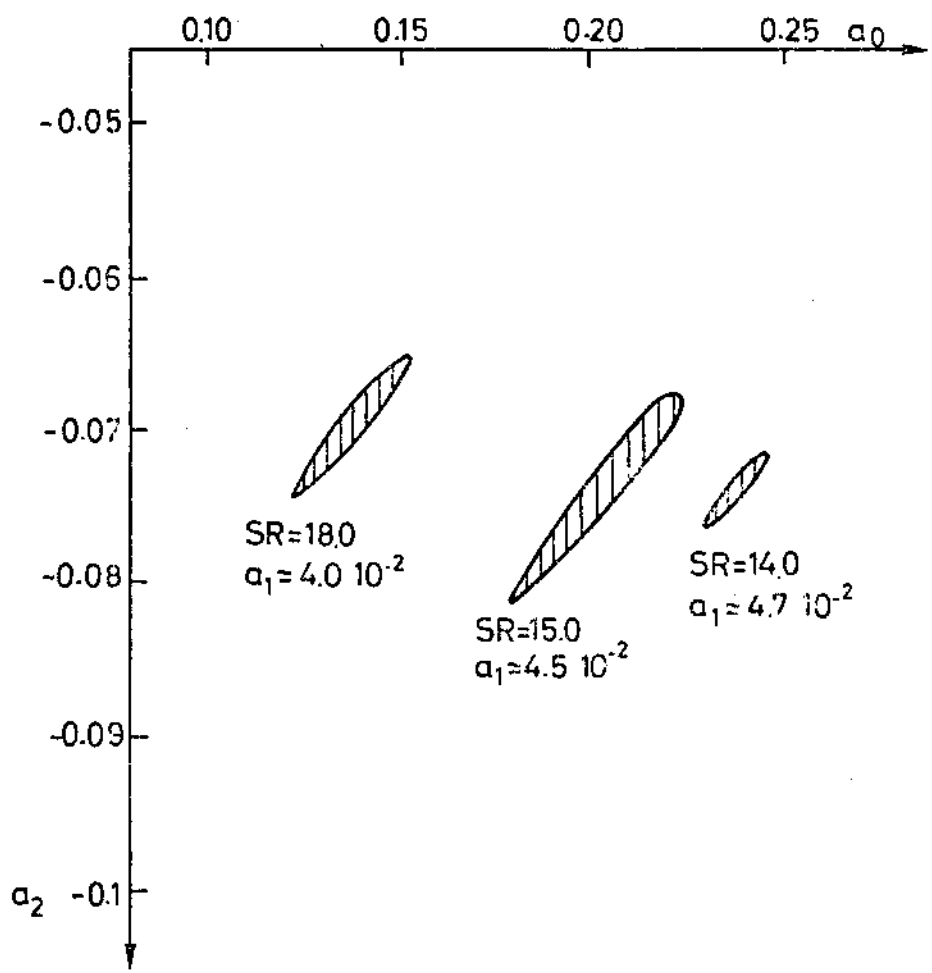
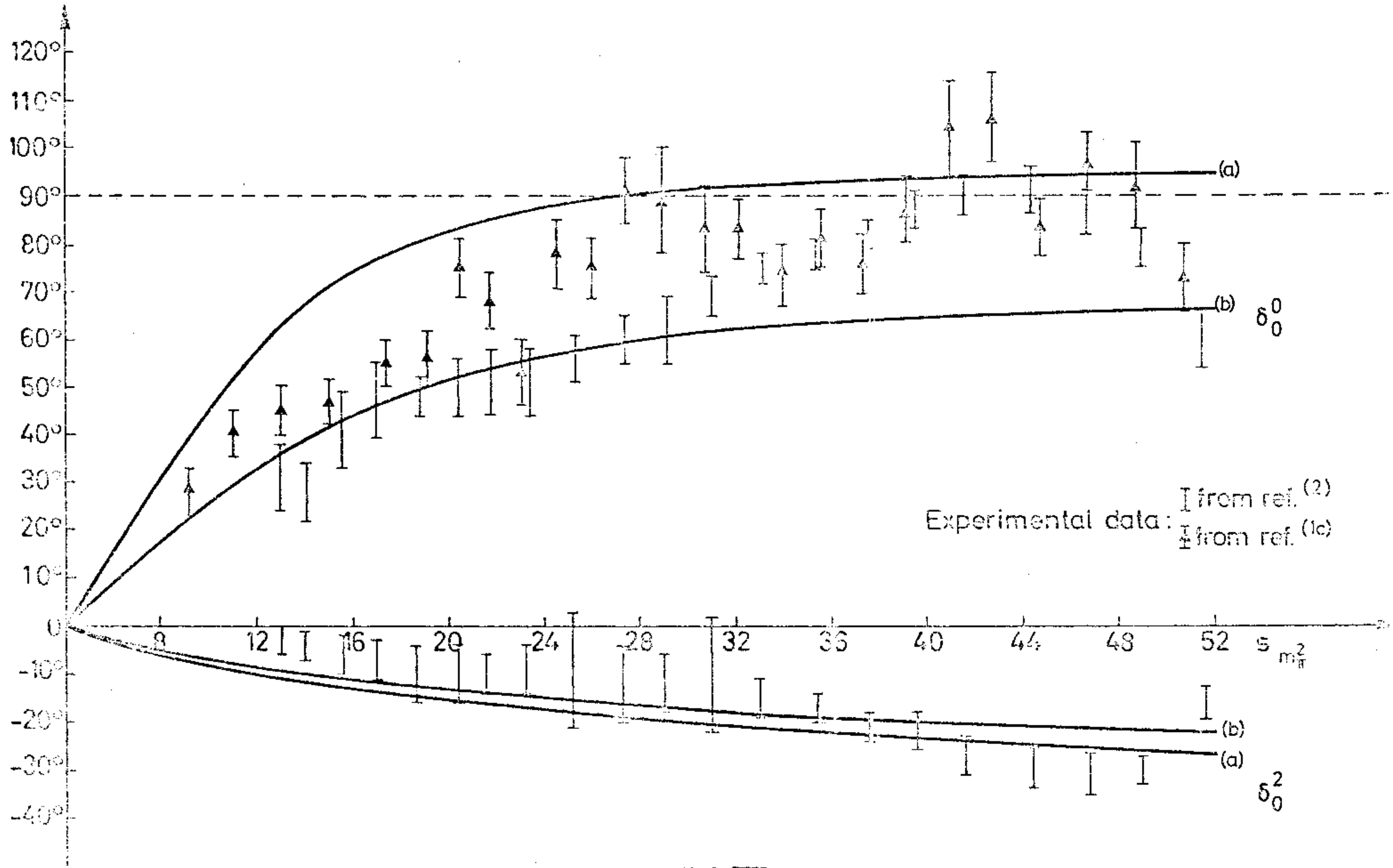


FIG. VI b)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s m_π^2
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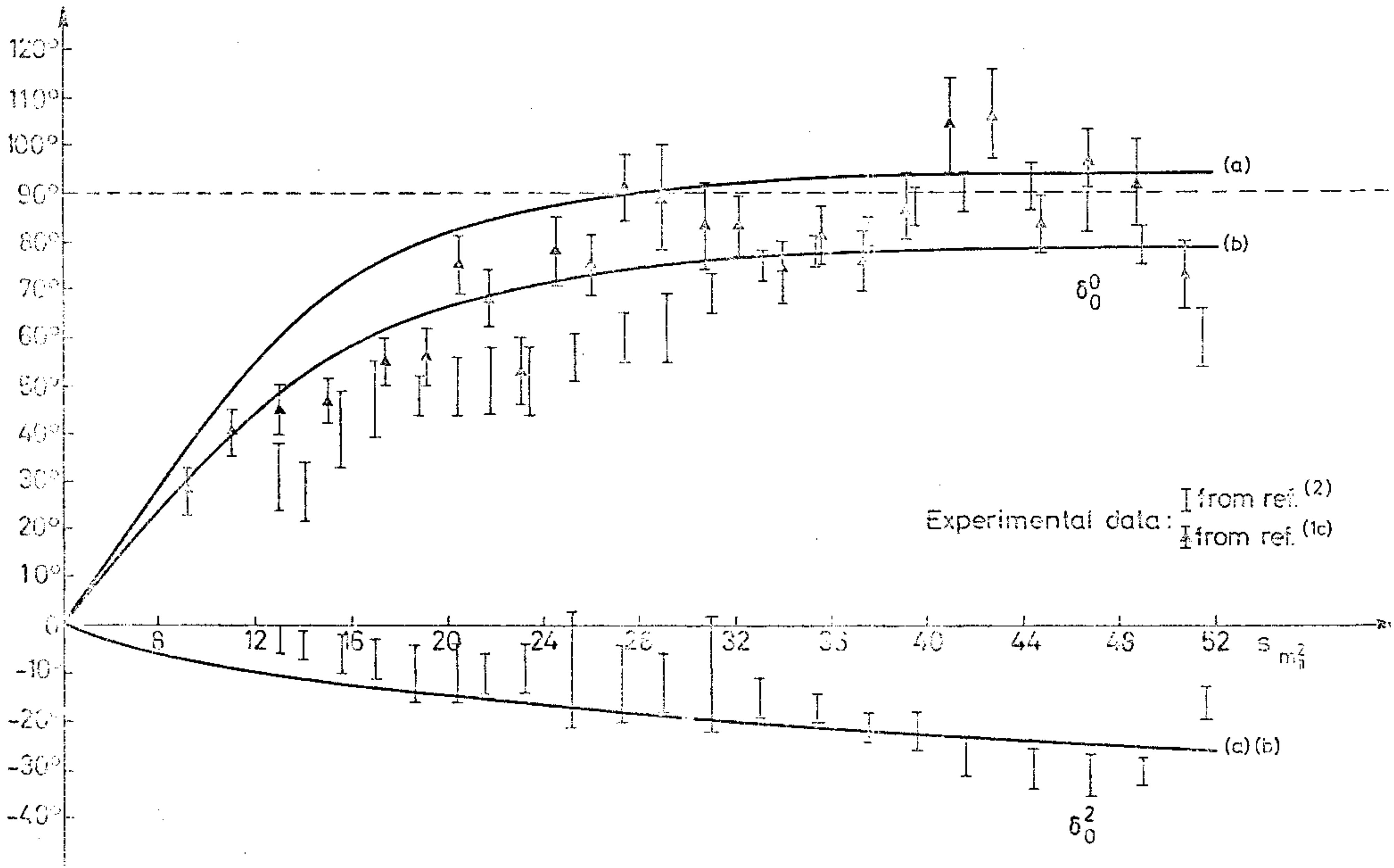
δ_0^2
Experimental data: from ref. (2)
from ref. (1c)

FIG. VII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Experimental data: from ref. (2)
from ref. (1c)

FIG. VIII